Proceedings of the Royal Society of Edinburgh, **149**, 533–560, *2019*
DOI:10.1017/prm.2018.38

Continuous solutions and approximating scheme for fractional Dirichlet problems on Lipschitz domains

Patricio Felmer
Departamento de Ingeniería Matemática and CMM (UMI 2807 CNRS),
Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile
(pfelmer@dim.uchile.cl)

Erwin Topp
Departamento de Matemática y C. C.,
Universidad de Santiago de Chile Casilla 307, Santiago, Chile
(erwin.topp@usach.cl)

(Received 5 July 2016; accepted 12 February 2017)

In this paper, we study the fractional Dirichlet problem with the homogeneous exterior data posed on a bounded domain with Lipschitz continuous boundary. Under an extra assumption on the domain, slightly weaker than the exterior ball condition, we are able to prove existence and uniqueness of solutions which are Hölder continuous on the boundary. In proving this result, we use appropriate barrier functions obtained by an approximation procedure based on a suitable family of zero-th order problems. This procedure, in turn, allows us to obtain an approximation scheme for the Dirichlet problem through an equicontinuous family of solutions of the approximating zero-th order problems on $\bar{\Omega}$. Both results are extended to an ample class of fully nonlinear operators.

Keywords: Elliptic equations; nonlocal operator; Dirichlet problem; Lipschitz domains; viscosity solutions

2010 *Mathematics subject classification:* 35D40; 35R10; 35B35; 35B50

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain and $f \in C(\bar{\Omega})$. In this paper, we are concerned with the study of the Dirichlet problem

$$(-\Delta)^\sigma u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c, \tag{1.1}$$

where, for $\sigma \in (0,1)$ fixed, the fractional Laplacian $(-\Delta)^\sigma$ is explicitly defined as

$$(-\Delta)^\sigma u(x) = -C_{N,\sigma} \, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x+z) - u(x)}{|z|^{N+2\sigma}} \mathrm{d}z,$$

where P.V. stands for the Cauchy principal value and $C_{N,\sigma} > 0$ is a normalizing constant, see [7].

Our goal in this paper is the study of existence and uniqueness of (viscosity) solutions to (1.1) which are Hölder continuous in the whole space $\mathbb{R}^N$, assuming the domain has Lipschitz regularity on the boundary together with a weak version of the exterior ball condition. The barrier functions for proving this result are obtained by approximating them through a family of barriers associated with non-local zero-th order problems. These barriers, in turn, allows us to study the behaviour of the solutions of the mentioned zero-th order problems as an approximating scheme for the fractional Dirichlet problem. The precise statements of the results and the hypothesis on the domain will be made rigorous below.

This type of problem has been addressed in several frameworks in the last decade. In the PDE setting, the fractional operator has an energy associated with fractional Sobolev spaces, and well-posedness in the weak formulation is possible to get under essentially no regularity on the boundary of the domain, see for example the work of Bellido and Mora–Corral [3], of Felsinger, Kassmann and Voigt [10] and references therein. However, the functional formulation lacks on information about the continuity of the solution (as a function in the whole space) at this level of generality. We also remark that this variational formulation can be carried out for f in a weaker functional space, see the work by Ros–Oton and Serra [14], and that an ad-hoc formulation allows to treat semi-linear problems [16], and nonlocal versions of the p-Laplacian [6].

In [1], Barles, Chasseigne and Imbert prove the well-posedness in the viscosity sense for (1.1), as a particular case of a large variety of integro-differential elliptic problems, in the case $\partial\Omega$ is of class C^2 and, under the same assumptions, the equivalence among weak and viscosity formulation is obtained by Servadei and Valdinoci [15]. In [14], this equivalence it obtained in the case of Lipschitz boundary regularity and the exterior ball condition. In an early result [9], the authors of this paper prove the continuous well-posedness of (1.1) in the viscosity sense when the boundary of the domain is of class C^1 and it satisfies the exterior ball condition. In this work, these two conditions are weakened in the sense of the following two definitions.

First, we have the Lipschitz regularity of the boundary of the domain Ω or *epi-Lipschitz property of* Ω, described through the following

DEFINITION 1.1. *We say that Ω has uniform Lipschitz boundary (or it is uniformly epi-Lipschitz) if there exist constants $\Lambda_0, r_0 > 0$ such that for each point $x \in \partial\Omega$ there exists a Lipschitz function $\phi : B'_{r_0} \subset \mathbb{R}^{N-1} \to \mathbb{R}$ with*

$$|\phi(x') - \phi(y')| \leqslant \Lambda_0 |x' - y'| \quad \text{for all } x', y' \in B'_{r_0}$$

such that, up to a rotation and translation, $B_{r_0} \cap \bar{\Omega} = B_{r_0} \cap \mathrm{Epi}(\phi)$. Here Epi *denotes the epigraph of a function and $B_{r_0} \subset \mathbb{R}^N$ is the ball centred at* 0 *and with radius r_0.*

This is basically the definition of an epi-Lipschitz set given by Clarke in [5], but where we have stressed on the uniformity of the Lipschitz constant. Notice

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

that this definition gives us immediately that if Ω is epi-Lipschitz then Ω^c is also epi-Lipschitz.

Next, we have a relaxation of the exterior ball condition for the boundary of the domain in the following

DEFINITION 1.2. *We say Ω satisfies the uniform exterior power condition if there exist $R > 0, c > 0$ and $\alpha \in (1, 2]$ such that, for any point $\hat{x} \in \partial\Omega$, there exists $\nu \in \mathbb{R}^N$ with $|\nu| = 1$ satisfying*

$$(z - \hat{x}) \cdot \nu < c|(z - \hat{x})'|^{\alpha} \quad \text{for all } z \in B_R(\hat{x}) \cap \Omega.$$

Here we have adopted the notation related to the orthogonal decomposition $y = y \cdot \nu + y'$ for any $y \in \mathbb{R}^N$.

In particular, it is easy to see that a domain satisfying the exterior ball condition satisfies the exterior power condition, just taking $\alpha = 2$. Now we state our first main theorem on the existence of Hölder continuous solutions to our Dirichlet problem.

THEOREM 1.3. *Assume Ω is a bounded open epi-Lipschitz domain, $f \in C(\bar{\Omega})$ and $\sigma \in (0, 1)$. If $\sigma \in [1/2, 1)$ assume additionally that Ω satisfies the uniform exterior power condition with $\alpha = 2\sigma'$, with $\sigma < \sigma' < 1$. Then the nonlocal linear Dirichlet problem (1.1) possesses a unique viscosity solution. Moreover, there exists $\beta_0 \in (0, \sigma)$ such that this solution is Hölder continuous with exponent β_0 in all $\mathbb{R}^N$.*

The existence of continuous solutions for equation (1.1) under stronger assumptions on the regularity of the boundary has been proved, for example in [**1**], assuming that the boundary is of class C^2 and in [**9**], assuming that the boundary is of class C^1 together with the exterior ball condition when $\sigma \in (1/2, 1)$. Further regularity for the solution up to the boundary is obtained in [**14**], where assuming that the boundary is Lipschitz continuous and it satisfies the exterior ball condition, they proved the solution is Hölder continuous of order σ. Hölder continuity of the solution in the interior of the domain is proved by Caffarelli and Silvestre in [**4**], see also the work by Barles, Chasseigne and Imbert in [**2**] and by Silvestre in [**17**]. In our theorem, we obtain Hölder regularity of the solution, merely assuming Lipschitz continuity on the boundary of Ω when $\sigma \in (0, 1/2)$ and additionally assuming the exterior power condition when $\sigma \in [1/2, 1)$.

The proof of Theorem 1.3 will be made for a more general linear operator, in order to prepare the arguments for the proof of Theorem 1.5 for non-linear operators. It is based on the construction of barriers attaining the boundary condition continuously and Perron's method, from which standard viscosity comparison principle provides the uniqueness. The main point is that these barriers are constructed as suitable powers of the distance function, and therefore they cannot be evaluated classically on the fractional Laplacian (at least in the case $\sigma \geqslant 1/2$), due to the weak assumption on the boundary. Nevertheless, we handle this by evaluating such barriers in the viscosity sense by an indirect approach, by 'approximating' the

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

fractional problem (1.1) by the following family of *zero-th order non-local problems*

$$-\mathcal{I}_\epsilon(u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \Omega^c, \tag{1.2}$$

where $\sigma \in (0,1)$ and $\epsilon > 0$. The operator $\mathcal{I}_\epsilon$ is defined as

$$\mathcal{I}_\epsilon(u,x) := C_{N,\sigma} \int_{\mathbb{R}^N} [u(x+z) - u(x)] \frac{\mathrm{d}z}{\epsilon^{N+2\sigma} + |z|^{N+2\sigma}}. \tag{1.3}$$

The integrability of the kernel defining $\mathcal{I}_\epsilon$ allows the evaluation of functions which are merely bounded. In particular, we can evaluate it in our barriers and then we can prove the right inequality by using a geometric estimate inspired in the work by Ishii and Nakamura [**11**], using the exterior power condition of the domain Ω, instead of the exterior ball condition. Thus, the strategy is to construct barriers for problem (1.2) with suitable compactness properties. It is easy to see that classical solutions, sub and super-solutions to (1.2) are at the same time viscosity solutions, sub and super-solutions for the same problem. Since $-\mathcal{I}_\epsilon$ approaches $(-\Delta)^\sigma$ as $\epsilon \to 0$, the result follows by standard stability results in the viscosity theory. As a consequence of the estimates for the solutions on the boundary, derived from these barriers, we can prove their Hölder regularity on the boundary following the strategy of Ros–Oton and Serra in [**14**].

