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Divergence of mock Fourier series for spectral measures

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In this paper, we study divergence properties of the Fourier series on Cantor-type fractal measure, also called the mock Fourier series. We give a sufficient condition under which the mock Fourier series for doubling spectral measure is divergent on a set of strictly positive measure. In particular, there exists an example of the quarter Cantor measure whose mock Fourier sums are not almost everywhere convergent.

Keywords: Mock Fourier series; spectral measure; divergence; quarter Cantor measure

2020 *Mathematics subject classification:* Primary: 28A80
Secondary: 42A20, 42B05

1. Introduction

This paper is a follow-up of Dutkay *et al.* [4] which studied divergence properties of the Fourier series on Cantor measures. They proved there are examples of continuous functions for which the mock Fourier series does not converge uniformly to itself. To prove this result, Dutkay *et al.* showed that the L^1-norm of the Dirichlet kernel can grow exponentially fast using Birkhoff's ergodic theorem. This approach is useful, and we show that if instead one assumes only that the function is integrable, then it is possible for the partial mock Fourier sums to diverge at a set of strictly positive measure for the example of Dutkay *et al.*

The proof given here and that of Dutkay *et al.* are both distinctive and connective from each other, which consists of the following elements: (i) the Christ [3] method of dyadic systems with doubling measure; (ii) the resulting discretization lemma for the mock Dirichlet summation operator and (iii) a variant of the Dutkay *et al.* discriminant value for these Dirichlet kernels, which allow us to apply the obtained lemma to certify the main result of this paper.

Let us briefly introduce some of the notations and results that we will use later. Consider a Borel probability measure μ on $\mathbb{R}^d$ with compact support. We say that μ is a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^d$ such that $E(\Lambda) := \{\mathrm{e}^{-2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu)$. Jorgensen and Pedersen [10] constructed the first singular, non-atomic spectral measure, namely the quarter

Cantor measure. Over 20 years, many other intriguing singular spectral measures on self-affine and Moran fractal sets have been constructed (see [1, 4, 5] and so on).

Given a spectral measure μ with a spectrum Λ, for $L^1(\mu)$ function f, we define coefficients $c_\lambda(f) = \int f(y)\mathrm{e}^{-2\pi i\lambda\cdot y}\mathrm{d}\mu(y)$ and the mock Fourier series $\sum_{\lambda\in\Lambda} c_\lambda(f)\mathrm{e}^{2\pi i\lambda\cdot x}$. There is a natural sequence of finite subsets Λ_n increasing to Λ as $n \to \infty$, and we define the partial sums of the mock Fourier series by

$$S_n(f)(x) = \sum_{\lambda\in\Lambda_n} c_\lambda(f)\mathrm{e}^{2\pi i\lambda\cdot x}.$$

We will use $(S_n,\, \Lambda_n)$ to denote *the mock Dirichlet summation operator* S_n *with* Λ_n.

As an analogue to classical Fourier analysis, an extremely natural question is whether $S_n(f)$ converges to f as $n \to \infty$. The answer has an added piquancy since: not only does it depend on the determining what the function space f is belonged to, but it also depends critically on how one defines 'convergence'.

Recalling that if μ is the Lebesgue measure on $[0,\, 1]^d$, the assertion of uniform convergence of classical Fourier series is not true for some continuous functions [13, p. 83]. By contrast, Strichartz [14] showed that it is correct for a large family of singular continuous spectral measures with the standard spectrums. On the other hand, different spectrums may have different convergence for a given spectral measure. As we have already mentioned, Dutkay *et al.* [6] proved there is a continuous function f whose $(S_n(f),\, \Lambda_n)$ does not even converge pointwise to f if one changes the spectrum from standard to some non-standard. Yet it is worth noting that Dutkay *et al.* [6] did not say if it could diverge on a set of strictly positive measure, we will show this is the case.

To conveniently state our main results, we briefly introduce the related concepts, and their well-posedness will be given in § 2. Let $(M,\, \rho)$ be a metric space and suppose that μ is a positive locally finite Borel measure on M. We call μ a doubling measure if μ satisfies the doubling condition

$$\mu(B(x, 2r)) \leqslant A_1\mu(B(x, r)) < \infty$$

for all $x \in M$ and $r > 0$, where A_1 is constant and independent of x, r. Here $B(x,\, r)$ denotes the closed ball $B(x,\, r) = \{y \in M : \rho(y,\, x) \leqslant r\}$.

Recall the finite discrete measure defined on a measure space $(X,\, \mathcal{A},\, \mu)$ has the form $\nu = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$ for every finite collection $x_1,\, x_2,\, \ldots, x_N \in X$ not necessarily pairwise different points, where δ_x is the Dirac measure concentrated at the point $x \in X$. Given a mock Dirichlet summation operator S_n with Λ_n, we formally write

$$S_n(\nu)(x) = \frac{1}{N}\sum_{\lambda\in\Lambda_n}\sum_{j=1}^{N}\mathrm{e}^{2\pi i\lambda\cdot(x-x_j)}.$$

Now we state our main result.

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THEOREM 1.1. *Let μ be a doubling spectral measure and let (S_n, Λ_n) be the mock Dirichlet summation operator. If*

$$\lim_{\alpha\to\infty} \sup_{\nu=\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}} \mu\Big(\Big\{x \in X : \sup_n |S_n(\nu)(x)| > \alpha\Big\}\Big) > 0,$$

then there exists an integrable function such that the mock Fourier series diverges on a set of strictly positive μ-measure.

As an application, we find the example in [6] diverges on a set of strictly positive measure.

