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Zariski dense orbits for regular self-maps on split semiabelian varieties

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Abstract. We provide a direct proof of the Medvedev–Scanlon's conjecture from Medvedev and Scanlon (Ann. Math. Second Series **179**(2014), 81–177) regarding Zariski dense orbits under the action of regular self-maps on split semiabelian varieties defined over a field of characteristic 0. Besides obtaining significantly easier proofs than the ones previously obtained in Ghioca and Scanlon (Trans. Am. Math. Soc. **369**(2017), 447–466; for the case of abelian varieties) and Ghioca and Satriano (Trans. Am. Math. Soc. **371**(2019), 6341–6358; for the case of semiabelian varieties), our method allows us to exhibit numerous starting points with Zariski dense orbits, which the methods from Ghioca and Scanlon (Trans. Am. Math. Soc. **369**(2017), 447–466) and Ghioca and Satriano (Trans. Am. Math. Soc. **371**(2019), 6341–6358) could not provide.

1 Introduction

1.1 Notation

Throughout this paper, we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the set of nonnegative integers. As always in arithmetic dynamics, we denote by Φ^n the nth iterate of the self-map Φ acting on some ambient variety X. For each point x of X, we denote its orbit under Φ by

$$\mathcal{O}_\Phi(x) := \{\Phi^n(x) : n \in \mathbb{N}_0\}.$$

1.2 A conjecture about Zariski dense orbits

In the early 1990s, Zhang formulated a far-reaching set of conjectures in arithmetic dynamics in parallel to famous questions in arithmetic geometry, hence the genesis of the dynamical Manin–Mumford and dynamical Bogomolov conjectures, which generated a lot of research in the past 20 years (for example, see [GT21] and the references therein). At that time, Zhang also formulated a very interesting conjecture regarding the existence of Zariski dense orbits under the action of a polarizable endomorphism of a projective variety defined over a number field (which appeared in print as [Zha06, Conjecture 4.1.6]). Later, both Amerik and Campana [AC08] and Medvedev and Scanlon [MS14] formulated a refinement of Zhang's original question regarding Zariski dense orbits as follows.

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Conjecture 1.1 *Given a variety X defined over an algebraically closed field K of characteristic* 0*, endowed with a dominant endomorphism Φ, then we have the following dichotomy:*

(A) *either there exists a point $x \in X(K)$ whose orbit $\mathcal{O}_\Phi(x)$ is Zariski dense in X; or*
(B) *there exists a nonconstant rational map $f : X \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.*

It is immediate to see that if condition (B) above holds, then no orbit can be Zariski dense; so, the entire difficulty of the conjecture advanced by Zhang, Medvedev and Scanlon, and Amerik and Campana is to prove that in the absence of condition (B), there must exist a Zariski dense orbit.

Amerik and Campana [AC08] (see also [BGR17]) proved Conjecture 1.1 under the assumption that K is uncountable; essentially, in the absence of condition (B) above, the orbit of a very general point (which lies outside countably many special proper subvarieties of X) would have a Zariski dense orbit. However, the case of a countable algebraically closed field K remains open and quite difficult, because, *a priori*, the method of both [AC08] and [BGR17] does not guarantee the existence of a K-point outside the union of those countably many special proper subvarieties of X. In the past 10 years, several partial results were obtained (for example, see [Xie] and the references therein).

1.3 Our results

We prove Conjecture 1.1 for regular self-maps of split semiabelian varieties. We recall the definition of a *split semiabelian variety* G (defined over some algebraically closed field K), which is a connected group variety isogenous to a direct product $\mathbb{G}_m^N \times A$ for some $N \in \mathbb{N}_0$ and some abelian variety A. Also, we recall (see [NW14, Theorem 5.1.37]) that any regular self-map on a semiabelian variety G is a composition of a translation with a group endomorphism.

