*Adv. Appl. Prob.* **53,** 107–132 (2021)
*doi:*10.1017/apr.2020.49

# LOAD-SHARING RELIABILITY MODELS WITH DIFFERENT COMPONENT SENSITIVITIES TO OTHER COMPONENTS' WORKING STATES

TOMASZ RYCHLIK,* *Polish Academy of Sciences*
FABIO SPIZZICHINO,** *Sapienza University of Rome*

## Abstract

We study the distributions of component and system lifetimes under the time-homogeneous load-sharing model, where the multivariate conditional hazard rates of working components depend only on the set of failed components, and not on their failure moments or the time elapsed from the start of system operation. Then we analyze its time-heterogeneous extension, in which the distributions of consecutive failure times, single component lifetimes, and system lifetimes coincide with mixtures of distributions of generalized order statistics. Finally we focus on some specific forms of the time-nonhomogeneous load-sharing model.

*Keywords:* Multivariate conditional hazard rate; linear breakdown model; Markov property of failure process; decomposition of reliability function; generalized order statistics; exchangeability; uniform supporting ability; uniform frailty

2010 Mathematics Subject Classification: Primary 62N05
Secondary 62G30

## 1. Introduction

We consider a coherent system composed of $n$ components with a structure function $\varphi : \{0, 1\}^n \mapsto \{0, 1\}$. Here 0 and 1 represent failure and operation, respectively, of a component or of the system. The components are labeled by the numbers from 1 to $n$, and the value of the $k$th coordinate of the structure function $\varphi$ represents the working status of the $k$th component. The value of $\varphi$ at a given sequence of zeros and ones shows whether the system is working when the elements of a fixed subset of components are working and the others are not. We say that the system is *coherent* if its structure function is nondecreasing and it does not contain redundant components. The first condition means that the failure of a component cannot improve the system state. The second implies that every component affects the operation of the system; i.e., the failure of each component implies the failure of the system for some working states of the other components. This in particular implies $\varphi(0, \ldots, 0) = 0$ and $\varphi(1, \ldots, 1) = 1$.

Received 16 April 2019; revision received 22 June 2020.

* Postal address: Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00 656 Warsaw, Poland. Email address: trychlik@impan.pl
** Postal address: Sapienza University of Rome, Rome, Italy. Email address: fabio.spizzichino@fondazione.uniroma1.it

It is convenient to represent the system structure function as a function of the subsets of the set $\{1, \ldots, n\}$, which is further denoted by $[n]$ for brevity. This is written as

$$\varphi(A) = \varphi(\mathbf{1}_A(1), \ldots, \mathbf{1}_A(n)), \qquad A \subset [n], \tag{1}$$

where $\mathbf{1}_A(k)$ denotes the indicator function of the set $A$, and equals 1 when $k$ belongs to $A$ and 0 when $k$ does not belong to $A$. In particular, $\varphi(\emptyset) = 0$ and $\varphi([n]) = 1$. Although the function (1) formally differs from the original function, because it is defined on a different domain, we do not introduce a different notation for it.

The component lifetimes $T_1, \ldots, T_n$ are nonnegative random variables all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume here that they are jointly absolutely continuous. The lifetime of every component depends both on its individual durability and on the states of the other components cooperating in the system, because the failure of some components increases the load acting on the still working components. The most intuitive notion for describing the tendency of the components in the system to fail is the multivariate conditional hazard rate, defined as

$$\lambda_j(t|A, t_1, \ldots, t_k) = \lim_{\triangle t \searrow 0} \frac{1}{\triangle t} \mathbb{P}(T_j \le t + \triangle t | T_i = t_i,\ i \in A,\ T_i > t,\ i \in A^c), \tag{2}$$

where $A$ is a subset of $[n]$ of cardinality $|A|$ for some $1 \le k < n$ such that $j \in A^c = [n] \setminus A$, and $0 < t_1, \ldots, t_k < t$. To be precise, this is the failure rate of the $j$th component at time $t$ under the condition that the components from the set $I$ have failed before $t$, at some prescribed time moments $t_1, \ldots, t_k$, and the remaining components are still working at $t$. By analogy, we can define the multivariate conditional hazard rates of the components under the condition that no components have failed by the given time $t$:

$$\lambda_j(t|\emptyset) = \lim_{\triangle t \searrow 0} \frac{1}{\triangle t} \mathbb{P}(T_j \le t + \triangle t | T_i > t,\ i \in [n]). \tag{3}$$

If we adopt the convention that $\emptyset \subset [n]$, the formula (3) becomes a special case of (2). It can be shown that the family of multivariate conditional hazard rates $\lambda_j(t|A, t_1, \ldots, t_k)$, $j \notin A \subset [n]$, $0 \le |A| = k \le n-1$, $0 < t_1, \ldots, t_k < t$, uniquely characterizes the joint distribution of $T_1, \ldots, T_n$. For more details on the concept of the multivariate conditional hazard rate see, e.g., [33], [34], [35], [40], and references cited therein.

Here we consider some special cases of models described by means of multivariate conditional hazard rates. We say that the component lifetimes are distributed according to the *load-sharing* model if

$$\lambda_j(t|A, t_1, \ldots, t_k) = \lambda_j(t|A);$$

i.e., the aging properties of a given component depend merely on its age and on the set $A$ of failed components at given $t$, but do not depend on the time distances from the failure times of the damaged components. A special case of the load-sharing model gives rise to the following *linear breakdown rule*, for which the multivariate conditional hazard rates have the forms

$$\lambda_j(t|A) = \lambda_j(A) L(t). \tag{4}$$

In this case the failure rates have two multiplicative factors, one depending on the running time, and the other representing the load acting on a living component owing to the failures of the components from the set $A$. The most natural example of (4) is

$$\lambda_j(t|A) = \frac{L(t)}{n - |A|}.$$

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

This condition reflects the situation that there is an overall load $L(t)$ acting on the whole system, possibly varying in time, and this load is uniformly distributed among the still living components. We further say that the load-sharing model is *time-homogeneous* if the conditional hazard rates are independent of time, and depend only on the failed components

$$\lambda_j(t|A) = \lambda_j(A), \qquad j \notin A \subsetneq [n].$$

Obviously, the time-homogeneous load-sharing model automatically results in a special case of the linear breakdown rule. We show below that it can be seen as a natural generalization of the conditions of independent and exponentially distributed inter-failure times. Furthermore, we prove that a simple transformation of the time-homogeneous load-sharing model fully characterizes the linear breakdown rule model with multivariate conditional hazard rate (4).

In different contexts, and possibly under a variety of terminologies, the interest in general load-sharing-type models dates back a long way; see in particular [13], [15], [16], [19], and [29]. Moreover, the term 'load-sharing' has been used with somewhat different meanings in the field of reliability; see, e.g., [14], [41], and [43]. The linear breakdown rule has been studied in [36]. The definition of multivariate conditional hazard rate functions, in the form recalled here, makes sense when the joint distribution of lifetimes is absolutely continuous. Thus load-sharing models that are defined in terms of the behavior of such functions are necessarily of absolutely continuous type. However, within the theory of point processes (cf., e.g., [7]), more general definitions of dynamic description of dependence can naturally be given, and corresponding models of load-sharing can be defined; see in particular [2], [3], and [26]. Recently, probabilistic models somehow related to the idea of load-sharing have been studied in [11]. The term 'load-sharing' emerges furthermore in many papers in engineering or physics, not strictly related to systems reliability; see [20], [27], [32], and [42], among others.

The study of reliability properties of coherent systems also has a very long tradition in the field of applied probability. However, in the recent past some new methods and ideas have been developed using the concept of the *signature*, introduced by Samaniego in [30] (see also [28]). The signature is especially useful under the assumption of exchangeability. Its application to cases of non-exchangeability is less direct. Further related references are listed in Section 4. Essentially, the concept of the signature paves the way to constructing, for an arbitrary coherent system, convenient decompositions of the reliability function as a convex combination of survival functions of order statistics.

Such an idea can be, in some way, extended to cases of non-exchangeability at the cost of considering more detailed decompositions. In [40], it is argued that such an approach can be conveniently employed under the assumption of time-homogeneous load-sharing for the component lifetimes, and some aspects concerning the computation of system reliability are presented therein.

In the present paper, we aim to detail some more probabilistic aspects of the time-homogeneous case and to extend results to nonhomogeneous cases. We also specifically analyze some relevant special cases for the function $L(t)$ and the parameters $\lambda_j(I)$. This treatment also offers the opportunity to establish connections with the theory of generalized order statistics.

More precisely, the plan of the paper is as follows. In Section 2 we investigate more thoroughly the time-homogeneous load-sharing model. In Section 3 we extend the study to nonhomogeneous cases. Section 4 is devoted to analysis of special cases of assumptions on the model parameters. One implies exchangeability of the component lifetimes, and the two other describe some generalizations of exchangeability. Finally, in Section 5 we specify the results for some parent risk distributions commonly used in reliability analysis, and we present a numerical example showing the benefits of properly allocating components in a system working under the load-sharing model.

## 2. Time-homogeneous load-sharing model

In order to distinguish this special case, we denote the component lifetimes by $S_1, \ldots, S_n$. The respective order statistics $S_{1:n}, \ldots, S_{n:n}$ coincide with the consecutive failure times of the system components. The joint distribution of $S_1, \ldots, S_n$ is described by the family of positive parameters

$$\mathcal{L} = \{\lambda_j(A) : A \subsetneq [n],\ j \in A^c\} \tag{5}$$

of cardinality $n2^{n-1}$. Indeed,

$$|\mathcal{L}| = \sum_{k=0}^{n-1} \binom{n}{k} (n-k) = n \sum_{k=0}^{n-1} \binom{n-1}{k} = n2^{n-1}.$$

Each $\lambda_j(A)$ represents the constant multivariate conditional failure rate,

$$\lambda_j(t|A, t_1, \ldots, t_k) = \lambda_j(A),$$

of the $j$th component under the condition that all the components from the set $A \subsetneq [n]$ have failed previously, while those from $A^c$ (which contains $j$) are still operating. The notation

$$\Lambda(A) = \sum_{j \in A^c} \lambda_j(A)$$

will be frequently used below.

Let $I_0, \ldots, I_n$ denote the set-valued variables such that $I_k$ is the label set of the first $k$ failed components. Certainly $I_0 = \emptyset$, $I_n = [n]$, $|I_k| = k$, and $|I_k \setminus I_{k-1}| = 1$. Our first claim is the following.

**Theorem 1.** *The sequence* $(S_{k:n}, I_k)$, $k = 1, \ldots, n$, *forms a two-dimensional finite Markov chain with the initial probability rule*

$$\mathbb{P}(S_{1:n} > s,\ I_1 = \{j_1\}) = \exp(-s\Lambda(\emptyset)) \frac{\lambda_{j_1}(\emptyset)}{\Lambda(\emptyset)}, \qquad s > 0,\ j_1 \in [n], \tag{6}$$

*and the transition laws*

$$\begin{aligned} &\mathbb{P}(S_{k:n} > s,\ I_k = A_{k-1} \cup \{j_k\} | S_{k-1:n} = s_{k-1},\ I_{k-1} = A_{k-1}) \\ &= \exp(-(s - s_{k-1})\Lambda(A_{k-1})) \frac{\lambda_{j_k}(A_{k-1})}{\Lambda(A_{k-1})}, \qquad s > s_{k-1},\ j_k \in A^c_{k-1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad |A_{k-1}| = k-1,\ k = 2, \ldots, n. \end{aligned}$$

*Proof.* When all the system components are working, the component lifetimes $S_1, \ldots, S_n$ have constant failure rates $\lambda_1(\emptyset), \ldots, \lambda_n(\emptyset)$, respectively. None of the $\lambda_k(\emptyset)$ depends in any way on the failure rates of the other components. This means that $S_1, \ldots, S_n$ are independent

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

exponential, with possibly different distributions determined by the respective failure rates. It is well known that $S_{1:n} = \min\{S_1, \dots, S_n\}$ is exponential with the failure rate

$$\Lambda(\emptyset) = \sum_{i=1}^{n} \lambda_i(\emptyset),$$

while

$$\mathbb{P}(S_i = S_{1:n}) = \frac{\lambda_i(\emptyset)}{\Lambda(\emptyset)}, \qquad i = 1, \dots, n,$$

and this holds independently of the value of $S_{1:n}$. This proves the formula (6). We may also refer to a more general result of De Santis *et al.* [12, Proposition 2], which asserts that for lifetime random variables $X_1, \dots, X_n$ with general multivariate conditional hazard rates (2) and (3) there exist independent random variables $Z_1, \dots, Z_n$ with failure rates $r_j(t) = \lambda_j(t|\emptyset)$, $j = 1, \dots, n$, such that the joint distributions of $(X_{1:n}, \mathbf{1}_{\{X_{1:n}=X_j\}})$ and $(Z_{1:n}, \mathbf{1}_{\{Z_{1:n}=Z_j\}})$, $j = 1, \dots, n$, are identical.

