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The Values of Modular Functions and Modular Forms

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Abstract. Let Γ_0 be a Fuchsian group of the first kind of genus zero and Γ be a subgroup of Γ_0 of finite index of genus zero. We find universal recursive relations giving the q_r-series coefficients of j_0 by using those of the q_{h_s}-series of j, where j is the canonical Hauptmodul for Γ and j_0 is a Hauptmodul for Γ_0 without zeros on the complex upper half plane $\mathfrak{H}$ (here $q_\ell := e^{2\pi i z/\ell}$). We find universal recursive formulas for q-series coefficients of any modular form on $\Gamma_0^+(p)$ in terms of those of the canonical Hauptmodul j_p^+.

1 Introduction

Let $j_{(N)}$ be the canonical Hauptmodul for a Hecke subgroup $\Gamma_0(N)$ of $SL_2(\mathbb{Z})$ of genus zero. By using Norton and Koike's idea, Kim and Koo [5] derived a recursive formula for q-series coefficients of $j_{(N)}$ ($q = e^{2\pi i z}$ throughout). Let $\Gamma_1(N)$ be the congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ mod N ($N = 2, 6, 8, 10, 12$). Kim and Koo [6] also found recursive formulas for the q-series coefficients of the canonical Hauptmodul $j_{(1,N)}$ for $\Gamma_1(N)$.

Let Γ_0 be a Fuchsian group of the first kind of genus zero and Γ be a subgroup of Γ_0 of finite index of genus zero. Let j be the canonical Hauptmodul for Γ and j_0 be a Hauptmodul for Γ_0. In Section 2, by using Bruinier, Kohnen and Ono's idea in [2] we find universal recursive relations giving the q_r-series coefficients of j_0 in terms of the q_{h_s}-series of j, where $q_l = e^{2\pi i z/l}$ and $l = h_s$ or $l = r$ throughout (see Theorem 2.2).

Let $J = 1/q + 744 + 196884q + \cdots$ be the usual elliptic modular function on $SL_2(\mathbb{Z})$. For every positive integer n, let j_n be the unique modular function which is holomorphic on $\mathfrak{H}$ whose Fourier expansion at ∞ is of the form $j_n = 1/q^n + \sum_{m=1}^{\infty} c_n(m)q^m$. Bruinier, Kohnen and Ono [2] considered the sums of the values of elliptic modular functions j_n over divisors of meromorphic modular function on $SL_2(\mathbb{Z})$, where $j_1 = J - 744$. They showed that the "trace" of these values dictates the properties of modular forms on $SL_2(\mathbb{Z})$. They provided a very useful link relating the values of J to the arithmetic of the Fourier coefficients of modular forms, that is, there are universal recursive formulas for the Fourier coefficients of every modular form on $SL_2(\mathbb{Z})$. They studied the action of Ramanujan's theta-operator defined by

$$\theta\Big(\sum_{n=n_0}^{\infty} a(n)q^n\Big) := \sum_{n=n_0}^{\infty} na(n)q^n$$

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on meromorphic modular forms on $SL_2(\mathbb{Z})$ in relation to the values of a certain sequence of modular functions. If $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ is a weight k meromorphic modular form on $SL_2(\mathbb{Z})$, then

$$\theta(f) = \frac{\widetilde{f} + kfE_2}{12}$$

and $\widetilde{f}$ is a weight $k+2$ meromorphic modular form where

$$E_2(z) := 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n$$

is the Eisenstein series. E_2 is not a modular form, but has a twisted transformation law that we will use in Section 3. They found an explicit formula for $\widetilde{f}$ which yields the universal recursive formulas mentioned above.

