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Composition of bi-Sobolev homeomorphisms

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We give sharp conditions under which the composition of two homeomorphisms of finite distortion is of finite distortion and has integrable distortion. As an application, we obtain a generalization of the classical uniqueness theorem of homeomorphic solution to the measurable Riemann mapping problem.

1. Introduction

Let Ω, Ω', Ω'' be domains in $\mathbb{R}^n$. The main theme running throughout this paper is homeomorphisms $f\colon \Omega \xrightarrow{\text{onto}} \Omega'$ and $g\colon \Omega' \xrightarrow{\text{onto}} \Omega''$ and their composition $h = g \circ f\colon \Omega \to \Omega''$. The term $\mathcal{W}^{1,p}_{\text{loc}}(\Omega, \Omega')$-homeomorphism refers to a continuous bijection $f\colon \Omega \to \Omega'$ whose components belong to the Sobolev space $\mathcal{W}^{1,p}_{\text{loc}}(\Omega)$, $1 \leqslant p \leqslant \infty$. If the exponent p need not be spelled out, we simply say that f is a *Sobolev homeomorphism*. Recall the following concept, originally proposed in [15].

Definition 1.1. A homeomorphism $f\colon \Omega \xrightarrow{\text{onto}} \Omega'$ is called a bi-Sobolev mapping if $f \in \mathcal{W}^{1,p}_{\text{loc}}(\Omega, \Omega')$ and its inverse $f^{-1} \in \mathcal{W}^{1,p}_{\text{loc}}(\Omega', \Omega)$, for some $1 \leqslant p \leqslant \infty$.

When the Sobolev exponent of f is essential, we shall emphasize it by saying that f is a $\mathcal{W}^{1,p}_{\text{loc}}(\Omega, \Omega')$ *bi-Sobolev map*. It is well known [32] that the Sobolev regularity of homeomorphisms in $\mathcal{W}^{1,p}_{\text{loc}}(\Omega, \Omega')$ is preserved under a bi-Lipschitz change of variables in the domain Ω. Another useful class of change of variables for Sobolev functions is furnished by quasiconformal mappings, which are a natural

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generalization of conformal mappings. By virtue of the classical analytic definition, a Sobolev homeomorphism $f\colon \Omega \xrightarrow{\text{onto}} \Omega'$ is K-quasiconformal, $1 \leqslant K < \infty$, if

$$|Df(x)|^n \leqslant K J_f(x) \quad \text{for almost every } x \in \Omega. \tag{1.1}$$

Hereafter, $|Df(x)|$ denotes the operator norm of the differential matrix and $J_f(x) = J(x,f) = \det Df(x)$ is the Jacobian determinant. If f is K-quasiconformal, then, for any $\varphi \in \mathcal{W}^{1,n}_{\mathrm{loc}}(\Omega')$, the composition $\varphi \circ f$ belongs to $\mathcal{W}^{1,n}_{\mathrm{loc}}(\Omega)$ [2]. Then the chain rule shows that, whenever $g\colon \Omega' \to \mathbb{R}^n$ is K'-quasiconformal for some $K' \geqslant 1$, the composition $g \circ f\colon \Omega \to \mathbb{R}^n$ is $K{\cdot}K'$-quasiconformal.

Further developments of geometric function theory are concerned with non-injective mappings, also allowing K to depend on x. The following conditions are necessary for a viable theory of such mappings.

Definition 1.2. A mapping $f \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is said to have finite distortion if there exists a measurable function $K\colon \Omega \to [1,\infty)$ such that

$$|Df(x)|^n \leqslant K(x) J_f(x). \tag{1.2}$$

Moreover, we assume that $J_f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Note that in the case of a homeomorphism the assumption on local integrability of the Jacobian determinant is redundant. As a matter of fact, (bona fide) local $\mathcal{L}^1$-integrability of the Jacobian holds for every Sobolev homeomorphism. The above definition, introduced in [1,16,19], was worked out and thoroughly developed in [4,9, 17,21,22]. However, the concept of mappings of finite distortion can be traced back to the work of Vodop′janov and Gol'dšteĭn [31] and Iwaniec and Šverák [18]. We take an opportunity here to explain the essence of mappings of finite distortion. First of all, note that the existence of a measurable function K finite almost everywhere (a.e.) and satisfying (1.2) amounts to saying that

$$J_f(x) = 0 \quad \Longrightarrow \quad Df(x) = 0 \quad \text{a.e.} \tag{1.3}$$

This condition makes it possible to consider the *distortion quotient*

$$\frac{|Df(x)|^n}{J_f(x)} \quad \text{for almost every } x \in \Omega. \tag{1.4}$$

Hereafter, the undetermined ratio $0/0$ is understood to be equal to 1 for x in the zero set of the Jacobian

$$K_f(x) = \begin{cases} \dfrac{|Df(x)|^n}{J_f(x)} & \text{if } J_f(x) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{1.5}$$

In other words, K_f is the smallest function greater than or equal to 1 for which (1.2) holds a.e. A part of the study of mappings of finite distortion that is vital to us is the regularity of the inverse of a Sobolev homeomorphism [3, 6, 12, 14, 15, 25]. In particular, we recall the following result from [3]. If f is a homeomorphism in $\mathcal{W}^{1,n-1}_{\mathrm{loc}}$ with finite distortion, then f^{-1} is in $\mathcal{W}^{1,1}_{\mathrm{loc}}$ and has finite distortion. In [13] the following question was raised: when does a composition $g \circ f$ of two homeomorphisms $f\colon \Omega \to \Omega'$ and $g\colon \Omega' \to \Omega''$ of finite distortion also have finite distortion? The major difficulty lies in the fact that f^{-1} need not satisfy the N-*condition of Lusin*. In other words, the image of a null set (in terms of Lebesgue measure) under f^{-1} may fail to be measurable. This poses serious problems concerning the measurability of the composition $g \circ f$. For this reason we assume that f^{-1} satisfies the N-condition of Lusin. Concerning the composition map, the next result can be easily deduced by following [10]. Let $f\colon \Omega \xrightarrow{\text{onto}} \Omega'$ and $g\colon \Omega' \xrightarrow{\text{onto}} \Omega''$ be homeomorphisms, with f^{-1} and g $\mathcal{W}^{1,n}_{\text{loc}}$-regular and of finite distortion. Then $g \circ f$ belongs to $\mathcal{W}^{1,1}_{\text{loc}}$ by [10, theorem 1.1]. Recently, in [29] it was observed that the above argument yields also that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ belongs to $\mathcal{W}^{1,1}_{\text{loc}}$, that is, $g \circ f$ is a bi-Sobolev mapping. Hence, in dimension $n = 2$, the composition has finite distortion, by a new strategic characterization of Sobolev homeomorphisms of finite distortion [15]. We give more details in § 3. Here we only remark that, in general, the Jacobian of a homeomorphism of finite distortion may vanish on a set of positive measure [16], but such sets must have no interior. Let us mention here an alarming recent discovery that $\mathcal{W}^{1,p}_{\text{loc}}(\Omega, \Omega')$-homeomorphisms, with $1 \leqslant p < n$, may have vanishing Jacobian determinant a.e. in Ω. Such amazing mappings have been constructed by Hencl [11].

In [13] Hencl and Koskela studied the integrability properties of the distortion of the composition map $g \circ f$. On this subject, we state the following theorem.