Assume now that the $(k-1)$th consecutive failure happened at time $s_{k-1}$, and components $j_1, \dots, j_{k-1} \in A_{k-1}$ had failed by then. From this moment on, each of the $n+1-k$ working components from the set $A^c_{k-1}$ get constant failure rates $\lambda_i(A_{k-1})$, $i \in A^c_{k-1}$. This means that, independently of the particular value of $S_{k-1:n} = s_{k-1} > 0$, the random variables $S_i - s_{k-1}$, $i \in A^c_{k-1}$, are independent exponential with corresponding failure rates $\lambda_i(A_{k-1})$, $i \in A^c_{k-1}$. It follows that their minimum $S_{k:n} - S_{k-1:n}$ is exponential with failure parameter $\Lambda(A_{k-1}) = \sum_{i \in A^c_{k-1}} \lambda_i(A_{k-1})$, and it is independent of the value of $S_{k-1:n} = s_{k-1}$. The minimum is achieved by $j_k \in A^c_{k-1}$ with probability

$$\begin{aligned}\mathbb{P}(I_k \setminus I_{k-1} = A_k \setminus A_{k-1} = \{j_k\} | S_{k-1:n} = s_{k-1},\ I_{k-1} = A_{k-1})\\ = \frac{\lambda_{j_k}(A_{k-1})}{\Lambda(A_{k-1})}, \qquad j_k \in A^c_{k-1}.\end{aligned}$$ □

**Remark 1.** Conditionally on $(S_{k-1:n}, I_{k-1})$, the variables $S_{k:n} - S_{k-1:n}$ and $I_k$ are independent, with distributions

$$\mathbb{P}(S_{k:n} > s | S_{k-1:n} = s_{k-1}, I_{k-1} = A_{k-1}) = \exp(-(s - s_{k-1})\Lambda(A_{k-1})),\ s > s_{k-1},$$

$$\mathbb{P}(I_k = A_{k-1} \cup \{j_k\} | S_{k-1:n} = s_{k-1}, I_{k-1} = A_{k-1}) = \frac{\lambda_{j_k}(A_{k-1})}{\Lambda(A_{k-1})}.$$

**Corollary 1.** *The joint density function of* $(S_{1:n}, \dots, S_{k:n}, I_1, \dots, I_k)$, $k = 1, \dots, n$, *with respect to the product of k-dimensional Lebesgue measure and k-dimensional counting measure is*

$$\begin{aligned}&f_{\mathcal{L}; S_{1:n}, \dots, S_{k:n}, I_1, \dots, I_k}(s_1, s_2, \dots, s_k, \{j_1\}, \{j_1, j_2\}, \dots, \{j_1, \dots, j_k\})\\ &= \prod_{i=1}^{k} \lambda_{j_i}(\{j_1, \dots, j_{i-1}\}) \exp\left(-\sum_{i=1}^{k} (s_i - s_{i-1})\Lambda(\{j_1, \dots, j_{i-1}\})\right),\\ &s_0 = 0 < s_1 < \dots < s_k, \quad \{j_1, \dots, j_k\} \subset [n].\end{aligned}$$

**Corollary 2.** *The sequence* $I_k$, $k = 1, \dots, n$, *is a Markov chain with initial probability*

$$\mathbb{P}(I_1 = \{j_1\}) = \frac{\lambda_{j_1}(\emptyset)}{\Lambda(\emptyset)}$$

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

*and transition probabilities*

$$\mathbb{P}(I_k = I_{k-1} \cup \{j_k\} | I_{k-1} = A_{k-1}) = \frac{\lambda_{j_k}(A_{k-1})}{\Lambda(A_{k-1})}, \qquad k = 2, \ldots, n.$$

*Consequently,*

$$\mathbb{P}(I_1 = \{j_1\}, \ldots, I_k = \{j_1, j_2, \ldots, j_k\}) = \prod_{i=1}^{k} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})},$$

*and*

$$\mathbb{P}(I_k = A_k) = \sum_{\sigma(A_k)} \prod_{i=1}^{k} \frac{\lambda_{\sigma_i(A_k)}(\{\sigma_1(A_k), \sigma_2(A_k), \ldots, \sigma_{i-1}(A_k)\})}{\Lambda(\{\sigma_1(A_k), \sigma_2(A_k), \ldots, \sigma_{i-1}(A_k)\})},$$

*where $\sigma(A_k)$ denotes a permutation of the elements of set $A_k$, and $\sigma_i(A_k)$ stands for its ith coordinate.*

**Remark 2.** Without the imposition of restrictive assumptions on the set of parameters (5), the sequence $S_{k:n}$, $k = 1, \ldots, n$, itself is not a Markov chain (see Section 4 below, where specific restrictions on the parameters guarantee Markovianity of consecutive component failures as well as exchangeability of component lifetimes).

It is sometimes convenient to replace a sequence of set-valued random variables $I_1, \ldots, I_n$ by an equivalent sequence of integer-valued variables $J_1, \ldots, J_n$ defined as $J_i = j_i$ if and only if $j_i \in I_i \setminus I_{i-1}$, with the obvious convention $I_0 = \emptyset$.

**Corollary 3.** *Let $V_1, \ldots, V_k$ for some $1 \le k \le n$ denote independent standard exponential random variables with the common mean* 1. *Then*

$$\begin{aligned}
&(S_{1:n}, S_{2:n}, \ldots, S_{k:n} | I_1 = \{j_1\}, I_2 = \{j_1, j_2\} \ldots, I_{k-1} = \{j_1, j_2, \ldots, j_{k-1}\}) \\
&\stackrel{\mathrm{d}}{=} (S_{1:n}, S_{2:n}, \ldots, S_{k:n} | J_1 = j_1, J_2 = j_2 \ldots, J_{k-1} = j_{k-1}) \\
&\stackrel{\mathrm{d}}{=} \left( \frac{V_1}{\Lambda(\emptyset)}, \frac{V_1}{\Lambda(\emptyset)} + \frac{V_2}{\Lambda(\{j_1\})}, \ldots, \frac{V_1}{\Lambda(\emptyset)} + \frac{V_2}{\Lambda(\{j_1\})} + \ldots + \frac{V_k}{\Lambda(\{j_1, \ldots, j_{k-1}\})} \right),
\end{aligned} \tag{7}$$

*and, in particular,*

$$\begin{aligned}
&(S_{k:n} | I_1 = \{j_1\}, I_2 = \{j_1, j_2\} \ldots, I_{k-1} = \{j_1, j_2, \ldots, j_{k-1}\}) \\
&\stackrel{\mathrm{d}}{=} (S_{k:n} | J_1 = j_1, J_2 = j_2 \ldots, J_{k-1} = j_{k-1}) \\
&\stackrel{\mathrm{d}}{=} \frac{V_1}{\Lambda(\emptyset)} + \frac{V_2}{\Lambda(\{j_1\})} + \ldots + \frac{V_k}{\Lambda(\{j_1, \ldots, j_{k-1}\})}.
\end{aligned} \tag{8}$$

In other words, the conditional distribution of $S_{k:n}$ is identical with the convolution of independent exponential random variables with intensity parameters $\Lambda(\emptyset), \Lambda(I_1), \ldots, \Lambda(I_k)$. The respective density functions can be easily calculated for fixed parameters $\Lambda(\emptyset), \Lambda(\{j_1\}), \ldots, \Lambda(\{j_1, \ldots, j_{k-1}\})$, but their particular forms essentially depend on the multiplicities of the parameters, and so we do not present them here. For brevity of notation we denote them by $f_{\mathcal{L};S_{k:n}}(s|j_1, \ldots, j_{k-1})$. Similarly, we use the notation

$$f_{\mathcal{L};S_{1:n},\ldots,S_{k:n}}(s_1, \ldots, s_k | j_1, \ldots, j_{k-1})$$

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

for the joint conditional density of the first $k$ component failure times $S_{1:n}, S_{2:n}, \ldots, S_{k:n}$. The conditional dependence between $S_{i:n}$ and $S_{j:n}$, expressed by means of (7), lies in the fact that both of them depend on the same $V_m$ for $1 \le m \le \min\{i, j\}$. The formula (7) can be used for calculating the joint marginal density function of several arbitrarily selected order statistics.

**Proposition 1.** *The joint unconditional density function of $S_{1:n}, S_{2:n}, \ldots, S_{k:n}$ has the form*

$$f_{\mathcal{L};S_{1:n},\ldots,S_{k:n}}(s_1, \ldots, s_k) = \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n])} f_{\mathcal{L};S_{1:n},\ldots,S_{k:n}}(s_1, \ldots, s_k|j_1, \ldots, j_{k-1}) \times \prod_{i=1}^{k-1} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})}.$$

*Similarly, the unconditional density of a single $S_{k:n}$ is*

$$f_{\mathcal{L};S_{k:n}}(s) = \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n])} f_{\mathcal{L};S_{k:n}}(s|j_1, \ldots, j_{k-1}) \prod_{i=1}^{k-1} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})}. \tag{9}$$

*Accordingly,*

$$\mathbb{E}S_{k:n} = \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n])} \left(\sum_{i=1}^{k} \frac{1}{\Lambda(\{j_1, \ldots, j_{i-1}\})}\right) \prod_{i=1}^{k-1} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})}. \tag{10}$$

The notation $(j_1, \ldots, j_{k-1}) \in \Pi([n])$ means that the summations in the above formulae run over all $(k-1)$-permutations of the set $[n]$, i.e., the vectors of length $k-1$ built from the different elements in $[n]$. We adhere to this notation further on. In this notation we do not specify the length of the vectors in $\Pi([n])$, because it always follows from the context. In other words, the distribution of the $k$th component failure of the system working in the time-homogeneous load-sharing model is the mixture of $k$-fold convolutions of independent exponential distributions. As shown in the next proposition, the same holds for the distributions of the system and all its components.

**Proposition 2.** *The density function of the ith component lifetime $S_i$ is the following convex combination of the convolutions of exponential random variables:*

$$f_{\mathcal{L};S_i}(s) = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n]\setminus\{i\})} f_{\mathcal{L};S_{k:n}}(s|j_1, \ldots, j_{k-1}) \prod_{i=1}^{k-1} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})} \times \frac{\lambda_i(\{j_1, j_2, \ldots, j_{k-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{k-1}\})}. \tag{11}$$

*Moreover,*

$$\mathbb{E}S_i = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n]\setminus\{i\})} \left(\sum_{i=1}^{k} \frac{1}{\Lambda(\{j_1, \ldots, j_{i-1}\})}\right) \prod_{i=1}^{k-1} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})} \times \frac{\lambda_i(\{j_1, j_2, \ldots, j_{k-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{k-1}\})}. \tag{12}$$

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

The difference between the expressions (9)–(10) and (11)–(12) lies in that the latter contain one summation more; the second summations in (11)–(12) are performed over the sequences of length $k-1$ of elements of $[n]$ other than $i$, and each mixing coefficient in (11) and (12) has one more factor

$$\frac{\lambda_i(\{j_1, j_2, \ldots, j_{k-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{k-1}\})}$$

than the corresponding terms in the first sums of (9)–(10). To justify (11) we note that the density function of the $i$th component lifetime is identical with that of the $k$th consecutive failure under the condition that $(j_1, \ldots, j_{k-1}, i)$ is the sequence of consecutively failed components for arbitrary $(j_1, \ldots, j_{k-1}) \in \Pi([n] \setminus \{i\})$ and for some $k = 1, \ldots, n$. The probability of the above sequence of failures is

$$\prod_{i=1}^{k-1} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})} \frac{\lambda_i(\{j_1, j_2, \ldots, j_{k-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{k-1}\})}.$$

Hence the representation (11) follows from the law of total probability, and (12) is its consequence.

A similar construction can be accomplished for the system lifetime. For this purpose we first introduce the function

$$\chi(j_1, \ldots, j_n) = \sum_{k=1}^{n} k\, [\varphi(\{j_1, \ldots, j_k\}) - \varphi(\{j_1, \ldots, j_{k-1}\})]$$

acting on all the permutations of the set $[n]$. If $(j_1, \ldots, j_n)$ represents the sequence of subsequently failing components, then $\chi(j_1, \ldots, j_n)$ is just the number of consecutively failed components that causes the failure of the system.