A further study of the approximating family of problems (1.2) gives rise to the second part of this paper. As it is stated in [**9**], problem (1.2) has a unique classical solution u_ϵ which can be found via Fixed Point arguments and in which the regularity of the boundary plays no role. In the context of a domain with the boundary of class C^1 and with the exterior ball condition, it is proven in [**9**] that the family $\{u_\epsilon\}$ is compact in $C(\bar{\Omega})$. Hence, a natural question here is if such a compactness property for the family $\{u_\epsilon\}$ of solutions to (1.2) still holds true under the current weaker assumptions on $\partial\Omega$. The answer is positive and it is stated in the following

Theorem 1.4. *Assume Ω is a bounded open epi-Lipschitz domain, $f \in C(\bar{\Omega})$ and $\sigma \in (0,1)$. If $\sigma \in [1/2, 1)$ assume additionally that Ω satisfies the uniform exterior power condition with $\alpha = 2\sigma'$, for $\sigma' > \sigma$. For $\epsilon \in (0,1)$, let u_ϵ be a solution to the problem (1.2). Then, there is a modulus of continuity m depending only on f, such that*

$$|u_\epsilon(x) - u_\epsilon(y)| \leqslant m(|x-y|), \quad \text{for } x, y \in \overline{\Omega}. \tag{1.4}$$

The proof of Theorem 1.4 is obtained by combining the translation invariance of $\mathcal{I}_\epsilon$ and the comparison principle, following the Ishii–Lions method as in the proof of Theorem 1.1 in [**9**]. However, the construction of barriers to manage the discontinuities that u_ϵ may have on $\partial\Omega$, and the understanding of the evolution of u_ϵ, as ϵ approaches zero, pose the main difficulties. They are overcome by using a power of the distance function corrected by adding ϵ, similarly to the proof of Theorem 1.3. In the case $\sigma \geqslant 1/2$, the use of the distance function poses extra difficulties in controlling the estimates of the operator $\mathcal{I}_\epsilon$ evaluated on the barriers, independent of ϵ.

Still the estimates are quite delicate and we need to use the co-area formula, requiring the Implicit Function Theorem in the case of locally Lipstchitz continuous functions. The point here is to prove the non-degeneracy of the generalized

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

derivative of the distance function, which is not an obvious fact. In order to do this, we need to prove a result on tangent cones associated with the Lipschitz boundary and the distance function, see proposition 3.1. We did not find such a result in the literature and we think it may be useful in dealing with other PDE problems where only Lipschitz regularity is assumed on the boundary of the domain.

We further mention that our approach allows us to obtain the results stated in theorems 1.3 and 1.4 for fully non-linear operators. Associated with our linear operators $\mathcal{I}_\epsilon$ and $(-\Delta)^\sigma$, there is an ample class of fully nonlinear operators appearing in many problems related with stochastic control and stochastic game theory, associated with jump Lévy processes.

A class for these operators is obtained by considering a family $\mathcal{K}$ of functions $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}_+$ satisfying the *ellipticity* assumption

$$\gamma \leqslant K(z) \leqslant \Gamma \quad \text{and} \quad K(z) = K(-z), \tag{1.5}$$

for all $z \neq 0$, where γ, Γ are fixed constants satisfying $0 < \gamma < \Gamma < +\infty$.

Given a fixed K satisfying (1.5) and for $\epsilon \geqslant 0$, we consider the following notation

$$K_\epsilon^\sigma(z) = \frac{K(z)}{\epsilon^{N+\sigma} + |z|^{N+2\sigma}}, \tag{1.6}$$

and we write $K^\sigma = K_0^\sigma$. If $\epsilon > 0$, we denote by $\mathcal{I}_{\epsilon,K}$ the linear operator

$$\mathcal{I}_{\epsilon,K}(u,x) := \int_{\mathbb{R}^N} [u(x+z) - u(x)] K_\epsilon^\sigma(z) \mathrm{d}z, \tag{1.7}$$

and when $\epsilon = 0$, the limit operator as

$$\mathcal{I}_K(u,x) = \mathcal{I}_{0,K}(u,x) := \text{P.V.} \int_{\mathbb{R}^N} [u(x+z) - u(x)] K^\sigma(z) \mathrm{d}z, \tag{1.8}$$

each time the integral makes a sense for u.

In order to consider Isaacs type operators in our study, we assume throughout this paper that the family $\mathcal{K}$ can be expressed as $\mathcal{K} = \{K_{\alpha\beta}\}_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}}$ where $\mathcal{A}, \mathcal{B}$ are index sets. Thus, the corresponding *Isaacs Operator* is given by

$$F_\epsilon(u,x) = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \mathcal{I}_{K_{\alpha\beta},\epsilon}(u,x). \tag{1.9}$$

We also consider the limit operator defined as $F(u) := F_0(u)$, for a sufficiently smooth bounded function u. The operator F so defined is uniformly elliptic in the sense of Caffarelli and Silvestre [4]. Now we state the results given in theorems 1.3 and 1.4 for non-linear operators as follows

Theorem 1.5. *Under the assumptions of theorem 1.3, the exists a unique viscosity solution to the problem*

$$-F(u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c, \tag{1.10}$$

and this solution is Hölder continuous with exponent $\tilde{\beta}_0 > 0$ in all $\mathbb{R}^N$.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

In addition, the approximating problem

$$-F_\epsilon(u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c \tag{1.11}$$

has a unique solution u_ϵ *and the family* $\{u_\epsilon\}$ *is compact in* $C(\bar{\Omega})$.

The proof of this result follows the same ideas presented in the proofs of Theorems 1.3 and 1.4. In fact, we present the proofs of these theorems in a slightly more general setting in order to make easier the justification of theorem 1.5. In particular, this implies that the exponent of the Hölder regularity of the solution to the non-linear problem is the same as the exponent for the linear problem, although it is expected that the later has 'better' regularity.

This paper is organized as follows: In § 2, we prove theorem 1.3. In § 3, we prove proposition 3.1 as a crucial step for proving theorem 1.4. In § 4, we prove theorem 1.4. Finally, we provide the main lines to get theorem 1.5 in § 5.

Basic Notation: For $x \in \mathbb{R}^N$ and $r > 0$, we write $B_r(x)$ for the open ball of centre x and radius r, and we simply put $B(x)$ if $r = 1$ and B if in addition $x = 0$.

For $\delta > 0$ we write $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$.

2. Existence, uniqueness and regularity for the linear problem

In this section, we are concerned with the proof of Theorem 1.3, though we are going to prove a slightly more general result which is gong to be useful for the treatment of the nonlinear version of this theorem. More specifically, our interest is the following

THEOREM 2.1. *Let* Ω, f *satisfying the assumptions of theorem* 1.3. *Consider* K *satisfying* (1.5) *and let* $\mathcal{I}_K$ *as in* (1.8). *Then, the problem*

$$-\mathcal{I}_K(u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c \tag{2.1}$$

posseses a unique viscosity solution $u \in C^{\beta_0}(\mathbb{R}^N)$, *for some* $\beta_0 \in (0, \sigma)$.

Notice that theorem 1.3 is a corollary of the previous result by considering $\gamma = \Gamma = C_{N,\sigma}$ in (1.5).

Then, throughout this section, we fix K satisfying (1.5) and for $\epsilon > 0$, we denote $\mathcal{I}_\epsilon = \mathcal{I}_{\epsilon,K}$ as in (1.7), and $\mathcal{I} = \mathcal{I}_K$ as in (1.8), that is, we omit the dependence on K for simplicity of the notation. For $\epsilon \geqslant 0$, u a bounded function and $A \subset \mathbb{R}^N$ measurable, we write

$$\mathcal{I}_\epsilon[A](u, x) = \int_A [u(x+z) - u(x)] K_\epsilon^\sigma(z) \mathrm{d}z, \tag{2.2}$$

each time the integral as a sense (in the case $\epsilon = 0$ we must consider PV whenever it is necessary).