Theorem 1.3. *Let $f\colon \Omega \xrightarrow{onto} \Omega'$ and $g\colon \Omega' \xrightarrow{onto} \Omega''$ be homeomorphisms of finite distortion. Assume that*

$$K_g \in \operatorname{Exp}_{\text{loc}}(\Omega'), \tag{1.6}$$

$$K_f \in \mathcal{L}^n_{\text{loc}}(\Omega). \tag{1.7}$$

Then

$$g \circ f\colon \Omega \to \Omega'' \quad \textit{is a mapping of finite distortion} \tag{1.8}$$

and

$$K_{g \circ f} \in \mathcal{L}^1_{\text{loc}}(\Omega). \tag{1.9}$$

Actually, in § 4 we present a sharp result (see theorem 4.1) that is more general than theorem 1.3.

As is well known, the case of dimension $n = 2$ is quite special. It is rather extraordinary that bi-Sobolev homeomorphisms are exactly those that have finite distortion [15]. Quasiconformal mappings provide a particularly useful class, which lies between homeomorphisms and diffeomorphisms. They are more flexible than bi-Lipschitz homeomorphisms. Bi-Sobolev mappings are even more flexible. For a bi-Sobolev map $f\colon \Omega \to \Omega'$ we shall examine the *distortion tensor*; that is, a Borel measurable matrix field

$$G_f(x) = \begin{cases} \dfrac{D^t f(x) Df(x)}{J_f(x)} & \text{if } J_f(x) > 0, \\ I & \text{otherwise.} \end{cases}$$

Hence, G_f is a symmetric matrix with $\det G_f \equiv 1$. Note that, for all $\xi \in \mathbb{R}^2$ and for almost every $x \in \Omega$, we have

$$\frac{|\xi|^2}{K_f(x)} \leqslant \langle G_f(x)\xi, \xi\rangle \leqslant K_f(x)|\xi|^2.$$

As an application of our results in §§ 3 and 4, in § 5 we obtain the following uniqueness result.

THEOREM 1.4. *Let Ω and Ω' be planar domains. Let $g, h\colon \Omega \xrightarrow{onto} \Omega'$ be $\mathcal{W}^{1,2}$-homeomorphisms of finite distortion and assume that*

$$G_g(x) = G_h(x) \tag{1.10}$$

for almost every $x \in \Omega$. Then the mapping

$$\varphi = g \circ h^{-1} \quad \text{is conformal.} \tag{1.11}$$

2. Preliminary results

2.1. Notation

Given a square matrix A, we denote by $|A|$ its operator norm, that is,

$$|A| = \sup\{|A\xi|\colon \xi \in \mathbb{R}^n,\ |\xi| = 1\}.$$

The adjugate $\operatorname{adj} A$ is the transpose of the cofactor matrix. So, we have the formula

$$A(\operatorname{adj} A) = (\operatorname{adj} A)A = \boldsymbol{I} \det A,$$

where $\boldsymbol{I}$ denotes the identity matrix. Thus, if A is non-singular,

$$\frac{1}{\det A} \operatorname{adj} A = A^{-1}. \tag{2.1}$$

The well-known Hadamard inequality implies

$$|\operatorname{adj} A| \leqslant |A|^{n-1}.$$

2.2. Some function spaces

Our main source here is [16, § 4.12]. We need to consider the Zygmund space $\mathcal{L}^p \log^\alpha \mathcal{L}(\Omega)$ for $1 \leqslant p < \infty$, $\alpha \in \mathbb{R}$ ($\alpha \geqslant 0$ for $p = 1$) and $\Omega \subset \mathbb{R}^n$. This is the Orlicz space generated by the function

$$\Phi(t) = t^p \log^\alpha(a + t), \quad t \geqslant 0,$$

where $a > 0$ is a suitably large constant, so that Φ is increasing and convex on $[0, \infty[$. The choice of a is immaterial, as we shall always consider these spaces on bounded domains. Thus, more explicitly, for a measurable function u on Ω, $u \in \mathcal{L}^p \log^\alpha \mathcal{L}(\Omega)$ simply means that

$$\int_\Omega |u|^p \log^\alpha(a + |u|)\,\mathrm{d}x < \infty.$$

As an example, for $\alpha = 0$ we have the ordinary Lebesgue spaces. We consider in $\mathcal{L}^p \log^\alpha \mathcal{L}(\Omega)$ the Luxemburg norm

$$\|u\|_{\mathcal{L}^p \log^\alpha \mathcal{L}} = \inf\left\{\lambda > 0\colon \int_\Omega \Phi(|u|/\lambda)\,\mathrm{d}x \leqslant 1\right\}.$$

The following Hölder-type inequality for Zygmund spaces will be important:

$$\|u_1 \cdots u_k\|_{\mathcal{L}^p \log^\alpha \mathcal{L}} \leqslant C\|u_1\|_{\mathcal{L}^{p_1} \log^{\alpha_1} \mathcal{L}} \cdots \|u_k\|_{\mathcal{L}^{p_k} \log^{\alpha_k} \mathcal{L}}, \tag{2.2}$$

where $p_i > 1$, $\alpha_i \in \mathbb{R}$, for $i = 1, \dots, k$, and

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_k}, \qquad \frac{\alpha}{p} = \frac{\alpha_1}{p_1} + \cdots + \frac{\alpha_k}{p_k}.$$

The positive constant C in (2.2) is independent of u_i. We write

$$u \in \mathcal{L}^p \log^\alpha \mathcal{L}_{\mathrm{loc}}(\Omega) \quad \text{if } u \in \mathcal{L}^p \log^\alpha \mathcal{L}(E),$$

for every compact subset E of Ω.

The exponential class $\operatorname{Exp}(\Omega)$ is formed by measurable functions u on Ω for which there exists $\lambda = \lambda(u) > 0$ such that

$$\exp(\lambda|u|) \in \mathcal{L}^1(\Omega).$$

The space $\operatorname{Exp}_{\mathrm{loc}}(\Omega)$ is defined in a similar way to above.

We shall need the following elementary inequality.

Lemma 2.1. *Fix $\lambda > 0$ and $\alpha > 0$. Then, for all $a \geqslant 0$, $b \geqslant 0$, we have*

$$a^\alpha b \leqslant C[\exp(\lambda a) + b\log^\alpha(\mathrm{e} + b)], \tag{2.3}$$

where

$$C = \left(\frac{\mathrm{e} + \alpha}{\lambda \mathrm{e}}\right)^\alpha.$$

Proof. If

$$a \leqslant \frac{\mathrm{e} + \alpha}{\lambda \mathrm{e}} \log(\mathrm{e} + b),$$

then the inequality is trivial. In the opposite case, we have

$$a^\alpha b \leqslant \exp\left[\alpha \log a + \frac{\lambda \mathrm{e}}{\mathrm{e} + \alpha} a\right]$$

and it is easily seen that the right-hand side does not exceed $C\exp(\lambda a)$. Indeed, this is equivalent to

$$\frac{\mathrm{e} + \alpha}{\lambda \mathrm{e}} \exp\left[\frac{\lambda}{\mathrm{e} + \alpha} a - \log a\right] \geqslant 1,$$

and the minimum of the expression in the left-hand side for $a > 0$ is exactly 1. □

2.3. Differentiability properties

We decompose the domain Ω of a given mapping f as follows:

$$\Omega = \mathcal{R}_f \cup \mathcal{Z}_f \cup \mathcal{E}_f,$$

where

$$\begin{aligned}\mathcal{R}_f &= \{x \in \Omega\colon f \text{ is differentiable at } x \text{ and } J_f(x) \neq 0\},\\ \mathcal{Z}_f &= \{x \in \Omega\colon f \text{ is differentiable at } x \text{ and } J_f(x) = 0\},\\ \mathcal{E}_f &= \{x \in \Omega\colon f \text{ is not differentiable at } x\}.\end{aligned}$$

Differentiability is understood in the classical sense. These are Borel sets if f is a homeomorphism. Moreover, $f(\mathcal{R}_f) = \mathcal{R}_{f^{-1}}$ and, for all $x \in \mathcal{R}_f$,

$$Df^{-1}(f(x)) = (Df(x))^{-1}, \qquad J_{f^{-1}}(f(x)) = \frac{1}{J_f(x)}. \tag{2.4}$$

A Sobolev homeomorphism f is known to be differentiable a.e. in Ω if $|Df| \in \mathcal{L}^p_{\mathrm{loc}}$ with $p > n-1$ [30]. For such a map $|\mathcal{E}_f|$ vanishes and either $J_f(x) \geqslant 0$ or $J_f(x) \leqslant 0$ a.e. We will assume $J_f \geqslant 0$. Moreover, Df is a Borel function and is the differential also in the sense of distributions.