**Proposition 3.** *The lifetime $S$ of the system with structure $\varphi$ and component lifetimes satisfying the assumptions of the time-homogeneous load-sharing model with parameters* (5) *has the density function*

$$f_{\mathcal{L};S}(s) = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} f_{\mathcal{L};S_{k:n}}(s|j_1, \ldots, j_{k-1}) \prod_{i=1}^{n-1} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})} \tag{13}$$

*and expectation*

$$\mathbb{E}S = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} \left( \sum_{i=1}^{k} \frac{1}{\Lambda(\{j_1, \ldots, j_{i-1}\})} \right) \prod_{i=1}^{n-1} \frac{\lambda_{j_i}(\{j_1, j_2, \ldots, j_{i-1}\})}{\Lambda(\{j_1, j_2, \ldots, j_{i-1}\})}. \tag{14}$$

The system lifetime density function coincides with that of the $k$th consecutive failure in the sequence of all consecutively failed components $(j_1, \ldots, j_n)$ if and only if $\chi(j_1, \ldots, j_n) = k$, i.e., for this sequence of component failures the system fails at the $k$th consecutive failure. The probability of the particular failure sequence $(j_1, \ldots, j_n)$ is represented by the product in (13), and (13) itself results from the total probability rule.

**Corollary 4.** *The density functions of $k$-out-of-$n$ systems are given by* (9)*, and their expectations are given by* (10).

**Remark 3.** In many cases it is natural in practice to assume that

$$\lambda_j(\{j_1, \ldots, j_{k-1}\}) \leq \lambda_j(\{j_1, \ldots, j_{k-1}, j_k\}), \quad j \in \{j_1, \ldots, j_{k-1}, j_k\}^c, \; k = 1, \ldots, n-1, \tag{15}$$

which means that a failure of any component $j_k \in \{j_1, \ldots, j_{k-1}\}^c$ implies that the conditional failure rates of all living components $j \in \{j_1, \ldots, j_{k-1}, j_k\}^c$ increase (or at least do not decrease) with the failure of any component $j_k \in \{j_1, \ldots, j_{k-1}\}^c$, $j_k \neq j$. Moreover, it is practically justified to have similar relations for the cumulative hazard rates

$$\Lambda(\emptyset) \leq \Lambda(\{j_1\}) \leq \ldots \leq \Lambda(\{j_1, \ldots, j_{n-1}\}). \tag{16}$$

We get the classic load-sharing model if $\Lambda(\emptyset) = \ldots = \Lambda(\{j_1, \ldots, j_{n-1}\})$, which means that there is a constant load acting on the system, and this is shared by its components. Then the conditional distribution of each $S_{k:n}$ is the Erlang (gamma) distribution with shape parameter $k$. However, it is more realistic to assume that the aggregate hazard rate does not depend linearly on the burden that the system undergoes, but rather increases as the number of failed components increases. The conditions $\Lambda(\emptyset) < \ldots < \Lambda(\{j_1, \ldots, j_{n-1}\})$ imply that the conditional distribution of each $S_{k:n}$ described in (8) is a linear combination of $k$ exponential distributions with different scales and some possibly negative coefficients summing to 1. In the most general case this is a linear combination of Erlang distributions with possibly different shapes (see, e.g., Smaili *et al.* [37, Theorem 1]).

## 3. Time-heterogeneous load-sharing model

The model presented above has a number of convincing motivations. On one hand, it realistically depicts the situation that the system components have different strengths, and these also depend on the circumstances under which they are working: the failures of some components may not necessarily imply the failure of the system, yet may make the still living components work harder, because of the increased burden on the whole system. On the other hand, it is assumed that the still working components have stable aging tendencies, expressed by constant failure rates. Clearly, the latter assumption is sometimes violated in practice. Below we present a generalization of the model given in Section 2 which admits variations of the inter-failure hazard rates in time, but still inherits some natural and useful properties of the time-homogeneous load-sharing model.

Let $F$ be an absolutely continuous lifetime distribution function strictly increasing on its support interval $[a, b] \subset [0, +\infty)$, with density function $f$. We say that the component lifetimes $T_1, \ldots, T_n$ satisfy the assumptions of the *time-heterogeneous load-sharing model*, with baseline distribution function $F$ and parameters (5), if the respective order statistics satisfy

$$\begin{aligned} &(T_{k:n} | (J_1, \ldots, J_{k-1}) = (j_1, \ldots, j_{k-1})) \\ &\overset{\mathrm{d}}{=} (F^{-1}(1 - \exp(-S_{k:n})) | (J_1, \ldots, J_{k-1}) = (j_1, \ldots, j_{k-1})), \quad k = 1, \ldots, n, \end{aligned} \tag{17}$$

with $S_{k:n}$, $k = 1, \ldots, n$, being the order statistics coming from the time-homogeneous load-sharing model with parameters (5), while

$$\mathbb{P}(T_i = T_{k:n} | (J_1, \ldots, J_{k-1}) = (j_1, \ldots, j_{k-1})) = \frac{\lambda_i(\{j_1, \ldots, j_{k-1}\})}{\Lambda(\{j_1, \ldots, j_{k-1}\})}, \quad i \in \{j_1, \ldots, j_{k-1}\}^c. \tag{18}$$

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

Actually, we could replace $T_{k:n}$ and $S_{k:n}$ in (17) by $T_i$ and $S_i$, respectively, but the formulae with order statistics are more convenient for our further calculations.

**Proposition 4.** *The sequence* $(T_{k:n}, I_k)$, $k = 1, \ldots, n$, *is a Markov chain with initial and transition probabilities*

$$\mathbb{P}(T_{1:n} > t,\ I_1 = \{j_1\}) = [1 - F(t)]^{\Lambda(\emptyset)} \frac{\lambda_{j_1}(\emptyset)}{\Lambda(\emptyset)}, \qquad a < t < b,\ j_1 \in [n], \tag{19}$$

*and*

$$\begin{aligned} &\mathbb{P}(T_{k:n} > t,\ I_k = A_{k-1} \cup \{j_k\} | T_{k-1:n} = t_{k-1},\ I_{k-1} = A_{k-1}) \\ &= \left[\frac{1 - F(t)}{1 - F(t_{k-1})}\right]^{\Lambda(A_{k-1})} \frac{\lambda_{j_k}(A_{k-1})}{\Lambda(A_{k-1})}, \qquad t_{k-1} < t < b,\ j_k \in A_{k-1}^c, \\ &\qquad\qquad |A_{k-1}| = k - 1,\ k = 2, \ldots, n, \end{aligned} \tag{20}$$

*respectively.*

*Proof.* Since $t \mapsto F^{-1}(1 - \exp(-t))$ is a one-to-one continuous strictly increasing transformation of $S_{k:n} > S_{k-1:n}$ onto $T_{k:n} \in (T_{k-1:n}, b)$, in view of the conditional independence of $S_{k:n}$ and $I_k$, it suffices to calculate

$$\begin{aligned} \mathbb{P}(T_{1:n} > t) &= \mathbb{P}(F^{-1}(1 - \exp(-S_{1:n}) > t) = \mathbb{P}(S_{1:n} > -\ln[1 - F(t)]) \\ &= \exp(\ln[1 - F(t)]\, \Lambda(\emptyset)) = [1 - F(t)]^{\Lambda(\emptyset)} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}(T_{k:n} > t | T_{k-1:n} = t_{k-1},\ I_{k-1} = A_{k-1}) \\ &\quad = \mathbb{P}(S_{k:n} > -\ln[1 - F(t)] | S_{k-1:n} = -\ln[1 - F(t_{k-1})],\ I_{k-1} = A_{k-1}) \\ &\quad = \exp(-\{-\ln[1 - F(t)] + \ln[1 - F(t_{k-1})]\}\, \Lambda(A_{k-1})) = \left[\frac{1 - F(t)}{1 - F(t_{k-1})}\right]^{\Lambda(A_{k-1})}. \end{aligned}$$

□

**Corollary 5.** *The joint density function of* $(T_{1:n}, \ldots, T_{k:n}, I_1, \ldots, I_k)$, $k = 1, \ldots, n$, *with respect to the product of k-dimensional Lebesgue measure on* $(a, b)^k$ *and k-dimensional counting measure on* $[n]^k$ *has the form*

$$\begin{aligned} &f_{\mathcal{L}F;T_{1:n},\ldots,T_{k:n},I_1,\ldots,I_k}(t_1, t_2, \ldots, t_k, \{j_1\}, \{j_1, j_2\}, \ldots, \{j_1, \ldots, j_k\}) \\ &= \prod_{i=1}^{k} \lambda_{j_i}(\{j_1, \ldots, j_{i-1}\}) f(t_i) \prod_{i=1}^{k-1} [1 - F(t_i)]^{\Lambda(\{j_1,\ldots,j_{i-1}\}) - \Lambda(\{j_1,\ldots,j_i\}) - 1} \\ &\quad \times [1 - F(t_k)]^{\Lambda(\{j_1,\ldots,j_{k-1}\}) - 1} \end{aligned} \tag{21}$$

*for* $a < t_1 < \ldots < t_k < b$ *and* $\{j_1, \ldots, j_k\} \subset [n]$.

**Proposition 5.** *Let* $V_1, \ldots, V_k$ *for some* $1 \le k \le n$ *denote independent standard exponential random variables with the common mean* 1. *Let* $U_1, \ldots, U_k$ *stand for independent standard*

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

*uniform random variables supported on* (0,1). *Then the conditional distribution of* $T_{k:n}$ *with respect to* $I_1, \ldots, I_{k-1}$ *has the representations*

$$\begin{aligned}
&(T_{k:n}|I_1=\{j_1\}, I_2=\{j_1,j_2\}\ldots, I_{k-1}=\{j_1,j_2,\ldots,j_{k-1}\}) \\
&\quad \stackrel{\mathrm{d}}{=} (T_{k:n}|J_1=j_1, J_2=j_2,\ldots,J_{k-1}=j_{k-1}) \\
&\quad \stackrel{\mathrm{d}}{=} F^{-1}\left(1-\exp\left(-\sum_{i=1}^{k}\frac{V_i}{\Lambda(\{j_1,\ldots,j_{i-1}\})}\right)\right) \\
&\stackrel{\mathrm{d}}{=} F^{-1}\left(1-\prod_{i=1}^{k} U_i^{\frac{1}{\Lambda(\{j_1,\ldots,j_{i-1}\})}}\right).
\end{aligned} \tag{22}$$

*Similarly, the joint conditional distribution of* $(T_{i:n})_{i=1}^k$ *can be written as*

$$\begin{aligned}
&((T_{i:n})_{i=1}^k|I_1=\{j_1\}, I_2=\{j_1,j_2\}\ldots, I_{k-1}=\{j_1,j_2,\ldots,j_{k-1}\}) \\
&\quad \stackrel{\mathrm{d}}{=} ((T_{i:n})_{i=1}^k|J_1=j_1, J_2=j_2,\ldots,J_{k-1}=j_{k-1}) \\
&\quad \stackrel{\mathrm{d}}{=} \left(F^{-1}\left(1-\exp\left(-\sum_{m=1}^{i}\frac{V_m}{\Lambda(\{j_1,\ldots,j_{m-1}\})}\right)\right)\right)_{i=1}^{k} \\
&\quad \stackrel{\mathrm{d}}{=} \left(F^{-1}\left(1-\prod_{m=1}^{i} U_m^{\frac{1}{\Lambda(\{j_1,\ldots,j_{m-1}\})}}\right)\right)_{i=1}^{k}.
\end{aligned}$$

*Proof.* The first representation in (22) immediately follows from the definition and Corollary 2 (see (8)). In order to get the last one, we first rewrite the exponential function in (22) as

$$\exp\left(-\sum_{i=1}^{k}\frac{V_i}{\Lambda(\{j_1,\ldots,j_{i-1}\})}\right)=\prod_{i=1}^{k}[\exp(-V_i)]^{\frac{1}{\Lambda(\{j_1,\ldots,j_{k-1}\})}}.$$

Note that random variables

$$\tilde{U}_i = 1-\exp(-V_i), \qquad i=1,\ldots,k,$$

are independent uniformly distributed on (0,1), and so are $U_i = 1-\tilde{U}_i$, $i=1,\ldots,k$. Therefore the two representations in (22) are equivalent. Similarly one obtains the representation of the joint conditional distribution of the first $k$ components failures. □

The relations between $(T_{k:n}|J_1,\ldots,J_{k-1})$ and $(T_{k-1:n}|J_1,\ldots,J_{k-2})$ are nicely illustrated by the formulae

$$\begin{aligned}
&(T_{k:n}|J_1,\ldots,J_{k-1}) \\
&\quad \stackrel{\mathrm{d}}{=} \left(F^{-1}\left(1-[1-F(T_{k-1:n})]\exp\left(\frac{-V_k}{\Lambda(\{J_1,\ldots,J_{k-1}\})}\right)\right)\middle| J_1,\ldots,J_{k-2}\right) \\
&\quad \stackrel{\mathrm{d}}{=} \left(F^{-1}\left(1-[1-F(T_{k-1:n})]\, U_k^{\frac{1}{\Lambda(\{J_1,\ldots,J_{k-1}\})}}\right)\middle| J_1,\ldots,J_{k-2}\right),
\end{aligned}$$

easily deduced from (22).