2.1. Well-posedness

The main technical result for the existence result is the following

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

PROPOSITION 2.2. *Assume hypotheses of theorem 2.1 hold. For $\beta \in (0,1)$ consider the function*

$$\psi(x) = (\epsilon + d(x))^{\beta} \mathbf{1}_{\bar{\Omega}}(x), \tag{2.3}$$

where $d = d_{\Omega}$ is the distance function to $\partial\Omega$. Then, there exists $\bar{\varrho}, c^*, \beta_0 > 0$ such that for all $\beta \in (0, \beta_0)$, we have

$$\mathcal{I}_{\epsilon}(\psi, x) \leqslant -c^*(\epsilon + d(x))^{\beta - 2\sigma} \quad \text{for all } x \in \Omega_{\bar{\varrho}}.$$

This estimate leads us to the

Proof of Theorem 2.1 – Existence and uniqueness. Notice that for each $x \in \bar{\Omega}$, we see that

$$\mathcal{I}_{\epsilon}(\mathbf{1}_{\bar{\Omega}}, x) = -\int_{\Omega^c - x} \frac{K(z)\mathrm{d}z}{\epsilon^{N+\sigma} + |z|^{N+2\sigma}} \leqslant -\gamma \int_{B^c_{\mathrm{diam}(\Omega)}} \frac{\mathrm{d}z}{1 + |z|^{N+2\sigma}} =: -\tilde{C},$$

where $\tilde{C} > 0$ defined above just depends on N, γ and $\sigma > 0$.

Once we have proposition 2.2 at hand, by using that minimum of supersolutions is a supersolution in the framework of nonlocal problems, we have the existence of constants $C_1, C_2 > 0$ just depending on $\tilde{C}, \bar{\varrho}$ and c^* in the last proposition to get that the function

$$\Psi^{\epsilon}_{+}(x) = C_1 ||f||_{\infty} \min\{C_2 \psi(x), (\epsilon + \bar{\varrho}/4)^{\beta}\} \mathbf{1}_{\bar{\Omega}}(x), \quad x \in \mathbb{R}^N \tag{2.4}$$

is a supersolution to (1.2). By linearity, a subsolution to the same problem is obtained with $\Psi^{\epsilon}_{-} = -\Psi^{\epsilon}_{+}$.

At this point, we notice that $\Psi^{\epsilon}_{+} \to \Psi^{0}_{+}$ uniformly on $\bar{\Omega}$ and the nonlocal Dirichlet problem associated with $\mathcal{I}_{\epsilon}$ approaches (2.1) in the viscosity sense. Then, stability results imply that Ψ^{0}_{+} is a supersolution to (2.1) attaining the boundary data pointwise. Similarly, we can get Ψ^{0}_{-} as a subsolution to the same problem and we clearly have that $\Psi^{0}_{-} \leqslant \Psi^{0}_{+}$ on $\bar{\Omega}$. Hence, applying Perron's method as it was established in [**1**], we conclude the existence of a solution to (2.1). Uniqueness comes as a consequence of standard viscosity comparison principle for sub and supersolutions ordered on Ω^c. □

In what follows, we concentrate in the details of proposition 2.2, which is carried out by a direct computation of the integral $\mathcal{I}_{\epsilon}(\psi, x)$. To simplify the exposition, we present the most difficult estimates through the following two lemmas below, and for this, we require to introduce some preliminaries concerning the main assumptions over the boundary of the domain.

We start with some properties and definitions concerning the exterior power condition. Let $x \in \Omega$ and $\hat{x} \in \partial\Omega$ with $d(x) = |x - \hat{x}|$ (i.e., a projection to the boundary). We claim that if ν denotes the direction of the axis of the exterior power condition at $\hat{x}$, then

$$\nu = (\hat{x} - x)/|\hat{x} - x|. \tag{2.5}$$

In what follows, we adopt the notation $e_x = (\hat{x} - x)/|\hat{x} - x|$. We prove (2.5) by contradiction, assuming that $\langle \nu, e_x \rangle =: a \in (0,1)$ (the case $a \in [-1, 0]$ follows the

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

same lines). For simplicity, we also assume that $\hat{x} = 0$ since the general case can be reduced from this one by translation.

Let $v \in \mathbb{R}^N$ with $|v| = 1$ be the unique vector in the plane formed by e_x, ν and the origin, such that $\langle v, \nu \rangle = 0$ and $\langle v, e_x \rangle = -\sqrt{1 - a^2} < 0$.

Then, for $t, b > 0$, we consider $y = t(v + b\nu)$. Notice that

$$\langle e_x, y \rangle = t(-\sqrt{1 - a^2} + ba),$$

from which, taking b small enough in terms of a, we get that $\langle e_x, y \rangle < 0$ for all $t > 0$. Then, for all t small enough, we see that $y \in B_{d(x)}(x)$ and, therefore, we conclude $y \in \Omega$. However, notice that

$$\langle y, \nu \rangle = tb \quad \text{and} \quad |y'| = |tv| = t,$$

from which, taking t small in terms of c, b and α we conclude $\langle y, \nu \rangle > c|y'|^\alpha$, which contradicts the exterior power condition assumption.

Assume Ω, such that $x = (0', 1 + \rho) \in \Omega$, $d(x) = \rho$ for some $\rho > 0$, and such that we can chose $\hat{x} = (0', 1)$ as a projection of x to the boundary. In view of the above discussion, the equation characterizing the exterior power condition becomes

$$y_n = 1 - c|y'|^\alpha, \quad \text{for } (y', y_n) \in \mathbb{R}^N. \tag{2.6}$$

Consider $z \in B_{\rho/2}(x)$. The line joining z and the origin crosses the surface defined by (2.6) at the point $\bar{y} = (\bar{y}', \bar{y}_n) = \lambda(z)(z', z_n)$, where $\lambda(z) \in (0, 1)$ satisfies

$$\lambda(z) z_n = 1 - c\lambda^\alpha(z)|z'|^\alpha. \tag{2.7}$$

Notice that $d(x) = (1 - \lambda(x))|x|$ and since $\bar{y} \in \Omega^c$ we remark the fundamental inequality

$$d(z) \leqslant |z| - |(\bar{y}', \bar{y}_n)| = (1 - \lambda(z))|z|. \tag{2.8}$$

We see that the function $\lambda(z)$, implicitly defined in (2.7) is differentiable with $D\lambda$ Hölder continuous with power $\alpha - 1$.

The above discussion is used in the following result, whose aim is to control the portion of the integral defining $\mathcal{I}_\epsilon(\psi, x)$ close to the origin.

Lemma 2.3. *For $\epsilon > 0$ and $x \in \Omega$, we denote*

$$I_1(x) := \mathcal{I}_\epsilon[B_{d(x)/2}](\psi, x), \tag{2.9}$$

where ψ is defined in (2.3). Then, under the assumptions of theorem 2.1, there exists $\bar{\varrho}, C > 0$ such that

$$I_1(x) \leqslant C\beta(\epsilon + d(x))^{\beta - 2\sigma} \quad \text{for all } x \in \Omega_{\bar{\varrho}}.$$

Proof. Let $x \in \Omega$ close to the boundary and $\hat{x} \in \partial\Omega$ a choice of projection. Without loss of generality, we can assume $x = (0', 1 + d(x))$ and $\hat{x} = (0', 1)$. In fact, if we denote $\mathcal{R}_x$ the rotation matrix making $\mathcal{R}_x x = \mathcal{R}_x \hat{x} + d(x) e_N$ and defining $\tilde{\Omega}_x =$

$\mathcal{R}_x^{-1}\Omega - (\hat{x} - (0', 1))$, then $(0', 1 + d(x)) \in \tilde{\Omega}_x$, its projection to $\partial\tilde{\Omega}_x$ is $(0', 1)$ and by the symmetry of the domain of integration and the kernel K_ϵ, we see that

$$I_1(x) = \mathcal{I}_\epsilon[B_{d(x)/2}](\tilde{\psi}_x, (0', 1 + d(x))),$$

where $\tilde{\psi}_x(y) = (\epsilon + d_{\partial\tilde{\Omega}_x}(y))^\beta$.