It is natural to investigate analogues of this work for modular forms on more general Fuchsian groups of the first kind of genus zero. In Section 3, we consider the problem for groups $\Gamma_0^+(p)$ generated by a Hecke subgroup $\Gamma_0(p)$ of $SL_2(\mathbb{Z})$ and Fricke involution $\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$ with genus zero. In the case $p = 1$, one has $\Gamma_0^+(p) = SL_2(\mathbb{Z})$. If Φ is the set of primes p for which $\Gamma_0^+(p)$ has genus zero, then (see [4])

$$\Phi = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

For each $p \in \Phi$, we give an explicit formula for the action of the Ramanujan's theta-operator on $\Gamma_0^+(p)$: if f is a weight k meromorphic form on $\Gamma_0^+(p)$, then $\theta(f) = (\widetilde{f} + kfE_2 + kpE_2(pz))/24$ and $\widetilde{f}$ is a weight $k+2$ meromorphic modular form on $\Gamma_0^+(p)$. In Section 3, we find an explicit formula for $\widetilde{f}$ in terms of the values of a certain sequence of modular functions (see Theorem 3.2). As a consequence, we obtain recurrence relations for Fourier coefficients of modular forms on these groups (see Theorem 3.1). Finally, we mention that two recent and forthcoming papers [1, 3] consider similar problems with respect to Hecke subgroups of $SL_2(\mathbb{Z})$.

2 Universal Recurrence Relations for Fourier Coefficients of a Hauptmodul

Let $\mathfrak{H}$ be the complex upper half plane. Let Γ_0 be a Fuchsian group of the first kind of genus zero. Let r be the unique positive real number such that $(\Gamma_0)_\infty \cdot \{\pm 1\} = \{\pm \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)^m \mid m \in \mathbb{Z}\}$. Let j_0 be a Hauptmodul for Γ_0. As a Hauptmodul for Γ_0, $j_0(z)$ has a Fourier expansion at ∞ in the form

$$j_0(z) = \frac{1}{q_r} + \sum_{n=0}^{\infty} a(n)q_r^n \quad (q_r = e^{2\pi iz/r}).$$

Let Γ be a subgroup of Γ_0 of finite index of genus zero. Let P_Γ be the set of all cusps of Γ and $\mathfrak{H}^* = \mathfrak{H} \cup P_\Gamma$. For each cusp $s \in P_\Gamma$, take $\sigma \in SL_2(\mathbb{R})$ such that $\sigma\infty = s$. Then there exists a unique positive real number h_s such that

$$\sigma^{-1}\Gamma_s\sigma \cdot \{\pm 1\} = \{\pm \left(\begin{smallmatrix} 1 & h_s \\ 0 & 1 \end{smallmatrix}\right)^m \mid m \in \mathbb{Z}\}.$$

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For convenience we write h for h_∞. Let j be the Hauptmodul for Γ whose Fourier expansion at ∞ is of the form

$$j(z) = \frac{1}{q_h} + \sum_{n=1}^{\infty} a_j(n) q_h^n \quad (q_h = e^{2\pi i z/h}).$$

For each $m \in \mathbb{N}$, there exists a modular function j_m for Γ which is holomorphic on $\mathfrak{H}^* - \Gamma\infty$ and has Fourier expansion at ∞ in the form

$$j_m(z) = \frac{1}{q_h^m} + \sum_{n=1}^{\infty} c_m(n) q_h^n.$$

Indeed, j_m is a polynomial in j with coefficients in $\mathbb{Z}[a_j(1), a_j(2), \dots, a_j(m-1)]$. Since $j_m(z)$ is holomorphic at $s \in P_\Gamma - \Gamma\infty$, it has a Fourier expansion at s in the form

$$j_m(\sigma z) = \sum_{n=0}^{\infty} \alpha_n q_{h_s}^n.$$

The constant term $\alpha_0 = j_m(s)$ is independent of the choice of σ.

For the purposes of the following lemma, let $F(z)$ be any meromorphic modular form for Γ of weight 2. We define the action of $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by

$$(F|_2\sigma)(z) = (\det \sigma) \cdot (cz+d)^{-2} \cdot F(\sigma z).$$

Then $F(z)$ has a Fourier expansion at each cusp $s \in P_\Gamma$ as follows

$$(F|_2\sigma)(z) = \sum_{n \geq N_0} a_n q_{h_s}^n.$$

We consider

$$\omega := F(z)dz$$

as a differential on $\Gamma\backslash\mathfrak{H}^*$ using the canonical quotient map $\pi\colon \mathfrak{H}^* \to \Gamma\backslash\mathfrak{H}^*$. Let $1/e_\tau$ be the cardinality of $\pm\Gamma_\tau/\{\pm 1\}$ for each $\tau \in \mathfrak{H}$.