Before stating our result, we define a notion useful for our Theorem 1.2: given two points α and β of some algebraic group G, we say that α is *linearly independent over* $\mathrm{End}(G)$ *from* β if for any two (group) endomorphisms ϕ_1 and ϕ_2 of G, we have that $\phi_1(\alpha) = \phi_2(\beta)$, then ϕ_1 must be the trivial map. Also, for any point β of the algebraic group G, we let $\tau_\beta : G \longrightarrow G$ be the translation-by- β map on G. Finally, we denote by Id the identity map on G.

Theorem 1.2 *Let G be a split semiabelian variety defined over an algebraically closed field K of characteristic* 0*. Let $\Phi : G \longrightarrow G$ be a dominant, regular self-map; we let $\Phi = \tau_\beta \circ \varphi$, where $\beta \in G(K)$ and φ is a group endomorphism of G. Then, the following statements are equivalent:*

(i) *there exists a nonconstant rational function $f : G \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$;*
(ii) *there exists no $\alpha \in G(K)$ such that $\mathcal{O}_\Phi(\alpha)$ is Zariski dense in G;*
(iii) *there exists a nonconstant group endomorphism $\Psi : G \longrightarrow G$ and there exists $r \in \mathbb{N}$ such that $\Psi \circ (\varphi^r - \mathrm{Id}) = 0$ in $\mathrm{End}(G)$ and also $\Psi\left(\sum_{j=0}^{r-1} \varphi^j(\beta)\right) = 0$.*

Furthermore, if none of the above conditions hold, then for each point $\alpha \in G(K)$, which is linearly independent over $\mathrm{End}(G)$ *from β, we have that $\mathcal{O}_\Phi(\alpha)$ is Zariski dense in G.*

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We note that according to [Vil08, Theorem 5], we can always find algebraic points α in any semiabelian variety G, which are linearly independent over $\mathrm{End}(G)$ from any given point β of G, because the group $G(K)$ has infinite rank, while $\mathrm{End}(G)$ is a finite $\mathbb{Z}$-module. We also observe (see Remark 2.2) that Theorem 1.2 holds with the same proof verbatim for abelian varieties G defined over an algebraically closed field K of characteristic p, assuming $\mathrm{Tr}_{K/\overline{\mathbb{F}_p}}(G)$ is trivial.

Conjecture 1.1 was previously proved in [GS19] for regular self-maps of arbitrary semiabelian varieties defined over an algebraically closed field of characteristic 0 (see also [GS17] for the proof in the case of abelian varieties). However, our current proof is much more direct (and simpler); furthermore, our Theorem 1.2 provides explicit points whose orbit is Zariski dense, which is in stark contrast with the results of [GS17, GS19] in which there was no explicit information about the points with Zariski dense orbits. For example, the "furthermore" statement in our Theorem 1.2 yields that for any finitely generated subfield $L \subset K$ for which the group $G(L)$ has sufficiently high rank (note that the group $G(K)$ has infinite rank for an algebraically closed field K), then we can find a point $\alpha \in G(L)$ with a Zariski dense orbit under Φ (assuming Φ does not leave invariant a nonconstant rational function). In particular, if $G = \mathbb{G}_m^N$, then our Theorem 1.2 yields that if the equivalent conditions (i)–(iii) do not hold for a regular self-map Φ on $\mathbb{G}_m^N$, then there exist infinitely many multiplicatively independent points $\alpha \in \mathbb{G}_m^N(\mathbb{Q})$ with a Zariski dense orbit under Φ. Conversely, our proof of Theorem 1.2 allows us also to construct a very explicit rational function, which is left invariant by Φ when conditions (i)–(iii) hold (see the proof of the implication (iii) ⇒(i) in Theorem 1.2, especially equation (2.13)).