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

Proposition 5 shows that the joint distribution of $T_{1:n}, \ldots, T_{n:n}$ under the condition that $J_1 = j_1, J_2 = j_2, \ldots, J_{n-1} = j_{n-1}$ is just the distribution of generalized order statistics with parent distribution function $F$ and parameters $(\gamma_1, \gamma_2, \ldots, \gamma_n) = (\Lambda(\emptyset), \Lambda(\{j_1\}), \ldots, \Lambda(\{j_1, j_2, \ldots, j_{n-1}\})$. The generalized order statistic is a distributional concept introduced by Kamps [18] and applied for a unified description of distributions of various ordered random variables. It is defined using a single distribution function $F$ and a set of positive parameters $\gamma_i$. The joint density function of the first $r$ generalized order statistics, with distribution function $F$ and density $f$, and parameters $\gamma_1, \ldots, \gamma_r$, has the form

$$f^*_{\gamma_1,\ldots,\gamma_r,F}(x_1, \ldots, x_r) = \left(\prod_{j=1}^{r} \gamma_j\right)\left(\prod_{j=1}^{r-1} [1 - F(x_j)]^{\gamma_j - \gamma_{j+1} - 1} f(x_j)\right)[1 - F(x_r)]^{\gamma_r - 1} f(x_r)$$

for $F^{-1}(0+) < x_1 < \ldots < x_r < F^{-1}(1-)$. For various choices of parameters, this describes the joint density function of, e.g., order statistics from independent and identically distributed (i.i.d.) samples with parent $F$, upper records in i.i.d. sequences with marginal $F$, their generalizations to $k$th records (which are new $k$th maxima in the sequences), Type-II censored samples, and progressively Type-II censored samples, where after consecutive failures fixed numbers of randomly selected living items are removed from the experiment for ethical or economic reasons. Since the publication of the book [18], the theory and applications of generalized order statistics have been developed in hundreds of papers. In particular, there have been attempts to employ the notion in reliability theory (see, e.g., [8], [9], and [17]).

It follows that the time-heterogeneous load-sharing model generalizes the model of generalized order statistics. Its weak point, however, is that it cannot be sequentially extended to an arbitrary dimension as the standard generalized order statistics can. In order to get marginal density functions of a single or several order statistics in the model, it suffices to integrate properly the joint density (21). However, we cannot expect closed and nice formulae in general. Cramer and Kamps [10] proved that in the case of the standard uniform distribution function $U$, the $k$th generalized order statistic with parameters $(\gamma_1, \gamma_2, \ldots, \gamma_k)$ has the density function

$$g^*_{k;\gamma_1,\ldots,\gamma_k,U}(t) = \left(\prod_{i=1}^{k} \gamma_i\right) \sum_{j=1}^{l} \sum_{m=0}^{d_j - 1} \frac{K_{jm}}{m!}(1-t)^{\delta_j - 1} \frac{[-\ln(1-t)]^{d_j - m - 1}}{(d_j - m - 1)!}, \qquad 0 < t < 1,$$

where $\delta_1 < \ldots < \delta_l$ represent the different values in the sequence $(\gamma_1, \ldots, \gamma_k)$, and $d_1, \ldots, d_j$ are their respective multiplicities, while

$$K_{j0} = \prod_{\substack{q=1 \\ q \neq j}}^{l} \frac{1}{(\delta_q - \delta_j)^{d_q}},$$

$$K_{jm} = \sum_{p=0}^{m-1} (-1)^{p+1} \frac{(m-1)!}{(m-1-p)!} K_{j,m-1-p} \sum_{\substack{q=1 \\ q \neq j}}^{l} \frac{d_q}{(\delta_q - \delta_j)^{p+1}}, \qquad j \geq 1.$$

This density function does not depend on $n$ and $\gamma_i$ with $i > k$. If the generalized order statistics have the baseline distribution $F$, then the marginal density function of the $k$th generalized order statistic $T_{k:n}$ takes the form

$$g^*_{k;\gamma_1,\ldots,\gamma_k,F}(t) = f(t)\, g^*_{k;\gamma_1,\ldots,\gamma_k,U}(F(t)), \qquad a < t < b.$$

The $k$th order statistic in the time-heterogeneous load-sharing model with parent distribution function $F$ and parameter set (5) has the conditional density function

$$f_{\mathcal{L},F;T_{k:n}}(t|j_1,\ldots,j_{k-1}) = g^*_{k;\Lambda(\emptyset),\Lambda(\{j_1\}),\ldots,\Lambda(\{j_1,j_2,\ldots,j_{k-1}\}),F}(t), \qquad a < t < b,$$

with respect to the random event $(J_1, J_2, \ldots, J_{k-1}) = (j_1, j_2, \ldots, j_{k-1})$. The unconditional distribution of the $k$th subsequent component failure in our model is a convex combination of $\frac{n!}{(n-k)!}$ distributions of $k$th generalized order statistics with common $F$ and various parameters.

**Corollary 6.** *The unconditional density of $T_{k:n}$ is given by*

$$f_{\mathcal{L},F;T_{k:n}}(t) = \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n])} g^*_{k;\Lambda(\emptyset),\Lambda(\{j_1\}),\ldots,\Lambda(\{j_1,j_2,\ldots,j_{k-1}\}),F}(t)\prod_{i=1}^{k-1}\frac{\lambda_{j_i}(\{j_1,j_2,\ldots,j_{i-1}\})}{\Lambda(\{j_1,j_2,\ldots,j_{i-1}\})}.$$

Similarly, the density functions of every component and of the system are also convex combinations of the density functions of generalized order statistics.

**Corollary 7.** *The density function of the ith component lifetime $T_i$ has the form*

$$f_{\mathcal{L},F;T_i}(t) = \sum_{k=1}^{n}\sum_{(j_1,\ldots,j_{k-1})\in\Pi([n]\setminus\{i\})} g^*_{k;\Lambda(\emptyset),\Lambda(\{j_1\}),\ldots,\Lambda(\{j_1,j_2,\ldots,j_{k-1}\}),F}(t)$$
$$\times\prod_{i=1}^{k-1}\frac{\lambda_{j_i}(\{j_1,j_2,\ldots,j_{i-1}\})}{\Lambda(\{j_1,j_2,\ldots,j_{i-1}\})}\frac{\lambda_i(\{j_1,j_2,\ldots,j_{k-1}\})}{\Lambda(\{j_1,j_2,\ldots,j_{k-1}\})}.$$

**Corollary 8.** *The lifetime $T$ of the system with structure $\varphi$ and component lifetimes $T_1,\ldots,T_n$ satisfying the assumptions of the time-heterogeneous load-sharing model with parent distribution function $F$ and parameters* (5) *has the density function*

$$f_{\mathcal{L},F;T}(t) = \sum_{k=1}^{n}\sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} g^*_{k;\Lambda(\emptyset),\Lambda(\{j_1\}),\ldots,\Lambda(\{j_1,j_2,\ldots,j_{k-1}\}),F}(t)\prod_{i=1}^{n-1}\frac{\lambda_{j_i}(\{j_1,j_2,\ldots,j_{i-1}\})}{\Lambda(\{j_1,j_2,\ldots,j_{i-1}\})}.$$

**Proposition 6.** *In the time-heterogeneous load-sharing model based on the distribution function $F$ and parameters $\mathcal{L}$, the multivariate conditional hazard rates of component lifetimes do not depend on the particular failure times of the components that have failed previously, and can be written as*

$$\lambda_j(t|A_{k-1},t_1,\ldots,t_{k-1}) = \lambda_j(A_{k-1})\frac{f(t)}{1-F(t)}, \quad j\in A^c_{k-1},\ |A_{k-1}| = k-1,\ a<t<b. \tag{23}$$

Formally, the formula (23) holds for $a < t_1 < \ldots < t_{k-1} < t$, but $t_1,\ldots,t_{k-1}$ could be located arbitrarily close to a, and so (23) has to be defined for all $t > a$.

*Proof.* We first determine the multivariate conditional failure rates at subsequent failure times. By (19) and (20), the conditional survival function of the $k$th order statistic

$$\mathbb{P}(T_{k:n} > t|T_{k-1:n} = t_{k-1}, I_{k-1} = A_{k-1}) = \left[\frac{1-F(t)}{1-F(t_{k-1})}\right]^{\Lambda(A_{k-1})}, \qquad k = 1,\ldots,n,$$

depends only on the labels of the already failed components $A_{k-1}$ and the last preceding failure time $t_{k-1}$. It is independent of the moments of other earlier failures and of the order of consecutively failing components. The corresponding conditional density is

$$f_{\mathcal{L},F;T_{k:n}}(t|T_{k-1:n}=t_{k-1}, I_{k-1}=A_{k-1})=\frac{\Lambda(A_{k-1})[1-F(t)]^{\Lambda(A_{k-1})-1}f(t)}{[1-F(t_{k-1})]^{\Lambda(A_{k-1})}}.$$

Writing the respective hazard rate

$$\lambda_{\mathcal{L},F;T_{k:n}}(t|T_{k-1:n}=t_{k-1}, I_{k-1}=A_{k-1})=\Lambda(A_{k-1})\,\frac{f(t)}{1-F(t)}=\lambda_{\mathcal{L},F;T_{k:n}}(t|I_{k-1}=A_{k-1}),$$

we also eliminate the dependence on $t_{k-1}$ and get a linear transformation of the failure rate $\lambda_F(t)=\frac{f(t)}{1-F(t)}$ of the model distribution function $F$. For the component lifetime, we have

$$\begin{aligned}\lambda_{\mathcal{L},F;T_j}(t|I_{k-1}=A_{k-1})&=\lim_{\triangle t\searrow 0}\frac{1}{\triangle t}\mathbb{P}(T_j\le t+\triangle t|I_{k-1}=A_{k-1}, T_{k:n}>t)\\&=\lim_{\triangle t\searrow 0}\frac{1}{\triangle t}\mathbb{P}(T_j=T_{k:n}, T_{k:n}\le t+\triangle t|I_{k-1}=A_{k-1}, T_{k:n}>t).\end{aligned}$$

In view of (18), under the condition $I_{k-1}=A_{k-1}$, the probability that $T_j=T_{k:n}$ is independent of the distribution of $T_{k:n}$. Therefore

$$\begin{aligned}\lambda_{\mathcal{L},F;T_j}(t|I_{k-1}=A_{k-1})&=\mathbb{P}(T_j=T_{k:n}|I_{k-1}=A_{k-1})\\&\quad\times\lim_{\triangle t\searrow 0}\frac{1}{\triangle t}\mathbb{P}(T_{k:n}\le t+\triangle t|I_{k-1}=A_{k-1}, T_{k:n}>t)\\&=\frac{\lambda_j(A_{k-1})}{\Lambda(A_{k-1})}\,\Lambda(A_{k-1})\,\frac{f(t)}{1-F(t)}=\lambda_j(A_{k-1})\lambda_F(t),\end{aligned}$$

as desired. □

Proposition 6 shows that our time-heterogeneous load-sharing model based on some fixed distribution function $F$ and parameters (5) follows the linear breakdown rule (4). The form of the multivariate conditional hazard rates (23) refers to some intuitive arguments. We assume that the components have some immanent durability properties, expressed by means of the failure rate $\lambda_F$ of the parent model distribution, which has fixed shape and monotonicity properties. After the system begins operation, and then after the working status of other components changes, the level of aging intensity of each component changes abruptly without violating its further variability. The degree of change depends on the location of the component in the system and the working status of the remaining system components.

We claim that the only joint distributions of the component lifetimes that follow the linear breakdown rule are those satisfying the assumptions of the time-heterogeneous load-sharing model with some baseline distribution function $F$ and parameters $\lambda_j(A)$, $j\notin A\subsetneq[n]$.