Thus, from now on we assume $x = (0', 1 + d(x))$ and $\hat{x} = (0', 1)$ is a choice of projection of x to the boundary. Using the symmetry of K and the domain of integration, we can write

$$I_1(x) = \frac{1}{2}\int_{B_{\mathrm{d}(x)/2}} [\psi(x + z) + \psi(x - z) - 2\psi(x)] K_\epsilon^\sigma(z)\mathrm{d}z.$$

Then, defining

$$\phi(z) = (\epsilon + (1 - \lambda(z))|z|)^\beta,$$

in view of (2.8), we can write

$$I_1(x) \leqslant \frac{1}{2}\int_{B_{\mathrm{d}(x)/2}} [\phi(x + z) + \phi(x - z) - 2\phi(x)] K_\epsilon^\sigma(z)\mathrm{d}z.$$

From this and Fundamental Theorem of Calculus we have

$$I_1(x) \leqslant \frac{1}{2}\int_{B_{\mathrm{d}(x)/2}} \int_0^1 [\nabla\phi(x + tz) - \nabla\phi(x - tz)] \cdot z \,\mathrm{d}t K_\epsilon^\sigma(z)\mathrm{d}z.$$

Next, using (2.7), we differentiate λ to obtain

$$\frac{\partial\lambda}{\partial z_n} = \frac{-\lambda}{z_n + c\alpha|z'|^\alpha\lambda^{\alpha-1}}; \quad \text{and} \quad \frac{\partial\lambda}{\partial z'} = \frac{-c\alpha\lambda^\alpha|z'|^{\alpha-2}}{z_n + c\alpha|z'|^\alpha\lambda^{\alpha-1}} z'.$$

Notice that these derivatives are Hölder continuous with power $\alpha - 1$, in particular, they are bounded. With this computation, we can write

$$\begin{aligned}\nabla\phi(z) &= \beta(\epsilon + (1 - \lambda(z))|z|)^{\beta-1}\left((1 - \lambda(z))\frac{z}{|z|} - \nabla\lambda(z)|z|\right)\\ &= \xi(x)\varphi(z),\end{aligned}$$

where $\xi(z)$ and $\varphi(z)$ satisfy, for $\tilde{t} \in (-1, 1)$ and $C > 0$

$$\begin{aligned}|\xi(x + tz) - \xi(x - tz)| &\leqslant C\beta(1 - \beta)(\epsilon + (1 - \lambda(x + \tilde{t}z)|x + \tilde{t}z|)^{\beta-2}|z|\\ &\leqslant C\beta(1 - \beta)(\epsilon + d(x + \tilde{t}z))^{\beta-2}|z|\\ &\leqslant C\beta(1 - \beta)\left(\epsilon + \frac{1}{2}d(x)\right)^{\beta-2}|z| \leqslant C(\epsilon + d(x))^{\beta-2}|z|\end{aligned}$$

and

$$|\varphi(x + tz) - \varphi(x - tz)| \leqslant C|z|^\alpha.$$

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

Consequently, we have

$$\begin{aligned} &I_1(x) \\ &\leqslant C\beta \int_{B_{d(x)/2}} \int_0^1 |\xi(x+tz) - \xi(x-tz)||\varphi(x+tz)| \\ &+ |\xi(x-tz)||\varphi(x+tz) - \varphi(x-tz)|K_\epsilon^\sigma(z)\mathrm{d}t\mathrm{d}z \\ &\leqslant C(\epsilon + d(x))^{\beta-1} \int_{B_{d(x)/2}} [(\epsilon + d(x))^{-1}|z|^2 + |z|^\alpha] K_\epsilon^\sigma(z)\mathrm{d}z \\ &\leqslant C(\epsilon + d(x))^{\beta-1} \int_{B_{d(x)/2}} [(\epsilon + d(x))^{-1}|z|^{2-(n+2\sigma)} + |z|^{\alpha-(n+2\sigma)}]\mathrm{d}z \\ &\leqslant C(\epsilon + d(x))^{\beta-2\sigma} + C(\epsilon + d(x))^{\beta-1+\alpha-2\sigma} \leqslant C(\epsilon + d(x))^{\beta-2\sigma}, \end{aligned}$$

since $\alpha = 2\sigma' > 2\sigma$. Here we have chosen $\beta < 1$ small and C is a constant depending on the data and the Hölder seminorm of the gradient of the function λ. □

Now we present some preliminaries concerning the epi-Lipschitz condition. For a point $z \in \mathbb{R}^N$, we write $z = (z', z_N)$ with $z' \in \mathbb{R}^{N-1}$ and $z_N \in \mathbb{R}$. By the Lipschitz property assumption over $\partial\Omega$, for all $y \in \partial\Omega$, we consider $\phi = \phi_y$ as in definition 1.1 and, eventually up to a rotation and translation, we can write $\phi(0') = 0$ and

$$\begin{aligned} (\Omega - y) \cap B_{r_0} &= \{(z', z_N) \in B_{r_0} : z_N > \phi(z')\} \\ \partial(\Omega - y) \cap B_{r_0} &= \{(z', z_N) \in B_{r_0} : z_N = \phi(z')\}. \end{aligned} \tag{2.10}$$

The above discussion is useful in the proof of the following

Lemma 2.4. *For $\epsilon > 0$ and $x \in \Omega$, we denote*

$$I_2(x) := \mathcal{I}_\epsilon[B \cap (\Omega^c - x)](\psi, x),$$

with ψ defined in (2.3). Under the assumptions of theorem 2.1, there exists $\bar{\varrho} > 0$ just depending on Ω and $c > 0$ not depending on ϵ such that

$$I_2(x) \leqslant -c(\epsilon + d(x))^{\beta-2\sigma}, \quad \text{for all } x \in \Omega_{\bar{\varrho}}.$$

Proof. Let $\hat{x}$ be a projection of x to $\partial\Omega$. Similarly, as in the proof of Lemma 2.3, we can assume $\hat{x} = 0$. Let $\phi = \phi_{\hat{x}}$ be the local chart associated with $\hat{x}$, consider the function $\Phi(z', r) = (z', \phi(z') + r)$ for $z' \in B'_{r_0} \subset \mathbb{R}^{N-1}$, $r \in (-r_0, 0)$, and define the set

$$\mathcal{C}_- = \Phi(B'_{r_0} \times (-r_0, 0)) \subset \Omega^c.$$

Notice that $\Phi : B_{r_0} \times (-r_0, 0) \to \mathbb{R}^N$ is injective and is differentiable a.e. in its domain of definition because ϕ is Lipschitz continuous. Moreover, direct computation shows that $|\mathrm{Det}\ D\Phi(z', r)| = 1$ for a.a. (z', r). Shortening r_0 if it is necessary (in terms of Λ_0) and taking $\bar{\varrho}$ small in terms of r_0, we can assume that $\mathcal{C}_- \subset B_1(x)$. Then, we see that

$$\begin{aligned} I_2(x) &= -(\epsilon + \mathrm{d}(x))^\beta \int_{\Omega^c \cap B_1(x)} K_\epsilon^\sigma(z-x)\mathrm{d}z \\ &\leqslant -(\epsilon + \mathrm{d}(x))^\beta \int_{\mathcal{C}_-} K_\epsilon^\sigma(z-x)\mathrm{d}z =: -(\epsilon + \mathrm{d}(x))^\beta \tilde{I}_2(x). \end{aligned}$$

Using the Lipschitz version of the Change of Variables Formula (see [8]), we can write

$$\tilde{I}_2(x) = \int_{-r_0}^0 \int_{B'_{r_0}} K_\epsilon^\sigma(\Phi(z', r) - x)\mathrm{d}z'\mathrm{d}r,$$

and since

$$\begin{aligned} |\Phi(z', r) - x| &\leqslant |x| + |\Phi(z', r)| \\ &= d(x) + |\Phi(z', r)| \\ &\leqslant d(x) + \sqrt{2}(|z'| + |\phi(z')| + |r|) \\ &\leqslant d(x) + \sqrt{2}((1 + \Lambda_0)|z'| + |r|), \end{aligned}$$

there exists a constant $c > 0$ depending on Λ_0, γ, N and σ such that

$$\tilde{I}_2(x) \geqslant c \int_0^{r_0} \int_{B'_{r_0}} \frac{\mathrm{d}z'\mathrm{d}r}{\epsilon^{N+2\sigma} + |z'|^{N+2\sigma} + (d(x) + r)^{N+2\sigma}},$$

from which, by a direct computation, we obtain

$$\tilde{I}_2(x) \geqslant c(\epsilon + d(x))^{-2\sigma},$$

for some contant $c > 0$ not depending on ϵ nor $d(x)$.