THEOREM 1.2. *Let μ_4 be the quarter Cantor measure and let $(S_n, 17\Lambda_n)$ be the mock Dirichlet summation operator with*

$$17\Lambda_n = \left\{17\sum_{j=0}^{n} 4^j l_j : \ l_j \in \{0, 1\}, \ n \in \mathbb{N}\right\}.$$

Then there exists an integrable function $f \in L^1(\mu_4)$ whose mock Fourier series $S_n(f)(x)$ diverges on a set of strictly positive μ-measure.

We organize our paper as follows. In § 2, we first present a brief overview of the relationship between the continuity of maximal operators and convergence almost everywhere. Succeeded by, we introduce the main tool for our proof of theorem 1.1, i.e. the dyadic cube analysis constructed by Christ [3]. In § 3, as an application of theorem 1.1, we consider the self-affine measures generated by Hadamard triples. Under some technical conditions about the spectrum, we give a criterion on there exists an integrable function whose mock Fourier series diverges on a set of strictly positive measure. The criterion can be applied to cover theorem 1.2.

2. The proof of theorem 1.1

Let $(X, \mathcal{A}, \mu)$ be a complete finite measure space with a σ-field $\mathcal{A}$. To recall some basic facts firstly, the space of (equivalence classes of) all measurable functions on$(X, \mathcal{A}, \mu)$ is denoted by $L^0(\mu)$. It is endowed with the topology of convergence in measure by the metric

$$d(f, g) = \int_X \frac{|f-g|}{1+|f-g|}\mathrm{d}\mu.$$

It is not difficult to show that $(L^0(\mu), d)$ is a complete metric space.

A mapping $T : (M, d_1) \to (L^0(\mu), d)$ from a metric space M to $L^0(\mu)$ is said to be continuous at $x \in M$, if for any sequence $\{x_n\} \subset M$, $n \geqslant 1$, we have $d(Tx_n, Tx) \longrightarrow 0$ whenever $d_1(x_n, x) \longrightarrow 0$. We call that a mapping T is continuous if it is continuous at every point of M.

We first recall the following theorem on the almost everywhere finiteness of the maximal operator due to Guzman [8].

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THEOREM 2.1 [8, p. 10]. *Assume $(X, \mathcal{A}, \mu)$ is a complete measure space and $T_k : L^1(\mu) \to (L^0(\mu), d)$ is a sequence of sub-linear operators with $\mu(X) < \infty$. If each T_k is continuous and that the maximal operator T^* defined for $f \in L^1(\mu)$ and $x \in X$ as*

$$T^* f(x) := \sup_k |T_k f(x)| < \infty \quad \mu - a.e.$$

Then T^* is also continuous at 0, and therefore

$$\lim_{\alpha\to\infty} \phi(\alpha) := \lim_{\alpha\to\infty} \sup_{\|f\|_{L^1(\mu)}\leqslant 1} \mu(\{x \in X : T^* f(x) > \alpha\}) = 0.$$

Notice that the bounded linear operator is always continuous. By theorem 2.1, if $\{T_k\}$ is a sequence of bounded linear operators, then $\lim_{\alpha\to\infty} \phi(\alpha) > 0$ implies there exists an integrable function g such that $T_k(g)$ is not almost everywhere convergent.

COROLLARY 2.2. *Let $(X, \mathcal{A}, \mu)$ be a complete measure space and $T_k : L^1(\mu) \to (L^0(\mu), d)$ be a sequence of bounded linear operators with $\mu(X) < \infty$. If*

$$\lim_{\alpha\to\infty} \sup_{\|f\|_{L^1(\mu)}\leqslant 1} \mu(\{x \in X : T^* f(x) > \alpha\}) > 0, \tag{2.1}$$

then a function g existing in $L^1(\mu)$ can be obtained that its $T_k(g)$ diverges on a set of strictly positive μ-measure.

However, it is difficult to verify (2.1). Thus, in comparison with testing on the integrable functions, we consider acting over a sum of Dirac measure, which is better handled in most cases. Concretely, let $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{A}, \nu)$ be complete Borel measure spaces defined on a Hausdorff space X, and $\mathcal{B}(X)$ be the space of locally finite Borel space measure on X. Consider a sequence k_j of kernels satisfying the following two properties:

(i) Each $k_j : X \times X \to X$ is a measurable function such that $k_j(\cdot, y) \in L^1(\mu)$.

(ii) For each j there exists O_j such that

$$\|k_j(\cdot, y)\|_{L^1(\mu)} \leqslant O_j \quad \text{for every } y \in X.$$

We write

$$K_j f(x) = \int_X k_j(x, y) f(y) \mathrm{d}\mu(y) \quad \text{for } f \in X.$$

Using Fubini–Tonelli's theorem, the second property of kernels makes the maximal operator sensible as follows:

$$K^* f(x) = \sup_j |K_j f(x)|.$$

Moreover, if $k_j(x, y)$ is a continuous function with compact support on $X \times X$ for any $j \in \mathbb{N}$, then each K_j is a bounded linear operator from $L^1(\mu)$ to $L^\infty(\mu)$. Such an operator has a natural extension to a bounded linear operator from $\mathcal{B}(X)$ to $L^\infty(\mu)$, which we denote by K_j again, namely

$$K_j\nu(x) = \int_X k_j(x, y)\mathrm{d}\nu(y), \quad K^*\nu(x) = \sup_j |K_j\nu(x)|. \tag{2.2}$$

Especially, choose a sum of Dirac measure $\nu = \frac{1}{H}\sum_{h=1}^H \delta_{x_h}$ for $x_1, \ldots, x_H \in X$, then

$$K_j\nu(x) = \frac{1}{H}\sum_{h=1}^H k_j(x, x_h), \quad K^*\nu(x) = \frac{1}{H}\sup_j \left|\sum_{h=1}^H k_j(x, x_h)\right|.$$

In what follows, our main aim is to extend the 'pointillist principle' of Carena [**2**, theorem 1], then a similar conclusion can be obtained under slightly different conditions. To certify theorem 1.1, recall the dyadic cubes constructed by Christ in [**3**], which is extremely important for extending results from classical harmonic analysis to the metric space setting.