Lemma 2.1 *We have*

(i) $\operatorname{Res}_{\pi(s)} \omega = \frac{h_s}{2\pi i} a_0$ *for* $s \in P_\Gamma$ *and*
(ii) $\operatorname{Res}_{\pi(\tau)} \omega = e_\tau \operatorname{Res}_\tau F(z)$ *for* $\tau \in \mathfrak{H}$.

Proof By simple calculation we obtain the assertion. ■

For every integer $n > 1$, define a polynomial

$$F_{n-1}(x_1, \dots, x_{n-1}) \in \mathbb{Q}[x_1, \dots, x_{n-1}]$$

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by

$$\sum_{\substack{m_1,\dots,m_{n-1}\geq 0\\ m_1+2m_2+\cdots+(n-1)m_{n-1}=n}} (-1)^{m_1+\cdots+m_{n-1}} \cdot \frac{(m_1+\cdots+m_{n-1}-1)!}{m_1!\dots m_{n-1}!} \cdot x_1^{m_1}\cdots x_{n-1}^{m_{n-1}}.$$

The first few polynomials F_n are

$$F_1(x_1) = \frac{1}{2}x_1^2,$$

$$F_2(x_1,x_2) = -\frac{1}{3}x_1^3 + x_1\cdot x_2,$$

$$F_3(x_1,x_2,x_3) = x_1\cdot x_3 - x_1^2\cdot x_2 + \frac{1}{2}x_2^2.$$

Let $j_0(z)$ have a Fourier expansion at ∞ (as a modular function for Γ) as follows

$$j_0(z) = \sum_{n=-l}^{\infty} b(n)q_h^n.$$

Here we take $l = \frac{h}{r} \in \mathbb{N}$. We then have $a(n) = b(nl)$ and $b(n) = 0$ if $l \nmid n$.

For any modular function f for Γ whose Fourier expansion at a cusp s is of the form

$$f(\sigma z) = \sum_{n=n_s} a_f(s,n)q_{h_s}^n \text{ with } a_f(s,n_s) \neq 0,$$

we call n_s the order of vanishing of f at s and denote it by $\operatorname{ord}_s f$. Moreover, $\operatorname{ord}_\tau g$ denotes the standard order of vanishing of g at the point $\tau \in \mathfrak{H}$ when g is a meromorphic function on $\mathfrak{H}$ (throughout).

Theorem 2.2 *We have that for $n \geq 2$,*

$$\begin{aligned} b(n-l) = {} & F_{n-1}(b(1-l),\dots,b(n-1-l)) \\ & - \frac{1}{n}{\sum}' \operatorname{ord}_s j_0 \cdot j_n(s) - \frac{1}{n}\sum_{\tau\in\Gamma\backslash\mathfrak{H}} e_\tau \cdot \operatorname{ord}_\tau j_0 \cdot j_n(\tau) \end{aligned}$$

and that

$$b(1-l) = -{\sum}' \operatorname{ord}_s j_0 \cdot j_1(s) - \sum_{\tau\in\Gamma\backslash\mathfrak{H}} e_\tau \cdot \operatorname{ord}_\tau j_0 \cdot j_1(\tau).$$

Here the ${\sum}'$ means we sum over representatives s of the cusps of Γ not including the cusp at ∞.

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Proof For $m \in \mathbb{N}$ let $G_m = (h \cdot j_m(z) \cdot \frac{dj_0(z)}{dz})/(2\pi i j_0(z))$ and $\omega_m = G_m(z)dz$. Then ω_m is a 1-form on $\Gamma\backslash\mathfrak{H}^*$. We calculate the residue of ω_m at each point $\pi(\tau)(\tau \in \mathfrak{H}^*)$. First we consider cusps $s \in P_\Gamma$.

