Proceedings of the Royal Society of Edinburgh, **130A**, 855–875, *2000*

Timoshenko's beam equation as limit of a nonlinear one-dimensional von Kármán system

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(MS received 3 August 1998; accepted 27 April 1999)

We consider a dynamical one-dimensional nonlinear von Kármán model depending on one parameter $\varepsilon > 0$ and study its weak limit as $\varepsilon \to 0$. We analyse various boundary conditions and prove that the nature of the limit system is very sensitive to them. We prove that, depending on the type of boundary condition we consider, the nonlinearity of Timoshenko's model may vanish.

1. Introduction

A widely accepted dynamical model describing large deflections of thin plates is the von Kármán system of equations. There is a large literature on this model specially in the last ten years or so, when several authors considered problems of existence, uniqueness, asymptotic behaviour in time (when some damping effect is considered) as well as some other important properties (see [2, 4, 7] and the references therein).

In a recent work, Lagnese and Leugering [5] considered a one-dimensional version of the von Kármán system describing the planar motion of a uniform prismatic beam of length L. More precisely, in [5] the following system was considered:

$$\left.\begin{aligned} v_{tt} - [v_x + \tfrac{1}{2}w_x^2]_x = 0, \\ w_{tt} + w_{xxxx} - hw_{xxtt} - [w_x(v_x + \tfrac{1}{2}w_x^2)]_x = 0, \end{aligned}\right\} \tag{1.1}$$

where $0 < x < L$ and $t > 0$. In (1.1), subscripts mean partial derivatives and $h > 0$ is a parameter related to the rotational inertia of the beam. The quantities $v = v(x, t)$ and $w = w(x, t)$ represent, respectively, the longitudinal and transversal displacement of the point x at time t. In [5], suitable dissipative boundary conditions at $x = 0$, $x = L$ and initial conditions at $t = 0$ were given and the stabilization problem was studied.

On the other hand, when we consider a uniform beam of length L and study the transverse deflections (represented by $u = u(x, t)$) of its centreline at the point x

https://doi.org/10.1017/S0308210500000470 Published online by Cambridge University Press

at time t, then the following model can be deduced (Timoshenko's equation):

$$u_{tt} + u_{xxxx} - h\, u_{xxtt} - \frac{1}{2L}\left(\int_0^L u_x^2 \, \mathrm{d}x\right) u_{xx} = 0 \tag{1.2}$$

(see [1,11] and the references therein).

Evidently, since both models (1.1) and (1.2) describe approximately the same phenomenon, there should be certain 'proximity' between (1.1) and the solution $u = u(x,t)$ of (1.2). This paper is devoted to the analysis of the convergence of (1.1) towards (1.2) or its variations when an appropriate parameter tends to zero. To be more precise, we need to recall briefly our previous work [8].

In [8] we considered the following problem. Let $\varepsilon > 0$ and $v = v^\varepsilon$, $w = w^\varepsilon$ be solutions of the problem

$$\varepsilon v_{tt} - [v_x + \tfrac{1}{2} w_x^2]_x = 0, \tag{1.3}$$

$$w_{tt} + w_{xxxx} - h w_{xxtt} - [w_x(v_x + \tfrac{1}{2} w_x^2)]_x = 0, \tag{1.4}$$

in the interval $\Omega = (0, L)$ and $t > 0$, with boundary conditions

$$\left.\begin{aligned} &v(0,t) = v(L,t) = 0 && \forall\, t > 0 \\ &w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0 && \forall\, t > 0 \end{aligned}\right\} \tag{1.5}$$

and initial conditions

$$\left.\begin{aligned} v(x,0) &= v_0(x), & w(x,0) &= w_0(x), \\ v_t(x,0) &= v_1(x), & w_t(x,0) &= w_1(x). \end{aligned}\right\} \tag{1.6}$$

The following result was proved in [8].

Assume that $(v_0, v_1, w_0, w_1) \in H_0^1(0,L) \times L^2(0,L) \times H_0^2(0,L) \times H_0^1(0,L)$*. Then*

$$(w^\varepsilon, w_t^\varepsilon) \overset{\varepsilon \to 0}{\rightharpoonup} (u, u_t)$$

weakly in $L^2(0,T; H_0^2(0,L)) \times L^2((0,L) \times (0,T))$*, where* u *solves (1.2) with* $u(x,0) = w_0(x)$, $u_t(x,0) = w_1(x)$ *and*

$$u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = 0 \quad \forall\, t > 0.$$

Here, $H^m(0,L)$ and $H_0^m(0,L)$ denote the usual Sobolev spaces.

The above result guarantees that, as the velocity of propagation of longitudinal vibrations tends to infinity, solutions of (1.3)–(1.4) converge weakly to the solutions of Timoshenko's beam model under Dirichlet boundary conditions (1.5).

In simple situations, the above result could be expected. For example, suppose that v in (1.3) only depends on x, i.e. $v = v(x)$. Then (1.3) implies that $v_x + \frac{1}{2} w_x^2 = \eta(t)$ for some function $\eta = \eta(t)$. Integration (in x) from zero to L and boundary conditions (1.5) give us that

$$\frac{1}{2}\int_0^L w_x^2 \, \mathrm{d}x = L\eta(t). \tag{1.7}$$

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Substitution in (1.4) gives us that

$$w_{tt} + w_{xxxx} - hw_{xxtt} - [w_x \eta(t)]_x = 0,$$

that is,

$$w_{tt} + w_{xxxx} - hw_{xxtt} - \frac{1}{2L}\left(\int_0^L w_x^2 \, \mathrm{d}x\right) w_{xx} = 0. \tag{1.8}$$

In the above motivation we can see how crucial were the boundary conditions (1.5) on v, since they can affect identity (1.7). The boundary conditions play also a key role in the rigorous proof of [8].

The main purpose of this paper is to study the general case, in which, of course, $v = v(x,t)$, and see how sensitive to the boundary conditions is this limit process as $\varepsilon \to 0$.

In this paper we consider mainly the following two types of boundary conditions (although some other cases will be briefly discussed at the end of the paper as well).

(I) Neumann conditions on v and clamped ends for w.

(II) Dirichlet boundary conditions on v and hinged ends for w.

Let us briefly describe each of these cases.

In case I, we consider the problem (1.3)–(1.4) with initial conditions (1.6) and Neumann boundary conditions for v and clamped end conditions for w. Then, as $\varepsilon \to 0$, the weak limit problem turns out to be a linear equation. More precisely, the nonlinearity of the problem vanishes when passing to the limit. A similar conclusion was noticed in a paper due to Cimetière *et al.* [3] in a quite different context (static case and convergence with respect to a geometric parameter) for nonlinear three-dimensional elastic straight slender rods.

In case II, we consider Dirichlet boundary conditions for v and hinged end conditions for w. The limit problem turns out to be the Timoshenko equation (1.8) in agreement with the result of [8] described above (valid in the case of Dirichlet boundary conditions for v and clamped end conditions on w).

Let us now briefly explain the methods we employ. Classical energy estimates provide easily uniform (with respect to ε) bounds on the solutions. The main difficulty when passing to the limit is the identification of the limit of the nonlinear term. This is done by using ad hoc test functions which depend on the boundary conditions on a sensitive way and that, as indicated above, may led to rather drastic changes on the nature of the limit system.

We point out that many other important situations could be treated using the main ideas of this paper. For example, instead of the boundary conditions (1.5) or the ones worked out in this paper, we could also consider the following ones.

$$\left.\begin{aligned} &v(0,t) = v_x(L,t) = 0 && \forall\, t > 0,\\ &w(0,t) = w(L,t) = w_{xx}(0,t) = w_{xx}(L,t) = 0 && \forall\, t > 0, \end{aligned}\right\} \tag{III}$$

or

$$\left.\begin{aligned} &v(0,t) = v_x(L,t) = 0 && \forall\, t > 0,\\ &w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0 && \forall\, t > 0. \end{aligned}\right\} \tag{IV}$$

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The results presented in this paper are an attempt to give a precise mathematical justification (at least in the one-dimensional case) to statements usually claimed in the engineering literature (see [10, pp. 501–506]) and known as Berger's approximation.