2.4. Area formula

Let $f\colon \Omega \to \mathbb{R}^n$ be a mapping defined in a domain of $\mathbb{R}^n$. We say that f satisfies the Lusin N-condition if the implication

$$|E| = 0 \quad\Longrightarrow\quad |f(E)| = 0$$

holds for any set $E \subset \Omega$. Here, $|E|$ denotes the Lebesgue measure of E. For a homeomorphism f, the N-condition holds if $f \in \mathcal{W}^{1,n}$ [27], but may fail if $f \in \mathcal{W}^{1,p}$ with $p < n$ [26]. Sharp results ensuring the N-condition can be found in [21,22]. Let $f \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism and let η be a non-negative Borel measurable function on $\mathbb{R}^n$. We have the inequality

$$\int_B \eta(f(x))|J_f(x)|\,\mathrm{d}x \leqslant \int_{f(B)} \eta(y)\,\mathrm{d}y \tag{2.5}$$

for every $B \subset \Omega$ Borel set [5, theorem 3.1.8]. We note the following consequence of (2.5). If $B' \subset f(\Omega)$ is a Borel subset with $|B'| = 0$, then $J_f(x) = 0$ for almost every $x \in f^{-1}(B')$. Indeed,

$$\int_{f^{-1}(B')} |J_f(x)|\,\mathrm{d}x \leqslant \int_{B'} \mathrm{d}y = |B'| = 0.$$

For example, if f^{-1} is differentiable a.e. on $f(\Omega)$, then $J_f(x) = 0$, for almost every $x \in f^{-1}(\mathcal{E}_{f^{-1}})$. We say that the area formula holds for f on B if (2.5) is valid as an equality, that is,

$$\int_B \eta(f(x))|J_f(x)|\,\mathrm{d}x = \int_{f(B)} \eta(y)\,\mathrm{d}y, \tag{2.6}$$

for all η. It is well known that there exists a set $\tilde{\Omega} \subset \Omega$ of full measure such that the area formula holds for f on $\tilde{\Omega}$. As a consequence, if f is a Sobolev homeomorphism with f^{-1} satisfying the N-condition, then $J_f(x) > 0$ for almost every $x \in \Omega$. Indeed, by the area formula (2.6) with $B = \tilde{\Omega}$,

$$|f(\{x \in \tilde{\Omega}\colon J_f(x) = 0\})| = 0$$

and hence, by N-condition for f^{-1}, and since $\tilde{\Omega}$ has full measure,

$$|\{x \in \Omega\colon J_f(x) = 0\}| = |\{x \in \tilde{\Omega}\colon J_f(x) = 0\} \cup (\Omega \setminus \tilde{\Omega})| = 0.$$

Moreover, the area formula holds on each set B on which f satisfies the N-condition. Note that the area formula holds on the set $\mathcal{R}_f \cup \mathcal{Z}_f$, where f is differentiable. In particular, we have the following version of the Sard theorem

$$|f(\mathcal{Z}_f)| = 0. \tag{2.7}$$

Therefore, if f is a homeomorphism differentiable a.e. and satisfying the N-condition, then f^{-1} is also differentiable a.e. In fact, f^{-1} is differentiable in $f(\mathcal{R}_f)$, which is a subset of full measure of $f(\Omega)$, since

$$f(\Omega) \setminus f(\mathcal{R}_f) = f(\mathcal{Z}_f) \cup f(\mathcal{E}_f)$$

has measure zero by (2.7) and the N-condition, as $|\mathcal{E}_f| = 0$ by assumptions.

2.5. Distortion functions

There are several distortion functions of interest in geometric function theory. We refer the reader to [16] for a comprehensive treatment. Here, in addition to the outer distortion already introduced in (1.5), we shall need to consider the *inner distortion*. A mapping $f \in \mathcal{W}^{1,n-1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ has finite inner distortion if J_f is strictly positive a.e. on the set where $\operatorname{adj} Df \neq 0$. We also assume that the Jacobian is locally integrable. For such a map, we call inner distortion of f the smallest function $K^I_f \geqslant 1$ such that

$$|\operatorname{adj} Df(x)|^n \leqslant K^I_f(x) J_f(x)^{n-1}, \tag{2.8}$$

for almost every $x \in \Omega$. Clearly, a map of finite outer distortion has also finite inner distortion and $K^I_f \leqslant (K_f)^{n-1}$, as a consequence of the Hadamard inequality. In dimension $n = 2$ the two notions coincide.

2.6. Radial stretching

Many critical examples are provided by radial stretchings

$$f(x) = \frac{x}{|x|}\rho(|x|). \tag{2.9}$$

In what follows, we assume that ρ is an absolutely continuous and strictly increasing function on the interval $[0,1]$ satisfying $\rho(0) = 0$ and $\rho(1) = 1$. As a consequence, the map defined by (2.9) is a Sobolev homeomorphism of the unit ball $\boldsymbol{B}$ onto itself, the inverse mapping being of course

$$f^{-1}(y) = \frac{y}{|y|}\rho^{-1}(|y|). \tag{2.10}$$

Moreover, we can easily find (setting $r = |x|$)

$$Df(x) = \frac{\rho(r)}{r}\boldsymbol{I} + \left[\rho'(r) - \frac{\rho(r)}{r}\right]\frac{x\otimes x}{r^2}, \qquad J_f(x) = \rho'(r)\left[\frac{\rho(r)}{r}\right]^{n-1}.$$

Hence, $J_f(x) \geqslant 0$, for almost every $x \in \boldsymbol{B}$. If we also assume that $r \mapsto \rho(r)/r$ is increasing, then $|Df(x)| = \rho'(r)$, and hence f has finite distortion $K = K_f$ given by

$$K(x) = K(r) = \left[\frac{r\rho'(r)}{\rho(r)}\right]^{n-1}. \tag{2.11}$$

Moreover, we also find

$$\operatorname{adj} Df(x) = \left[\frac{\rho(r)}{r}\right]^{n-2}\left\{\rho'(r)\boldsymbol{I} + \left[\frac{\rho(r)}{r} - \rho'(r)\right]\frac{x\otimes x}{r^2}\right\} \tag{2.12}$$

and the inner distortion is

$$K^I(x) = \frac{r\rho'(r)}{\rho(r)} = K(x)^{1/(n-1)}. \tag{2.13}$$

We can immediately express ρ in terms of K from (2.11):

$$\rho(r) = \exp\left[\int_1^r K(t)^{1/(n-1)}\frac{\mathrm{d}t}{t}\right]. \tag{2.14}$$

Conversely, given a function $K \geqslant 1$ with $K^{1/(n-1)}$ locally integrable on $]0,1]$, formula (2.14) yields the function ρ verifying (2.11), $\rho(0) = 0$, $\rho(1) = 1$, and such that $r \mapsto \rho(r)/r$ is increasing.