Case 1 $s = \infty$. Since $G_m(z) =$ higher terms in $q_{-h} - e(m) +$ higher terms in q_h (we are defining $e(m)$ in (1) so that the following residue calculation holds), we have

$$\mathrm{Res}_{\pi(\infty)}\, \omega_m = \frac{-h}{2\pi i} \cdot e(m).$$

Case 2 $s \in P_\Gamma - \Gamma\infty$. Since $j_m(z)$ is holomorphic at s and

$$\Bigl(h \frac{dj_0(\sigma z)}{dz}\Bigr) / (2\pi i j_0(\sigma z)) = h \cdot \frac{\mathrm{ord}_s\, j_0}{h_s} + \text{ higher terms in } q_{h_s},$$

we obtain

$$\begin{aligned}(G_m|_2\sigma)(z) &= \Bigl(h \cdot j_m(\sigma z) \cdot \frac{dj_0(\sigma z)}{dz}\Bigr) / (2\pi i j_0(\sigma z)) \\ &= h \cdot \frac{\mathrm{ord}_s\, j_0}{h_s} \cdot j_m(s) + \text{ higher terms in } q_{h_s}.\end{aligned}$$

This implies

$$\mathrm{Res}_{\pi(s)}\, \omega_m = \frac{h}{2\pi i} \cdot \mathrm{ord}_s\, j_0 \cdot j_m(s).$$

Case 3 $\tau \in \mathfrak{H}$. j_0 is holomorphic on $\mathfrak{H}$ and $j_m(z)$ is holomorphic on $\mathfrak{H}$. These imply

$$\mathrm{Res}_\tau\, G_m(z) = \frac{h}{2\pi i} \cdot \mathrm{Res}_\tau \frac{\frac{dj_0(z)}{dz}}{j_0(z)} j_m(z) = \frac{h}{2\pi i} \cdot \mathrm{ord}_\tau\, j_0 \cdot j_m(\tau).$$

Consequently the residue theorem $\bigl(\sum_{\tau \in \Gamma\backslash\mathfrak{H}^*} \mathrm{Res}_{\pi(\tau)}\, \omega_m = 0\bigr)$ shows

$$e(m) = \sum{}' \mathrm{ord}_s\, j_0 \cdot j_m(s) + \sum_{\tau \in \Gamma\backslash\mathfrak{H}} e_\tau \cdot \mathrm{ord}_\tau\, j_0 \cdot j_m(\tau).$$

Here the $\sum'$ means we sum over representatives s of the cusps of Γ not including the cusp at ∞. On the other hand, consider $J_0(q_h) := \sum_{n=-l}^{\infty} b(n) q_h^n$ as a meromorphic function in a neighborhood of $q_h = 0$. Arguing as in [2, Propositon 2.1], we have that the $e(n)$ are the q_h-series coefficients of the logarithmic derivative of $J_0(q_h)$:

$$\text{(1)} \qquad \frac{q_h J_0'(q_h)}{J_0(q_h)} = -l - \sum_{n=1}^{\infty} e(n) q_h^n \text{ with } e(n) \in \mathbb{C}.$$

Hence we obtain

$$\sum_{n \geq -l} nb(n) q_h^n = \bigl(-l - \sum_{n=1}^{\infty} e(n) q_h^n\bigr) \Bigl(\sum_{n \geq -l} b(n) q_h^n\Bigr)$$

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which implies

$$e(1) = -b(1-l)$$

and

$$e(n) + e(n-1)b(1-l) + \cdots + e(1)b(n-l-1) + nb(n-l) = 0 \ (n \geq 2).$$

Let σ_k be the elementary symmetric function in $x_1, \dots, x_n$ and s_k be the power function in these variables, That is, $\sigma_1 = x_1 + \cdots + x_n, \sigma_2 = x_1x_2 + \cdots + x_{n-1}x_n, \dots, \sigma_n = x_1x_2\cdots x_n$ and $s_k = x_1^k + \cdots + x_n^k$.