Recalling that $I_2(x) \leqslant -(\epsilon + d(x))^\beta \tilde{I}_2(x)$, we conclude the result. □

Now we are in position to provide the

Proof of Proposition 2.2. Without loss of generality, we can assume $\bar{\varrho}$ of the previous lemmas is the same one, and let $x \in \Omega$ with $d(x) < \bar{\varrho}$. Recalling the notation (2.2), we write $\mathcal{I}_\epsilon(\psi, x) = \sum_{i=0}^3 I_i(x)$ with

$$\begin{aligned} I_0(x) = {} & \mathcal{I}_\epsilon[B^c](\psi, x); \quad I_1(x) = \mathcal{I}_\epsilon[B_{d(x)/2}](\psi, x); \\ I_2(x) = {} & \mathcal{I}_\epsilon[B \cap (\Omega^c - x)](\psi, x); \quad I_3(x) = \mathcal{I}_\epsilon[B \cap (\Omega - x) \setminus B_{d(x)/2}](\psi, x) \end{aligned}$$

Notice that the estimates for I_1 and I_2 are given by lemmas 2.3 and 2.4, respectively. It remains to estimate I_0 and I_3.

For I_0, by the boundedness of ψ independent of ϵ, β when $\epsilon, \beta \in (0, 1)$, we can write

$$I_0(x) \leqslant C, \tag{2.11}$$

where $C > 0$ depends only on Ω and N.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

It remains to estimate $I_3(x)$. Denoting $\mathcal{D} = B \cap (\Omega - x) \setminus B_{d(x)/2}$ for simplicity, using the Lipschitz continuity of the distance function (with Lipschitz constant 1), we can write

$$\begin{aligned} I_3(x) &= (\epsilon + d(x))^\beta \int_{\mathcal{D}} \left[\left(\frac{\epsilon + d(x+z)}{\epsilon + d(x)} \right)^\beta - 1 \right] K_\epsilon^\sigma(z) \mathrm{d}z \\ &\leqslant (\epsilon + d(x))^\beta \int_{\mathcal{D}} \left[\left(1 + \frac{|z|}{\epsilon + d(x)} \right)^\beta - 1 \right] K_\epsilon^\sigma(z) \mathrm{d}z \\ &\leqslant (\epsilon + d(x))^\beta \int_{B \setminus B_{d(x)/2}} \left[\left(1 + \frac{|z|}{\epsilon + d(x)} \right)^\beta - 1 \right] K_\epsilon^\sigma(z) \mathrm{d}z \end{aligned}$$

and thus, there exists a constant just depending on N and Γ such that

$$I_3(x) \leqslant C(\epsilon + d(x))^\beta \int_{d(x)/2}^1 [(1 + r/(d(x) + \epsilon))^\beta - 1] \frac{r^{N-1} \mathrm{d}r}{\epsilon^{N+2\sigma} + r^{N+2\sigma}}.$$

Now, defining $\tau = \epsilon/(\epsilon + d(x)) \in (0, 1)$ and applying the change of variables $t = r/(\epsilon + d(x))$ in the last integral, we conclude

$$I_3(x) \leqslant C(\epsilon + d(x))^{\beta - 2\sigma} \int_{(1-\tau)/2}^{+\infty} \frac{((1+t)^\beta - 1) t^{N-1} \mathrm{d}t}{\tau^{N+2\sigma} + t^{N+2\sigma}},$$

and from here it is possible to take β small not depending on ϵ nor $d(x)$ in order to obtain

$$I_3(x) \leqslant c(\epsilon + d(x))^{\beta - 2\sigma}/2,$$

where $c > 0$ is the constant given in lemma 2.4. Joining the estimates for $I_i, i = 0, 1, 2, 3$, we conclude the result. □

2.2. Regularity

We start with the following interior Hölder regularity result which is basically contained in [2], but we provide here a proof for completeness, stressing on its dependence with respect to the data since this is going to be crucial in the extension of the regularity up to the boundary.

THEOREM 2.5. *Assume K satisfies (1.5), let $\mathcal{I} = \mathcal{I}_K$ as in (1.8) and assume $u \in L^\infty \cap C(\mathbb{R}^N)$ is a viscosity solution to the problem*

$$-\mathcal{I}(u, x) = f \quad \text{in } B_2. \tag{2.12}$$

Then, for all $\beta \in (0, \min\{1, \sigma\})$ there exists $C > 0$ such that

$$[u]_{C^\beta(\bar{B}_{1/4})} \leqslant C(||f||_{L^\infty(B_2)} + ||u||_{L^\infty(B_1)} + ||uw_\sigma||_{L^1(\mathbb{R}^N)}),$$

where $w_\sigma(y) = (1 + |y|)^{-(N+2\sigma)}$ for all $y \in \mathbb{R}^N$.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

Proof. We start considering a smooth, bounded, even function $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$, with $0 < \tilde{\phi}(t) < 3/2$ for all $t > 0$, $\tilde{\phi}(0) = 0$, increasing in $[0, +\infty)$ and such that $\tilde{\phi}(t) \geqslant 1$ for $t \geqslant 1/4$.

We fix $x_0 \in B_{1/4}$, denote $\phi(x) := \mathrm{osc}_{B_1}(u)\tilde{\phi}(|x - x_0|)$ and for $L > 0$ to be fixed, we consider the function

$$\Phi(x, y) = u(x) - u(y) - L|x - y|^\beta - \phi(x), \qquad x, y \in \bar{B}_1. \tag{2.13}$$

If we prove that for $L > 0$ large enough not depending on x_0, we get that $\max_{\bar{B}_1^2} \Phi \leqslant 0$, Hölder regularity holds. Thus, we proceed by contradiction by assuming that there exists $(\bar{x}, \bar{y}) \in \bar{B}_1^2$ such that

$$\Phi(\bar{x}, \bar{y}) = \max_{\bar{B}_1^2} \Phi > 0,$$

and from this it is direct to verify that

(i) $\bar{x} \neq \bar{y}$.

(ii) $|\bar{x} - \bar{y}| \leqslant (L^{-1}\mathrm{osc}_{B_1}(u))^{1/\beta}$.

(iii) $|\bar{x} - x_0| \leqslant 1/4$ for all $L > 0$ by construction of ϕ. Moreover, if $L \geqslant 4^{1/\beta}\mathrm{osc}_{B_1}(u)$, then $\bar{y} \in \bar{B}_{3/4}$ using *(ii)*.

Then, denoting $h(x, y) = L|x - y|^\beta + \phi(x)$, we have that $\bar{x}$ is a local maximum point to

$$x \mapsto u(x) - u(\bar{y}) - h(x, \bar{y})$$

in $B_{1/4}(\bar{x})$ and $\bar{y}$ is a local minimum point for

$$y \mapsto u(y) - u(\bar{x}) + h(\bar{x}, y)$$

in $B_{1/4}(\bar{y})$. Thus, the corresponding viscosity inequalities can be written for all $0 < \delta < |\bar{x} - \bar{y}|$ as

$$\begin{aligned} -\mathcal{I}[B_\delta](h(\cdot, \bar{y}), \bar{x}) - \mathcal{I}[B_\delta^c](u, \bar{x}) &\leqslant f(\bar{x}) \\ \mathcal{I}[B_\delta](h(\bar{x}, \cdot), \bar{y}) - \mathcal{I}[B_\delta^c](u, \bar{y}) &\geqslant f(\bar{y}). \end{aligned}$$

Now, using the radiality of the kernel, we get that

$$J_1(\delta) - J_2(\delta) \leqslant f(\bar{x}) - f(\bar{y}), \tag{2.14}$$

where, denoting $\bar{a} = \bar{x} - \bar{y}$, we have introduced the notation

$$\begin{aligned} J_1(\delta) &= 2L\mathcal{I}[B_\delta](|\cdot|^\beta, \bar{a}) - \mathcal{I}[B_\delta](\phi, \bar{x}), \\ J_2(\delta) &= \int_{B_\delta^c} [u(\bar{x} + z) - u(\bar{y} + z) - (u(\bar{x}) - u(\bar{z}))]K^\sigma(z)\mathrm{d}z. \end{aligned}$$

Using the maximality of $(\bar{x}, \bar{y})$, for all $z \in B_{1/4}$, we can write

$$\begin{aligned} & u(\bar{x}+z) - u(\bar{y}+z) - (u(\bar{x}) - u(\bar{y})) \leqslant \phi(\bar{x}+z) - \phi(\bar{x}) \\ & u(\bar{x}+z) - u(\bar{x}) \leqslant L(|\bar{a}+z|^\beta - |\bar{a}|^\beta) + \phi(\bar{x}+z) - \phi(\bar{x}) \\ & u(\bar{y}+z) - u(\bar{y}) \geqslant -L(|\bar{a}-z|^\beta - |\bar{a}|^\beta). \end{aligned} \tag{2.15}$$

Notice that by chosing L as in (iii) above, we have that $|\bar{a}| \leqslant 1/4$. At this point, we consider $\rho, \eta \in (0,1)$ and the set

$$\mathcal{C} = \{z \in B_{\rho|\bar{a}|} : |\langle \bar{a}, z\rangle| \geqslant (1-\eta)|\bar{a}||z|\}.$$

Using the first inequality of (2.15) for $z \in B_{1/4} \setminus \mathcal{C}$, and the second and third inequalities of (2.15) for $z \in \mathcal{C}$, we can write

$$J_2(\delta) \leqslant L\mathcal{I}[\mathcal{C} \setminus B_\delta](|\cdot|^\beta, \bar{a}) + \mathcal{I}[B_{1/4} \setminus (\mathcal{C} \cup B_\delta)](\phi, \bar{x}) + J_2(1/4).$$

Recalling that $\phi = \mathrm{osc}_{B_1}(u)\tilde{\phi}$, by the smoothness of $\tilde{\phi}$ we see that

$$J_2(\delta) \leqslant L\mathcal{I}[\mathcal{C} \setminus B_\delta](|\cdot|^\beta, \bar{a}) + C\mathrm{osc}_{B_1}(u) + J_2(1/4),$$

for some universal constant $C > 0$.