Our notations in this paper are standard and can be found in the book of Lions [6].

Let us briefly describe all sections in this paper. In all sections we will consider the coupled system (1.3), (1.4) with initial conditions (1.6). In § 2 we study the well-posedness of system (1.3), (1.4) for both classes of boundary conditions I and II. In § 3 we briefly recall from [8] the main steps to prove the (weak) convergence as $\varepsilon \to 0$ in the case of Dirichlet conditions on v and clamped ends for w. In § 4 we pass to the limit in the case of boundary conditions of type I. In § 5 we analyse the asymptotic behaviour for boundary conditions of type II. In § 6 we analyse other boundary conditions and formulate an open problem. We end up with § 7 devoted to present some closely related results and open problems.

2. Global well-posedness: existence and uniqueness of solutions

As we mentioned in the introduction, in this section we analyse the existence and uniqueness of solutions of system (1.3), (1.4) with initial conditions (1.6) subject to boundary conditions of type I and II. Let us write explicitly these boundary conditions.

(I) Neumann conditions on v and clamped ends for w.

$$\left.\begin{aligned} v_x(0,t) = v_x(L,t) = 0 \qquad\qquad\qquad\qquad\quad & \forall\, t > 0,\\ w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0 \quad & \forall\, t > 0. \end{aligned}\right\} \tag{2.1}$$

(II) Dirichlet conditions on v and hinged ends for w.

$$\left.\begin{aligned} v(0,t) = v(L,t) = 0 \qquad\qquad\qquad\qquad\quad & \forall\, t > 0,\\ w(0,t) = w(L,t) = w_{xx}(0,t) = w_{xx}(L,t) = 0 \quad & \forall\, t > 0. \end{aligned}\right\} \tag{2.2}$$

In order to study the well-posedness of system (1.3), (1.4), (1.6), with either one of the above boundary conditions, we formulate the system as an abstract evolution equation in a suitable Hilbert space.

Local (in-time) existence will be obtained using standard semigroup theory. Global existence will be deduced as a consequence of the conservation of energy.

This section is divided in two subsections, devoted, respectively, to boundary conditions of type I and II.

2.1. Boundary conditions of type I

We consider the problem (1.3), (1.4), (1.6), with $\varepsilon > 0$, $h > 0$ and boundary conditions (2.1). We introduce the Hilbert space

$$X = V \times H \times H_0^2(0,L) \times H_0^1(0,L),$$

where

$$H = \left\{ \varphi \in L^2(0,L) : \int_0^L \varphi \,\mathrm{d}x = 0 \right\}, \qquad V = H^1(0,L) \cap H.$$

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The choice of space X is justified by the conservation law

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^L v\,\mathrm{d}x = 0,$$

which is obtained by integrating (1.3) with respect to x.

The norm in X is given by

$$\|(v,y,w,z)\|_X^2 = \|v_x\|^2 + \varepsilon\|y\|^2 + \|w_{xx}\|^2 + \|z\|^2 + h\|z_x\|^2$$

for any $(v,y,w,z) \in X$. Here, $\|\cdot\|$ denotes the norm in $L^2(0,L)$. We write our problem in the form

$$\left.\begin{aligned} DU_t &= AU + N(U),\\ U(0) &= U_0 \in X, \end{aligned}\right\} \tag{2.3}$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (1 - h(\partial^2/\partial x^2)) \end{bmatrix}, \qquad U = \begin{bmatrix} v \\ y \\ w \\ z \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \partial^2/\partial x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\partial^4/\partial x^4 & 0 \end{bmatrix}, \qquad U_0 = \begin{bmatrix} v_0 \\ v_1 \\ w_0 \\ w_1 \end{bmatrix}$$

and

$$N(U) = \begin{bmatrix} 0 \\ \frac{1}{2}(w_x^2)_x \\ 0 \\ [w_x(v_x + \frac{1}{2}w_x^2)]_x \end{bmatrix}.$$

It is easy to see that $D^{-1}A$, with domain

$$\mathcal{D}(D^{-1}A) = H_1 \times V \times [H^3(0,L) \cap H_0^2(0,L)] \times H_0^2(0,L),$$

where

$$H_1 = \{\varphi \in H^2(0,L) \cap V,\ \varphi_x = 0 \text{ at } x = 0, L\},$$

is the infinitesimal generator of a group of isometries in X. In fact,

$$\begin{aligned}\langle D^{-1}AU, U\rangle_X = \int_0^L y_x v_x\,\mathrm{d}x + \int_0^L v_{xx} y\,\mathrm{d}x + \int_0^L w_{xx} z_{xx}\,\mathrm{d}x \\ - \int_0^L (1 - h\partial_x^2)^{-1}\partial_x^4 w z\,\mathrm{d}x - h\int_0^L \partial_x (1 - h\partial_x^2)^{-1}\,\partial_x^4 w \partial_x z\,\mathrm{d}x = 0\end{aligned}$$

for any $U \in \mathcal{D}(D^{-1}A)$.

On the other hand, given $F = (f,g,j,k) \in X$, system

$$D^{-1}AU = F$$

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admits a unique solution $U \in \mathcal{D}(D^{-1}A)$. Indeed, this system is reduced to

$$\begin{gathered} y = f, \\ (1/\varepsilon)\partial_x^2 v = g, \quad v_x = 0, \quad x = 0,\, L, \\ z = j, \\ -(1 - h\,\partial_x^2)^{-1}\partial_x^4 w = k, \quad w = w_x = 0, \quad x = 0,\, L. \end{gathered}$$

Obviously, $y = f \in V$. Moreover, the equation of v admits an unique solution $v \in H_1$, since $g \in H$. On the other hand, $z = j \in H_0^2(0, L)$ and the equation satisfied by w is equivalent to

$$\partial_x^4 w = -(1 - h\partial_x^2)k, \quad w = w_x = 0, \quad x = 0,\, L,$$

which admits an unique solution $w \in H^3 \cap H_0^2(0, L)$, since $(1 - h\partial_x^2)k \in H^{-1}(0, L)$ because of the fact that $k \in H_0^1(0, L)$. Observe that here $(1 - h\partial_x^2)^{-1}$ denotes the inverse of the operator $1 - h\partial_x^2$ with Dirichlet boundary conditions.

This implies that, in order to show the local existence of solutions to the problem (2.3), it is enough to show that $D^{-1}N(U)$ is locally Lipschitz continuous in X. Clearly, if

$$U = \begin{pmatrix} v \\ y \\ w \\ z \end{pmatrix} \quad \text{and} \quad \tilde{U} = \begin{pmatrix} \tilde{v} \\ \tilde{y} \\ \tilde{w} \\ \tilde{z} \end{pmatrix}$$

belong to X, then

$$D^{-1}[N(U) - N(\tilde{U})] = \begin{pmatrix} 0 \\ f \\ 0 \\ g \end{pmatrix},$$

where

$$f = \frac{1}{2\varepsilon}[w_x^2 - \tilde{w}_x^2]_x \quad \text{and} \quad g = \left(1 - h\frac{\partial^2}{\partial x^2}\right)^{-1}[w_x(v_x + \tfrac{1}{2}w_x^2) - \tilde{w}_x(\tilde{v}_x + \tfrac{1}{2}\tilde{w}_x^2)]_x.$$

Consequently,

$$\|D^{-1}[N(U) - N(\tilde{U})]\|_X^2 = \varepsilon\|f\|_{L^2(0,L)}^2 + \|g\|_{L^2(0,L)}^2 + h\|g_x\|_{L^2(0,L)}^2.$$