Consider the fact (see [7]) that

$$s_n - s_{n-1}\sigma_1 + \cdots + (-1)^{n-1}s_1\sigma_{n-1} + (-1)^n n\sigma_n = 0.$$

By evaluating these identities at $x_k = q(k,n)$, where the $q(k,n)$ are the roots of the polynomial $x^n + b(1-l)x^{n-1} + \cdots + b(n-l)$, we obtain

$$e(n) = n \cdot \sum_{\substack{m_1,\dots,m_n \geq 0 \\ m_1+2m_2+\cdots+nm_n=n}} (-1)^{m_1+\cdots+m_n} \cdot \frac{(m_1+\cdots+m_n-1)!}{m_1!\cdots m_n!} \cdot b(1-l)^{m_1}\cdots b(n-l)^{m_n}$$

because (see [7])

$$s_i = i \cdot \sum_{\substack{m_1,\dots,m_n \geq 0 \\ m_1+2m_2+\cdots+nm_n=i}} (-1)^{m_2+m_4+\cdots} \frac{(m_1+\cdots+m_n-1)!}{m_1!\cdots m_n!} \cdot \sigma_1^{m_1}\cdots\sigma_n^{m_n}.$$

Therefore we obtain the assertion. ■

Example 2.3 Let $j(z)$ be the canonical Hauptmodul for $\Gamma_1(8)$. In [6] we see

$$j(z) = \frac{1}{q} + 3\cdot q + 2\cdot q^2 + q^3 - 2\cdot q^4 - 4\cdot q^5 - 4\cdot q^6 + 0\cdot q^7 + 6\cdot q^8 + \cdots .$$

Then we have

$$\begin{aligned}
j_1(z) &= j(z) \\
j_2(z) &= j(z)^2 - 6 \\
j_3(z) &= j(z)^3 - 9\cdot j(z) - 6 \\
j_4(z) &= j(z)^4 - 12\cdot j(z)^2 - 8\cdot j(z) + 14 \\
j_5(z) &= j(z)^5 - 15\cdot j(z)^3 - 10\cdot j(z)^2 + 40\cdot j(z) + 100 \\
&\vdots
\end{aligned}$$

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Table 1: The values of $j(z)$ at all inequivalent cusps of $\Gamma_1(8)$

cusp s	∞	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{8}$
$j(s)$	∞	$1+2\sqrt{2}$	-1	-3	$1-2\sqrt{2}$	-2
$\operatorname{ord}_s j_0(z)$	-1	-1	1	1	0	0

Let Γ_0 be the group generated by $\Gamma_1(8)$ and a Fricke involution $\left(\begin{smallmatrix} 0 & -1/(2\sqrt{2}) \\ 2\sqrt{2} & 0 \end{smallmatrix}\right)$. By simple calculation we know that

$$j_0(z) := \frac{j(z)^2 + 4j(z) + 3}{j(z) - 2\sqrt{2} - 1}$$

is a Hauptmodul for Γ_0. In this case we have $l = 1$. By easy calculation we obtain Table 1. Then by Theorem 2.2 and Table 1, we have $a(0) = 5 + 2\sqrt{2}$ and

$$a(n-1) = F_{n-1}(a(0), a(1), \dots, a(n-2)) + \frac{1}{n}\big(j_n(0) - j_n(\tfrac{1}{2}) - j_n(\tfrac{1}{4})\big)\ (n > 1)$$

which implies

$$j_0(z) = \frac{1}{q} + (5 + 2\sqrt{2}) + (19 + 12\sqrt{2}) \cdot q + (56 + 44\sqrt{2}) \cdot q^2 \\ + (167 + 160\sqrt{2}) \cdot q^3 + (612 + 356\sqrt{2}) \cdot q^4 + \cdots .$$