Now, recalling that $\bar{x}, \bar{y} \in \bar{B}_1$ we arrive at

$$\begin{aligned} J_2(1/4) &\leqslant C\ \Gamma\mathrm{osc}_{B_1}(u) + \int_{B_{1/4}^c} [u(\bar{x}+z) - u(\bar{y}+z)]K^\sigma(z)\mathrm{d}z \\ &\leqslant C\ \Gamma\mathrm{osc}_{B_1}(u) + C\Gamma \int_{\mathbb{R}^N} |u(y)|(1+|y|)^{-(N+2\sigma)}\mathrm{d}y, \end{aligned}$$

from which we deduce the existence of $C > 0$ just depending on N, σ and Γ such that

$$J_2(\delta) \leqslant L\mathcal{I}[\mathcal{C} \setminus B_\delta](|\cdot|^\beta, \bar{a}) + C\mathrm{osc}_{B_1}(u) + C||uw_\sigma||_{L^1(\mathbb{R}^N)}.$$

Since it is direct to see that $|J_1(\delta)| \to 0$ as $\delta \to 0$ and using the last inequality into (2.14), we obtain that

$$-L\mathcal{I}[\mathcal{C}](|\cdot|^\beta, \bar{a}) \leqslant C(1+\Gamma)\Big(\mathrm{osc}_{B_1}(u) + ||uw_\sigma||_{L^1(\mathbb{R}^N)}\Big) + (f(\bar{x}) - f(\bar{y})), \tag{2.16}$$

where the term in the left-hand side is understood in the principal value sense because of the symmetry of the domain of integration defining it. Next, we concentrate on estimating this term.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

Following the lines of [2], using the definition of $\mathcal{C}$, we get that

$$\begin{aligned}&\mathcal{I}[\mathcal{C}](|\cdot|^\beta,\bar{a})\\&\quad=\frac{\beta}{2}\int_0^1(1-s)\int_{\mathcal{C}}|\bar{a}+sz|^{\beta-2}\left((\beta-2)|\langle\bar{a}+sz/|\bar{a}+sz|,z\rangle|^2+|z|^2\right)K^\sigma(z)\mathrm{d}z\\&\quad\leqslant\frac{\beta}{2}\int_0^1(1-s)\int_{\mathcal{C}}|\bar{a}+sz|^{\beta-2}\left((\beta-2)(1-\eta-\rho)^2/(1+\rho)^2+1\right)K^\sigma(z)\mathrm{d}z,\end{aligned}$$

and since $\beta<1$, there exists $\rho,\eta>0$ small just depending on $2-\beta>1$ such that

$$\mathcal{I}[\mathcal{C}](|\cdot|^\beta,\bar{a})\leqslant C(\beta-1)\beta\gamma|\bar{a}|^{\beta-2}\int_{\mathcal{C}}|z|^{2-(N+2\sigma)}\mathrm{d}z,$$

from which we conclude that there exists $c>0$ such that

$$\mathcal{I}[\mathcal{C}](|\cdot|^\beta,\bar{a})\leqslant c(\beta-1)\beta\gamma|\bar{a}|^{\beta-2\sigma}.$$

Hence, denoting $c^*=-(\beta-1)\beta\gamma c>0$ and replacing this into (2.16), we arrive at

$$c^*L|\bar{a}|^{\beta-2\sigma}\leqslant C(1+\Gamma)\Big(\mathrm{osc}_{B_1}(u)+||uw_\sigma||_{L^1(\mathbb{R}^N)}\Big)+\mathrm{osc}_{B_2}(f).$$

From this inequality, we arrive at a contradiction after taking

$$L=\bar{C}(\mathrm{osc}_{B_1}(u)+||uw_\sigma||_{L^1(\mathbb{R}^N)}+\mathrm{osc}_{B_2}(f))$$

with C large in terms of $C,c^*,\beta,\sigma,\gamma,\Gamma$. □

Now we are in a position to provide the

Proof of Theorem 1.3 – *Regularity.* Interior regularity is a direct consequence of theorem 2.5. For the boundary regularity, we follow closely the steps of [14]. We consider $x_0\in\Omega$ close enough to the boundary and denote $r=d(x_0)/2$.

We consider the scaled function $\tilde{u}(y)=u(x_0+ry),\ y\in\mathbb{R}^N$. By the barriers applied in the existence proof, we can see that

$$||\tilde{u}||_{L^\infty(B_2)}\leqslant Cr^{\beta_0},$$

with $\beta_0<\min\{1,\sigma\}$ given in proposition 2.2. Moreover, using this and the L^∞ bounds for u obtained in the existence proof, we see that

$$|\tilde{u}(y)|\leqslant Cr^{\beta_0}(1+|y|)^{\beta_0},\quad \text{for all } y\in\mathbb{R}^N,$$

where $C>0$ depends on $||f||_\infty$ and $\mathrm{diam}(\Omega)$.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

Now, for K as in (1.5), we denote $K_r(z) = K(rz)$, $z \in \mathbb{R}^N$, which still satisfies (1.5), and the nonlocal operator

$$\tilde{\mathcal{I}}(\phi, x) = \text{P.V.} \int_{\mathbb{R}^N} [\phi(x+z) - \phi(x)] K_r^\sigma(z) \mathrm{d}z.$$

A direct computation implies that $\tilde{u}$ solves

$$-\tilde{\mathcal{I}}(\tilde{u}, y) = r^{2\sigma} f(x_0 + ry), \quad y \in B_2,$$

for which the interior regularity estimates given by theorem 2.5 apply. Then, we get that

$$[\tilde{u}]_{C^\beta}(\bar{B}_{1/4}) \leqslant C\Big(r^{\beta_0}||uw_{\sigma-\beta_0}||_{L^1(\mathbb{R}^N)} + r^{2\sigma}||f||_{L^\infty(B_1)} + Cr^{\beta_0}\Big) \leqslant \bar{C} r^{\beta_0}.$$

with $\bar{C} > 0$ just depending on the data. Then, using the homogeneity of the Hölder seminorm, we see that

$$[u]_{C^\beta(B_{r/4}(x_0))} = r^{-\beta}[\tilde{u}]_{C^\beta(B_{1/4})},$$

from which we get uniformly bounded estimates for $[u]_{C^\beta(B_{r/4}(x_0))}$ with respect to $x_0 \in \Omega$ close to the boundary when $\beta \leqslant \beta_0$. From this point, we follow the lines of the proof of Proposition 1.1. of [**14**] to get the Hölder regularity up to the boundary (with exponent β_0). The proof is complete. □

3. Preliminaries for theorem 1.4: Non-degeneracy of the generalized derivative of the distance function

As we mentioned in the introduction, the compactness result for the linear nonlocal problem (1.2) given by theorem 1.4 is a consequence of the construction of appropriate barriers to control the modulus of continuity of the solution to (1.2) on $\bar{\Omega}$ which is independent of $\epsilon \in (0, 1)$. As it can be seen in the next section, these barriers use in a significant way the distance function, and therefore, we require a careful analysis of the behaviour of this function near the boundary.

Thus, the purpose of this section is to provide a significant property of the distance function, coming from the Lipschitz assumption on the boundary of the domain, and which is crucial to construct the mentioned barriers. See proposition 3.1 bellow.