THEOREM 2.3 [**3**, theorem 11]. *Let (X, ρ) be a metric space and suppose that μ is a regular doubling measure on X. Then there exists a collection of open subsets*

$$\{Q_\alpha^k \subset X : k \in \mathbb{Z},\ \alpha \in I_k\}$$

satisfying the following properties:

(i) *For each integer k,*

$$\mu\left(X \backslash \bigcup\nolimits_\alpha Q_\alpha^k\right) = 0.$$

(ii) *Each Q_α^k has a centre $z_{Q_\alpha^k}$ such that*

$$B(z_{Q_\alpha^k}, C_1\delta^k) \subseteq Q_\alpha^k \subseteq B(z_{Q_\alpha^k}, C_2\delta^k),$$

where C_1, C_2 and δ are positive constants depending only on the doubling constant A_1 of the measure μ and independent of Q_α^k.

(iii) *For each (k, α) and each $l < k$, there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$.*

(iv) *For any k, α and $t > 0$, there exist constants $\delta \in (0, 1)$, $C_3 < \infty$, $\eta > 0$ depending only on μ such that*

$$\mu\left\{x \in Q_\alpha^k : \rho(x, X\backslash Q_\alpha^k) \leqslant t\delta^k\right\} \leqslant C_3 t^\eta \mu(Q_\alpha^k).$$

Note that I_k denotes some index set. Dyadic cubes are constructed by $\bigcup\limits_{k\in\mathbb{Z},\, \alpha\in I_k} \{Q_\alpha^k\}$. We also assume that the centre of dyadic cubes satisfies maximal δ^k-distance disperse condition, that is

$$\rho(z_{Q_\alpha^k}, z_{Q_\beta^k}) \geqslant \delta^k \quad \text{for any } \alpha \neq \beta. \tag{2.3}$$

In this context, maximality means that no new points of the space X can be added to the set $\{z_{Q_\alpha^k}\}$ such that (2.3) remains valid. One other factor indeed, the last condition says that the area near the boundary of a 'cube' Q_α^k is small.

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The main result is as follows.

LEMMA 2.4. *Let (X, ρ) be a metric space and let μ be a positive regular Borel measure satisfying the doubling condition on X. Let ν be a measure such that $\mathrm{d}\nu = g\mathrm{d}\mu$ with $g \in L^1_{loc}(X, \rho, \mu)$. Denote*

$$\psi_\alpha(f) := \nu(\{x \in X : |f(x)| > \alpha\})$$

for a measurable function f defined on (X, ρ, ν). If each kernel $k_j(x, y)$ is a continuous function with compact support on $X \times X$ and K^* is defined in (2.2), then

$$\lim_{\alpha\to\infty} \sup_{\|f\|_{L^1(\nu)}\leqslant 1} \psi_\alpha(K^* f) = 0 \iff \lim_{\alpha\to\infty} \sup_{\omega=\frac{1}{H}\sum_{h=1}^{H}\delta_{a_h}} \psi_\alpha(K^*\omega) = 0 \tag{2.4}$$

for every finite collection $a_1, a_2, \ldots, a_H \in X$ not necessarily pairwise different points.

Proof. For pairwise different points $a_1, a_2, \ldots, a_H \in X$, we firstly denote following sets of measures as elements:

$$\mathcal{M}_{\mathbb{N}} = \left\{\omega = \frac{\sum_{h=1}^{H} c_h\delta_{a_h}}{\sum_{h=1}^{H} c_h} : c_h \in \mathbb{N}^+\right\},$$

$$\mathcal{M}_{\mathbb{Q}} = \left\{\omega = \frac{\sum_{h=1}^{H} c_h\delta_{a_h}}{\sum_{h=1}^{H} c_h} : c_h \in \mathbb{Q}^+\right\},$$

$$\mathcal{M}_{\mathbb{R}} = \left\{\omega = \frac{\sum_{h=1}^{H} c_h\delta_{a_h}}{\sum_{h=1}^{H} c_h} : c_h \in \mathbb{R}^+\right\},$$

and set of functions

$$\mathcal{F}_{\mathcal{R}} = \left\{f = \frac{\sum_{h=1}^{H} c_h\chi_{Q_h}}{\sum_{i=1}^{H} c_h\nu(Q_h)} : c_h \in \mathbb{R}^+,\ Q_i \cap Q_j = \emptyset\right\},$$

where Q_h is a dyadic cube constructed in theorem 2.3 and χ_{Q_h} is the characteristic function.

Necessity. We divide the proof of

$$\lim_{\alpha\to\infty} \sup_{\omega=\frac{1}{H}\sum_{h=1}^{H}\delta_{a_h}} \psi_\alpha(K^*\omega) = 0 \Longrightarrow \lim_{\alpha\to\infty} \sup_{\|f\|_{L^1(\nu)}\leqslant 1} \psi_\alpha(K^* f) = 0$$

into four steps.

Step 1. In the first step, we verify $\lim_{\alpha\to\infty}\sup_{\omega\in\mathcal{M}_{\mathbb{Q}}}\psi_\alpha(K^*\omega) = 0$. Consider an element ω in the class of all linear combinations of Dirac deltas with positive integer coefficients $\mathcal{M}_{\mathbb{N}}$. Then $\lim_{\alpha\to\infty}\sup_{\omega\in\mathcal{M}_{\mathbb{N}}}\psi_\alpha(K^*\omega) = 0$ by the assumption.