3 The Divisor of a Modular Form on $\Gamma_0^+(p)$

In this section we agree that p is a prime number contained in

$$\Phi = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$$

and $\Gamma_0^+(p)$ is the group generated by a Hecke subgroup $\Gamma_0(p)$ of $SL_2(\mathbb{Z})$ and Fricke involution $\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. We let $\mathfrak{H}^* = \mathfrak{H} \cup P_{\Gamma_0^+(p)}$ and t_p^+ be the canonical Hauptmodul for $\Gamma_0^+(p)$. Then t_p^+ has the Fourier expansion at ∞ in the form

$$t_p^+(z) = \frac{1}{q} + \sum_{n \geq 1} a_n q^n.$$

Let

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

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be the Eisentein series, where $\sigma_1(n) = \sum_{d|n} d$. We define Ramanujan's theta-operator by:

$$\theta\Big(\sum_{n=n_0}^{\infty} a(n)q^n\Big) := \sum_{n=n_0}^{\infty} na(n)q^n.$$

If f is a weight k meromorphic modular form on $\Gamma_0^+(p)$, then $\theta(f) - (kE_2(z) + kpE_2(pz))f(z))/24$ is a weight $k+2$ meromorphic modular form on $\Gamma_0^+(p)$. This follows from the transformation formula for E_2:

$$E_2(\gamma z) = (cz+d)^2 E_2(z) - \frac{6ci}{\pi}(cz+d), \qquad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

Let j_m be the modular function for $\Gamma_0^+(p)$ which is holomorphic on $\mathfrak{H}^* - \Gamma_0^+(p)\infty$ and has the Fourier expansion at ∞ in the form

$$j_m(z) = \frac{1}{q^m} + c_m(1)q + c_m(2)q^2 + \cdots \qquad \text{for each } m \in \mathbb{N}.$$

Then j_m is a polynomial in t_p^+.

For each integer $n > 1$, define a polynomial $H_n(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ by

$$\sum_{\substack{m_1,\dots,m_{n-1} \geq 0 \\ m_1 + 2m_2 + \cdots + (n-1)m_{n-1} = n}} (-1)^{m_1 + \cdots + m_{n-1}} \cdot \frac{(m_1 + \cdots + m_{n-1} - 1)!}{m_1! \cdots m_{n-1}!} \cdot x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} - \frac{1}{n} \cdot x_n \cdot \sigma_1(n) - \frac{p}{n} \cdot x_n \cdot \sigma_1\Big(\frac{n}{p}\Big).$$

Here $\sigma_1\big(\frac{n}{p}\big)$ is zero if $n \not\equiv 0 \bmod p$. Let $1/e_\tau$ be the cardinality of $\Gamma_0^+(p)_\tau/\{\pm 1\}$.

Theorem 3.1 *For any weight k meromorphic modular form f on $\Gamma_0^+(p)$ which has Fourier expansion at ∞*

$$f(z) = q^h + \sum_{n \geq h+1} a_f(n)q^n$$

we have

$$a_f(h+1) = -\sum_{\tau \in \Gamma_0^+(p)\backslash\mathfrak{H}} e_\tau \cdot \operatorname{ord}_\tau f \cdot j_1(\tau).$$

Furthermore, for each integer $n \geq 2$,

$$a_f(h+n) = H_n(a_f(h+1), \dots, a_f(h+n-1), k) - \frac{1}{n} \sum_{\tau \in \Gamma_0^+(p)\backslash\mathfrak{H}} e_\tau \cdot \operatorname{ord}_\tau f \cdot j_n(\tau).$$

Proof For each $m \in \mathbb{N}$ let $M_m(z) = j_m(z) \cdot (\theta(f)/f - (kE_2(z) + kpE_2(pz))/24)$ and $\omega_m = M_m(z)dz$. Then ω_m is a 1-form on $\Gamma_0^+(p)\backslash\mathfrak{H}^*$.