We start with some definitions, referring to the book of F.H. Clarke [**5**]. For more insights on this topic, see also [**12**, **13**]. For a Lipschitz function $f : A \subset \mathbb{R}^N \to \mathbb{R}$ (A open) and $x \in A$, we define the *generalized gradient* of f at x as

$$\partial f(x) = \text{co}\{\lim_{k\to\infty} Df(x_k) : x_k \to x, f \text{ is differentiable at } x_k\}, \tag{3.1}$$

where co denotes the convex hull of a set. For a closed set $C \in \mathbb{R}^N$, we denote by δ_C the usual distance function to C

$$\delta_C(x) = \inf_{c\in C} ||x - c||, \quad x \in \mathbb{R}^N,$$

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

and we also consider the *signed distance function* (which is nonnegative in C) defined as

$$d_C(x) = \delta_{C^c}(x) - \delta_C(x). \tag{3.2}$$

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set such that $\bar{\Omega}$ is epi-Lipschitz in the sense of definition 1.1, and let $d = d_\Omega : \mathbb{R}^N \to \mathbb{R}$ the signed distance function relavive to $\bar{\Omega}$. Then, there exists $a > 0$ such that for each $x \in \partial\Omega$ there exists a rotation matrix $\mathcal{R} = \mathcal{R}_x$ satisfying*

$$v_N > a \quad \text{for all } v = (v', v_N) \in \mathcal{R}^{-1}\partial d(x).$$

The main point here is the uniformity of the constant a, which is a consequence of the assumed uniformity of the Lipschitz bounds of the local parametrizations of $\partial\Omega$ given by definition 1.1.

We require further definitions and previously known results in the theory of nonsmooth analysis to prove this proposition. We start with the following

DEFINITION 3.2 (Tangent Cone). *Let $C \subset \mathbb{R}^N$ a closed set and $x \in C$. We say that v is in the tangent cone to C at x, denoted by $T_C(x)$, if for each sequence $x_k \to x$ with $x_k \in C$ and each $\tau_k \searrow 0$, there exists a sequence $v_k \to v$ such that $x_k + \tau_k v_k \in C$ for all k large enough.*

It is known that $T_C(x)$ is a closed convex cone containing 0. We notice that the definition given above is not exactly the definition given by Clarke in [**5**], but it is equivalent to it in virtue of theorem 2.4.5 in [**5**].

The following is the main technical result of this section.

LEMMA 3.3. *Let $\Omega \subset \mathbb{R}^N$ be an epi-Lipschitz domain, and $x \in \partial\Omega$. Then*

$$T_{\bar{\Omega}}(x) = -T_{\Omega^c}(x).$$

Proof. Up to a translation, we may assume that $x = 0$ and up to a rotation, an $\partial\Omega-$neigbourhood of x can be seen as the graph of a Lipschitz function $\phi : B_{r_0}(0)' \subset \mathbb{R}^{N-1} \to \mathbb{R}$ with Lipschitz constant $\Lambda_0 > 0$.

It is sufficient to prove that

$$T_{\bar{\Omega}}(x) \subseteq -T_{\Omega^c}(x), \tag{3.3}$$

since the arguments below are local and then the boundedness of $\partial\Omega$ does not play any role. In fact, if we denote by $\mathcal{O} = int(\Omega^c)$, then the last inclusion leads us to

$$T_{\bar{\mathcal{O}}}(x) \subseteq -T_{\mathcal{O}^c}(x),$$

which implies $T_{\Omega^c}(x) \subseteq -T_{\bar{\Omega}}(x)$, and subsequently the reverse inclusion in (3.3).

We proceed by considering $v \in (T_{\bar{\Omega}}(x))^c$ and prove that $v \in (-T_{\Omega^c}(x))^c$. Since the complementary of a cone is also as cone, then it is sufficient to prove the property by assuming that $||v|| = 1$.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

By definition 3.2, $v \in (T_{\bar{\Omega}}(x))^c$ implies the existence of sequences $x_k \to x$ with $x_k \in \bar{\Omega}$ and $\tau_k \searrow 0$ such that, for all $v_k \to v$, we have

$$x_k + \tau_k v_k \in (\bar{\Omega})^c \quad \text{for infinitely many } k's, \tag{3.4}$$

and in the particular case of the constant sequence $v_k = v$, for all k, we denote $y_k := x_k + \tau_k v$.

In addition, we claim that we can assume that $x_k \in \partial\Omega$. In fact, let us denote l_k the line joining x_k and y_k and consider $\hat{x}_k \in \partial\Omega$ a choice of the intersection between l_x and $\partial\Omega$, and for this choice, we also denote $\hat{\tau}_k = \tau_k - |x_k - \hat{x}_k| \in (0, \tau_k]$. Clearly, we have that $\hat{\tau}_k \searrow 0$. Then, for $w_k \to v$ arbitrary, we can write

$$\begin{aligned} \hat{x}_k + \hat{\tau}_k w_k &= x_k + (\hat{x}_k - x_k) + \hat{\tau}_k w_k \\ &= x_k + (\tau_k - \hat{\tau}_k) v + \hat{\tau}_k w_k \\ &= x_k + \tau_k (v + \hat{\tau}_k/\tau_k (w_k - v)), \end{aligned}$$

and since $0 < \hat{\tau}_k/\tau_k \leqslant 1$, we see that $v + \hat{\tau}_k/\tau_k(w_k - v) \to v$. Hence, we get the claim by (3.4) replacing x_k by $\hat{x}_k$ and τ_k by $\hat{\tau}_k$, but from this point we omit the superscript '$\wedge$' for simplicity.

Since $y_k \to x$ and $y_k \in (\bar{\Omega})^c$, for all k large enough, we see that y_k must belong to the hypograph of ϕ. Then, by the standard notation, we write $y_k = (y_k', y_k^N)$ and define $\hat{y}_k = (y_k', \hat{y}_k^N)$ as

$$\hat{y}_k := (y_k', \phi(y_k')),$$

from which we immediately have that $\hat{y}_k \in \partial\Omega$ for all k. Since $y_k \in (\bar{\Omega})^c$, we have $y_k^N < \hat{y}_k^N$, and more precisely, there exists a constant $\theta_0 > 0$ not depending on k such that, up to a subsequence, $|\hat{y}_k - y_k| = \hat{y}_k^N - y_k^N \geqslant \theta_0 \tau_k$. In fact, if this does not hold, we can write $|\hat{y}_k - y_k| = o_k(1)\tau_k$ with $o_k(1) \to 0$ and therefore, defining

$$v_k := \tau_k^{-1}(\hat{y}_k - x_k),$$

we see that $v_k = \tau_k^{-1}(y_k - x_k) + o_k(1) \to v$ as $k \to \infty$, and that

$$x_k + \tau_k v_k = x_k + \hat{y}_k - x_k = \hat{y}_k \in \bar{\Omega} \quad \text{for all } k,$$

but this contradicts (3.4).

In what remains we prove that $\hat{y}_k, \tau_k$ above chosen are suitable to get the desired conclusion $v \notin -T_{\Omega^c}(x)$. For this, we define $z_k = \hat{y}_k - \tau_k v$. By the parallelogram rule, we see that $z_k' = x_k'$ and $z_k^N > \phi(x_k')$, from which we get that z_k is in the (strict) epigraph of ϕ. Now, the epi-Lipschitzian property of the boundary of Ω implies that

$$\Theta_k := x_k + B_{r_0/2} \cap \{(z', z^N) : z^N \geqslant 2\Lambda_0 |z'|\}$$

is a subset of Epi(ϕ) and therefore, $\Theta_k \subset \Omega$. Notice that since z_k is on the 'vertical axis' of Θ_k, we have

$$\operatorname{dist}(z_k, \Theta_k^c) \geqslant c_0 \tau_k \tag{3.5}$$

for all k large enough, where $c_0 > 0$ depends on Λ_0 and θ_0, but not on k.

Now, considering $w_k \to v$ arbitrary, we can write

$$\hat{y}_k - \tau_k w_k = \hat{y}_k - \tau_k v + \tau_k(v - w_k) = z_k + \tau_k o_k(1),$$

with $o_k(1) \to 0$ as $k \to 0$, and using (3.5), we get that $\hat{y}_k - \tau_k w_k \in \Omega$ for infinitely many $k's$, leading to $-v \notin T_{\Omega^c}(x)$, completing the proof. □

Next, we describe a uniformity property coming from the uniformity of the Lipschitz constant in definition 1.1.