3. Composition of Sobolev homeomorphisms

Under the assumption (1.6) that g has locally exponentially integrable distortion, we can easily obtain $|Dg| \in \mathcal{L}^n \log^{-1}\mathcal{L}_{\mathrm{loc}}$, but without any additional condition we cannot deduce that $g \in \mathcal{W}^{1,n}_{\mathrm{loc}}$. So [13, theorem 1.1] does not apply to showing that the composition $g \circ f$ belongs to $\mathcal{W}^{1,1}_{\mathrm{loc}}$. On the other hand, (1.7) implies $|Df^{-1}| \in \mathcal{L}^n \log^{1/(n-1)}\mathcal{L}_{\mathrm{loc}}$ (see corollary 4.5). To take advantage of this regularity of f^{-1} and compensate for the lack of regularity of g, we need to extend [10, theorem 1.1] concerning the composition of Sobolev mappings to the case of derivatives in Zygmund classes.

Theorem 3.1. *Let $r > n-1$ and $\alpha \in \mathbb{R}$ be given numbers and set $q = r/(r-n+1)$. Let $f\colon \Omega \to \mathbb{R}^n$ be a homeomorphism with f^{-1} of finite distortion, $u \in \mathcal{W}^{1,1}_{\mathrm{loc}}(f(\Omega))$, and assume that*

$$|Df^{-1}| \in \mathcal{L}^r \log^{\alpha}\mathcal{L}_{\mathrm{loc}}(f(\Omega)), \qquad |\nabla u| \in \mathcal{L}^q \log^{-\alpha(q-1)}\mathcal{L}_{\mathrm{loc}}(f(\Omega)). \tag{3.1}$$

Moreover, for $q > n$, or $q = n$ and $\alpha < -1$, assume also that u is continuous. Then $u \circ f \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\Omega)$.

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As is well known, for $q > n$ there is a continuous representative of u. This is true also if $q = n$ and $\alpha < -1$, so that $\beta = -\alpha(q-1) > n-1$ [20]. Indeed, this can also be deduced easily using Hölder's inequality in Zygmund spaces (2.2). Fixing a ball $B \Subset \Omega$ and denoting as usual by u_B the integral mean of u over B, for almost every $x \in B$, we have

$$\begin{aligned} |u(x) - u_B| &\leqslant C \int_B |x-y|^{1-n} |\nabla u(y)| \, \mathrm{d}y \\ &\leqslant C \| \, |x - \cdot|^{-1} \|^{n-1}_{\mathcal{L}^n \log^\alpha \mathcal{L}(B)} \|\nabla u\|_{\mathcal{L}^n \log^\beta \mathcal{L}(B)}. \end{aligned} \tag{3.2}$$

The first inequality in (3.2) is well known, while the second follows by (2.2). Since the function $y \mapsto 1/|y|$ belongs to $\mathcal{L}^n \log^\alpha \mathcal{L}_{\mathrm{loc}}(\mathbb{R}^n)$, by a routine argument, (3.2) implies, for example, that the approximation of u by standard mollification converges locally uniformly on Ω. The choice of the continuous representative of u avoids problems in defining $u \circ f$. On the other hand, if $q < n$, or $q = n$ and $\alpha \geqslant -1$, then f^{-1} satisfies the N-condition of Lusin [21], and hence $u \circ f$ does not depend on the representative of u.

Proof of theorem 3.1. Consider first the case $u \in C^\infty(f(\Omega))$. Then, u being locally Lipschitz continuous and f continuous, we have $u \circ f \in \mathcal{W}^{1,1}_{\mathrm{loc}}$ and

$$\nabla(u \circ f)(x) = \nabla u(f(x)) Df(x).$$

Moreover, f has finite distortion [14, 25], and hence

$$J_f(x) = 0 \implies \nabla(u \circ f)(x) = 0.$$

Let us prove that, for any ball $B \Subset \Omega$, we have

$$\int_B |\nabla(u \circ f)| \, \mathrm{d}x \leqslant \int_{f(B)} |\nabla u(y)| |Df^{-1}(y)|^{n-1} \, \mathrm{d}y. \tag{3.3}$$

Recall [30] that f^{-1} is differentiable a.e., i.e., $|\mathcal{E}_{f^{-1}}| = 0$, thus $J_f(x) = 0$ for almost every $x \in f^{-1}(\mathcal{E}_{f^{-1}})$ (see § 2.4). Furthermore, by Sard's lemma, $|f^{-1}(\mathcal{Z}_{f^{-1}})| = 0$ and therefore $\nabla(u \circ f)(x) = 0$ for almost every $x \in \Omega \setminus f^{-1}(\mathcal{R}_{f^{-1}})$. On the other hand, for all $y \in \mathcal{R}_{f^{-1}}$ we have

$$J_f(f^{-1}(y)) = \frac{1}{J_{f^{-1}}(y)}, \qquad Df(f^{-1}(y)) = (Df^{-1}(y))^{-1}. \tag{3.4}$$

Now, defining the Borel set $A = B \cap f^{-1}(\mathcal{R}_{f^{-1}})$, by using area formula (2.5) and (3.4) we compute

$$\begin{aligned} \int_B |\nabla(u \circ f)| \, \mathrm{d}x &\leqslant \int_A |\nabla u(f(x))| \frac{|Df(x)|}{J_f(x)} J_f(x) \, \mathrm{d}x \\ &\leqslant \int_{f(A)} |\nabla u(y)| \frac{|Df(f^{-1}(y))|}{J_f(f^{-1}(y))} \, \mathrm{d}y \\ &\leqslant \int_{f(B)} |\nabla u(y)| \, |\mathrm{adj}\, Df^{-1}(y)| \, \mathrm{d}y, \end{aligned} \tag{3.5}$$

which implies (3.3). Using Hölder's inequality in Zygmund spaces (2.2), we deduce from (3.3) that

$$\int_B |\nabla(u \circ f)| \, \mathrm{d}x \leqslant C \|\nabla u\|_{\mathcal{L}^q \log^{-\alpha(q-1)} \mathcal{L}(f(B))} \|Df^{-1}\|^{n-1}_{\mathcal{L}^r \log^\alpha \mathcal{L}(f(B))}.$$

Now let u be an arbitrary function in $\mathcal{W}^{1,1}_{\mathrm{loc}}(f(\Omega))$ satisfying the assumptions. As in [10], by the estimate (3.3) we see that, if u_j, $j = 1, 2, \dots$, is an approximation of u by standard mollification, then $\nabla(u_j \circ f)$ is a Cauchy sequence in $\mathcal{L}^1(B)$. □

Corollary 3.2. *Let* $f\colon \Omega \xrightarrow{onto} \Omega'$ *and* $g\colon \Omega' \xrightarrow{onto} \Omega''$ *be homeomorphisms, with* f^{-1} *and* g *of finite distortion. If*

$$|Df^{-1}| \in \mathcal{L}^n \log^\alpha \mathcal{L}_{\mathrm{loc}} \quad \text{and} \quad |Dg| \in \mathcal{L}^n \log^{-\alpha(n-1)} \mathcal{L}_{\mathrm{loc}},$$

with $\alpha \geqslant 0$, *then* $h = g \circ f \in \mathcal{W}^{1,1}_{\mathrm{loc}}$ *and has finite distortion. Moreover,*

$$K_h(x) \leqslant K_g(f(x)) K_f(x) \quad \text{for almost every } x \in \Omega. \tag{3.6}$$