Write $c_h = \frac{n_h}{m_h}$ with $n_h, m_h \in \mathbb{N}^+$ for $c_h \in \mathbb{Q}^+$. Since

$$\frac{\sum_{h=1}^{H} c_h\delta_{a_h}}{\sum_{h=1}^{H} c_h} = \frac{\sum_{h=1}^{H} \bar{c}_h\delta_{a_h}}{\sum_{h=1}^{H} \bar{c}_h}, \quad \text{where } \bar{c}_h = n_h \prod_{j=1, j\neq h}^{H} m_j \in \mathbb{N}^+,$$

we know $\mathcal{M}_{\mathbb{N}} = \mathcal{M}_{\mathbb{Q}}$. Hence $\lim_{\alpha\to\infty}\sup_{\omega\in\mathcal{M}_{\mathbb{Q}}}\psi_\alpha(K^*\omega) = 0$.

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Step 2. In this step, we want to prove that $\lim_{\alpha\to\infty}\sup_{\omega\in\mathcal{M}_{\mathbb{R}}}\psi_\alpha(K^*\omega)=0$. Let us start by defining maximal truncated operator acting on functions and distributions:

$$K_N^*f(x)=\max_{1\leqslant j\leqslant N}|K_jf(x)|,\quad K_N^*\nu(x)=\frac{1}{H}\sup_{1\leqslant j\leqslant N}\left|\sum_{h=1}^{H}k_j(x,x_h)\right|.$$

We next claim that, for each integer N, real numbers α, $\varepsilon>0$, $0<\beta<\alpha$ and $\omega\in\mathcal{M}_{\mathbb{R}}$, we can find a finite discrete measure $\varpi\in\mathcal{M}_{\mathbb{Q}}$ satisfying the inequality

$$\psi_\alpha(K_N^*\omega)\leqslant\psi_{\alpha-\beta}(K_N^*\varpi)+2\varepsilon.$$

In fact, for $\omega\in\mathcal{M}_{\mathbb{R}}$, take $d_h\in\mathbb{Q}^+$ such that $c_h=d_h+r_h$, where $r_h>0$ will be determined later. If we write $\varpi=\sum_{h=1}^H d_h\delta_{d_h}/\sum_{h=1}^H d_h$, then, for $0\leqslant\beta\leqslant\alpha$,

$$\psi_\alpha(K_N^*\omega)\leqslant\psi_{\alpha-\beta}(K_N^*\varpi)+\psi_\beta(K_N^*(\omega-\varpi)).$$

Hence

$$\begin{aligned}&\psi_\beta(K_N^*(\omega-\varpi))\\&\quad\leqslant\frac{2}{\beta}\sum_{j=1}^{N}\sum_{h=1}^{H}|r_h|\int_X|k_j(x,a_h)|\mathrm{d}\nu\left(\frac{1}{\sum_{h=1}^H d_h}+\frac{\sum_{h=1}^H r_h}{\sum_{h=1}^H d_h\sum_{h=1}^H c_h}\right).\end{aligned}$$

Applying the properties (i) of kernels, we can choose small r_h such that the right-hand side of the above inequality are all less than arbitrary $\varepsilon>0$. Thus our claim is proved.

Hence we use $K_N^*\leqslant K^*$ to get $\psi_\alpha(K_N^*\omega)\leqslant\sup_{\varpi\in\mathcal{M}_{\mathbb{Q}}}\psi_{\alpha-\beta}(K^*\varpi)+2\varepsilon$. Taking the maximum in the measure family $\mathcal{M}_{\mathbb{R}}$ and letting $\beta\to0$, $\varepsilon\to0$, we have

$$\sup_{\varpi\in\mathcal{M}_{\mathbb{Q}}}\psi_\alpha(K^*\varpi)\leqslant\sup_{\omega\in\mathcal{M}_{\mathbb{R}}}\psi_\alpha(K_N^*\omega)\leqslant\lim_{\alpha_0\to\alpha^-}\sup_{\varpi\in\mathcal{M}_{\mathbb{Q}}}\psi_{\alpha_0}(K^*\varpi).$$

Since $\sup_{\varpi\in\mathcal{M}_{\mathbb{Q}}}\psi_\alpha(K^*\varpi)$ monotonically decreases with α and $\lim_{\alpha\to\infty}\sup_{\varpi\in\mathcal{M}_{\mathbb{Q}}}\psi_\alpha(K^*\varpi)=0$, one obtain $\lim_{\alpha\to\infty}\sup_{\varpi\in\mathcal{M}_{\mathbb{R}}}\psi_\alpha(K^*\varpi)=0$.

Step 3. Next, we show $\lim_{\alpha\to\infty}\sup_{f\in\mathcal{F}_{\mathbb{R}}}\psi_\alpha(K^*f)=0$. It will suffice to show that for each integer N, real numbers α, $\varepsilon>0$, $0<\beta<\alpha$ and $\omega\in\mathcal{M}_{\mathbb{R}}$, there exists a function $f\in\mathcal{F}_{\mathbb{R}}$ satisfying the inequality $\psi_\alpha(K_N^*\omega)\leqslant\psi_{\alpha-\beta}(K_N^*f)+\varepsilon$. Once again, the desired limiting behaviour will follow by reasoning as in the proof of step 2.