**Theorem 2.** *Let $T_1,\dots,T_n$ be the component lifetimes with the multivariate conditional hazard rate function of a form as in (4) for all $t$ in some $(a,b)\subset(0,+\infty)$. Then there exists an absolutely continuous distribution function $F$ supported on $(a, b)$ with the failure rate $\lambda_F(t)=L(t)$, $a<t<b$, such that the joint distribution of $T_1,\dots,T_n$ is identical with that of the component lifetimes in the time-heterogeneous load-sharing model with parent distribution function $F$ and parameters (5).*

In the proof we apply the following auxiliary result.

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

**Proposition 7.** *Suppose that $T_1, \dots, T_n$ have a joint distribution determined by the multivariate conditional hazard rates (2) and (3). Then for every $k = 0, \dots, n-1$, $j_1, \dots, j_k \in [n]$ and $0 < t_1 < \dots < t_k$, the random variables $T_i - t_k$, $i \in \{j_1, \dots, j_k\}^c$, and independent positive random variables $Z_i$, $i \in \{j_1, \dots, j_k\}^c$, with respective failure rates*

$$r_i(t) = \lambda_i(t + t_k | j_1, \dots, j_k, t_1, \dots, t_k), \qquad t > 0,$$

*satisfy*

$$\begin{aligned}&\mathbb{P}(T_i - t_k = \min\{T_j - t_k : j \in \{j_1, \dots, j_k\}^c\} \in B | T_{j_1} = t_1, \dots, T_{j_k} = t_k < T_{k+1:n}) \\ &\quad = \mathbb{P}(Z_i = \min\{Z_j : j \in \{j_1, \dots, j_k\}^c\} \in B)\end{aligned}$$

*for every $i \in \{j_1, \dots, j_k\}^c$ and $B$ an arbitrary Borel subset of $\mathbb{R}_+$.*

This result immediately follows from De Santis et al. [12, Proposition 1]. The special case with $k = 0$ was formulated in Proposition 2 of [12]. In our more general version, we consider conditional distributions of the residual lifetimes $T_i - t_k$, $i \in \{j_1, \dots, j_k\}^c$, given $\{T_{j_1} = t_1, \dots, T_{j_k} = t_k, T_i > t_k, i \in \{j_1, \dots, j_k\}^c\}$. The assumption of absolute continuity of $T_1, \dots, T_n$ ensures the possibility of constructing the multivariate conditional hazard rates $\lambda_\cdot(\cdot \mid \cdot)$ on one hand, and the existence of the joint conditional density function of the aforementioned conditional distribution on the other. The latter implies that there exist multivariate conditional hazard rates $\tilde{\lambda}_\cdot(\cdot \mid \cdot)$ of the conditional distribution. In view of the definitions, we immediately notice that the relations

$$\tilde{\lambda}_j(t|\emptyset) = \lambda_j(t + t_k | j_1, \dots, j_k, t_1, \dots, t_k),$$

with $j \in \{j_1, \dots, j_k\}^c$ and $t > 0$, and

$$\tilde{\lambda}_j(t | i_1, \dots, i_\ell, s_1, \dots, s_\ell) = \lambda_j(t + t_k | j_1, \dots, j_k, i_1, \dots, i_\ell, t_1, \dots, t_k, t_k + s_1, \dots, t_k + s_\ell),$$

with $\ell = 0, \dots, n-k-1$, $\{i_1, \dots, i_\ell, j\} \subset \{j_1, \dots, j_k\}^c$, and $0 < s_1 < \dots s_\ell < t$, hold.

*Proof of Theorem 2.* We first show that the function $L(t)$ is the failure rate of a lifetime distribution function with a positive density on $(a, b)$. The function is certainly positive on $(a, b)$, because so is $\lambda_j(A)$ for every $j$ and $A$. Consequently, it suffices to show that $\int_a^b L(t) dt = +\infty$.

The multivariate conditional failure rates of the $j$th component lifetime have the forms

$$\begin{aligned}&\lambda_j(t | J_1 = j_1, \dots, J_k = j_k, T_{j_1} = t_1 < \dots < T_{j_k} = t_k < t < \min_{i \in \{j_1, \dots, j_k\}^c} T_i) \\ &\quad = \lambda_j(\{j_1, \dots, j_k\}) L(t), \qquad a < t < b, \quad j \in \{j_1, \dots, j_k\}^c. \qquad (24)\end{aligned}$$

In view of the fact that such conditional failure rates do not depend on the particular times of the previous failures, we use the total probability law to calculate the unconditional version of the failure rate,

$$\lambda_j(t) = \sum_{k=0}^{n-1} \sum_{(j_1, \dots, j_k) \in \Pi([n] \setminus \{j\})} \lambda_j(\{j_1, \dots, j_k\}) \mathbb{P}(T_{j_1} < \dots < T_{j_k} < t < \min_{i \in \{j_1, \dots, j_k\}^c} T_i) L(t),$$

which satisfies

$$\begin{aligned}+\infty = \int_a^b \lambda_j(t)\, dt = &\sum_{k=0}^{n-1} \sum_{(j_1, \dots, j_k) \in \Pi([n] \setminus \{j\})} \lambda_j(\{j_1, \dots, j_k\}) \\ &\times \int_a^b \mathbb{P}(T_{j_1} < \dots < T_{j_k} < t < \min_{i \in \{j_1, \dots, j_k\}^c} T_i)\, L(t)\, dt\end{aligned}$$

if and only if $\int_a^b L(t) dt = +\infty$.

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

Now we prove the relations (18). We fix $0 \le k \le n-1$, $j_1, \ldots, j_k \in [n]$, and $0 < t_1 < \ldots < t_k$. Under the notation of Proposition 7, the independent random variables $Z_i$, $i \in \{j_1, \ldots, j_k\}^c$, have failure rates

$$r_i(t) = \lambda_i(\{j_1, \ldots, j_k\})L(t + t_k), \qquad 0 < t < b - t_k.$$

Their survival and density functions are

$$\bar{G}_i(t) = \exp\left(-\lambda_i(\{j_1, \ldots, j_k\}) \int_0^t L(u + t_k)\, du\right),$$

$$g_i(t) = \lambda_i(\{j_1, \ldots, j_k\})L(t + t_k)\ \exp\left(-\lambda_i(\{j_1, \ldots, j_k\}) \int_0^t L(u + t_k)\, du\right),$$

respectively. Then

$$\begin{aligned}
&\mathbb{P}(Z_i = \min\{Z_j : j \in \{j_1, \ldots, j_k\}^c\}) = \mathbb{P}(Z_i \le \min\{Z_j : j \in \{j_1, \ldots, j_k, j\}^c\}) \\
&= \int_0^{b-t_k} \left[\prod_{j \in \{j_1, \ldots, j_k, j\}^c} \bar{G}_j(t)\right] g_i(t)\, dt = \int_0^{b-t_k} \lambda_i(\{j_1, \ldots, j_k\})L(t + t_k) \\
&\quad \times \exp\left(-\sum_{j \in \{j_1, \ldots, j_k\}^c} \lambda_j(\{j_1, \ldots, j_k\}) \int_0^t L(u + t_k)\, du\right) dt \\
&= \frac{\lambda_i(\{j_1, \ldots, j_k\})}{\sum_{j \in \{j_1, \ldots, j_k\}^c} \lambda_j(\{j_1, \ldots, j_k\})}.
\end{aligned}$$

By Proposition 7, independently of the particular values of $T_{j_1} = t_1 < \ldots < T_{j_k} = t_k < T_{k+1:n}$, we obtain

$$\mathbb{P}(T_i = T_{k+1:n} | J_1 = j_1 \ldots, J_k = j_k) = \frac{\lambda_i(\{j_1, \ldots, j_k\})}{\sum_{j \in \{j_1, \ldots, j_k\}^c} \lambda_j(\{j_1, \ldots, j_k\})}.$$

Summing up, under the condition (4) the function $L(t)$, $a < t < b$, is the failure rate of a unique distribution function $F$, say, supported on $(a, b)$. The representation (4) implies (18) as well. It follows that the assumptions of the time-heterogeneous load-sharing model with baseline distribution function $F$ and parameters (5) are satisfied. Since the family of multivariate conditional hazard rate functions uniquely determines the joint distribution of $(T_1, \ldots, T_n)$, the assumption (4) describes a time-heterogeneous load-sharing model as defined at the beginning of this section. □

The time-heterogeneous load-sharing model distribution uniquely determines the respective set of parameters $\mathcal{L}$ and $F$ up to the transformations $\mathcal{L}_\theta = \{\theta\lambda_j(A) : A \subsetneqq [n],\ j \in A^c\}$ and $F_\theta(t) = 1 - [1 - F(t)]^{1/\theta}$ for $\theta > 0$. Note that the multivariate conditional hazard rate functions have the same form (4) for all $\theta > 0$.

**Remark 4.** Schechner [36] assumed that the parameters of the linear breakdown model (4) satisfy conditions analogous to (15), and additionally that $\sum_{j \in A^c} \lambda_j(A) = 1$, $A \subsetneqq [n]$. Our approach allows us to remove the latter condition. What is more, this assumption seems to be rather idealistic in view of our investigations. It implies that the conditional hazard rates of consecutive

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

failure times $T_{1:n}, \ldots, T_{n:n}$ are just equal to $L(t)$, and do not depend on the number and types of the failures before $t$. A realistic assumption, however, is that after each failure the sum of the particular conditional failure rates of the still operating components increases (see (16)). This means that when the components fail in the order $j_1, j_2, \ldots, j_n$, the conditional hazard rate of the $(k+1)$th subsequent failure jumps up abruptly from $\Lambda(\{j_1, \ldots, j_{k-1}\})L(T_{k+1:n}-)$ to $\Lambda(\{j_1, \ldots, j_k\})L(T_{k+1:n})$ at the $(k+1)$th failure moment $t = T_{k+1:n}$ (when $L$ is continuous, we simply have $L(T_{k+1:n}-) = L(T_{k+1:n})$). Here the function $L(t)$ may be treated as the failure rate of a component which does not undergo sudden jumps of the load level while working in the system. The working regime in the system changes the levels of the component lifetime hazard rates, but it preserves its time variability properties in the inter-failure times.

## 4. Special cases of the model parameters

In this section we analyze the time-heterogeneous load-sharing model in the cases when the set of model parameters (5) takes on simplified forms. The first one is equivalent to the exchangeability of component lifetimes, and the other two provide some generalizations of the exchangeable case.

*Exchangeable components.* Here we assume that the behavior of any system component is indistinguishable from that of the others; i.e. the joint density $f_{T_1,\ldots,T_n}(t_1, \ldots, t_n)$ of the component lifetimes is invariant under permutations of its arguments $t_1, \ldots, t_n$. In terms of the multivariate conditional hazard rates, this assumption can be equivalently expressed by the condition that for any pair $(j, A)$ with $A \subsetneq [n]$ and $j \in A^c$, the load-sharing parameter $\lambda_j(A)$ is independent of $j$, and depends on $A$ only through its cardinality $|A|$. In other words, we assume existence of positive numbers $\alpha(i)$, $i = 1, \ldots, n-1$, such that

$$\lambda_j(A) = \alpha(|A|), \qquad A \subsetneq [n], \quad j \in A^c. \tag{25}$$

This implies that $\Lambda(A) = (n - |A|)\alpha(|A|)$, $A \subsetneq [n]$. Note that in the exchangeable submodel, the number of parameters is reduced from $n2^{n-1}$ to $n$. The equivalence between exchangeability and (25) can be proven by resorting to the formulae which express the multivariate conditional hazard rate functions in terms of the joint density and vice versa. See also, e.g., the arguments in [38], and [40]. By the exchangeability assumption and Corollary 2, the random sequence of consecutive subsets of failed components $I_1, \ldots, I_n$ ($I_k \subset I_{k+1}$, $|I_k| = k$) forms a Markov chain with initial probability $\mathbb{P}(I_1 = \{j_1\}) = \frac{1}{n}$ and transition probability law

$$\mathbb{P}(I_{k+1} = A_k \cup \{j_{k+1}\} | I_k = A_k) = \frac{1}{n+1-k}, \qquad k = 1, \ldots, n,$$

for $A_k$ denoting a value of the random variable $I_k$, i.e., the set of the first $k$ failed components. This implies that

$$\mathbb{P}(I_1 = A_1, \ldots, I_k = A_k) = \frac{(n-k)!}{n!}$$

and

$$\mathbb{P}(I_k = A_k) = \frac{1}{\binom{n}{k}}$$

for $k = 1, \ldots, n$. In particular, all the sequences of labels of consecutively failed components have the same probability $\mathbb{P}(I_1 = A_1, \ldots, I_n = A_n) = \frac{1}{n!}$.