Proof. Note that, for $r = n$, the number q defined in theorem 3.1 equals n, hence $h \in \mathcal{W}^{1,1}_{\mathrm{loc}}$. Furthermore, the chain rule is valid, as f and g are differentiable a.e. This follows directly by [30] for g and f^{-1}, and then also for f, as f^{-1} verifies the N-condition (see § 2.4). The map h is differentiable at every point x in the set of full measure

$$E = f^{-1}(\mathcal{R}_{f^{-1}} \cap (\mathcal{R}_g \cup \mathcal{Z}_g)),$$

and we have

$$Dh(x) = Dg(f(x)) Df(x), \qquad J_h(x) = J_g(f(x)) J_f(x). \tag{3.7}$$

From these formulae we can deduce that

$$J_h(x) = 0 \quad \Longrightarrow \quad Dh(x) = 0, \tag{3.8}$$

for almost every $x \in \Omega$, that is, the composition map h has finite distortion. To this end, recall that since g has finite distortion, there exists a set $E' \subset \Omega'$ such that $|\Omega' \setminus E'| = 0$ and

$$J_g(y) = 0 \quad \Longrightarrow \quad Dg(y) = 0 \quad \text{for every } y \in E'.$$

By (3.7), $J_h(x) = 0$ can only happen on E for

$$x \in f^{-1}(\mathcal{R}_{f^{-1}} \cap \mathcal{Z}_g),$$

so that $J_g(f(x)) = 0$, hence also $Dg(f(x)) = 0$ and in turn $Dh(x) = 0$, if $f(x) \in E'$. Therefore, (3.8) holds at every point x in the set of full measure

$$f^{-1}(\mathcal{R}_{f^{-1}} \cap (\mathcal{R}_g \cup \mathcal{Z}_g) \cap E').$$

The above argument also gives

$$K_h(x) = \frac{|Dh(x)|^n}{J_h(x)} \leqslant \frac{|Dg(f(x))|^n}{J_g(f(x))} \cdot \frac{|Df(x)|^n}{J_f(x)} = K_g(f(x)) K_f(x)$$

on $f^{-1}(\mathcal{R}_{f^{-1}} \cap \mathcal{R}_g)$, and $K_h(x) = 1$ a.e. on the complementary. Thus, inequality (3.6) follows. □

4. Integrability of the distortion of the composition map

In [13, theorem 6.3] some integrability properties of the distortion of the composition map $h = g \circ f$ are proved under suitable integrability assumptions on the distortion functions K_f and K_g. We prove the optimal integrability of K_h in the following theorem.

THEOREM 4.1. *Given $p \geqslant n-1$ and $\alpha > 0$, define*

$$q = \frac{\alpha p}{\alpha + n - 1}. \tag{4.1}$$

Let $f\colon \Omega \xrightarrow{onto} \Omega'$ and $g\colon \Omega' \xrightarrow{onto} \Omega''$ be homeomorphisms of finite distortion. Assume that

$$K_g^\alpha \in \mathrm{Exp}_{\mathrm{loc}}(\Omega'), \tag{4.2}$$

$$K_f \in \mathcal{L}^p_{\mathrm{loc}}(\Omega), \tag{4.3}$$

and, if $\alpha \leqslant 1$ and $\alpha(p-n+1) < 1$, also

$$|Dg| \in \mathcal{L}^n \log^{-p+n-1} \mathcal{L}_{\mathrm{loc}}(\Omega). \tag{4.4}$$

Then the composition $g \circ f\colon \Omega \to \Omega''$ has finite distortion verifying

$$K^q_{g \circ f} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega). \tag{4.5}$$

Note that the above statement reduces to theorem 1.3 for $p = n$ and $\alpha = 1$. The integrability property (4.5) is optimal in view of examples 4.6 and 4.7. To prove theorem 4.1 we need to deduce regularity of f^{-1} as a consequence of integrability assumptions on the distortion K_f. We give a sharp statement in terms of the inner distortion.

LEMMA 4.2. *Let $n \geqslant 2$, let $1 \leqslant q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a domain. If $f \in \mathcal{W}^{1,n-1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ is a homeomorphism of finite inner distortion, with $K^I_f \in \mathcal{L}^q_{\mathrm{loc}}(\Omega)$, then $f^{-1} \in \mathcal{W}^{1,n}_{\mathrm{loc}}(f(\Omega); \mathbb{R}^n)$ has finite distortion and*

$$J_{f^{-1}} \log^q(\mathrm{e} + J_{f^{-1}}) \in \mathcal{L}^1_{\mathrm{loc}}(f(\Omega)), \tag{4.6}$$

$$|Df^{-1}|^n \log^{q-1}(\mathrm{e} + |Df^{-1}|) \in \mathcal{L}^1_{\mathrm{loc}}(f(\Omega)). \tag{4.7}$$

Proof. By [6, theorem 2.3] we know that $f^{-1} \in \mathcal{W}^{1,1}_{\mathrm{loc}}$ has finite (outer) distortion. Moreover, by [6, equation (2.7)] and the area formula (2.5), we get $f^{-1} \in \mathcal{W}^{1,n}_{\mathrm{loc}}$, since $K^I_f \in \mathcal{L}^1_{\mathrm{loc}}$. In particular, we have (4.7) for $q = 1$. As a matter of fact, (4.7) and (4.6) are equivalent to each other, for all $q \geqslant 1$. Indeed, (4.7) implies (4.6) with no conditions on the distortion, assuming merely $J_{f^{-1}} \geqslant 0$ a.e., by higher integrability of the Jacobian determinant [8]. On the other hand, (4.6) is equivalent to $J_{f^{-1}} \log^q(\mathrm{e} + |Df^{-1}|) \in \mathcal{L}^1_{\mathrm{loc}}$ by (2.3), and hence implies (4.7) by (4.15) (for $\alpha = q$). Note also that f^{-1} is differentiable a.e. [30] and satisfies the N-condition [27], hence $J_f(x) > 0$ for almost every $x \in \Omega$ and also f is differentiable a.e. (see § 2.4). Assume now that $q > 1$. There is an interesting iterative argument which proves that

$$|Df^{-1}|^n \log^{\alpha-1}(\mathrm{e} + |Df^{-1}|) \in \mathcal{L}^1_{\mathrm{loc}}(f(\Omega)) \tag{4.8}$$

for any α such that

$$1 \leqslant \alpha < q. \tag{4.9}$$

We briefly describe this argument now; more details will be given later. Let $\gamma = 1 - 1/q$. Assume that

$$\log^{\beta}\left(\mathrm{e} + \frac{1}{J_f}\right) \in \mathcal{L}^1_{\mathrm{loc}} \tag{4.10}$$

for some $\beta \geqslant 0$. Then, arguing as in the proof of [13, lemma 6.2], essentially using the area formula, we can show that

$$|Df^{-1}|^n \log^{\gamma\beta}(\mathrm{e} + |Df^{-1}|) \in \mathcal{L}^1_{\mathrm{loc}}. \tag{4.11}$$

By the higher integrability of the Jacobian determinant, (4.11) implies

$$J_{f^{-1}} \log^{\gamma\beta+1}(\mathrm{e} + J_{f^{-1}}) \in \mathcal{L}^1_{\mathrm{loc}}. \tag{4.12}$$

By the area formula again, as in the proof of theorem 6.1 of [12], we then have

$$\log^{\gamma\beta+1}\left(\mathrm{e} + \frac{1}{J_f}\right) \in \mathcal{L}^1_{\mathrm{loc}}. \tag{4.13}$$