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We calculate the residue of ω_m at each point $\pi(\tau)$ for $\tau \in \mathfrak{H}^*$. As in [2, Proposition 2.1], we have

$$\frac{\theta(f)}{f} = h - \sum_{n=1}^{\infty} v(n)q^n \text{ with } v(n) \in \mathbb{C}. \tag{2}$$

Then $M_m(z) =$ higher terms in $q_{-1} - v(m) + k\sum_{d|m} d + pk\sum_{p|m,d|\frac{m}{p}} d +$ higher terms in q. Hence we have

$$\operatorname{Res}_{\pi(\infty)} M_m(z)dz = \frac{1}{2\pi i} \cdot (k\sigma_1(m) + pk\sigma_1(m/p) - v(m)).$$

For each $\tau \in \mathfrak{H}$, we observe

$$\operatorname{Res}_{\tau} M_m(z) = \operatorname{Res}_{\tau} \frac{\theta(f)}{f} j_m(z) = \frac{1}{2\pi i} \cdot \operatorname{ord}_{\tau} f \cdot j_m(\tau)$$

because $E_2(z), E_2(pz)$ and $j_m(z)$ are holomorphic on $\mathfrak{H}$. Hence we have

$$\operatorname{Res}_{\pi(\tau)} M_m(z)dz = \frac{e_\tau}{2\pi i} \cdot \operatorname{ord}_{\tau} f \cdot j_m(\tau).$$

Now by the residue theorem we obtain

$$v(m) = \sum_{\tau \in \Gamma_0^+(p)\backslash\mathfrak{H}} e_\tau \cdot \operatorname{ord}_\tau f \cdot j_m(\tau) + k\sigma_1(m) + pk\sigma_1(m/p). \tag{3}$$

On the other hand, we obtain that

$$a_f(1+h) = -v(1),$$

and for each integer $n \geq 2$

$$na_f(h+n) + v(n) = n \cdot \sum_{\substack{m_1,\dots,m_{n-1}\geq 0 \\ m_1+2m_2+\cdots+(n-1)m_{n-1}=n}} (-1)^{m_1+\cdots+m_{n-1}} \cdot \frac{(m_1+\cdots+m_{n-1}-1)!}{m_1!\cdots m_{n-1}!} \cdot \\ \cdot a_f(h+1)^{m_1} \cdots a_f(h+n-1)^{m_{n-1}}$$

by the recurrence (of the usual complete symmetric functions and sum) which is used in the proof of Theorem 2.2. By combining these relations with (3) we obtain the assertion. ■

Let f be a weight k meromorphic modular form on $\Gamma_0^+(p)$ and define two functions

$$H_\tau(z) := 1 + \sum_{n=1}^{\infty} e_\tau \cdot j_n(\tau) \cdot q^n (z, \tau \in \mathfrak{H})$$

and

$$f_\theta(z) := \frac{k(p+1)}{24} - \operatorname{ord}_\infty f + \sum_{\tau \in \Gamma_0^+(p)\backslash\mathfrak{H}} \operatorname{ord}_\tau f \cdot (H_\tau(z) - 1).$$

Then we obtain the following theorem

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Theorem 3.2 *For a weight k meromorphic modular form f on $\Gamma_0^+(p)$, we have*

(i) *$H_\tau(z)$ and $f_\theta(z)$ are weight 2 meromorphic modular forms on $\Gamma_0^+(p)$.*

(ii) *$\theta(f) = \big(-f_\theta + \frac{k}{24}E_2(z) + \frac{kp}{24}E_2(pz)\big)\, f(z)$.*

Proof For a fixed $\tau \in \mathfrak{H}$ let $h(z)$ be a modular form of weight zero on $\Gamma_0^+(p)$ such that $\operatorname{ord}_\infty h(z) = 1, \operatorname{ord}_\tau h(z) = -1$ and $\operatorname{ord}_\mu h(z) = 0$ for all $\mu \in \mathfrak{H} - \Gamma_0^+(p)\tau$. By replacing f by $h(z)$ in (2) and (3) we obtain $\theta(h(z))/h(z) = H_\tau(z)$. Hence $H_\tau(z)$ is a weight 2 meromorphic modular form on $\Gamma_0^+(p)$. From (2) and (3) we see that

$$\frac{\theta(f)}{f} - \frac{k}{24}E_2(z) - \frac{kp}{24}E_2(pz) = -f_\theta.$$

This proves the assertion (ii) and the rest of assertion (i). ■

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