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

By Proposition 4, the sequence of pairs $(T_{k:n}, I_k)$, $k = 1, \ldots, n$, is a two-dimensional Markov chain with probability measure determined by

$$\mathbb{P}(T_{1:n} > t, I_1 = \{j_1\}) = [1 - F(t)]^{n\alpha(0)} \frac{1}{n}, \qquad t > 0,$$

and

$$\begin{aligned}&\mathbb{P}(T_{k:n} > t, I_k = A_{k-1} \cup \{j_k\} | T_{k-1:n} = t_{k-1}, I_{k-1} = A_{k-1}) \\ &= \left[\frac{1 - F(t)}{1 - F(t_{k-1})}\right]^{(n+1-k)\alpha(k-1)} \frac{1}{n+1-k}, \qquad t > t_{k-1},\ k = 2, \ldots, n.\end{aligned}$$

It follows that the sequence of order statistics $T_{k:n}$, $k = 1, \ldots, n$, itself is a Markov chain with initial and transition probability laws

$$\mathbb{P}(T_{1:n} > t) = [1 - F(t)]^{n\alpha(0)}, \qquad t > 0,$$

$$\mathbb{P}(T_{k:n} > t | T_{k-1:n} = t_{k-1}) = \left[\frac{1 - F(t)}{1 - F(t_{k-1})}\right]^{(n+1-k)\alpha(k-1)}, \quad t > t_{k-1},\ k = 2, \ldots, n,$$

respectively. This implies that the distribution of consecutive failures $T_{1:n}, \ldots, T_{n:n}$ is identical with the distribution of $n$ generalized order statistics with baseline distribution function $F$ and parameters $\gamma_k = (n + 1 - k)\alpha(k - 1)$, $k = 1, \ldots, n$. Therefore the unconditional density function of the $k$th system component failure can be written as

$$f_{\mathcal{L},F,T_{k:n}}(t) = g^*_{k;n\alpha(0),\ldots,(n+1-k)\alpha(k-1),F}(t).$$

By Corollary 7, we also have

$$\begin{aligned}f_{\mathcal{L},F,T_i}(t) &= \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!}\, g^*_{k;n\alpha(0),\ldots,(n+1-k)\alpha(k-1),F}(t)\, \frac{(n-k)!}{n!} \\ &= \frac{1}{n} \sum_{k=1}^{n} g^*_{k;n\alpha(0),\ldots,(n+1-k)\alpha(k-1),F}(t) = \frac{1}{n} \sum_{k=1}^{n} f_{\mathcal{L},F,T_{k:n}}(t).\end{aligned}$$

This illustrates the classic property of exchangeable random variables that the marginal distribution of any single variable is the uniform mixture of the distributions of order statistics. By Corollary 8, we get

$$f_{\mathcal{L},F,T}(t) = \sum_{k=1}^{n} \frac{|\chi^{-1}(k)|}{n!} g^*_{k;n\alpha(0),\ldots,(n+1-k)\alpha(k-1),F}(t) = \sum_{k=1}^{n} \frac{|\chi^{-1}(k)|}{n!} f_{\mathcal{L},F,T_{k:n}}(t), \tag{26}$$

which is the mixture of density functions of order statistics with the coefficient vector

$$\mathbf{s} = (s_1, \ldots, s_n) = \left(\frac{|\chi^{-1}(1)|}{n!}, \ldots, \frac{|\chi^{-1}(n)|}{n!}\right),$$

dependent merely on the structure function of the system. The vector $\mathbf{s}$ is called the structural Samaniego signature of the system. The representation (26) was proven in Samaniego [30] in the case of i.i.d. component lifetimes, and it was extended to the exchangeable case by Navarro *et al.* [24] (see also [23]). Note that the Samaniego signature has also the probabilistic meaning $s_k = p_k = \mathbb{P}(T = T_{k:n})$, $k = 1, \ldots, n$. Hence (26) can be rewritten as

$$f_{\mathcal{L},F,T}(t) = \sum_{k=1}^{n} \mathbb{P}(T = T_{k:n}) f_{\mathcal{L},F,T_{k:n}}(t),$$

which means that under the assumption of exchangeability of $T_1, \ldots, T_n$ the component failure times are independent of the system structure. The vector $\mathbf{p} = (p_1, \ldots, p_n)$ is called the probabilistic signature of the system; under violation of the exchangeability assumption it usually differs from the structural one. The time-heterogeneous load-sharing model with exchangeable components is also called the failure-dependent proportional hazard model, because the multivariate conditional hazard rates here,

$$\lambda_j(t|A_{k-1}, t_1, \ldots, t_{k-1}) = \alpha(k-1)\,\frac{f(t)}{1-F(t)}, \qquad k = 1, \ldots, n,\ t > 0$$

(cf. (23)), are proportional to the baseline hazard rate $f(t)/(1 - F(t))$, and change only at the failure times of the components. The model was studied by Hollander and Pẽna [17], Aki and Hirano [1], Burkschat [8], and Burkschat and Rychlik [9], among others.

*Components with uniform supporting ability.* Assume now that the model parameters $\lambda_j(A)$, $j \notin A \subsetneq [n]$, depend on the number of failed components, but do not depend on their particular labels. The submodel is thus characterized by $n^2$ parameters

$$\beta_j(k) = \lambda_j(A) \qquad \text{iff} \qquad |A| = k,\ j = 1, \ldots, n,\ k = 0, \ldots, n-1.$$

It is natural, but not necessary, to assume that for fixed $j$ all the sequences $\beta_j(k)$ are nondecreasing.

The quantity

$$\varsigma_{i,j}(A) = \lambda_j(A \cup \{i\}) - \lambda_j(A), \qquad i \neq j \in A^c,\ A \subsetneq [n],$$

which can be regarded as a measure of the ability of component $i$ to reduce the stress suffered by component $j$ when all the components in $A$ have failed, does not depend on the label of the supporting component, but only on the label of the supported one. This can be interpreted as the uniformity of the ability of operating components to reduce the stress on other operating components.

In contrast to the exchangeable case, it appears here that despite a significant reduction of the number of parameters, the distributional properties of component and system lifetimes and respective formulae do not simplify much compared to those in the general model. We have

$$f_{\mathcal{L},F,T_{k:n}}(t) = \sum_{(j_1,\ldots,j_{k-1}) \in \Pi([n])} g^{*}_{k;B(\emptyset),\ldots,B(\{j_1,\ldots,j_{k-1}\}),F}(t) \prod_{i=1}^{k} \frac{\beta_{j_i}(i-1)}{B(\{j_1, \ldots, j_{i-1}\})},$$
$$k = 1, \ldots, n,$$

$$f_{\mathcal{L},F,T_i}(t) = \sum_{k=1}^{n} \beta_i(k-1) \sum_{(j_1,\ldots,j_{k-1}) \in \Pi([n]\setminus\{i\})} \frac{g^{*}_{k;B(\emptyset),\ldots,B(\{j_1,\ldots,j_{k-1}\}),F}(t) \prod_{i=1}^{k-1} \beta_{j_i}(i-1)}{\prod_{i=1}^{k} B(\{j_1, \ldots, j_{i-1}\})},$$
$$i = 1, \ldots, n,$$

$$f_{\mathcal{L},F,T}(t) = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n) \in \chi^{-1}(k)} g^{*}_{k;B(\emptyset),\ldots,B(\{j_1,\ldots,j_n\}),F}(t) \prod_{i=1}^{n} \frac{\beta_{j_i}(i-1)}{B(\{j_1, \ldots, j_{i-1}\})}$$

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

for $B(A) = \Lambda(A) = \sum_{j\in A^c} \beta_j(|A|)$, $A \subsetneq [n]$, denoting the cumulative hazard rate of living components when those of $A$ have failed.

As in the general model, $(T_{k:n}, I_k)$ and $(I_k)$, $k = 1, \ldots, n$, are Markov chains, but $(T_{k:n})$, $k = 1, \ldots, n$, is not. We finally observe that

$$p_k = \mathbb{P}(T = T_{k:n}) = \sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} \prod_{i=1}^{n} \frac{\beta_{j_i}(i-1)}{B(\{j_1, \ldots, j_{i-1}\})}, \qquad k = 1, \ldots, n,$$

are different from the structural signature coefficients, and the conditional density functions

$$f_{\mathcal{L},F,T|T=T_{k:n}}(t) = \frac{\sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} g^*_{k;B(\emptyset),\ldots,B(\{j_1,\ldots,j_n\}),F}(t) \prod_{i=1}^{n-1} \frac{\beta_{j_i}(i-1)}{B(\{j_1,\ldots,j_{i-1}\})}}{\sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} \prod_{i=1}^{n-1} \frac{\beta_{j_i}(i-1)}{B(\{j_1,\ldots,j_{i-1}\})}},$$

$k = 1, \ldots, n$, strongly depend on the structure of the system, unlike in the exchangeable case.

*Components with uniform frailty.* In this subsection, we consider the case when the load-sharing parameters $\lambda_j(A)$ are independent of $j$ for any $A \subsetneq [n]$ such that $j \in A^c$. That is, we assume the existence of $2^n - 1$ positive numbers $\delta(A)$ satisfying

$$\lambda_j(A) = \delta(A), \qquad j \in A^c, \ A \subsetneq [n] .$$

The quantities $\delta(A)$ do not depend on the specific identity $j$ of each still surviving component, but they may depend on the set $A$ of already failed components. Unlike in the exchangeable setup we do not assume here the relation $|A'| = |A''| \Rightarrow \delta(A') = \delta(A'')$. The special form of the assumptions can be interpreted as uniform frailty or uniform sensitivity of all still operating components to the load acting on the system.

We first notice that

$$\Lambda(A) = \sum_{j\in A^c} \lambda_j(A) = (n - |A|)\, \delta(A),$$

$$\frac{\lambda_j(A)}{\Lambda\,(A)} = \frac{1}{n - |A|}, \qquad A \subsetneq [n].$$

This implies that the distributions of the sequences $(I_1, \ldots, I_k)$, $k = 1, \ldots, n$, are

$$\mathbb{P}(I_1 = \{j_1\}, \ldots, I_k = \{j_1, \ldots, j_k\}) = \frac{(n-k)!}{n!},$$

as in the exchangeable case. Applying Corollary 6, we conclude that

$$f_{\mathcal{L},F,T_{k:n}}(t) = \frac{(n+1-k)!}{n!} \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n])} g^*_{k;n\delta(\emptyset),\ldots,(n+1-k)\delta(\{j_1,\ldots,j_{k-1}\}),F}(t),$$

which means that the distribution of the $k$th consecutive component failure is the uniform mixture of $k$th generalized order statistics with parent distribution function $F$ and parameters $\gamma_i = (n+1-i)\delta(\{j_1, \ldots, j_{i-1}\})$, $i = 1, \ldots, k$, over all subsequences of different elements of $[n]$ of length $k-1$. Furthermore, we have

$$f_{\mathcal{L},F,T_i}(t) = \frac{1}{n} \sum_{k=1}^{n} \frac{(n-k)!}{(n-1)!} \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n]\setminus\{i\})} g^*_{k;n\delta(\emptyset),\ldots,(n+1-k)\delta(\{j_1,\ldots,j_{k-1}\}),F}(t),$$

$$f_{\mathcal{L},F,T}(t) = \frac{1}{n!} \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} g^*_{k;n\delta(\emptyset),\ldots,(n+1-k)\delta(\{j_1,\ldots,j_{k-1}\}),F}(t).$$

The first is the mean over all $k=1, \ldots, n$ of uniform mixtures of $k$th generalized order statistic densities with parameters $n\delta(\emptyset), \ldots, (n+1-k)\delta(\{j_1, \ldots, j_{k-1}\})$ and baseline $F$, with $(j_1, \ldots, j_{k-1})$ being the sequences of different elements of $[n]$ other than $i$. The latter is the average of all conditional density functions of system failure under all possible orders of consecutively failing components. Note that although the uniform frailty model has many more parameters than the uniform supporting-ability model, the distributions of component and system lifetimes have much simpler forms. Moreover, under the uniform frailty assumption, the structural and probabilistic signatures coincide. On the other hand, the conditional density functions

$$f_{\mathcal{L},F,T|T=T_{k:n}}(t) = \frac{1}{|\chi^{-1}(k)|} \sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} g^{*}_{k;n\delta(\emptyset),\ldots,(n+1-k)\delta(\{j_1,\ldots,j_{k-1}\}),F}(t)$$

evidently depend on the system structure, in contrast to their independence in the exchangeable model. Also, the sequence of consecutive failures does not form a Markov chain.