If $\beta < \gamma\beta + 1$, then (4.13) is stronger than the condition (4.10) we started with, and we can iterate the above argument. Clearly, (4.10) holds with $\beta = 0$; hence, we find in turn that

$$J_{f^{-1}} \log(\mathrm{e} + J_{f^{-1}}), \quad J_{f^{-1}} \log^{\gamma+1}(\mathrm{e} + J_{f^{-1}}), \quad J_{f^{-1}} \log^{\gamma(\gamma+1)+1}(\mathrm{e} + J_{f^{-1}}), \quad \dots,$$

are locally integrable. As

$$1 + \gamma + \gamma^2 + \cdots = \frac{1}{1-\gamma} = q,$$

obviously with a finite number of steps we get (4.8) for every fixed α satisfying (4.9). To prove (4.6) and (4.7) we need to make the above argument more precise. Let $B \Subset f(\Omega)$ and $\mu \in C_0^{\infty}(B)$, $\mu \geqslant 0$. We start with the following estimate:

$$\int_B \mu^n J_{f^{-1}} \log^{\alpha}(\mathrm{e} + \mu|Df^{-1}|)\,\mathrm{d}y \leqslant C\int_B F\,\mathrm{d}y + C\int_B \mu^n |Df^{-1}|^n \log^{\alpha-1}(\mathrm{e} + \mu|Df^{-1}|)\,\mathrm{d}y, \tag{4.14}$$

with

$$F = |f^{-1} \otimes \nabla\mu|(|f^{-1} \otimes \nabla\mu| + \mu|Df^{-1}|)^{n-1} \log^{q}(\mathrm{e} + |f^{-1} \otimes \nabla\mu| + \mu|Df^{-1}|).$$

Estimate (4.14) follows from corollary 3.2 and example 2.8 of [7]. Note that the constant $C = C(n, q) > 0$ in (4.14) can be chosen independent of α satisfying (4.9). Moreover, $F \in \mathcal{L}^1(B)$. Now we consider the last term in (4.14). Since f^{-1} has finite distortion, by Young's inequality, for $\varepsilon \in]0,1[$ we can write

$$\begin{aligned}\int_B \mu^n |Df^{-1}|^n \log^{\alpha-1}(\mathrm{e}+\mu|Df^{-1}|)\,\mathrm{d}y \\ &= \int_B \mu^n \frac{|Df^{-1}|^n}{(\varepsilon J_{f^{-1}})^{(\alpha-1)/\alpha}} (\varepsilon J_{f^{-1}})^{(\alpha-1)/\alpha} \log^{\alpha-1}(\mathrm{e}+\mu|Df^{-1}|)\,\mathrm{d}y \\ &\leqslant \varepsilon^{1-q} \int_B \mu^n \left(\frac{|Df^{-1}|^n}{J_{f^{-1}}}\right)^{\alpha} J_{f^{-1}}\,\mathrm{d}y \\ &\quad + \varepsilon \int_B \mu^n J_{f^{-1}} \log^{\alpha}(\mathrm{e}+\mu|Df^{-1}|)\,\mathrm{d}y. \end{aligned} \tag{4.15}$$

Inserting (4.15) into (4.14) and choosing ε so that $C\varepsilon = \frac{1}{2}$, we get

$$\int_B \mu^n J_{f^{-1}} \log^{\alpha}(\mathrm{e}+\mu|Df^{-1}|)\,\mathrm{d}y \leqslant C\int_B F\,\mathrm{d}y + C\int_B \mu^n \left(\frac{|Df^{-1}|^n}{J_{f^{-1}}}\right)^{\alpha} J_{f^{-1}}\,\mathrm{d}y. \tag{4.16}$$

Note that, on the left-hand side, we can absorb a term appearing on the right-hand side, since, by our iterative argument, we already know that it is converging. We now pass to the limit in (4.16) as $\alpha \to q$, using the monotone convergence theorem, and obtain

$$\int_B \mu^n J_{f^{-1}} \log^{q}(\mathrm{e}+\mu|Df^{-1}|)\,\mathrm{d}y \leqslant C\int_B F\,\mathrm{d}y + C\int_B \mu^n \left(\frac{|Df^{-1}|^n}{J_{f^{-1}}}\right)^{q} J_{f^{-1}}\,\mathrm{d}y. \tag{4.17}$$

We conclude by showing that the last integral in (4.17) is finite, under the assumption $K_f^I \in \mathcal{L}^q_{\mathrm{loc}}$. To this end, we use the area formula and (2.4). As

$$\frac{|Df^{-1}(f(x))|^n}{J_{f^{-1}}(f(x))} = J_f(x)|(Df(x))^{-1}|^n = \frac{|\mathrm{adj}\,Df(x)|^n}{J_f(x)^{n-1}} \leqslant K_f^I(x),$$

we find that

$$\int_B \mu^n(y)\left(\frac{|Df^{-1}(y)|^n}{J_{f^{-1}}(y)}\right)^{q} J_{f^{-1}}(y)\,\mathrm{d}y \leqslant \int_{f^{-1}(B)} \mu^n(f(x)) K_f^I(x)^q\,\mathrm{d}x$$

is finite. □

Remark 4.3. The case in which $n = 2$ is contained in [23].

The result of lemma 4.2 is optimal in the following sense.

Example 4.4. For every $q \geqslant 1$, there exists a Lipschitz homeomorphism f of finite inner distortion $K_f^I \in \mathcal{L}^q_{\mathrm{loc}}$ such that

$$J_{f^{-1}} \log^p(\mathrm{e}+J_{f^{-1}}) \quad \text{and} \quad |Df^{-1}|^n \log^{p-1}(\mathrm{e}+|Df^{-1}|)$$

are not locally integrable, for any $p > q$. From the proof of lemma 4.2, we can clearly equivalently show that $\log^p(\mathrm{e}+1/J_f)$ is not locally integrable. We can construct a homeomorphism with the required properties as a radial stretching (2.9) onto the

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unit ball $\boldsymbol{B}$ (cf. § 2.6); the inverse map is given by (2.10). As we have seen, we can prescribe the inner distortion function of f. We set (for $r = |x|$)

$$K_f^I(x) = \left(r \log \frac{\mathrm{e}}{r}\right)^{-n/q}, \tag{4.18}$$

so that clearly $K_f^I \in \mathcal{L}^q(\boldsymbol{B})$. According to (2.13) and (2.14), we find

$$\rho(r) = \exp\left[\int_1^r t^{-1-n/q}\left(\log \frac{\mathrm{e}}{t}\right)^{-n/q} \mathrm{d}t\right]. \tag{4.19}$$

By (4.19) we easily see that

$$\lim_{r\to 0} \frac{\rho(r)}{r} = \lim_{r\to 0} \rho'(r) = 0,$$

so that actually $f \in C^1(\bar{\boldsymbol{B}}; \bar{\boldsymbol{B}})$. Moreover, we have, as $r \to 0$,

$$|\log \rho(r)| \sim (r|\log r|)^{-n/q}. \tag{4.20}$$

For r close to 0, we have

$$\log \frac{1}{J_f(x)} = |\log \rho'(r)| + (n-1)|\log \rho(r)| - (n-1)|\log r|.$$

Therefore, since $r \mapsto |\log r|^p$ is integrable on $\boldsymbol{B}$, by (4.20) we see that $\log^p(\mathrm{e} + 1/J_f) \notin \mathcal{L}^1_{\mathrm{loc}}(\boldsymbol{B})$, as desired.