Using the embedding $H^1(0, L) \hookrightarrow L^\infty(0, L)$, we can easily show that

$$\|f\|_{L^2(0,L)} \leqslant c(\varepsilon, h)[1 + \|U\|_X + \|\tilde{U}\|_X]\|U - \tilde{U}\|_X, \tag{2.4}$$

for some constant $c(\varepsilon, h) > 0$. Since the operator

$$\left(1 - h\frac{\partial^2}{\partial x^2}\right)^{-1}\frac{\partial}{\partial x}$$

is bounded from $L^2(0, L) \to H_0^1(0, L)$, then

$$\|g\|_{H^1(0,L)} \leqslant c\|w_x(v_x + \tfrac{1}{2}w_x^2) - \tilde{w}_x(\tilde{v}_x + \tfrac{1}{2}\tilde{w}_x^2)\|_{L^2(0,L)}. \tag{2.5}$$

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Adding and subtracting the term $(v_x + \frac{1}{2}w_x^2)\tilde{w}_x$ (inside the norm in (2.5)), and using the triangle inequality, we obtain that

$$\begin{aligned}\|g\|_{H^1(0,L)} \leqslant c\|w_x - \tilde{w}_x\|_{L^\infty(0,L)}\|v_x + \tfrac{1}{2}w_x^2\|_{L^2(0,L)} \\ + c\|\tilde{w}_x\|_{L^\infty(0,L)}\{\|v_x - \tilde{v}_x\|_{L^2(0,L)}\|v_x + \tilde{v}_x\|_{L^\infty(0,L)} \\ + \tfrac{1}{2}\|w_x - \tilde{w}_x\|_{L^\infty(0,L)}\|w_x + \tilde{w}_x\|_{L^2(0,L)}\}. \quad (2.6)\end{aligned}$$

Again, we use the embedding $H^1(0,L) \hookrightarrow L^\infty(0,L)$ and deduce from (2.6) that

$$\|g\|_{H^1(0,L)} \leqslant c(\|U\|_X, \|\tilde{U}\|_X)\|U - \tilde{U}\|_X,$$

which together with (2.4) shows that $D^{-1}N(U)$ is locally Lipschitz continuous in X. In order to obtain global existence, we need an *a priori* estimate. In our case, this is not difficult because the total energy associated with the problem (1.3), (1.4) is conserved. Indeed, let

$$E_\varepsilon(t) = \frac{1}{2}\int_0^L \{\varepsilon v_t^2 + [v_x + \tfrac{1}{2}w_x^2]^2 + w_t^2 + w_{xx}^2 + hw_{xt}^2\}\,\mathrm{d}x. \quad (2.7)$$

Then we can easily verify that the time derivative of $E_\varepsilon(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t) = [w_t w_{xtt} + w_{xt}w_{xx} - w_t w_{xxx} + (v_x + \tfrac{1}{2}w_x^2)v_t + w_t w_x(v_x + \tfrac{1}{2}w_x^2)]_0^L,$$

which is identically equal to zero due to the boundary conditions (2.1). Consequently, the energy is conserved, i.e. we have that $E_\varepsilon(t) = E_\varepsilon(0)$ for all $t \geqslant 0$. This implies that $\|U(t)\|_X$ is bounded in each interval where the solution exists, since $E_\varepsilon(t)$ is equivalent to $\|U(t)\|_X^2$. Therefore, the solution exists globally in time. Uniqueness is proved in the usual way using Gronwall's inequality. Therefore, the following result holds.

THEOREM 2.1. *Let $\varepsilon > 0$, $h > 0$ and $(v_0, v_1, w_0, w_1) \in X$. Then problem (1.3), (1.4), (1.6), with boundary conditions (2.1), has a (unique) global weak solutions such that*

$$(v^\varepsilon, v_t^\varepsilon, w^\varepsilon, w_t^\varepsilon) \in C([0,\infty); X)$$

and the total energy $E_\varepsilon(t)$ given by (2.7) satisfies

$$E_\varepsilon(t) = E_\varepsilon(0) \quad \text{for all } t \geqslant 0.$$

REMARK 2.2. Theorem 2.1 guarantees the existence and uniqueness of finite-energy solutions for initial data $(v_0, v_1, w_0, w_1) \in X$. In particular, we assume that

$$\int_0^L v_0\,\mathrm{d}x = \int_0^L v_1\,\mathrm{d}x = 0. \quad (2.8)$$

This condition is not necessary to obtain a unique finite-energy solution. Indeed, as indicated above, integrating equation (1.3) with respect to x, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^L v(x,t)\,\mathrm{d}x = 0,$$

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and therefore

$$\int_0^L v(x,t)\,\mathrm{d}x = \int_0^L v_0(x)\,\mathrm{d}x + t\int_0^L v_1(x)\,\mathrm{d}x. \tag{2.9}$$

In view of (2.9), the unique solution of the system is easy to construct without the compatibility conditions (2.8). We set

$$\begin{aligned} v &= \frac{1}{L}\int_0^L v_0(x)\,\mathrm{d}x + \frac{1}{L}\int_0^L v_1(x)\,\mathrm{d}x\,t + \tilde{v}, \\ w &= \tilde{w}, \end{aligned}$$

where $(\tilde{v}, \tilde{w})$ is the unique solution provided by theorem 2.1 with the initial data $(\tilde{v}_0, \tilde{v}_1, w_0, w_1)$, where

$$\tilde{v}_0 = v_0 - \frac{1}{L}\int_0^L v_0\,\mathrm{d}x, \qquad \tilde{v}_1 = v_1 - \frac{1}{L}\int_0^L v_1\,\mathrm{d}x,$$

which, obviously, do satisfy conditions (2.8).

2.2. Boundary conditions of type II

We now consider the system (1.3), (1.4), (1.6), with boundary conditions (2.2). We introduce the Hilbert space

$$Y = H_0^1(0,L) \times L^2(0,L) \times [H^2 \cap H_0^1(0,L)] \times H_0^1(0,L). \tag{2.10}$$

The norm in Y is as follows:

$$\|(v,y,w,z)\|_Y^2 = \|v_x\|^2 + \varepsilon\|y\|^2 + \|w_{xx}\|^2 + \|z\|^2 + h\|z_x\|^2. \tag{2.11}$$

We write the problem in the form

$$\begin{aligned} DU_t &= AU + N(U), \\ U(0) &= U_0 \in Y, \end{aligned}$$

where A, D are as in § 2.1 above. The domain of $D^{-1}A$ is now

$$\mathcal{D}(D^{-1}A) = [H^2 \cap H_0^1(0,L)] \times H_0^1(0,L) \times H_2 \times [H^2 \cap H_0^1(0,L)],$$

where

$$H_2 = \{\varphi \in H^3(0,L) : \varphi = \varphi_{xx} = 0,\ x = 0, L\}.$$

Following the arguments of § 2.1 above it is easy to see that $D^{-1}A$ is the infinitesimal generator of a group of isometries in Y. It is also easy to check that $D^{-1}N$ is locally Lipschitz in Y. Moreover, the energy E_ε is also conserved in time.

As a consequence of all these facts, we deduce that the following holds.

THEOREM 2.3. *Let $\varepsilon > 0$ and $h > 0$. Then, for any $(v_0, v_1, w_0, w_1) \in Y$, problem (1.3), (1.4), (1.6), with boundary conditions (2.2), admits a unique global solution $(v, v_t, w, w_t) \in C([0,\infty); Y)$.*

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3. The asymptotic limit with Dirichlet boundary conditions

For the sake of completeness, we briefly recall the main steps of the convergence result in [8].