Generally, the partial results toward Conjecture 1.1 have employed various complicated techniques: from invariant theory (as in [GX18]), to methods from algebraic geometry akin to the study of higher dimensional varieties (as in [BGRS17]), to Diophantine arguments in the spirit of the famous theorem of Laurent [Lau84] (as in [GH18]), and to deep results regarding the algebraic dynamics on surfaces coupled with the so-called "p-adic arc lemma" (first introduced in the context of the dynamical Mordell–Lang conjecture; see [BGT16]), as recently employed by Xie [Xie] in his proof of Conjecture 1.1 for endomorphisms of surfaces. Also, the proofs of Conjecture 1.1 for regular self-maps on abelian varieties or, more generally, on semiabelian varieties (see [GS17, GS19]) were quite involved, employing nontrivial arithmetic and geometric results (besides the use of the famous theorems of Faltings [Fal94] and Vojta [Voj96] which solved the classical Mordell–Lang conjectures for abelian, respectively, and semiabelian varieties). Our proof of Theorem 1.2 only exploits the Poincaré Reducibility Theorem for abelian varieties (see [GS17, Fact 3.2]), avoiding all of the much more difficult arithmetic arguments present in the proofs from [GS17, GS19]. Because the Poincaré Reducibility Theorem also holds in the context of algebraic tori and therefore for split semiabelian varieties, we are able to prove the result from our Theorem 1.2. However, the failure of the Poincaré Reducibility Theorem for general semiabelian varieties means that one would still need to use more geometric and arithmetic arguments as in [GS19] in order to prove Conjecture 1.1 in this general case (for example, see [GS19, Section 3], especially the use of minimal dominating semiabelian subvarieties and the construction of topological generators). Also, following the suggestion of the referee (to whom we are indebted for their insightful remarks, which improved our paper), our proof even avoids the use of the classical Mordell–Lang theorems in arithmetic geometry of [Lau84, Fal94, Voj96] and instead we employ the very clever (but also more elementary) result of Pink–Rössler [PR04] regarding ψ-invariant subvarieties of semiabelian varieties (where ψ is an isogeny of a semiabelian variety).

2 Proof of our main result

Proof of Theorem 1.2 We note that for each nonnegative integer n and for each $\alpha \in G(K)$, we have that

$$\Phi^n(\alpha) = \left(\sum_{j=0}^{n-1} \varphi^j(\beta)\right) + \varphi^n(\alpha). \tag{2.1}$$

We prove the equivalence for the conditions (i)–(iii) by showing that (i) $\Rightarrow$(ii) $\Rightarrow$(iii) $\Rightarrow$(i); the "furthermore" statement from the conclusion of Theorem 1.2 follows as a consequence of our proof of the implication (ii) $\Rightarrow$(iii).

First, we note that (i) always implies (ii) as previously observed in [AC08, MS14, BGR17]; this statement holds very generally for any dominant, regular self-map on any variety.

Second, we assume (ii) holds and we prove (iii) must also hold. We establish this by proving that if condition (iii) does not hold, then the orbit of any point $\alpha \in G(K)$, which is linearly independent over $\mathrm{End}(G)$ from β, must be Zariski dense; in particular, this proves our "furthermore" statement from the conclusion of Theorem 1.2.

So, assume condition (iii) from Theorem 1.2 does not hold. We will employ the following technical claim in our proof.

Claim 2.1 *If condition* (iii) *does not hold for* (G, Φ)*, then there exists no positive dimensional semiabelian variety* G_0 *endowed with some automorphism* Φ_0 *of finite order, and also endowed with some dominant map* $\pi : G \longrightarrow G_0$ *such that* $\pi \circ \Phi = \Phi_0 \circ \pi$. ■

Proof of Claim 2.1 We prove that if such a Φ-invariant subquotient (G_0, Φ_0) were to exist, then actually condition (iii) would have to hold for (G, Φ).