COROLLARY 4.5. *Let $p \geqslant n-1$ and $f \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ be a homeomorphism of finite outer distortion $K_f \in \mathcal{L}^p_{\mathrm{loc}}(\Omega)$. Then*

$$J_{f^{-1}} \log^{p/(n-1)}(\mathrm{e} + J_{f^{-1}}) \in \mathcal{L}^1_{\mathrm{loc}}(f(\Omega)),$$
$$|Df^{-1}|^n \log^{p/(n-1)-1}(\mathrm{e} + |Df^{-1}|) \in \mathcal{L}^1_{\mathrm{loc}}(f(\Omega)).$$

Proof. It suffices to recall that $K_f^I \leqslant K_f^{n-1}$ and to remark that, by Hölder's inequality, as $J_f \in \mathcal{L}^1_{\mathrm{loc}}$, the assumption $K_f \in \mathcal{L}^p_{\mathrm{loc}}$ implies

$$|Df| \in \mathcal{L}^{np/(p+1)}_{\mathrm{loc}}.$$

□

Proof of theorem 4.1. Let us start by showing that (4.4) holds in each case. When not explicitly assumed, (4.4) is a consequence of (4.2). In fact, in the case $\alpha > 1$ we get $|Dg| \in \mathcal{L}^n_{\mathrm{loc}}$ by [7, theorem 4.1]. If $\alpha \leqslant 1$, then, by (2.3), we deduce $|Dg| \in \mathcal{L}^n \log^{-1/\alpha} \mathcal{L}_{\mathrm{loc}}$. Hence, (4.4) follows if $\alpha(p-n+1) \geqslant 1$. On the other hand, by corollary 4.5, assumption (4.3) implies

$$J_{f^{-1}} \in \mathcal{L} \log^{/p(n-1)} \mathcal{L}_{\mathrm{loc}}(\Omega'), \qquad |Df^{-1}| \in \mathcal{L}^n \log^{(p-n+1)/n-1} \mathcal{L}_{\mathrm{loc}}(\Omega').$$

Then, by corollary 3.2 we know that $h = g \circ f \in \mathcal{W}^{1,1}_{\mathrm{loc}}$ has finite distortion and that (3.6) holds. We only need to prove (4.5). By Hölder's inequality and (4.3), it clearly suffices to show that $(K_g \circ f)^{pq/(p-q)} \in \mathcal{L}^1_{\mathrm{loc}}$. To this end, we note that f^{-1} satisfies

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the N-condition and, hence, for a fixed compact subset A of Ω, by the area formula we have

$$\int_A K_g(f(x))^{pq/(p-q)}\,\mathrm{d}x = \int_{f(A)} K_g(y)^{pq/(p-q)} J_{f^{-1}}(y)\,\mathrm{d}y. \tag{4.21}$$

Moreover, by (4.2) we find $\lambda > 0$, so that

$$\int_{f(A)} \exp[\lambda K_g(y)^{\alpha}]\,\mathrm{d}y < \infty. \tag{4.22}$$

As

$$\frac{pq}{p-q}\frac{1}{\alpha} = \frac{p}{n-1},$$

using the elementary inequality (2.3) of lemma 2.1, we have

$$K_g{}^{pq/(p-q)} J_{f^{-1}} \leqslant C[\exp(\lambda K_g{}^{\alpha}) + J_{f^{-1}} \log^{p/(n-1)}(\mathrm{e} + J_{f^{-1}})],$$

concluding the proof. □

The following example shows that we cannot drop assumption (4.2) of exponential integrability of K_g.

Example 4.6. There exist two homeomorphisms of finite distortion $f\colon \boldsymbol{B} \to \boldsymbol{B}$, $g\colon \boldsymbol{B} \to \boldsymbol{B}$ such that $\exp(\lambda K_f) \in \mathcal{L}^1$ for all $\lambda < n$, $K_g \in \mathcal{L}^p$, for all $p < \infty$, but $K_h^q \notin \mathcal{L}^1$ for any $q > 0$. We consider two radial stretchings

$$f(x) = \frac{x}{|x|}\rho_1(|x|), \qquad g(x) = \frac{x}{|x|}\rho_2(|x|), \tag{4.23}$$

with $r \mapsto \rho_i(r)/r$ increasing, $i = 1, 2$ (compare with § 2.6). The composition mapping is

$$h(x) = g(f(x)) = \frac{x}{|x|}\rho_2(\rho_1(|x|)) \tag{4.24}$$

and, by (2.11), it follows that its distortion is

$$K_h(x) = K_h(r) = K_g(f(x))K_f(x). \tag{4.25}$$

We can prescribe the distortion function of f and of g. We set

$$K_f(r) = \log\frac{\mathrm{e}}{r}, \qquad K_g(r) = \exp\left[\left(\log\frac{\mathrm{e}}{r}\right)^{\vartheta}\right],$$

where ϑ satisfies

$$\frac{n-1}{n} < \vartheta < 1.$$

It may readily be checked that K_f and K_g have the stated properties. Let us show that K_h^q is not integrable. It clearly suffices to show that $(K_g \circ f)^q \notin \mathcal{L}^1$. From (2.14) we deduce

$$\rho_1(r) = \exp\left[\int_1^r \left(\log\frac{\mathrm{e}}{t}\right)^{1/(n-1)} \frac{\mathrm{d}t}{t}\right] = \exp\left\{\frac{n-1}{n}\left[1 - \left(\log\frac{\mathrm{e}}{r}\right)^{n/(n-1)}\right]\right\},$$

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and hence

$$K_g(f(x)) = \exp\left[\left(\log\frac{\mathrm{e}}{\rho_1(r)}\right)^{\vartheta}\right] = \exp\left\{\left[\frac{1}{n}+\frac{n-1}{n}\left(\log\frac{\mathrm{e}}{r}\right)^{n/(n-1)}\right]^{\vartheta}\right\},$$

which clearly implies the claim.

Our next example shows that the integrability of $K_h^q \in \mathcal{L}^1_{\mathrm{loc}}$ is optimal in dimension $n = 2$.

Example 4.7. Here, we consider the case $n = 2$. For every $p \geqslant 1$ and $\alpha > 0$, there exist two homeomorphisms of finite distortion $f\colon \boldsymbol{B} \to \boldsymbol{B}$, $g\colon \boldsymbol{B} \to \boldsymbol{B}$ such that $K_f \in \mathcal{L}^p$, $\exp(\lambda K_g^{\alpha}) \in \mathcal{L}^1$ for all $\lambda < 2$, but $K_h^s \notin \mathcal{L}^1_{\mathrm{loc}}$, for any $s > q$. As in example 4.6, we consider two radial stretchings given by (4.23), with $r \mapsto \rho_i(r)/r$ increasing, $i = 1, 2$. The composition mapping is given by (4.24) and its distortion by (4.25). We set

$$K_f(r) = \left(r\log\frac{\mathrm{e}}{r}\right)^{-2/p}, \qquad K_g(r) = \left(\log\frac{\mathrm{e}}{r}\right)^{1/\alpha}.$$

Then, we easily find $K_f \in \mathcal{L}^p(\boldsymbol{B})$ and $\exp(\lambda {K_g}^{\alpha}) \in \mathcal{L}^1(\boldsymbol{B})$ if $\lambda < 2$. On the other hand,

$$K_h(r) = (1-\log\rho_1(r))^{1/\alpha}\left(r\log\frac{\mathrm{e}}{r}\right)^{-2/p}.$$

Moreover, as in example 4.4, for $r \to 0$ we have

$$|\log\rho_1(r)| \sim \left(r\log\frac{\mathrm{e}}{r}\right)^{-2/p},$$

and hence

$$K_h(r) \sim \left(r\log\frac{\mathrm{e}}{r}\right)^{-2/p}\left(1+\frac{1}{\alpha}\right).$$

Since

$$\frac{1}{p}\left(1+\frac{1}{\alpha}\right) = \frac{1}{q},$$

clearly K_h^s is not locally integrable for $s > q$.