Let $\varepsilon > 0$, $h > 0$ and consider the initial boundary-value problem (1.3), (1.4), (1.5), (1.6), with initial conditions (v_0, v_1, w_0, w_1) belonging to the Hilbert space $X = H_0^1(0,L) \times L^2(0,L) \times H_0^2(0,L) \times H_0^1(0,L)$. A similar discussion as the one given in the previous section shows that the problem (1.3), (1.4), (1.5), (1.6) is globally well posed in the space X, and the total energy $E_\varepsilon(t)$ is given by

$$E_\varepsilon(t) = \frac{1}{2}\int_0^L [\varepsilon(v_t^\varepsilon)^2 + [v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2]^2 + (w_t^\varepsilon)^2 + (w_{xx}^\varepsilon)^2 + h(w_{xt}^\varepsilon)^2]\,\mathrm{d}x$$

is constant, i.e. $E_\varepsilon(t) = E_\varepsilon(0)$ for all $t \geqslant 0$. Here, $\{v^\varepsilon, w^\varepsilon\}$ denote the solution-pair of the system (1.3), (1.4), (1.5), (1.6). Thus the following sequences (in ε) remain bounded in $L^\infty(0,\infty; L^2(0,L))$:

$$\{\sqrt{\varepsilon}v_t^\varepsilon\}, \quad \{v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2\}, \quad \{w_t^\varepsilon\}, \quad \{w_{xt}^\varepsilon\}, \quad \{w_{xx}^\varepsilon\}.$$

Extracting subsequences (that we still denote by the index ε in order to simplify notations), we deduce that there exist $\xi(x,t)$, $\eta(x,t)$ and $u(x,t)$ such that

$$\sqrt{\varepsilon}v_t^\varepsilon \rightharpoonup \xi \quad \text{weakly}_* \text{ in} L^\infty(0,\infty; L^2(0,L)), \tag{3.1}$$

$$v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2 \rightharpoonup \eta \quad \text{weakly}_* \text{ in } L^\infty(0,\infty; L^2(0,L)) \tag{3.2}$$

and

$$w^\varepsilon \rightharpoonup u \quad \text{weakly}_* \text{ in } L^\infty(0,\infty; H^2(0,L)) \cap W^{1,\infty}(0,\infty; H_0^1(0,L)) \tag{3.3}$$

as $\varepsilon \to 0$.

Clearly, the weak convergence in (3.3) suffices to pass to the limit in the linear terms of (1.4). It remains to identify the weak limit of the nonlinear term

$$[w_x^\varepsilon(v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2)]_x$$

as $\varepsilon \to 0$.

Since $E_\varepsilon(t)$ is bounded, then $\{w^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in

$$L^\infty(0,\infty; H_0^2(0,L)) \cap W^{1,\infty}(0,\infty; H_0^1(0,L)).$$

Then we can use the Aubin–Lions compactness criteria to deduce that

$$w^\varepsilon \to u \quad \text{strongly in } L^\infty(0,T; H_0^{2-\delta}(\Omega)) \tag{3.4}$$

as $\varepsilon \to 0$, for any $\delta > 0$ and $T < \infty$. Combining (3.2) with (3.4), it follows that

$$w_x^\varepsilon[v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2] \rightharpoonup u_x\eta \quad \text{weakly in } L^2((0,L)\times(0,T)) \tag{3.5}$$

as $\varepsilon \to 0$ for any $T < \infty$.

To conclude our result, it suffices to identify the weak limit η in (3.2). Again, we use the boundedness of $E_\varepsilon(t)$ to observe that $\{v_x^\varepsilon\}$ is bounded in $L^2((0,L)\times(0,T))$. Consequently, we can extract a subsequence such that

$$v_x^\varepsilon \rightharpoonup \rho \quad \text{weakly in } L^2((0,L)\times(0,T)) \tag{3.6}$$

as $\varepsilon \to 0$, for some $\rho = \rho(x,t)$. From (3.4) and (3.6), we deduce that

$$v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2 \rightharpoonup \rho + \tfrac{1}{2}u_x^2 \quad \text{weakly in } L^2((0,L)\times(0,T)). \tag{3.7}$$

Together with (3.2), this implies that

$$\eta = \rho + \tfrac{1}{2}u_x^2. \tag{3.8}$$

We claim that η is independent of x. In fact, due to (3.1), we have that

$$\varepsilon v_{tt}^\varepsilon \rightharpoonup 0 \quad \text{weakly in } H^{-1}(0,T;L^2(0,L)) \tag{3.9}$$

as $\varepsilon \to 0$. From (1.3), (3.9) and (3.7), it follows that

$$\eta_x = [\rho + \tfrac{1}{2}u_x^2]_x = 0,$$

which proves our claim. Thus $\eta = \eta(t)$. Integrating identity (3.8) from $x = 0$ to $x = L$, we get

$$L\eta(t) = \int_0^L \rho \, \mathrm{d}x + \frac{1}{2}\int_0^L u_x^2 \, \mathrm{d}x = \frac{1}{2}\int_0^L u_x^2 \, \mathrm{d}x,$$

because

$$\int_0^L \rho \, \mathrm{d}x = 0.$$

Indeed,

$$\int_0^L \rho \, \mathrm{d}x = \lim_{\varepsilon\to 0}\int_0^L v_x^\varepsilon \, \mathrm{d}x = 0,$$

since $v^\varepsilon(0,t) = v^\varepsilon(L,t) = 0$ and (3.6) holds. Hence

$$\eta u_x = \left(\frac{1}{2L}\int_0^L u_x^2 \, \mathrm{d}x\right)u_x.$$

Consequently,

$$[w_x^\varepsilon(v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2)]_x \rightharpoonup \left(\frac{1}{2L}\int_0^L u_x^2 \, \mathrm{d}x\right)u_{xx} \quad \text{weakly in } L^2(0,T;H^{-1}(0,L))$$

as $\varepsilon \to 0$. The above convergences hold along suitable subsequences. However, taking into account that the limit u has been identified as the unique solution of

$$u_{tt} + u_{xxxx} - hu_{xxtt} - \left(\frac{1}{2L}\int_0^L u_x^2 \, \mathrm{d}x\right)u_{xx} = 0 \quad \text{in } \Omega\times(0,\infty), \tag{3.10}$$

$$u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = 0, \quad t > 0, \tag{3.11}$$

$$u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), \quad x \in (0,L) \tag{3.12}$$

we deduce that the whole family converges as $\varepsilon \to 0$.

We can summarize the above result as follows.

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THEOREM 3.1 (see [8]). *Let* $(v_0, v_1, w_0, w_1) \in X$, *where*

$$X = H_0^1(0, L) \times L^2(0, L) \times H_0^2(0, L) \times H_0^1(0, L),$$

be fixed. Let $h > 0$ *and consider the solution* $\{w^\varepsilon, v^\varepsilon\}$ *of the system (1.3), (1.4), (1.5), (1.6). Then the following convergences hold as* $\varepsilon \to 0$*:*

$$w^\varepsilon \rightharpoonup u \qquad \textit{weakly in } L^2(0, T; H_0^2(0, L)),$$

$$v_x^\varepsilon \rightharpoonup \frac{1}{2L}\int_0^L u_x^2 \, \mathrm{d}x - \tfrac{1}{2}u_x^2 \qquad \textit{weakly in } L^2(0, L \times (0, T))$$

for all $T < \infty$.

The function $u = u(x, t)$ satisfies (3.10), (3.11), (3.12).

4. The asymptotic limit with Neumann boundary conditions on v^ε and clamped end conditions on w^ε

In this section we consider $\varepsilon > 0$ ($h > 0$) and analyse the asymptotic behaviour as $\varepsilon \to 0$ of the global solution of the problem (1.3), (1.4), with boundary conditions (2.1) and initial conditions (1.6). The existence of solutions is guaranteed by theorem 2.1.

Our main purpose now is to study the asymptotic limit of $\{v^\varepsilon, w^\varepsilon\}$ as $\varepsilon \to 0$. We shall use the method of [8, § 3], just pointing out the extra steps needed in this case due to the new boundary conditions.