So, assume the existence of G_0, Φ_0, and π as in the conclusion of Claim 2.1. In particular, we assume that Φ_0^r is the identity morphism on G_0 for some $r \in \mathbb{N}$; therefore, we have that

$$\pi \circ \Phi^r = \pi. \tag{2.2}$$

Now, because G_0 is a quotient of the semiabelian variety G, which is assumed to be a split semiabelian variety, then the Poincaré Reducibility Theorem yields that G_0 is isogenous to a semiabelian subvariety of G and therefore (using also equation (2.2)), there exists a nonconstant endomorphism $\Psi : G \longrightarrow G$ such that

$$\Psi \circ \Phi^r = \Psi. \tag{2.3}$$

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Equation (2.3) yields that $\Psi(\varphi^r - \mathrm{Id}) = 0$ and also $\Psi\left(\sum_{i=0}^{r-1} \varphi^i(\beta)\right) = 0$, i.e., condition (iii) would have to hold, which is a contradiction.

This concludes our proof of Claim 2.1. ∎

Let $\alpha \in G(K)$ be linearly independent over $\mathrm{End}(G)$ from β (such points always exist, see [Vil08], for example). If $\mathcal{O}_\Phi(\alpha)$ is not Zariski dense, then its Zariski closure is a proper subvariety Z of G, which is mapped by the étale map Φ into itself. Therefore, as proved in [PR04, Theorem 3.4], Z is a finite union of translates of proper semiabelian subvarieties of G. Indeed, in order to see that the hypotheses of [PR04, Theorem 3.4] are met, we note that (G, Φ) is strictly mixed (as in [PR04, Definition 2.1(d)]), because K has characteristic 0 (and therefore, it cannot have a subquotient that is pure of positive weight as in [PR04, Definition 2.1(c)]); also, by Claim 2.1, we know that (G, Φ) does not possess a Φ-invariant subquotient of positive dimension that is pure of weight 0 (as in [PR04, Definition 2.1(a)]).

Now, because Z is a finite union of cosets of proper algebraic subgroups of G, by the pigeonhole principle, there exist integers $0 \le m < n$ and some proper algebraic subgroup H of G such that

$$\Phi^n(\alpha) - \Phi^m(\alpha) \in H. \tag{2.4}$$

The Poincaré Reducibility Theorem for abelian varieties (see [GS17, Fact 3.2]), which also holds for algebraic tori and therefore for split semiabelian varieties, yields that there exists a complement C of H in G, i.e., some algebraic subgroup $C \subset G$ such that $C + H = G$ and $C \cap H$ is finite. Thus, considering the projection $G \longrightarrow G/H$ composed with the isogeny $G/H \longrightarrow C$ and finally the embedding $C \hookrightarrow G$, we conclude that there exists a *nontrivial* endomorphism $\Psi_1 \in \mathrm{End}(G)$ such that $H \subseteq \ker(\Psi_1)$ (note that $\ker(\Psi_1)$ must be a proper subgroup of G, because H is a proper subgroup of G and the index of H in $\ker(\Psi_1)$ is finite). In particular, we have

$$\Psi_1\left(\Phi^n(\alpha) - \Phi^m(\alpha)\right) = 0,$$

which coupled with (2.1) yields that

$$\left(\Psi_1 \circ (\varphi^n - \varphi^m)\right)(\alpha) = \left(\Psi_1 \circ \left(-\sum_{j=m}^{n-1} \varphi^j\right)\right)(\beta). \tag{2.5}$$

Equation (2.5) coupled with the fact that α is linearly independent over $\mathrm{End}(G)$ from β yields that $\Psi_1 \circ (\varphi^n - \varphi^m) = 0$. Because Φ is dominant, then also φ must be dominant, and so, we conclude that actually,

$$\Psi_1 \circ (\varphi^{n-m} - \mathrm{Id}) = 0. \tag{2.6}$$

Moreover, using (2.5) and (2.6), we obtain that

$$\Psi_1\left(\sum_{j=m}^{n-1} \varphi^j(\beta)\right) = 0. \tag{2.7}$$

We let $\Psi := \Psi_1 \circ \varphi^m$ and also let $r := n - m$. Because φ is a dominant endomorphism and Ψ_1 is nontrivial, then Ψ is a nonconstant endomorphism of G. Then, equation (2.7)