5. A uniqueness theorem

In this section, we consider the case of planar mappings; that is, we assume $n = 2$. Given a matrix field $G = G(x)$ and a function $K = K(x) \geqslant 1$ that are Borel measurable in a domain Ω and satisfy

$$G(x) = G^{\mathrm{T}}(x), \qquad \det G(x) = 1, \tag{5.1}$$

and

$$\frac{|\xi|^2}{K(x)} \leqslant \langle G(x)\xi,\xi\rangle \leqslant K(x)|\xi|^2 \tag{5.2}$$

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for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^2$, the *measurable Riemann mapping problem* consists in finding a bi-Sobolev homeomorphism $f\colon \Omega \to f(\Omega)$ such that G is its distortion tensor, that is,

$$G_f(x) = G(x) \quad \text{for almost every } x \in \Omega, \tag{5.3}$$

$$K_f(x) \leqslant K(x) \quad \text{for almost every } x \in \Omega. \tag{5.4}$$

This is a difficult problem that was solved classically by Morrey [24] for $K \in \mathcal{L}^\infty$, and by David [4] for $K \in \mathrm{Exp}$ (see also [17]). In this section, using the results of previous sections, we address the question of the uniqueness of solution f to equation (5.3) in the following sense. Recall that a diffeomorphism φ is called conformal in Ω if, for every $x \in \Omega$, it preserves the angle between any pair of smooth curves passing through x. In our planar context here, an orientation-preserving conformal map φ is holomorphic, that is, satisfies the Cauchy–Riemann equations

$$\frac{\partial \varphi_1}{\partial x_1} = \frac{\partial \varphi_2}{\partial x_2}, \qquad \frac{\partial \varphi_1}{\partial x_2} = -\frac{\partial \varphi_2}{\partial x_1}, \tag{5.5}$$

at every point of Ω. Note that, by the Weyl lemma, it is enough that (5.5) holds in the sense of distributions to conclude that φ is holomorphic, and (5.5) actually holds at every point. The Cauchy–Riemann system (5.5) can be rewritten in various equivalent ways:

$$|D\varphi|^2 = J_\varphi, \qquad D^{\mathrm{T}}\varphi D\varphi = J_\varphi \boldsymbol{I}, \qquad D^{\mathrm{T}}\varphi = \operatorname{adj} D\varphi. \tag{5.6}$$

Furthermore, if φ is a mapping of finite distortion, then the Cauchy–Riemann system (5.5) is also equivalent to the validity of either of the following equations a.e.:

$$G_\varphi = \boldsymbol{I}, \qquad K_\varphi = 1. \tag{5.7}$$

Now, it is easy to see that post-composing a bi-Sobolev mapping with a conformal map does not change the distortion tensor. More precisely, if $h\colon \Omega \xrightarrow{\text{onto}} \Omega'$ is a bi-Sobolev mapping and $\varphi\colon \Omega' \to \Omega'$ is a conformal map, then $\varphi \circ h$ is bi-Sobolev and

$$G_{\varphi \circ h}(x) = G_h(x) \quad \text{for almost every } x \in \Omega. \tag{5.8}$$

Indeed, since φ is locally Lipschitz continuous and h is continuous, we have $\varphi \circ h \in \mathcal{W}^{1,1}_{\mathrm{loc}}$ and, for almost every $x \in \Omega$,

$$D(\varphi \circ h)(x) = D\varphi(h(x))Dh(x), \tag{5.9}$$

and hence

$$D^{\mathrm{T}}(\varphi \circ h)(x)D(\varphi \circ h)(x) = D^{\mathrm{T}}h(x)D^{\mathrm{T}}\varphi(h(x))D\varphi(h(x))Dh(x). \tag{5.10}$$

Therefore, using the characterization of conformality expressed by the second equality at (5.6), we immediately find

$$D^{\mathrm{T}}(\varphi \circ h)(x)D(\varphi \circ h)(x) = J_\varphi(h(x))D^{\mathrm{T}}h(x)Dh(x). \tag{5.11}$$

Moreover, by (5.9), we get $J_{\varphi \circ h}(x) = J_\varphi(h(x))J_h(x)$ and then conclude by (5.11) with the desired equality (5.8), simply dividing by the Jacobian $J_{\varphi \circ h}(x)$ and recalling that J_φ does not vanish at any point, as φ is injective. In this section we prove theorem 1.4, which is a uniqueness result for the solution of equation (5.3) modulo a post-composition with a conformal mapping. To the best of our knowledge, only the most general uniqueness theorem has been proved in [28, corollary 5.5]. Note that we only assume g and h in $\mathcal{W}^{1,2}_{\rm loc}(\Omega, \Omega')$, whereas in [28] it is also required that g^{-1} and h^{-1} belong to $\mathcal{W}^{1,2}_{\rm loc}(\Omega', \Omega)$.

Proof of theorem 1.4. Since $h \in \mathcal{W}^{1,2}_{\rm loc}$, it is differentiable a.e. and satisfies the N-condition. Also, h^{-1} is differentiable a.e., and $J_{h^{-1}}(y) > 0$, for almost every $y \in \Omega'$ (see § 2.4), that is, $\mathcal{R}_{h^{-1}}$ is a subset of Ω' of full measure, $|\Omega' \setminus \mathcal{R}_{h^{-1}}| = 0$. Similarly, g is differentiable a.e. in Ω as well. Therefore, we can find a Borel subset F of Ω, having full measure, $|\Omega \setminus F| = 0$, such that g is differentiable and (1.10) holds, for all $x \in F$. Recalling that g has finite distortion, we may also assume that

$$J_g(x) = 0 \quad \Longrightarrow \quad Dg(x) = 0, \tag{5.12}$$

for all $x \in F$. By corollary 3.2, the mapping

$$\varphi = g \circ h^{-1} \colon \Omega' \to \Omega'$$

belongs to $\in \mathcal{W}^{1,1}_{\rm loc}(\Omega', \Omega')$ and has finite distortion. Also, φ is differentiable at every point of the set of full measure $E' = \mathcal{R}_{h^{-1}} \cap h(F)$, and by the chain rule we have

$$D\varphi(y) = Dg(h^{-1}(y))Dh^{-1}(y), \qquad J_\varphi(y) = J_g(h^{-1}(y))J_{h^{-1}}(y). \tag{5.13}$$

Moreover, for all $y \in E'$,

$$Dh^{-1}(y) = (Dh(h^{-1}(y)))^{-1}, \qquad J_{h^{-1}}(y) = \frac{1}{J_h(h^{-1}(y))}, \tag{5.14}$$

and, by (1.10),

$$G_g(h^{-1}(y)) = G_h(h^{-1}(y)). \tag{5.15}$$

In order to compute G_φ, let us consider the set

$$Z = \{y \in h(F) \colon J_g(h^{-1}(y)) = 0\}.$$

For all $y \in E' \setminus Z$, we have

$$J_g(h^{-1}(y)) > 0, \qquad J_h(h^{-1}(y)) > 0, \qquad J_\varphi(y) > 0.$$

Therefore, by definition,

$$\begin{aligned} G_\varphi(y) &= \frac{D^t\varphi(y)D\varphi(y)}{J_\varphi(y)} \\ &= \frac{D^{\rm T}h^{-1}(y)D^t g(h^{-1}(y))Dg(h^{-1}(y))Dh^{-1}(y)}{J_g(h^{-1}(y))J_{h^{-1}}(y)} \\ &= \frac{1}{J_{h^{-1}}(y)} D^{\rm T}h^{-1}(y)G_g(h^{-1}(y))Dh^{-1}(y). \end{aligned}$$

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By (5.15),

$$G_\varphi(y) = \frac{1}{J_{h^{-1}}(y)} D^{\mathrm{T}} h^{-1}(y) G_h(h^{-1}(y)) D h^{-1}(y)$$
$$= \frac{D^{\mathrm{T}} h^{-1}(y) D^t h(h^{-1}(y)) D h(h^{-1}(y)) D h^{-1}(y)}{J_h(h^{-1}(y)) J_{h^{-1}}(y)}$$

and, by (5.14), it is easily seen that

$$G_\varphi(y) = \boldsymbol{I}. \tag{5.16}$$

On the other hand, if $y \in E' \cap Z$, then $J_g(h^{-1}(y)) = 0$ and we have (5.16) immediately by definition. Therefore, (5.16) holds for every $y \in E'$ and we conclude the proof. □

Remark 5.1. We only need to assume (1.10) at points x such that both $J_g(x) \neq 0$ and $J_h(x) \neq 0$.

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