The total energy $E_\varepsilon(t)$ given by (2.7) is constant for all $t \geqslant 0$. Therefore, the following sequences are bounded in $L^\infty(0, \infty; L^2(0, L))$:

$$\{\sqrt{\varepsilon} v_t^\varepsilon\}, \quad \{v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2\}, \quad \{w_x^\varepsilon\}, \quad \{w_{xx}^\varepsilon\} \quad \text{and} \quad \{w_{xt}^\varepsilon\}.$$

On the other hand, in view of (2.8), it follows that v^ε is bounded in

$$L^\infty(0, T; H^1(0, L))$$

for any finite $T > 0$. Extracting subsequences we deduce the existence of functions $\xi(x, t)$, $\eta(x, t)$ and $u(x, t)$ such that

$$\sqrt{\varepsilon} v_t^\varepsilon \rightharpoonup \xi \quad \text{weakly}_* \text{ in } L^\infty(0, \infty; L^2(0, L)), \tag{4.1}$$

$$v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon) \rightharpoonup \eta \quad \text{weakly}_* \text{ in } L^\infty(0, \infty; L^2(0, L)), \tag{4.2}$$

$$w^\varepsilon \rightharpoonup u \quad \text{weakly}_* \text{ in } L^\infty(0, \infty; H_0^2(0, L)) \cap W^{1,\infty}(0, \infty; H_0^1(0, L)) \tag{4.3}$$

as $\varepsilon \to 0$.

Using (4.3) we can pass to the limit in the linear terms of (1.4). It remains to identify the limit of the nonlinear term

$$[w_x^\varepsilon(v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2)]_x$$

as $\varepsilon \to 0$. Since $E_\varepsilon(t)$ is bounded, then $\{w^\varepsilon\}$ is uniformly bounded in

$$L^\infty(0, \infty; H_0^2(0, L)) \cap W^{1,\infty}(0, \infty; H_0^1(0, L)).$$

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Using the classical Aubin–Lions compactness lemma, we deduce that

$$w^{\varepsilon} \to u \quad \text{strongly in } L^{\infty}(0,T;H_0^{2-\delta}(0,L)) \tag{4.4}$$

as $\varepsilon \to 0$, for any $\delta > 0$ and $T < \infty$. Combining (4.2) with (4.4), we obtain that

$$w_x^{\varepsilon}[v_x^{\varepsilon} + \tfrac{1}{2}(w_x^{\varepsilon})^2] \rightharpoonup u_x\eta \quad \text{weakly in } L^2((0,L)\times(0,T)). \tag{4.5}$$

We want to identify η in (4.5). Taking into account that $\{v^{\varepsilon}\}$ is bounded in $L^{\infty}(0,T;H^1(0,L))$, we can extract a subsequence such that

$$v_x^{\varepsilon} \rightharpoonup \rho \quad \text{weakly in } L^2((0,L)\times(0,T)) \tag{4.6}$$

as $\varepsilon \to 0$, for some $\rho = \rho(x,t)$. From (4.4) and (4.6) it follows that

$$v_x^{\varepsilon} + \tfrac{1}{2}(w_x^{\varepsilon}) \rightharpoonup \rho + \tfrac{1}{2}u_x^2 \quad \text{weakly in } L^2((0,L)\times(0,T)) \tag{4.7}$$

as $\varepsilon \to 0$. Together with (4.2), this says that

$$\eta = \rho + \tfrac{1}{2}u_x^2.$$

In view of (4.1), we also know that

$$\varepsilon v_{tt}^{\varepsilon} \rightharpoonup 0 \quad \text{weakly in } H^{-1}(0,T;L^2(0,L)) \tag{4.8}$$

as $\varepsilon \to 0$. From (4.8), (1.3) and (4.7), it follows that

$$\eta_x = [\rho + \tfrac{1}{2}u_x^2]_x = 0,$$

i.e. $\eta = \eta(t)$. Let us identify $\eta(t)$. We take the derivative in x of (1.3) and multiply the result by $a(x) = \frac{1}{4}L^2 - (x - \frac{1}{2}L)^2$. Integration (in space) from zero to L followed by integration by parts gives us

$$\begin{aligned}\varepsilon \frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^L v_x^{\varepsilon} a(x)\,\mathrm{d}x &= \int_0^L [v_x^{\varepsilon} + \tfrac{1}{2}(w_x^{\varepsilon})^2]_{xx} a(x)\,\mathrm{d}x \\ &= -2\int_0^L [v_x^{\varepsilon} + \tfrac{1}{2}(w_x^{\varepsilon})^2]\,\mathrm{d}x. \end{aligned}\tag{4.9}$$

Note that, when integrating by parts, no boundary terms appear since $a = 0$ at $x = 0$, L and also because of the boundary conditions that v^{ε} and w^{ε} satisfy that guarantee that $v_x^{\varepsilon} + \frac{1}{2}|w_x^{\varepsilon}|^2 = 0$ at $x = 0$, $x = L$.

Letting $\varepsilon \to 0$ in (4.9) and using (4.7) we obtain that

$$\varepsilon \frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^L v_x^{\varepsilon} a(x)\,\mathrm{d}x \rightharpoonup -2L\eta(t). \tag{4.10}$$

On the other side, since $a \in L^2(0,L)$, then

$$\int_0^L v_x^{\varepsilon} a(x)\,\mathrm{d}x \rightharpoonup \int_0^L \rho(x)a(x)\,\mathrm{d}x \quad \text{weakly in } L^2(0,T),$$

therefore,

$$\varepsilon \frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^L v_x^{\varepsilon} a(x)\,\mathrm{d}x \rightharpoonup 0 \quad \text{in } \mathcal{D}'(0,T)$$

as $\varepsilon \to 0$. This information, together with (4.10), implies that

$$-2L\eta(t) = 0,$$

that is, $\eta(t) = 0$. Consequently,

$$v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2 \rightharpoonup 0 = \eta \quad \text{weakly in } L^2((0,L)\times(0,T))$$

as $\varepsilon \to 0$. Returning to (4.5), we conclude that

$$[w_x^\varepsilon(v_x^\varepsilon + \tfrac{1}{2}(w_x^\varepsilon)^2)]_x \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;H^{-1}(0,L))$$

as $\varepsilon \to 0$.

We have identified the weak limit of v_x^ε as $\rho = -\frac{1}{2}u_x^2$. However, in order to have a complete description of the limiting behaviour of v^ε, we have to use identity (2.8), which provides an exact formula for the average

$$\int_0^L v^\varepsilon \, \mathrm{d}x.$$

We have proved the following theorem.

THEOREM 4.1. *Let $(v_0, v_1, w_0, w_1) \in X$ and $\{w^\varepsilon, v^\varepsilon\}$ be the (unique) global solution of problem (1.3), (1.4), (2.1), (1.6). Then*

$$(w^\varepsilon, w_t^\varepsilon) \rightharpoonup (u, u_t) \quad \textit{weakly in } L^2(0,T;H_0^2(0,L)) \times L^2(0,T;H_0^1(0,L))$$

as $\varepsilon \to 0$, for all $T < \infty$, where $u = u(x,t)$ is the solution of

$$\left.\begin{aligned} u_{tt} + u_{xxxx} - h\,u_{xxtt} &= 0 \quad \text{in } (0,L)\times(0,\infty), \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) &= 0 \quad \forall\, t > 0, \\ u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), &\quad 0 < x < L. \end{aligned}\right\} \tag{4.11}$$

Moreover,

$$v_x^\varepsilon \rightharpoonup -\tfrac{1}{2}u_x^2 \quad \textit{weakly in } L^2((0,L)\times(0,T))$$

as $\varepsilon \to 0$ and

$$\int_0^L v^\varepsilon \, \mathrm{d}x = \int_0^L v_0 \, \mathrm{d}x + t\int_0^L v_1 \, \mathrm{d}x \quad \textit{for all } \varepsilon > 0.$$

REMARK 4.2. The final result of theorem 4.1 is kind of unexpected when compared with the result given in theorem 3.1 (see also [8]) in the case of Dirichlet boundary conditions for v^ε. In the present case, the limit system is completely linear.