Let $f=\sum_{h=1}^H c_h\chi_{Q_h}/\sum_{h=1}^H c_h\nu(Q_h)\in\mathcal{F}_{\mathbb{R}}$ and $\omega=\sum_{h=1}^H c_h\nu(Q_h)\delta_{z_{Q_h}}/\sum_{h=1}^H c_h\nu(Q_h)\in\mathcal{M}_{\mathbb{R}}$, where z_{Q_h} denotes the centre of the dyadic cube Q_h. Without loss of generality, we shall assume that every dyadic set Q_h such that $\mathrm{diam}(Q_h)<\eta$ for $h=1,2,\ldots,H$; $\eta>0$. This assumption is practicable since we can write, except on a set with ν-measure equal to zero, $f=\sum_{j=1}^W c_j\chi_{Q'_j}/\sum_{j=1}^W c_j\nu(Q'_j)$ with disjoint dyadic cubes Q'_j and $\mathrm{diam}(Q'_j)<\eta$ for all j if f and η is given. See properties (i) and (iii) in theorem 2.3. Thus we will keep writing $f=\sum_{h=1}^H c_h\chi_{Q_h}/\sum_{h=1}^H c_h\nu(Q_h)$ and suppose that the diameter of each Q_h is as little as we need.

If $0 \leqslant \beta \leqslant \alpha$, for fixed N, we obtain that

$$\begin{aligned}
&\psi_\beta(K_N^*(f-\omega)) \\
&\quad \leqslant \sum_{j=1}^N \frac{1}{\beta}\int_X |K_j(f-\omega)(x)|\mathrm{d}\nu(x) \\
&\quad \leqslant \frac{1}{\beta\left(\sum_{h=1}^H c_h\nu(Q_h)\right)}\sum_{j=1}^N\int_X\left(\sum_{h=1}^H c_h\int_{Q_h}|k_j(x,y)-k_j(x,z_{Q_h})|\mathrm{d}\nu(y)\right)\mathrm{d}\nu(x) \\
&\quad \leqslant \frac{1}{\beta\left(\sum_{h=1}^H c_h\nu(Q_h)\right)}\sum_{j=1}^N\sum_{h=1}^H c_h\int_{Q_h}\left(\int_{F_j}|k_j(x,y)-k_j(x,z_{Q_h})|\mathrm{d}\nu(x)\right)\mathrm{d}\nu(y),
\end{aligned}$$

where F_j denotes the projection of the support of $k_j(x,\, y)$. From the hypothesis, F_j is a bounded set with finite measure. Hence every $k_j(x,\, y)$ is a uniformly continuous function with compact support in $X \times X$, we can take small $\operatorname{diam}(Q_i)$ such that $\psi_\beta(K_N^*(f-\omega))$ is small enough.

Step 4. Finally, from the fact that the set of all real coefficients linear combinations of characteristic functions of dyadic sets is dense in $L^1(\mu)$, by the standard argument, one obtains

$$\lim_{\alpha\to\infty}\sup_{\|f\|_1\leqslant 1}\psi_\alpha(K^*f) = \lim_{\alpha\to\infty}\sup_{f\in\mathcal{F}_{\mathbb{R}}}\psi_\alpha(K^*f) = 0.$$

The proof of lemma 2.4 in one direction is complete.

Sufficiency. Conversely, we want to prove

$$\lim_{\alpha\to\infty}\sup_{\|f\|_1\leqslant 1}\psi_\alpha(K^*f) = 0 \Longrightarrow \lim_{\alpha\to\infty}\sup_{\omega\in\mathcal{M}_{\mathbb{N}}}\psi_\alpha(K^*\omega) = 0.$$

Clearly $\omega \in \mathcal{M}_{\mathbb{N}} \subset \mathcal{M}_{\mathbb{Q}}$. For pairwise different points $a_1,\, a_2,\, \ldots, a_H \in X$, we denote the metric by ρ and $d = \min\{\rho(x_i,\, x_j),\, x_i \neq x_j\}$. Now we apply properties (i) and (ii) in theorem 2.3, then there exists $Q_{i_h}^n$ such that $x_h \in \overline{Q_{i_h}^n}$. Let n be a large integer such that $C_2\delta^n < \frac{d}{4}$, where $C_2,\, \delta$ are constants mentioned in theorem 2.3. We claim the set in $\{Q_{i_h}^n\}_{h=1}^H$ is pairwise disjoint. In fact, if $x \in Q_{i_h}^n \cap Q_{i_m}^n$ for $h \neq m$. Since $Q_{i_h}^n \subset B(z_{Q_{i_h}^n},\, C_2\delta^n) \subset B(z_{Q_{i_h}^n},\, \frac{d}{4})$ and $Q_{i_m}^n \subset B(z_{Q_{i_m}^n},\, C_2\delta^n) \subset B(z_{Q_{i_m}^n},\, \frac{d}{4})$, using the triangle inequality, we have

$$\rho(x_h, x_m) \leqslant \rho(x_h, z_{Q_{i_h}^n}) + \rho(z_{Q_{i_h}^n}, x) + \rho(x, z_{Q_{i_m}^n}) + \rho(z_{Q_{i_m}^n}, x_m) < d.$$

This is a contradiction to $\rho(x_h,\, x_m) \geqslant d$. Apparently, let $c_h \in \mathbb{N}^+$ and

$$\omega = \sum_{h=1}^H c_h\delta_{x_h}, \quad f = \frac{1}{\sum_{h=1}^H c_h}\sum_{h=1}^H \frac{c_h}{\nu(Q_{i_h}^n)}\chi_{Q_{i_h}^n},$$