## 5. Expectations of system lifetimes for special baseline distributions

For some parent distribution functions $F$, one can establish analytic formulae for the expectations of component lifetimes, their order statistics, and system lifetimes. The most natural choice is the standard exponential distribution function $F(t)=1-\exp(-t)$, $t>0$, with the quantile function $F^{-1}(u)=-\ln(1-u)$, $0<u<1$. Plugging this into the model of Section 3 (see (17)), we obtain exactly the time-homogeneous load-sharing model with parameters $\mathcal{L}$. In consequence, for the standard exponential model distribution function $F$ we can simply recall all the establishments of Section 2. Location-scale transformations of the parent distribution result in analogous transformations of the random variables in the model.

Another example with frequent applications in the lifetime analysis is the Lomax distribution function $F_\alpha(t)=1-(1+t)^{-\alpha}$, $t>0$, $\alpha>0$. It has the inverse $F_\alpha^{-1}(u)=(1-u)^{-1/\alpha}-1$. By Proposition 5, we have the representation

$$(T_{k:n}|J_1=j_1, J_2=j_2 \ldots, J_{k-1}=j_{k-1}) \stackrel{\mathrm{d}}{=} \prod_{i=1}^{k} U_i^{\frac{-1}{\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})}} - 1$$

with the conditional expectation

$$\mathbb{E}(T_{k:n}|J_1=j_1, J_2=j_2 \ldots, J_{k-1}=j_{k-1}) = \prod_{i=1}^{k} \frac{\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})}{\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})-1} - 1,$$

which is finite under the assumption that all $\Lambda(\{j_1, \ldots, j_{i-1}\}, i=1, \ldots, k$, are greater than $\frac{1}{\alpha}$. In consequence, we have

$$\mathbb{E}T_{k:n} = \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n])} \frac{\alpha^k \Lambda(\{j_1,\ldots,j_{k-1}\}) \prod_{i=1}^{k-1} \lambda_{j_i}(\{j_1,\ldots,j_{i-1}\})}{\prod_{i=1}^{k} [\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})-1]} - 1,$$

$$\mathbb{E}T_i = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n]\setminus\{i\})} \frac{\alpha^k \lambda_i(\{j_1,\ldots,j_{k-1}\}) \prod_{i=1}^{k-1} \lambda_{j_i}(\{j_1,\ldots,j_{i-1}\})}{\prod_{i=1}^{k} [\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})-1]} - 1,$$

$$\mathbb{E}T = \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} \prod_{i=1}^{k} \frac{\alpha\lambda_{j_i}(\{j_1,\ldots,j_{i-1}\})}{\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})-1} \prod_{i=k+1}^{n} \frac{\lambda_{j_i}(\{j_1,\ldots,j_{i-1}\})}{\Lambda(\{j_1,\ldots,j_{i-1}\})} - 1, \tag{27}$$

under the condition that all the denominators are positive.

The last example of this type is provided by the reflected power distribution function $F_\alpha(t) = 1-(1-t)^\alpha$, $0<t<1$, $\alpha>0$, with the quantile function $F_\alpha^{-1}(u) = 1-(1-u)^{1/\alpha}$, $0<u<1$. By (22), this yields

$$(T_{k:n}|J_1 = j_1, J_2 = j_2 \ldots, J_{k-1} = j_{k-1}) \stackrel{\mathrm{d}}{=} 1 - \prod_{i=1}^{k} U_i^{\frac{1}{\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})}}.$$

Since the respective expectation is

$$\mathbb{E}(T_{k:n}|J_1 = j_1, J_2 = j_2 \ldots, J_{k-1} = j_{k-1}) = 1 - \prod_{i=1}^{k} \frac{\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})}{\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})+1},$$

we conclude the following:

$$\mathbb{E}T_{k:n} = 1 - \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n])} \frac{\alpha^k \Lambda(\{j_1,\ldots,j_{k-1}\}) \prod_{i=1}^{k-1} \lambda_{j_i}(\{j_1,\ldots,j_{i-1}\})}{\prod_{i=1}^{k} [\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})+1]},$$

$$\mathbb{E}T_i = 1 - \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_{k-1})\in\Pi([n]\setminus\{i\})} \frac{\alpha^k \lambda_i(\{j_1,\ldots,j_{k-1}\}) \prod_{i=1}^{k-1} \lambda_{j_i}(\{j_1,\ldots,j_{i-1}\})}{\prod_{i=1}^{k} [\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})+1]},$$

$$\mathbb{E}T = 1 - \sum_{k=1}^{n} \sum_{(j_1,\ldots,j_n)\in\chi^{-1}(k)} \prod_{i=1}^{k} \frac{\alpha\lambda_{j_i}(\{j_1,\ldots,j_{i-1}\})}{\alpha\Lambda(\{j_1,\ldots,j_{i-1}\})+1} \prod_{i=k+1}^{n} \frac{\lambda_{j_i}(\{j_1,\ldots,j_{i-1}\})}{\Lambda(\{j_1,\ldots,j_{i-1}\})}. \tag{28}$$

In the special case $\alpha = 1$ we obtain the formulae for $F$ being the standard uniform distribution function. The Lomax, exponential, and reflected power distributions together form the family of generalized Pareto distributions, which have multiple applications in the theory and practice of lifetime data analysis.

We complete the paper by presenting a numerical example.

**Example 1.** Consider the 5-component bridge system with the structure presented in Figure 1.

It has minimal path sets {1, 2}, {4, 5}, {1, 3, 5}, {2, 3, 4}, and minimal cut sets {1, 4}, {2, 5}, {1, 3, 5}, {2, 3, 4} (for the definitions of these notions, we refer the reader to, e.g., [5, p. 9]). Its Samaniego signature is $\left(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0\right)$, which means in particular that the system cannot fail because of one component failure, and cannot survive four component failures. This is also easily verified by a cursory look at the figure.

Another immediate conclusion of the system diagram analysis is the fact that the components in positions 1, 2, 4, and 5 are equally important to the functioning of the system, while the component in position 3 affects it less. We may support this claim more formally using component importance measures. For example, the Barlow–Proschan importance measure $I_{BP}(i)$ of the $i$th component, in the case of identical exchangeable components, is the proportion of the number of sequences of labels of consecutively failing components such that the failure of component $i$ implies the failure of the system to the total number of sequences (see [4]). For the bridge system we have $I_{BP}(3) = \frac{1}{15}$ and $I_{BP}(j) = \frac{7}{30}$, $j \neq 3$.

If we have four equally strong components and one weaker one, we certainly locate the weakest component at the least important position 3. Assume that this is reflected

https://doi.org/10.1017/apr.2020.49 Published online by Cambridge University Press

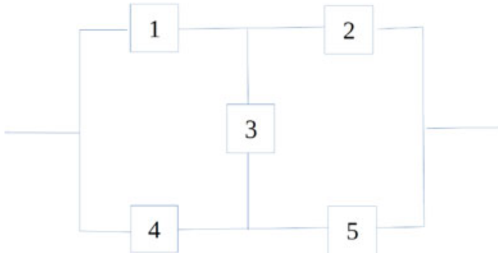


FIGURE 1: Bridge system.

by the following assumption on the parameters of the load-sharing model: for different $j_1, j_2, j_3, j_4 \in \{1, 2, 4, 5\}$, we set

$$\begin{aligned}
&\lambda_{j_1}(\emptyset) = 1 < \lambda_3(\emptyset) = \frac{4}{3} < \lambda_{j_1}(\{3\}) = \frac{5}{3} < \lambda_{j_1}(\{j_2\}) = 2 < \lambda_3(\{j_1\}) = \frac{7}{3} \\
&\quad < \lambda_{j_1}(\{3, j_2\}) = \frac{8}{3} < \lambda_{j_1}(\{j_2, j_3\}) = 3 < \lambda_3(\{j_1, j_2\}) = \frac{10}{3} < \lambda_{j_1}(\{3, j_2, j_3\}) = \frac{17}{3} \\
&\quad < \lambda_{j_1}(\{j_2, j_3, j_4\}) = 6 < \lambda_3(\{j_1, j_2, j_3\}) = \frac{19}{3}.
\end{aligned}$$

Since for system lifetime analysis we do not need to consider the last failure, we do not determine $\lambda_{j_1}(\{3, j_2, j_3, j_4\})$ nor $\lambda_3(\{j_1, j_2, j_3, j_4\})$. The above definition can be concisely written as

$$\lambda_j(A) = \begin{cases} |A| + 2\lfloor \frac{|A|}{3} \rfloor + 1, & j \in A^c \not\ni 3, \\ |A| + 2\lfloor \frac{|A|}{3} \rfloor + \frac{4}{3}, & j = 3 \in A^c, \\ |A| + 2\lfloor \frac{|A|}{3} \rfloor + \frac{2}{3}, & j \neq 3 \in A^c. \end{cases}$$

In particular, the natural condition $\lambda_i(A) < \lambda_j(B)$ holds for $|A| < |B|$ and $i \notin A, j \notin B$. Moreover, for the sets with identical cardinalities we have

$$\lambda_{j_1}(A \cup \{3\}) < \lambda_{j_1}(A \cup \{j_2\}) < \lambda_3(A \cup \{j_2\}), \qquad \{j_1, j_2, 3\} \cap A = \emptyset.$$

The latter inequality means that in any system state, component 3 is more liable to fail than any other component. The former asserts that a component has a possibility of living longer when the still operating components are different from component 3. Note that, in view of the system structure and the homogeneity of four of the components, the number of parameters is reduced from 80 to 11.

Using the definition, we calculate the cumulative hazard rates:

$$\begin{aligned}
&\Lambda(\emptyset) = \frac{16}{3} < \Lambda(\{3\}) = \frac{20}{3} < \Lambda(\{j_1\}) = \frac{23}{3} < \Lambda(\{3, j_1\}) = 8 \\
&\quad < \Lambda(\{j_1, j_2\}) = \frac{28}{3} < \Lambda(\{3, j_1, j_2\}) = \frac{34}{3} < \Lambda(\{j_1, j_2, j_3\}) = \frac{37}{3}.
\end{aligned}$$

It follows that $\Lambda$ increases with the increase of cardinality of its argument, and moreover $\Lambda(A \cup \{3\}) < \Lambda(A \cup \{j\})$, $3 \neq j \in A^c$. We determine the following probabilities for labels of consecutively failing components:

$$\mathbb{P}(J_1 = j_1, J_2 = j_2) = \frac{\lambda_{j_1}(\emptyset)}{\Lambda(\emptyset)} \frac{\lambda_{j_2}(\{j_1\})}{\Lambda(\{j_1\})} = \frac{9}{184},$$

$$\mathbb{P}(J_1 = 3, J_2 = j_2, J_3 = j_3) = \frac{\lambda_3(\emptyset)}{\Lambda(\emptyset)} \frac{\lambda_{j_2}(\{3\})}{\Lambda(\{3\})} \frac{\lambda_{j_3}(\{3, j_2\})}{\Lambda(\{3, j_2\})} = \frac{1}{48},$$

$$\mathbb{P}(J_1 = j_1, J_2 = 3, J_3 = j_3) = \frac{\lambda_{j_1}(\emptyset)}{\Lambda(\emptyset)} \frac{\lambda_3(\{j_1\})}{\Lambda(\{j_1\})} \frac{\lambda_{j_3}(\{j_1, 3\})}{\Lambda(\{j_1, 3\})} = \frac{7}{320},$$

$$\mathbb{P}(J_1 = j_1, J_2 = j_2, J_3 = 3) = \frac{\lambda_{j_1}(\emptyset)}{\Lambda(\emptyset)} \frac{\lambda_{j_2}(\{j_1\})}{\Lambda(\{j_1\})} \frac{\lambda_3(\{j_1, j_2\})}{\Lambda(\{j_1, j_2\})} = \frac{45}{2576},$$

$$\mathbb{P}(J_1 = j_1, J_2 = j_2, J_3 = j_3) = \frac{\lambda_{j_1}(\emptyset)}{\Lambda(\emptyset)} \frac{\lambda_{j_2}(\{j_1\})}{\Lambda(\{j_1\})} \frac{\lambda_{j_3}(\{j_1, j_2\})}{\Lambda(\{j_1, j_2\})} = \frac{81}{6808}.$$