5. The asymptotic limit: boundary conditions of type II

In this section we analyse the limiting behaviour as $\varepsilon \to 0$ of the system (1.3), (1.4), (1.6), with boundary conditions (2.2) of type II.

Our main result is as follows.

https://doi.org/10.1017/S0308210500000470 Published online by Cambridge University Press

THEOREM 5.1. *Let $(v_0, v_1, w_0, w_1) \in Y$ and $(v^\varepsilon, w^\varepsilon)$ be the unique global weak solution of (1.3), (1.4), (1.6) and (2.2). Then*

$$(w^\varepsilon, w_t^\varepsilon) \rightharpoonup (u, u_t) \quad \text{weakly in } L^2(0,T; H^2 \cap H_0^1(0,L)) \times L^2(0,T; H_0^1(0,L)) \tag{5.1}$$

as $\varepsilon \to 0$, for all $0 < T < \infty$, where u is the unique weak solution of

$$\left.\begin{aligned} u_{tt} + u_{xxxx} - u_{xxtt} - \frac{1}{2L}\left(\int_0^L u_x^2 \, \mathrm{d}x\right) u_{xx} = 0 \quad \text{in } (0,L) \times (0,\infty), \\ u(0,t) = u_{xx}(0,t) = u(L,t) = u_{xx}(L,t) = 0, \quad t > 0, \\ u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), \quad 0 < x < L. \end{aligned}\right\} \tag{5.2}$$

Moreover,

$$v_x^\varepsilon \rightharpoonup \frac{1}{2}\int_0^L u_x^2 \, \mathrm{d}x - \tfrac{1}{2} u_x^2 \quad \text{weakly in } L^2((0,L) \times (0,T)) \tag{5.3}$$

as $\varepsilon \to 0$ for all $T < \infty$.

REMARK 5.2. Note that in this case we obtain the nonlinear Timoshenko model in the limit as in § 3 above.

Proof of theorem 5.1. The proof of theorem 5.1 is similar to that of theorem 3.1. We give a sketch of the proof emphasizing the new developments.

By conservation of the energy we deduce that the sequences

$$\{\sqrt{\varepsilon} v_t^\varepsilon\}, \quad \{v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2\}, \quad \{w_t^\varepsilon\}, \quad \{w_{xx}^\varepsilon\} \quad \text{and} \quad \{w_{tx}^\varepsilon\}$$

are bounded. In view of the boundary conditions, we deduce that

$$\{(\sqrt{\varepsilon} v_t^\varepsilon, v^\varepsilon, w^\varepsilon, w_t^\varepsilon)\}_{\varepsilon > 0}$$
$$\text{is bounded in } L^\infty(0,\infty; L^2(0,L) \times H_0^1(0,L) \times [H^2 \cap H_0^1(0,L)] \times H_0^1(0,L)).$$

By extracting subsequences, we deduce that

$$\sqrt{\varepsilon}\, v_t^\varepsilon \rightharpoonup \xi \qquad \text{weakly}_* \text{ in } L^\infty(0,\infty; L^2(0,L)), \tag{5.4}$$

$$w^\varepsilon \rightharpoonup u \qquad \text{weakly}_* \text{ in } L^\infty(0,\infty; H^2 \cap H_0^1(0,L)) \cap W^{1,\infty}(0,\infty; H_0^1(0,L)), \tag{5.5}$$

$$v^\varepsilon \rightharpoonup v \qquad \text{weakly}_* \text{ in } L^\infty(0,\infty; H_0^1(0,L)). \tag{5.6}$$

Consequently,

$$v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2 \rightharpoonup v_x + \tfrac{1}{2}|u_x|^2 = \eta \quad \text{weakly}_* \text{ in } L^\infty(0,\infty; L^2(0,L)) \tag{5.7}$$

and

$$w_x^\varepsilon(v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2) \rightharpoonup \eta u_x \quad \text{weakly}_* \text{ in } L^\infty(0,\infty; L^2(0,L)).$$

The limit u satisfies the equation

$$\left.\begin{aligned} u_{tt} + u_{xxxx} - u_{xxtt} - (\eta u_x)\_x = 0 \quad \text{in } (0,L) \times (0,\infty), \\ u(0,t) = u_{xx}(0,t) = u(L,t) = u_{xx}(L,t) = 0, \quad t > 0, \\ u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), \quad 0 < x < L. \end{aligned}\right\} \tag{5.8}$$

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Therefore, to conclude the proof of the theorem it is sufficient to identify the limit η.

In view of (5.4), (5.7) and passing to the limit in equation (1.3), we deduce that $\eta_x = 0$ and therefore $\eta = \eta(t)$. In order to identity η we integrate the identity $\eta = v_x + \frac{1}{2}|u_x|^2$ in (5.7). Taking into account that $v(t) \in H_0^1(0,L)$ for all $t > 0$, we deduce that

$$L\eta(t) = \frac{1}{2}\int_0^L |u_x(x,t)|^2 \, \mathrm{d}x.$$

This concludes the proof of theorem 5.1. □

6. Other boundary conditions

The techniques developed in [8] and in the present paper allow us to pass to the limit under other boundary conditions as well. We describe here some of these cases. At the end of the paper we discuss the case where v satisfies Neumann boundary conditions and we impose hinged end conditions on w. In this case, the uniform boundedness of the sequence $(v^\varepsilon, w^\varepsilon)$ is an open problem.

6.1. Dirichlet–Neumann boundary conditions on v

Up to now we have only considered the case where v satisfies either Dirichlet or Neumann boundary conditions in both extremes $x = 0, L$. We now consider the system (1.3), (1.4), (1.6), with boundary conditions

$$\left.\begin{aligned} v(0,t) &= v_x(L,t) = 0, && t > 0, \\ w(0,t) &= w_x(0,t) = w(L,t) = w_x(L,t) = 0, && t > 0. \end{aligned}\right\} \tag{6.1}$$

In this case, global existence of solutions can be proved as in previous sections since, in view of the boundary conditions (6.1), the energy $E_\varepsilon(\cdot)$ is conserved in time.

Due to the conservation of energy, we also obtain uniform bounds on the solutions as $\varepsilon \to 0$. We may pass to the limit as in previous sections. As usual, the only difficulty to determine u, the weak limit of w^ε, is the identification of the limit of the nonlinear term $w_x^\varepsilon(v_x^\varepsilon + \frac{1}{2}|w_x^\varepsilon|^2)$. As usual, we have

$$w_x^\varepsilon(v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2) \rightharpoonup u_x(v_x + \tfrac{1}{2}|u_x|^2)$$

weakly in $L^2((0,L)\times(0,T))$ as $\varepsilon \to 0$, and $\eta = v_x + \frac{1}{2}|u_x|^2$ is independent of x. It remains to identify the value of $\eta(t)$.

Proceeding as in § 4, we have

$$\begin{aligned} \varepsilon \frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^L v_x^\varepsilon \, a(x)\,\mathrm{d}x &= \int_0^L (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)_{xx} a(x)\,\mathrm{d}x \\ &= \int_0^L (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)\partial_x^2 a(x)\,\mathrm{d}x \\ &\quad + (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)_x a(x)|_0^L - (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)\partial_x a(x)|_0^L, \end{aligned} \tag{6.2}$$

for any smooth function $a(x)$.

The left-hand side of (6.2) converges to zero in $\mathcal{D}'(0,T)$ as $\varepsilon \to 0$. On the other hand, taking $a = a(x)$ such that

$$a(0) = a(L) = 0, \quad \partial_x a(0) = 0, \tag{6.3}$$

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the right-hand side of (6.2) coincides with

$$\int_0^L (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)\partial_x^2 a(x)\,\mathrm{d}x,$$

which converges to

$$\eta(t)\int_0^L \partial_x^2 a(x)\,\mathrm{d}x.$$

We deduce that

$$0 = \eta(t)\int_0^L \partial_x^2 a(x)\,\mathrm{d}x = \eta(t)[\partial_x a(L) - \partial_x a(0)] = \eta(t)\partial_x a(L) \tag{6.4}$$

holds for any smooth function $a(x)$. But with $\partial_x a(L) \neq 0$, we deduce that $\eta \equiv 0$.