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then $\omega \in \mathcal{M}_{\mathbb{N}}$ and $\|f\|_1 = 1$. Fix $N \in \mathbb{N}^+$ and $\beta > 0$ we have

$$\begin{aligned}
&\psi_\beta(K_N^*(f-\omega)) \\
&\quad \leqslant \frac{1}{\sum_{h=1}^H \beta c_h} \sum_{h=1}^H \frac{c_h}{\nu\left(Q_{i_h}^n\right)} \sum_{j=1}^N \int_X \left(\int_{Q_{i_h}^n} |k_j(x,y) - k_j(x,x_h)| \mathrm{d}\nu(y) \right) \mathrm{d}\nu(x) \\
&\quad \leqslant \frac{1}{\sum_{h=1}^H \beta c_h} \sum_{h=1}^H \frac{c_h}{\nu\left(Q_{i_h}^n\right)} \sum_{j=1}^N \int_{F_j} \left(\int_{Q_{i_h}^n} |k_j(x,y) - k_j(x,x_h)| \mathrm{d}\nu(x) \right) \mathrm{d}\nu(y),
\end{aligned}$$

where F_j is the projection of the support of $k_j(x, y)$ once again. Repeating the arguments in the proof of step 3, we obtain that $\lim_{\alpha\to\infty} \sup_{\omega\in\mathcal{M}_{\mathbb{N}}} \psi_\alpha(K^*\omega) = 0$. This completes the proof of lemma 2.4. □

Combining with corollary 2.2 and lemma 2.4, we have the following corollary.

COROLLARY 2.5. *Let (X, ρ) be a metric space. Assume every kernel $k_j(x, y)$ is a continuous function with compact support on $X \times X$ and K^* is defined in (2.2). If μ is a totally finite complete measure satisfying the doubling condition and*

$$\lim_{\alpha\to\infty} \sup_{\omega=\frac{1}{H}\sum_{h=1}^H \delta_{a_h}} \mu(\{x \in X : K^*\omega(x) > \alpha\}) > 0,$$

then there exists a function $g \in L^1(\mu)$ such that $T_k(g)$ diverges on a set of positive μ-measure.

Next we establish theorem 1.1.

Proof. Let μ denote a spectral measure supported on a compact subset of $X \subset (\mathbb{R}^d, \rho)$, where $(\mathbb{R}^d, \rho)$ is the Euclidean space. Let $\{\mathrm{e}^{-2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ be an exponential orthonormal basis of $L^2(\mu)$. For a natural sequence of finite subsets Λ_n increasing to Λ as $n \to \infty$, consider the mock Dirichlet kernel as

$$k_n(x, y) = \sum_{\lambda\in\Lambda_n} \mathrm{e}^{2\pi i\lambda\cdot(x-y)}.$$

Then the mock Dirichlet summation operator S_n with Λ_n can be written as

$$S_n(f)(x) = \sum_{\lambda\in\Lambda_n} c_\lambda(f)\mathrm{e}^{2\pi i\lambda\cdot x} = \int_X k_n(x, y) f(y) \mathrm{d}\mu(y).$$

From corollary 2.5, the theorem follows immediately. □

3. Application to self-affine spectral measures

In this section, we apply our results to self-affine spectral measures. Recall that the self-affine measure is defined by iterated function system (IFS).

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DEFINITION 3.1 (Self-affine measure). *Let R be a $d \times d$ expansive matrix (all its eigenvalues have modulus strictly bigger than one). Let $B = \{b_1, b_2, \ldots, b_N\}$ be a finite subset of $\mathbb{R}^d$. We define the affine iterated function system*

$$\varphi_b(x) = R^{-1}(x + b) \quad \text{for } x \in \mathbb{R}^d \text{ and } b \in B.$$

The self-affine measure (with equal weights) is the unique probability measure satisfying

$$\mu(E) = \frac{1}{N} \sum_{b \in B} \mu(\varphi_b^{-1}(E)) \quad \text{for all Borel subsets } E \text{ of } \mathbb{R}^d.$$

We will use $\mu_{R,B}$ to denote it for convenience. This measure is supported on the attractor $T(R, B)$ which is the unique compact set that satisfies

$$T(R, B) = \bigcup_{b \in B} \varphi_b(T(R, B)).$$

The set $T(R, B)$ is also called the self-affine set associated with the IFS. It can also be described as

$$T(R, B) = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k : \ b_k \in B \right\}.$$

One can refer to [9] for a detailed exposition of the theory of iterated function systems. In this section, we will use μ_4 to denote the quarter Cantor measure which is the special case when $d = 1$, $R = 4$ and $B = \{0, 2\}$.

To the best of our knowledge, most of self-affine spectral measures are constructed by Hadamard triples.

DEFINITION 3.2 (Hadamard triple). *For a given expansive $d \times d$ matrix R with integer entries. Let $B, L \subset \mathbb{Z}^d$ be finite sets of integer vectors with the same cardinality $N \geqslant 2$. We say that the triple (R,B,L) forms a Hadamard triple if the matrix*

$$H = \frac{1}{\sqrt{N}} \left[\mathrm{e}^{2\pi i R^{-1} b \cdot l} \right]_{l \in L, b \in B}$$

is unitary, i.e. $H^* H = I$, where H^* denotes the conjugate transpose of H.

The system (R, B, L) forms a Hadamard triple if and only if the Dirac measure $\delta_{R^{-1}D} = \frac{1}{\#B} \sum_{b \in B} \delta_{R^{-1}b}$ is a spectral measure with the spectrum L. Moreover, this property is a key property in producing a spectrum of self-affine spectral measures. Łaba and Wang [11] and Dutkay *et al.* [5] eventually proved that the Hadamard triple generates the self-affine spectral measure in all dimensions.

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If $(R,\, B,\, L)$ forms a Hadamard triple, we let

$$\Lambda_n = L + R^t L + (R^t)^2 L + \cdots + (R^t)^{n-1} L = \sum_{k=0}^{n-1} (R^t)^k L,$$

and

$$\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n = \sum_{k=0}^{\infty} (R^t)^k L, \tag{3.1}$$

where R^t denotes the transpose of R.