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simply becomes

$$\Psi\left(\sum_{j=0}^{r-1} \varphi^j(\beta)\right) = 0. \tag{2.8}$$

Also, equation (2.6) yields

$$\Psi \circ (\varphi^r - \mathrm{Id}) = 0. \tag{2.9}$$

Equations (2.8) and (2.9) are exactly the desired conditions from (iii) in the conclusion of Theorem 1.2. This concludes our proof for the implication (ii) ⇒(iii); in addition, we see that if condition (iii) does not hold, then the orbit of any point $\alpha \in G(K)$, which is linearly independent over $\mathrm{End}(G)$ from β, must be Zariski dense in G.

Finally, we prove the implication (iii) ⇒(i). So, we assume there exists a nonconstant endomorphism Ψ of G such that for some positive integer r, we have that

$$\Psi \circ (\varphi^r - \mathrm{Id}) = 0 \tag{2.10}$$

and

$$\Psi\left(\sum_{j=0}^{r-1} \varphi^j(\beta)\right) = 0. \tag{2.11}$$

Let $\overline{G} := G/\ker(\Psi)$; then $\overline{G}$ is a positive dimensional semiabelian variety, because Ψ is a nontrivial endomorphism of G. Let $h : \overline{G} \dashrightarrow \mathbb{P}^1$ be a nonconstant rational function. Let $\pi : G \longrightarrow \overline{G}$ be the natural projection map and let $g := h \circ \pi$; then $g : G \dashrightarrow \mathbb{P}^1$ is a nonconstant rational function.

For each of the r fundamental symmetric functions τ_i (for $i = 1, \dots, r$) on r variables, we let $g_i : G \dashrightarrow \mathbb{P}^1$ be defined by

$$g_i(x) := \tau_i\left(g(x), g(\Phi(x)), \dots, g\left(\Phi^{r-1}(x)\right)\right).$$

We claim that for each $x \in G$, we have that

$$g(x) = g(\Phi^r(x)). \tag{2.12}$$

Indeed, using (2.1) and also equations (2.10) and (2.11), we get that $\Phi^r(x) - x \in \ker(\Psi)$ and therefore, by the definition of $g = h \circ \pi$, we obtain equality (2.12). Thus, for each of the r symmetric functions τ_i, we have that

$$g_i(\Phi(x)) = \tau_i\left(\Phi^1(x), \dots, \Phi^r(x)\right) = \tau_i\left(x, \dots, \Phi^{r-1}(x)\right) = g_i(x). \tag{2.13}$$

Now, if each of the rational functions g_i is constant (for $i = 1, \dots, r$), then because the τ_i's are all the fundamental symmetric functions based on r variables, we conclude that each rational function $G \dashrightarrow \mathbb{P}^1$ given by $x \mapsto g\left(\Phi^{j-1}(x)\right)$, for $j = 1, \dots, r$, must be constant. However, because Φ is a dominant endomorphism of G, this would mean that $g : G \dashrightarrow \mathbb{P}^1$ is a constant map, which is a contradiction. Therefore, there exists some nonconstant rational function $f := g_i$ (for some $i = 1, \dots, r$); using (2.13), we conclude that $f : G \dashrightarrow \mathbb{P}^1$ is a nonconstant rational function invariant under Φ.

This concludes our proof for the last implication (iii) ⇒(i) and also concludes our proof for Theorem 1.2.

Remark 2.2 Because [PR04, Theorem 3.4] is valid also in positive characteristic p, as long as there is no proper Φ-invariant subquotient of the semiabelian variety G for which the induced map is a power of the Frobenius, we have that our Theorem 1.2 holds with an identical proof also when K has characteristic p, assuming the trace of G over $\overline{\mathbb{F}_p}$ is trivial. Alternatively, one could employ the famous theorem of Hrushovski [Hru96] (instead of the main theorem of [PR04]) to infer the corresponding result in positive characteristic.

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