This shows that, as in § 4, the limit of the system is once again a linear beam equation for w.

6.2. A nonlinear boundary condition

In the context of the beam equations (1.3), (1.4) it is also natural to impose boundary conditions on the quantity $v_x + \frac{1}{2}|w_x|^2$, which is related to the variation of the length of the beam under the deformation (see [5]).

Let us consider, for instance, the boundary conditions

$$v(0,t) = w(0,t) = w_x(0,t) = 0, \quad t > 0, \tag{6.5}$$

$$[v_x + \tfrac{1}{2}w_x^2](L,t) = w(L,t) = w_{xx}(L,t) = 0, \quad t > 0. \tag{6.6}$$

With these boundary conditions, it is easy to prove the existence and uniqueness of finite-energy solutions. Moreover, the energy $E_\varepsilon(\cdot)$ is constant in time. This provides uniform bounds on the solutions $(v^\varepsilon, w^\varepsilon)$ which allow to pass to the limit. The only difficulty is once again to identity the limit of the nonlinear term. We have

$$w_x^\varepsilon(v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2) \rightharpoonup \eta u_x \quad \text{weakly in } L^2((0,L)\times(0,T))$$

as $\varepsilon \to 0$, where

$$\eta = v_x + \tfrac{1}{2}|u_x|^2$$

and $\eta = \eta(t)$.

Moreover,

$$\varepsilon\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^L v^\varepsilon a(x)\,\mathrm{d}x = \int_0^L (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)_x a(x)\,\mathrm{d}x \tag{6.7}$$

$$= -\int_0^L (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)\partial_x a(x)\,\mathrm{d}x, \tag{6.8}$$

provided $a(0) = 0$. The left-hand side in (6.7) tends to zero in $\mathcal{D}'(0,T)$ as $\varepsilon \to 0$. On other hand, the right-hand side converges to

$$-\eta(t)\int_0^L \partial_x a(x)\,\mathrm{d}x = -\eta(t)a(L).$$

Taking $a(L) \neq 0$, we deduce, once again, that the limit system is linear.

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6.3. An open problem

We now consider the system (1.3), (1.4), (1.6), with boundary conditions

$$\left.\begin{aligned} v_x &= 0, & x &= 0,\ L,\\ w = w_{xx} &= 0, & x &= 0,\ L. \end{aligned}\right\} \tag{6.9}$$

We introduce the Hilbert space

$$Z = H^1(0,L) \times L^2(0,L) \times [H^2 \cap H_0^1(0,L)] \times H_0^1(0,L). \tag{6.10}$$

The norm in Z is

$$\|(v,y,w,z)\|_Z^2 = \|v\|^2 + \|v_x\|^2 + \varepsilon\|y\|^2 + \|w_{xx}\|^2 + \|z\|^2 + h\|z_x\|^2. \tag{6.11}$$

We write the problem in the form

$$\left.\begin{aligned} DU_t &= AU + N(U),\\ U(0) &= U_0 \in Z, \end{aligned}\right\} \tag{6.12}$$

where A, D are as in § 2.1. The domain of $D^{-1}A$ is now

$$\mathcal{D}(D^{-1}A) = H_3 \times H^1(0,L) \times H_2 \times [H^2 \cap H_0^1(0,L)], \tag{6.13}$$

where H_2 is as in § 2.2 and $H_3 = \{u \in H^2(0,L) : \varphi_x = 0,\ x = 0, L\}$.

This operator $D^{-1}A$ is the infinitesimal generator of a continuous semigroup in Z.

On the other hand, proceeding as in § 2.1 above, it follows that $D^{-1}N$ is locally Lipschitz in Z.

Therefore, we deduce the following local existence result.

THEOREM 6.1. *Let $\varepsilon > 0$, $h > 0$. Then, for any $(v_0, v_1, w_0, w_1) \in Z$ the problem (1.3), (1.4), (1.6), with boundary conditions (6.9) admits a unique local weak solution. More precisely, there exists $T = T(\|(v_0, v_1, w_0, w_1)\|_Z) > 0$ such that*

$$(v, v_t, w, w_t) \in C([0,T); Z).$$

Moreover, the following alternative holds. Either $T = \infty$ or

$$\lim_{t \nearrow T} \|(v(t), v_t(t), w(t), w_t(t))\|_Z = \infty.$$

Note that theorem 6.1 does not guarantee global existence. In order to analyse the global existence issue, let us consider the energy

$$E_\varepsilon(t) = \frac{1}{2}\int_0^L [\varepsilon v_t^2 + (v_x + \tfrac{1}{2}|w_x|^2)^2 + w_t^2 + |w_{xx}|^2 + h|w_{xt}|^2]\,\mathrm{d}x.$$

In this case, according to boundary conditions (6.9), it follows that

$$\frac{\mathrm{d}E_\varepsilon}{\mathrm{d}t}(t) = w_x^2 v_t|_0^L = w_x^2(L,t)v_t(L,t) - w_x^2(0,t)v_t(0,t). \tag{6.14}$$

Obviously, this identity does not allow to obtain global (in-time) estimates. Therefore, our existence and uniqueness result remains to be of local nature.

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On the other hand, identity (6.14) does not allow to obtain uniform estimates on the (local) solutions as $\varepsilon \to 0$. Therefore, we may not pass to the limit on the system.

But let us assume for a moment that $(v^\varepsilon, w^\varepsilon)$ remains bounded in the energy space as $\varepsilon \to 0$ to see what the nature of the limit system should be.

As usual, the main problem is the identification of the weak limit $\eta(t)$ of $v_x^\varepsilon + \frac{1}{2}|w_x^\varepsilon|^2$. Going back to (1.3), we have

$$\varepsilon \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^L v_x^\varepsilon a(x)\,\mathrm{d}x = \int_0^L (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)_{xx}\, a(x)\,\mathrm{d}x. \tag{6.15}$$

The limit of the left-hand side of (6.15) is zero as usual. On the other hand,

$$\begin{aligned}\int_0^L (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)_{xx} a(x)\,\mathrm{d}x &= \int_0^L (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)\partial_x^2 a(x)\,\mathrm{d}s \\ &\quad + (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)_x a(x)|_0^L - (v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2)\partial_x a(x)|_0^L.\end{aligned} \tag{6.16}$$

Taking $a(x) = \frac{1}{4}L^2 - (x - \frac{1}{2}L)^2$ and passing to the limit on the right-hand side of (6.16), we deduce that

$$0 = \eta(t) \int_0^L \partial_x^2 a(x)\,\mathrm{d}x + L[|u_x(L,t)|^2 + |u_x(0,t)|^2].$$

This implies

$$\eta(t) = \tfrac{1}{2}[|u_x(L,t)|^2 + |u_x(0,t)|^2].$$

According to this fact, the limit system should be

$$u_{tt} + u_{xxxx} - u_{xxtt} - \tfrac{1}{2}[|u_x(0,t)|^2 + |u_x(L,t)|^2]u_{xx} = 0. \tag{6.17}$$

However, as we said above, these developments are formal, since we do not have uniform bounds on $(v^\varepsilon, w^\varepsilon)$.

The analysis of the asymptotic limit under the boundary conditions (6.9) together with a rigorous proof of how to obtain the limit system (6.17) is an open problem.

7. Further comments and results

In this section we describe some possible extensions of our results and also indicate open problems on the subject.