The set Λ forms an orthonormal set for the self-affine spectral measure $\mu := \mu_{R,B}$. But the set Λ can be incomplete (see [**7**, p. 4]). In this paper, we assume that the self-affine spectral measure μ generated by the Hadamard triple $(R,\, B,\, L)$ always has a spectrum like (3.1).

We say that the self-affine measure μ in definition 3.1 satisfies the no-overlap condition or the measure disjoint condition if

$$\mu(\varphi_b(T(R,B)) \cap \varphi_{b'} T(R,B)) = 0 \text{ for all } b \neq b^{'} \in B.$$

Dutkay *et al.* [**5**] proved that if the self-affine spectral measure is generated by a Hadamard triple, then the no-overlap condition is satisfied.

Following the work of Dutkay *et al.* [**6**], the encoding map plays the key role in linking a no-overlap self-affine measure and a code space. Let $\mathbb{N}^*$ denote the positive integer numbers. Recall the symbolic space $B^{\mathbb{N}^*}$ with the product probability measure dP where each digit in B has probability $1/N$, also known as the equally weighted Bernoulli measure space. The right shift defined by $T : B^{\mathbb{N}^*} \to B^{\mathbb{N}^*}$,

$$T(b_1 b_2 b_3 \cdots) = b_2 b_3 \cdots ,$$

is a dP-measure-preserving transformation. Also note that T is dP-ergodic.

For the attractor, we consider the map $\mathcal{R} : T(R,\, B) \to T(R,\, B)$,

$$\mathcal{R}\left(\sum_{i=1}^{\infty} R^{-i} b_i\right) = \sum_{i=1}^{\infty} R^{-i} b_{i+1}.$$

Dutkay *et al.* [**6**] proved the following proposition.

Proposition 3.3 (Dutkay *et al.* [**6**, proposition 1.11]). *Define the encoding map* $h : B^{\mathbb{N}^*} \to T(R,\, B)$ *by*

$$h(b_1 b_2 b_3 \cdots) = \sum_{i=1}^{\infty} R^{-i} b_i,$$

then h is onto, measure preserving and $hT = \mathcal{R}h$. Furthermore, if the self-affine measure $\mu_{R,B}$ satisfies the no-overlap condition, then h is one to one on a set of full measure.

This implies that the map h is an isomorphism of dynamical systems. By proposition 3.3, we immediately obtain the following corollary.

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Corollary 3.4. *If $\mu_{R,B}$ has the no-overlap condition, $\mu_{R,B}$ is an ergodic measure.*

For a self-affine spectral measure $\mu_{R,B}$ generated by a Hadamard triple $(R,\ B,\ L)$, let τ be an integer such that

$$\tau\Lambda = \tau\bigcup_{n=0}^{\infty}\Lambda_n = \tau\sum_{k=0}^{\infty}(R^t)^k L$$

is a spectrum of μ. Define the Dirichlet kernel

$$D_n(x) := \sum_{\lambda\in\tau\Lambda_n} \mathrm{e}^{2\pi i\lambda\cdot x} \quad (x\in\mathbb{R}^d).$$

For $f\in L^1(\mu)$, the mock Dirichlet summation operator

$$S_n(f)(x) = \sum_{\lambda\in\tau\Lambda_n}\left(\int_{T(R,B)} f(y)\mathrm{e}^{-2\pi i\lambda\cdot y}\mathrm{d}\mu(y)\right)\mathrm{e}^{2\pi i\lambda\cdot x}$$

can be written as

$$S_n(f)(x) = \int_{T(R,B)} f(y)D_n(x-y)\mathrm{d}\mu(y).$$

We record one obvious fact, and its proof is similar to [6, proposition 2.2]. This fact suggests that there is an explicit formula of the Dirichlet kernel that can be concisely estimated, which depends to a large extent on the division of the 'mock block', i.e. the choice of Λ_n.

Proposition 3.5. *Define trigonometric polynomials*

$$m_\tau(x) = \sum_{l\in L}\mathrm{e}^{2\pi i(\tau l)\cdot x} \quad (x\in\mathbb{R}^d).$$

Then the Dirichlet kernel satisfies the formula

$$D_n(x) = \prod_{k=0}^{n} m_\tau(R^k x). \tag{3.2}$$

Proof. The proof is by induction on n. The base case holds for $n=0$ since

$$D_0(x) = \sum_{\lambda\in\tau\Lambda_0}\mathrm{e}^{2\pi i\lambda\cdot x} = \sum_{l\in L}\mathrm{e}^{2\pi i(\tau l)\cdot x} = m_\tau(x).$$

Assume that the formula holds for n, and we prove that the statement holds for $n+1$. To do this, we need to prove

$$D_{n+1}(x) = m_\tau(x)D_n(Rx). \tag{3.3}$$

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Since $\Lambda_{n+1} = R^t\Lambda_n + L$, we see that every point λ_{n+1} in Λ_{n+1} will have a unique representation of the form $\lambda_{n+1} = R^t\lambda_n + l$ with $\lambda_n \in \Lambda_n$ and $l \in L$. This yields

$$\begin{aligned} D_{n+1}(x) &= \sum_{\lambda_n \in \Lambda_n} \sum_{l \in L} \mathrm{e}^{2\pi i\tau(R^t\lambda_n + l)\cdot x} \\ &= \sum_{l \in L} \mathrm{e}^{2\pi i(\tau l)\cdot x} \sum_{\lambda_n \in \Lambda_n} \mathrm{e}^{2\pi i\tau\lambda_n\cdot(Rx)} \\ &= m_\tau(x)D_n(Rx). \end{aligned}$$

Thus equation (3.2) follows by induction from equation (3.3). $\square$

Using above propositions and theorem 1.1, we shall prove the following lemma.