These can further be used to evaluate the system lifetime. For simplicity of calculation, we assume that the baseline model distribution is standard exponential, and apply (14). Analogous results can be established for the Lomax and reflected power distributions using (27) and (28), respectively. In the exponential case, we determine that

$$\begin{aligned}
\mathbb{E}T = {} & 4\left(\frac{1}{\Lambda(\emptyset)} + \frac{1}{\Lambda(\{j_1\})}\right) \mathbb{P}(J_1 = j_1, J_2 = j_2) \\
& + 8\left(\frac{1}{\Lambda(\emptyset)} + \frac{1}{\Lambda(\{3\})} + \frac{1}{\Lambda(\{3, j_2\})}\right) \mathbb{P}(J_1 = 3, J_2 = j_2, J_3 = j_3) \\
& + 8\left(\frac{1}{\Lambda(\emptyset)} + \frac{1}{\Lambda(\{j_1\})} + \frac{1}{\Lambda(\{j_1, 3\})}\right) \mathbb{P}(J_1 = j_1, J_2 = 3, J_3 = j_3) \\
& + 4\left(\frac{1}{\Lambda(\emptyset)} + \frac{1}{\Lambda(\{j_1\})} + \frac{1}{\Lambda(\{j_1, j_2\})}\right) \mathbb{P}(J_1 = j_1, J_2 = j_2, J_3 = 3) \\
& + 16\left(\frac{1}{\Lambda(\emptyset)} + \frac{1}{\Lambda(\{j_1\})} + \frac{1}{\Lambda(\{j_1, j_2\})}\right) \mathbb{P}(J_1 = j_1, J_2 = j_2, J_3 = j_3) \\
& + 4\left(\frac{1}{\Lambda(\emptyset)} + \frac{1}{\Lambda(\{3\})} + \frac{1}{\Lambda(\{3, j_2\})} + \frac{1}{\Lambda(\{3, j_2, j_3\})}\right) \mathbb{P}(J_1 = 3, J_2 = j_2, J_3 = j_3) \\
& + 4\left(\frac{1}{\Lambda(\emptyset)} + \frac{1}{\Lambda(\{j_1\})} + \frac{1}{\Lambda(\{j_1, 3\})} + \frac{1}{\Lambda(\{j_1, 3, j_3\})}\right) \mathbb{P}(J_1 = j_1, J_2 = 3, J_3 = j_3) \\
& + 4\left(\frac{1}{\Lambda(\emptyset)} + \frac{1}{\Lambda(\{j_1\})} + \frac{1}{\Lambda(\{j_1, j_2\})} + \frac{1}{\Lambda(\{j_1, j_2, 3\})}\right) \mathbb{P}(J_1 = j_1, J_2 = j_2, J_3 = 3).
\end{aligned}$$

The first term represents system failure caused by two component failures, which happens in the four cases when $I_2$ is equal to either $\{1, 4\}$ or $\{2, 5\}$, and the failure order is arbitrary. The last three expressions correspond to the case $T = T_{4:5}$ if $I_3 = \{1, 2, 3\}$ or $I_3 = \{3, 4, 5\}$, and component 3 is respectively the first, second, or third component to fail. The remaining summands are connected with $T$ coinciding with the third failure. The consecutive terms express the contribution of $\mathbb{E}T$ in the cases that component 3 fails respectively first, second, third, or later. Performing numerical calculations yields $\mathbb{E}T \approx 0.45588$.

In order to check whether the proper location of components yields any benefit, we compare these results with the performance of identical system components with an exchangeable joint distribution. To make the comparison fair, we assume that the inter-failure cumulative hazard rates $\Lambda_i$, $i = 0, 1, 2, 3$, in the exchangeable case are identical with their expected counterparts in the non-exchangeable case; i.e., $\Lambda_0 = \Lambda(\emptyset)$, and

$$\Lambda_i = \Lambda(A_{i-1} \cup \{3\})\mathbb{P}(I_i = A_{i-1} \cup \{3\}) + \Lambda(A_{i-1} \cup \{j\})[1 - \mathbb{P}(I_i = A_{i-1} \cup \{3\})]$$

for $j \neq 3$, $|A_{i-1}| = i - 1$, $|I_i| = i$, and $i = 1, 2, 3$. By elementary calculations we obtain $\Lambda_0 = 5\frac{1}{3}$, $\Lambda_1 = 7\frac{5}{12}$, $\Lambda_2 = 9\frac{4}{23}$, and $\Lambda_3 = 12\frac{14073}{51520}$. By the Samaniego formula, the corresponding expectation of the system lifetime,

$$\mathbb{E}\tilde{T} = \frac{1}{\Lambda_0} + \frac{1}{\Lambda_1} + \frac{4}{5\Lambda_2} + \frac{1}{5\Lambda_3} \approx 0.42581,$$

is 6.528% lower than $\mathbb{E}T$.

## Acknowledgements

The authors are grateful to two anonymous referees for careful reading of the paper and valuable remarks which allowed them to correct mistakes and improve the presentation. The first author was supported by the National Science Centre, Poland, Grant No. 2015/19/B/ST1/03100. The second author was supported by the Ateneo Sapienza Research Projects *Dipendenza, disuguaglianze e approssimazioni in modelli stocastici* (2015) and *Processi stocastici: Teoria e applicazioni* (2016).

## References

[1] Aki, S. and Hirano, K. (1997). Lifetime distributions of consecutive-$k$-out-of-$n$:F systems. *Nonlinear Anal.* **30**, 555–562.

[2] Arjas, E. (1981). The failure and hazard processes in multivariate reliability systems. *Math. Operat. Res.* **6**, 551–562.

[3] Arjas, E. and Norros, I. (1984). Life lengths and association: a dynamic approach. *Math. Operat. Res.* **9**, 151–158.

[4] Barlow, R. E. and Proschan, F. (1975). Importance of system components and fault tree events. *Stoch. Process. Appl.* **3**, 153–173.

[5] Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life-Testing*. Holt, Rinehart, and Winston, New York.

[6] Boland, P. J. (2001). Signatures of indirect majority systems. *J. Appl. Prob.* **38**, 597–603.

[7] Bremaud, P. (1981). *Point Processes and Queues: Martingale Dynamics*. Springer, New York.

[8] Burkschat, M. (2009). Systems with failure-dependent lifetimes of components. *J. Appl. Prob.* **46**, 1052–1072.

[9] Burkschat, M. and Rychlik, T. (2018). Sharp inequalities for quantiles of system lifetime distributions from failure dependent proportional hazard model. *TEST* **27**, 618–638.

[10] Cramer, E. and Kamps, U. (2003). Marginal distributions of sequential and generalized order statistics. *Metrika* **58**, 293–310.

[11] Dempsey, W. and McCullagh, P. (2017). Exchangeable Markov survival process and weak continuity of predictive distributions. *Electron. J. Statist.* **11**, 5406–5451.

[12] De Santis, E., Malinovsky, Y. and Spizzichino, F. (2020). Stochastic precedence and minima among dependent variables. *Methodology Comput. Appl. Prob. Available online*, DOI: 10.1007/s11009-020-09772-3.

[13] Deshpande, J. V., Dewan, I. and Naik-Nimbalkar, U. V. (2010). A family of distributions to model load sharing systems. *J. Statist. Planning Infer.* **140**, 1441–1451.

[14] Finkelstein, M. and Hazra, N. K. (2017). On stochastic comparisons for load-sharing series and parallel systems. *Prob. Eng. Inf. Sci.* **31**, 311–329.

[15] Freund, J. E. (1961). A bivariate extension of the exponential distribution. *J. Amer. Statist. Assoc.* **56**, 971–977.

[16] Grabski, F. (2003). The reliability of an object with semi-Markov failure rate. *Appl. Math. Comput.* **135**, 1–16.

[17] Hollander, M. and Peña, E. A. (1995). Dynamic reliability models with conditional proportional hazard rates. *Lifetime Data Anal.* **1**, 377–401.

[18] Kamps, U. (1995). *A Concept of Generalized Order Statistics*. Teubner, Stuttgart.
[19] Kopocińska, I. and Kopociński, B. (1980). On system reliability under random load of elements. *Appl. Math. (Warsaw)* **17**, 5–14.
[20] Li, S., Gleaton, J. and Lynch, J. (2019). What is the shape of a bundle? An analysis of Rosen's fibrous composites experiments using the chain-of-bundles model. *Scand. J. Statist.* **46**, 59–86.
[21] Marichal, J.-L. and Mathonet, P. (2011). Extensions of system signatures to dependent lifetimes: explicit expressions and interpretations. *J. Multivariate Anal.* **102**, 931–936.
[22] Marichal, J.-L., Mathonet, P. and Waldhauser, T. (2011). On signature-based expressions of system reliability. *J. Multivariate Anal.* **102**, 1410–1416.
[23] Navarro, J. and Rychlik, T. (2007). Reliability and expectation bounds for coherent systems with exchangeable components. *J. Multivariate Anal.* **98**, 102–113.
[24] Navarro, J., Samaniego, F. J., Balakrishnan, N. and Bhattacharya, D. (2008). On the application and extension of system signatures in engineering reliability. *Naval Res. Logistics* **55**, 313–327.
[25] Navarro, J., Spizzichino, F. and Balakrishnan, N. (2010). The role of average and projected systems in the study of coherent systems. *J. Multivariate Anal.* **101**, 1471–1482.
[26] Norros, I. (1985). System weakened by failures. *Stoch. Process. Appl.* **20**, 181–196.
[27] Rajan, V. P. and Curtin, W. A. (2016). Micromechanical design of hierarchical composites using global load sharing theory. *J. Mech. Phys. Solids* **90**, 1–17.
[28] Ross, S. M., Shahshahani, M. and Weiss, G. (1980). On the number of component failures in systems whose component lives are exchangeable. *Math. Operat. Res.* **5**, 358–365.
[29] Ross, S. M. (1984). A model in which component failure rates depend on the working set. *Naval Res. Logistics* **31**, 297–300.
[30] Samaniego, F. J. (1985). On closure of the IFR class under formation of coherent systems. *IEEE Trans. Reliab.* 34, 69–72.
[31] Samaniego, F. J. (2007). *System Signatures and Their Applications in Engineering Reliability*. Springer, New York.
[32] Scala, A. and De Sanctis Lucentini, P. G. (2016). The equal load-sharing model of cascade failures in power grids. *Physica A* **462**, 737–742.
[33] Shaked, M. and Shanthikumar, J. G. (1990). Multivariate stochastic orderings and positive dependence in reliability theory. *Math. Operat. Res.* **15**, 545–552.
[34] Shaked, M. and Shanthikumar, J. G. (2006). *Stochastic Orders*. Springer, New York.
[35] Shaked, M. and Shanthikumar, J. G. (2015). Multivariate conditional hazard rate functions: an overview. *Appl. Stoch. Models Business Industry* **31**, 285–296.
[36] Schechner, Z. (1984). A load-sharing model: the linear breakdown rule. *Naval Res. Logistics* **31**, 137–144.
[37] Smaili, K., Kadri, T. and Kadry, S. (2016). Finding the PDF of the hypoexponential random variable using the Kad matrix similar to the general Vandermonde matrix. *Commun. Statist. Theory Meth.* **45**, 1542–1549.
[38] Spizzichino, F. (2001). *Subjective Probability Models for Lifetimes*. Chapman and Hall/CRC, Boca Raton, FL.
[39] Spizzichino, F. (2008). The role of signature and symmetrization for systems with non-exchangeable components. In *Advances in Mathematical Modeling for Reliability*, eds. T. Bedford, J. Quigley, L Walls, B. Alkali, A. Daneshkhah, and G. Hardman, IOS, Amsterdam, pp. 138–148.
[40] Spizzichino, F. (2019). Reliability, signature, and relative quality functions for systems under time-homogeneous load-sharing models. *Appl. Stoch. Models Business Industry* **35**, 158–176.
[41] Sutar, S. and Naik-Nimbalkar, U. V. (2019). A load share model for non-identical components of a $k$-out-of-$m$ system. *Appl. Math. Modelling* **72**, 486–498.
[42] Swolfs, Y., McMeeking, R. M., Rajan, V. P., Zok, F. W., Verpoest, I. and Gorbatikh, L. (2015). Global load-sharing model for unidirectional hybrid fibre-reinforced composites. *J. Mech. Phys. Solids* **84**, 380–394.
[43] Zhang, Z. and Balakrishnan, N. (2017). Stochastic properties and parameter estimation for a general load-sharing system. *Commun. Statist. Theory Meth.* **46**, 747–760.