7.1. Thermoelastic beams

System (1.3), (1.4) could be considered under the presence of thermal effects. For example, we can consider the model

$$\begin{aligned} \varepsilon v_{tt} - [v_x + \tfrac{1}{2}w_x^2]_x &= 0,\\ w_{tt} + w_{xxxx} - hw_{xxtt} - [w_x(v_x + \tfrac{1}{2}(w_x)^2)]_x + \alpha\theta_{xx} &= 0,\\ \theta_t - \theta_{xx} - \alpha w_{xxt} &= 0 \end{aligned}$$

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in $\Omega \times (0,T)$, $\Omega = \{0 < x < L\}$, with boundary conditions

$$\begin{aligned} v_x(0,t) = v_x(L,t) = \theta(0,t) = \theta(L,t) = 0 \qquad \forall\, t > 0, \\ w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0 \quad \forall\, t > 0, \end{aligned}$$

and initial conditions

$$\begin{aligned} v(x,0) = v_0(x), \qquad & w(x,0) = w_0(x), \\ v_t(t,x) = v_1(x), \qquad & w_t(x,0) = w_1(x), \\ \theta(x,0) = \theta_0&(x). \end{aligned}$$

The arguments of § 4 allow us to describe the limit of $\{w^\varepsilon, v^\varepsilon, \theta^\varepsilon\}$ as $\varepsilon \to 0$. The limit (u,θ) of $(w^\varepsilon, \theta^\varepsilon)$ satisfies a linear system of equations modelling a thermoelastic beam.

The same problem can be analysed with other boundary conditions as well.

7.2. Strong convergence

One may show that when the initial data satisfy suitable compatibility conditions, the convergences in (for instance) theorem 4.1 hold in the strong topologies. Indeed, by weak lower semicontinuity of the L^2-norm, we have

$$\liminf_{\varepsilon \to 0} E_\varepsilon(t) \geqslant \frac{1}{2}\int_0^L \{|\xi|^2 + u_t^2 + u_{xx}^2 + hu_{xt}^2\}\,\mathrm{d}x, \tag{7.1}$$

where $E_\varepsilon(t)$ is given by (2.7), u solves (4.1) and ξ was found in (4.1). Using the conservation of energy, we also know that

$$\begin{aligned} E_\varepsilon(t) = E_\varepsilon(0) \xrightarrow{\varepsilon \to 0} \frac{1}{2}\int_0^L \biggl\{ \biggl[\frac{\partial v_0}{\partial x} + \frac{1}{2}\biggl(\frac{\partial w_0}{\partial x}\biggr)^2\biggr]^2 + w_1^2 + \biggl(\frac{\partial^2 w_0}{\partial x^2}\biggr)^2 + h\biggl(\frac{\partial w_1}{\partial x}\biggr)^2\biggr\}\,\mathrm{d}x \\ = E(0). \end{aligned} \tag{7.2}$$

The energy for the limit system (4.11) in theorem 4.1 is given by

$$F(t) = \frac{1}{2}\int_0^L [u_t^2 + u_{xx}^2 + hu_{xt}^2]\,\mathrm{d}x, \tag{7.3}$$

and it is conserved along time, i.e. $F(t) = F(0)$ for all $t > 0$. Combining (7.1) with (7.3), we deduce that

$$E(0) = \lim_{\varepsilon \to 0} E_\varepsilon(t) \geqslant F(t) + \frac{1}{2}\int_0^L |\xi|^2\,\mathrm{d}x = F(0) + \frac{1}{2}\int_0^L |\xi|^2\,\mathrm{d}x. \tag{7.4}$$

Suppose that the initial data $\{w_0, w_1\}$ are such that

$$E(0) = F(0). \tag{7.5}$$

Note that (7.5) holds if and only if

$$\frac{\partial v_0}{\partial x} + \frac{1}{2}\biggl|\frac{\partial w_0}{\partial x}\biggr|^2 = 0. \tag{7.6}$$

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Then, combining (7.4), (7.5), we would have $\xi \equiv 0$ and

$$\liminf_{\varepsilon\to 0} E_\varepsilon(t) = F(t) \quad \forall\, 0 \leqslant t \leqslant T. \tag{7.7}$$

As a consequence of (7.6), we deduce that

$$(\sqrt{\varepsilon} v_t^\varepsilon, v_x^\varepsilon + \tfrac{1}{2}|w_x^\varepsilon|^2) \to (0,0) \quad \text{strongly in } (L^2((0,L)\times(0,T)))^2 \tag{7.8}$$

and

$$(w^\varepsilon, w_t^\varepsilon) \to (u, u_t) \quad \text{strongly in } L^2(0,T; H_0^2(0,L)\times H_0^1(0,L)) \tag{7.9}$$

as $\varepsilon \to 0$.

When the compatibility condition (7.6) does not hold, we have

$$E(0) > F(0)$$

and, more precisely,

$$E(0) = F(0) + \frac{1}{2}\int_0^L (\partial_x v_0 + \tfrac{1}{2}|\partial_x w_0|^2)^2 \,\mathrm{d}x.$$

Consequently,

$$\lim_{\varepsilon\to 0} E_\varepsilon(t) = F(t) + \frac{1}{2}\int_0^L (\partial_x v_0 + \tfrac{1}{2}|\partial_x w_0|^2)^2 \,\mathrm{d}x$$

for all $t > 0$, and therefore strong convergences (7.8), (7.9) do not hold.

As a consequence of this analysis, we deduce that the convergences in theorem 4.1 hold in the corresponding strong topologies if and only if the compatibility condition (7.6) holds.

A similar discussion also works in all other cases we studied in the previous sections.

7.3. Varying initial data

In the proof of theorems 3.1, 4.1 and 5.1, we considered the case when the initial data (v_0, v_1, w_0, w_1) are fixed. The same results hold if we consider the case when they do depend on ε, provided we assume that $(v_0^\varepsilon, v_1^\varepsilon, w_0^\varepsilon, w_1^\varepsilon)$ are such that the energy $E_\varepsilon(0)$ remains bounded and $(w_0^\varepsilon, w_t^\varepsilon)$ converge weakly to (w_0, w_1) in the corresponding spaces.

7.4. Asymptotic limit of the Cauchy problem

The same problems may be analysed for the Cauchy problem on the whole line $\mathbb{R}$. When passing to the limit as $\varepsilon \to 0$, by weak lower semicontinuity of the energy in the limit one has $v_x + \frac{1}{2}|w_x|^2 \in L^\infty(0,\infty; L^2(\mathbb{R}))$. But passing to the limit in (1.3), one also gets that $v_x + \frac{1}{2}|w_x|^2$ is independent of x. This immediately implies that $v_x + \frac{1}{2}|w_x|^2 \equiv 0$. Therefore, the weak limit u of w^ε in this case satisfies the linear equation

$$u_{tt} + \partial_x^4 u - u_{xxtt} = 0 \quad \text{in } \mathbb{R}\times(0,\infty).$$

The same arguments apply when the spatial domain under consideration is the half-line $(0,\infty)$.

7.5. Two-dimensional models

Similar problems arise in two space dimensions. For instance, one may consider the full nonlinear von Kármán equations or related models (see [2, 4] and the references therein) and try to show that it remains 'close' to the two-dimensional Timoshenko's model,

$$u_{tt} + \Delta^2 u - h\Delta u_{tt} - \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}y\right)\Delta u = 0.$$

In the engineering literature, there is a formal procedure named Berger's approximation where such proximity is claimed (see, for instance, [10, § 7.6.1]). The mathematical justification of this limit has been proved in [9].

Acknowledgments

The first author expresses his thanks to the Departamento de Matemática Aplicada of the Universidad Complutense de Madrid for the kind hospitality and support while he was visiting (in November 1997) and where this joint research initiated. This work was partly supported by a grant of CNPq and PRONEX (MCT, Brasil) and by grants PB 96-0663 and the DGES (Spain), ERB FMRX CT 960033 of the European Union and partly by PRONEX (MCT, Brasil).

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(*Issued 28 July 2000*)