LEMMA 3.6. *Let R be an integer matrix and let $\mu := \mu_{R,B}$ be a self-affine spectral measure generated by a Hadamard triple $(R,\, B,\, L)$. Assume μ is a doubling measure with the spectrum $\tau\Lambda = \sum_{k=0}^{\infty} R^k\tau L$. Let*

$$\Delta(m_{\tau,b}) := \exp\left(\int_{T(R,B)} \log|m_\tau(x - (I - R^{-1})^{-1}b)|\mathrm{d}\mu(x)\right), \quad b \in B,$$

where $m_\tau(x)$ is defined in proposition 3.5. If $\Delta(m_{\tau,b}) > 1$ for some $b \in B$, then there exists an integrable function such that the mock Fourier series diverges on a set of strictly positive μ-measure set.

Proof. Recall that the points in $T(R,\, B)$ have the form $x = \sum_{i=1}^{\infty} R^{-i}b_i$ with $b_i \in B$, and the map $\mathcal{R}$ is

$$\mathcal{R}\left(\sum_{i=1}^{\infty} R^{-i}b_i\right) = \sum_{i=1}^{\infty} R^{-i}b_{i+1}.$$

Denote $\mathcal{R}^k = \mathcal{R}\cdots\mathcal{R}$ and $y = \sum_{i=1}^{\infty} R^{-i}c_i$ with $c_i \in B$, and we see that

$$(\mathcal{R}^k x - \mathcal{R}^k y) - R^k(x - y) = -\sum_{i=1}^{k} R^{k-i}(b_i - c_i) \in \mathbb{Z}.$$

Since $m_\tau(x)$ is $\mathbb{Z}$-periodic, we have

$$m_\tau(\mathcal{R}^k x - \mathcal{R}^k y) = m_\tau(R^k(x - y)) \tag{3.4}$$

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for all $x \in T(R, B)$ and $k \in \mathbb{N}$. Now note that $\sum_{i=0}^{\infty} R^{-i}b = (I - R^{-1})^{-1}b$ are fix points of the map $\mathcal{R}$ for any $b \in B$, thus proposition 3.5 and formula (3.4) gives

$$\begin{aligned} D_n(x-(I-R^{-1})^{-1}b) &= \prod_{k=0}^{n} m_\tau\left(R^k\left(x-\sum_{i=0}^{\infty}R^{-i}b\right)\right) \\ &= \prod_{k=0}^{n} m_\tau\left(\mathcal{R}^k x-\mathcal{R}^k\left(\sum_{i=0}^{\infty}R^{-i}b\right)\right) \\ &= \prod_{k=0}^{n} m_\tau(\mathcal{R}^k x-(I-R^{-1})^{-1}b). \end{aligned}$$

Combining with corollary 3.4 and Birkhoff's ergodic theorem, one has that for μ-a.e. x in $T(R, B)$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log|m_\tau(\mathcal{R}^k x-(I-R^{-1})^{-1}b)| = \log\Delta(m_{\tau,b}),$$

i.e.

$$\lim_{n\to\infty}\frac{1}{n}\log|D_{n-1}(x-(I-R^{-1})^{-1}b)| = \log\Delta(m_{\tau,b}).$$

Thus we can get a subset $A \subset T(R, B)$ with measure $\mu(A) > \frac{1}{2}$ such that the limit above is uniform on A. If $\Delta(m_{\tau,b}) > 1$ for some $b \in B$, for $x \in A$, taking $1 < \rho < \Delta(m_{\tau,b})$, then there exists n_ρ such that for $n > n_\rho$,

$$\frac{1}{n}\log|D_{n-1}(x-(I-R^{-1})^{-1}b)| > \log\rho.$$

For $x \in A$, it is easy to see

$$\begin{aligned} \sup_n|D_{n-1}(x-(I-R^{-1})^{-1}b)| &\geqslant \sup_{n>n_\rho}|D_{n-1}(x-(I-R^{-1})^{-1}b)| \\ &\geqslant \sup_{n>n_\rho}\rho^n = +\infty. \end{aligned}$$

Hence for any $\alpha \geqslant 0$, the mock Dirichlet summation operator S_n acting on $\delta_{(I-R^{-1})^{-1}b}$ satisfies

$$\mu\left(\left\{x \in T(R,B) : \sup_n|S_n(\delta_{(I-R^{-1})^{-1}b})(x)| > \alpha\right\}\right) \geqslant \mu(A) \geqslant \frac{1}{2}.$$

The proof is complete by theorem 1.1. □

Recall the discriminant value defined by Dutkay *et al.* [6, theorem 2.3]

$$\Delta_{Nm_L} := \exp\left(\int_{T(R,B)}\log|Nm_L(x)|\mathrm{d}\mu\right)$$

where $m_L(x) = \frac{1}{N}\sum_{l\in L}\mathrm{e}^{2\pi i l\cdot x}$. Our discriminant value $\Delta(m_{\tau,0})$ may be reduced into Δ_{Nm_L} if $d = 1$ and B contains the origin.

It is noteworthy that the proper separation condition of IFS implies the regularity of the measure, and in fact, Mauldin and Urbański proved that if the IFS satisfies the open set condition, then $\mu_{R,B}$ has the no-overlap condition, and thus the associated self-affine is doubling on its support $T(R, B)$. See [**12**, lemma 3.14]. This means that if the discriminant of Dutkay is greater than 1, not only there exists a continuous functions such that the mock Fourier series at zero is unbounded, but also there exists an integrable function such that the mock Fourier series diverges on a set of strictly positive μ-measure.

Proof. By lemma 3.6, we only need to gauge whether Δ_{2m_L} is larger than 1. But this fact has been shown in [**6**, example 2.5] by numerical approximation. □

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