LEMMA 3.4. *Assume $\mathcal{O}$ is open and epi-Lipschitz boundary in the sense of definition (1.1). Then, there exists $\epsilon_0 > 0$ such that, for each $x \in \partial\mathcal{O}$, there exists $\xi \in T_{\bar{\mathcal{O}}}(x)$ with $|\xi| = 1$ such that $B_{\epsilon_0}(\xi) \subset T_{\bar{\mathcal{O}}}(x)$.*

We sketch the proof. For $x \in \partial\mathcal{O}$, we can consider a local parametrization ϕ such that the epigraph of ϕ coincides with $\mathcal{O}$ in a neighbourhood of x. Then, by using the characterization given by definition 3.2 for the tangent cone, it is possible to prove that

$$\{(z', z^N) \in \mathbb{R}^N : z^N \geqslant 2\Lambda_0|z'|\} \subseteq T_{\bar{\mathcal{O}}}(x),$$

from which the property holds with $\xi = e_N$ and ϵ_0 small in terms of Λ_0.

Now we are in position to provide the

Proof of Proposition 3.1. Using (3.2) and applying algebraic properties of the generalized gradient (see propositions 2.3.1 and 2.3.3 in [**5**]), we can write

$$\partial d(x) \subseteq \partial\delta_{\Omega^c}(x) - \partial\delta_{\bar{\Omega}}(x).$$

Then, using the characterization of tangent cones in term of generalized gradients of the distance function δ given by proposition 2.4.2 in [**5**], we conclude that

$$\partial d(x) \subset (T_{\Omega^c}(x))^* + (-T_{\bar{\Omega}}(x))^*,$$

where for $A \subset \mathbb{R}^N$, we denote $A^* = \{x \in \mathbb{R}^N : \langle x, v\rangle \leqslant 0 \text{ foreach } v \in A\}$, that is the polar set relative to A.[1]

From here, lemma 3.3 and using the fact that $C + C = C$ for a convex cone C, we conclude that $\partial d(x) \subset T^*_{\Omega^c}(x)$. Hence, by lemma 3.4 (with $\bar{\mathcal{O}} = \Omega^c$), we have $\epsilon_0 \in (0,1)$ and $\xi = \xi(x)$ with $||\xi|| = 1$ such that

$$\mathcal{B} := \{\lambda z : z \in B_{\epsilon_0}(\xi),\ \lambda \geqslant 0\} \subset T_{\Omega^c}(x).$$

From this, denoting Π_ξ the normal plane to ξ we have the existence of $\tilde{a} > 0$ (depending on ϵ_0) such that $\langle v, w\rangle > \tilde{a}$ for all $v \in \mathcal{B}^*$ and $w \in \Pi_\xi$. Hence, we conclude the result by fixing $0 < a < \tilde{a}$ small enough, noticing that $\partial d(x) \subseteq T^*_{\Omega^c}(x) \subseteq \mathcal{B}^*$ and considering the rotation matrix $\mathcal{R}$ making $\mathcal{R}\xi/|\xi| = -e_N$. □

[1]The mentioned characterization is described via *normal cones* in [**5**]

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

4. Approximation scheme for the linear problem

As for theorem 1.3 in § 2, we present theorem 1.4 in a slightly more general way

THEOREM 4.1. *Let* Ω *and* f *satisfying the assumptions of theorem 1.4. Consider* K *satisfying (1.5), and for each* $\epsilon \in (0,1)$, *we consider* $\mathcal{I}_{K,\epsilon}$ *as in (1.7). Let* $u_\epsilon \in C(\bar{\Omega})$ *be the unique solution to the problem*

$$-\mathcal{I}_{K,\epsilon}(u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \Omega^c. \tag{4.1}$$

Then, there exists a modulus of continuity m *depending only on* f *and* K *such that*

$$|u_\epsilon(x) - u_\epsilon(y)| \leqslant m(|x-y|) \quad \text{for } x, y \in \bar{\Omega}.$$

Clearly, theorem 1.4 is a consequence of theorem 4.1 above. The existence and uniqueness of u_ϵ can be directly obtained by adapting proposition 2.2 in [9] to the current setting. Throughout this section, we are going to think of K satisfying (1.5) is fixed and for $\epsilon > 0$, we write $\mathcal{I}_\epsilon = \mathcal{I}_{K,\epsilon}$.

We fix $y \in \mathbb{R}^N, y \neq 0$ small enough and define the sets

$$\mathcal{O} = \mathcal{O}(y) = \Omega \setminus \Omega_{|y|}; \quad \mathcal{U} = \mathcal{U}(y) = \{x \in \mathbb{R}^N : -|y| \leqslant d_\Omega(x) < |y|\}, \tag{4.2}$$

and the function

$$w(x) = w_{y,\epsilon}(x) = u_\epsilon(x+y) - u_\epsilon(x), \quad x \in \mathbb{R}^N. \tag{4.3}$$

Thanks to the linearity and invariance under translation of the operator $\mathcal{I}_\epsilon$, we see that w satisfies the inequalities

$$\begin{aligned} -\mathcal{I}_\epsilon(w, x) &\leqslant m_f(|y|) \quad \text{for } x \in \bar{\mathcal{O}} \\ w(x) &\leqslant C_0(\epsilon + |y|)^{\beta_0}\mathbf{1}_{\mathcal{U}}(x) \quad \text{for } x \in \bar{\mathcal{O}}^c, \end{aligned} \tag{4.4}$$

where m_f is a modulus of continuity for f, and the upper bound in $\bar{\mathcal{O}}^c$ follows by proposition 2.2 (see the begining of the proof of the existence part of theorem 1.3).

To give a precise reference of the barriers involved, we introduce some notation. We consider C_0, ϱ, β_0 as in proposition 2.2 and $\eta, \zeta : \mathbb{R}^N \to \mathbb{R}$ the following functions defined for each $x \in \mathbb{R}^N$

$$\begin{aligned} \zeta(x) &= \min\{(\epsilon + \varrho - |y|)^\epsilon, (\epsilon + d_{\mathcal{O}}(x))^\epsilon\}\mathbf{1}_{\bar{\mathcal{O}}}(x), \\ \eta(x) &= C_0(\epsilon + |y|)^{\beta_0}\mathbf{1}_{\mathcal{U}}(x). \end{aligned} \tag{4.5}$$

Recalling that m_f represents a modulus of continuity for f in $\bar{\Omega}$, by replacing it by $m_f(s) + Cs^\theta$ with $C > 0$, $0 < \theta < \beta_0$, we assume that

$$m_f(s) \geqslant Cs^\theta, \quad \text{for all } s \geqslant 0. \tag{4.6}$$

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

Proposition 4.2. *Let Ω be as in theorem 1.4 and assume (4.6) holds. Then, there exists $A > 0$ large enough such that for all $|y|, \epsilon$ small we have the function*

$$W(x) := \eta(x) + A m_f(|y|)\zeta(x), \quad x \in \mathbb{R}^N \tag{4.7}$$

satisfies the inequality

$$-\mathcal{I}_\epsilon(W, x) \geqslant m_f(|y|) \quad \text{for all } x \in \bar{\mathcal{O}}_{\bar{\varrho}}.$$

The proof of this result is accomplished by obtaining appropriate estimates for $\mathcal{I}_\epsilon(\zeta)$ and $\mathcal{I}_\epsilon(\eta)$, given in lemmas 4.3 and 4.5 below, respectively.

We start remarking that the relevant assumptions over Ω are also satisfied by the inner domain $\mathcal{O}$ in order to use previous results. In fact, it is straightforward that $\mathcal{O}$ is an epi-Lipschitz domain. To see that it satisfies the exterior power condition, we fix $x_0 \in \partial\mathcal{O}$, choose $\hat{x}_0 \in \partial\Omega$ such that $|x_0 - \hat{x}_0| = |y|$ and consider ν an exterior normal to $\partial\Omega$ at $\hat{x}_0$. Thus, by the analysis driving to (2.5), we see that in this setting, we necessarily have that $\nu = (\hat{x}_0 - x_0)/|\hat{x}_0 - x_0|$. We claim that $\partial\mathcal{O}$ satisfies the exterior power condition at x_0 with the same α, c, ν for which $\partial\Omega$ satisfies it at $\hat{x}_0$, but with R replaced by $R/4$ when $|y| \leqslant R/4$. In fact, traslating and rotating, we can assume $\hat{x}_0 = 0$ and $\nu = -e_N$. Hence, $x_0 = (0', |y|)$ and $\mathcal{O} \cap B_{R/2}(x_0) \subset \Omega \cap B_R(\hat{x}_0)$. Let $z \in \mathcal{O} \cap B_{R/2}(x_0)$ and define $z_y = z - |y|\nu$. It is direct to see that $z_y \in \Omega \cap B_R(\hat{x}_0)$ and that $z - x_0 = z_y - \hat{x}_0$. Then, using the exterior power condition for $\partial\Omega$ at $\hat{x}_0$, we get

$$(z - x_0) \cdot \nu = (z_y - \hat{x}_0) \cdot \nu < c|(z_y - \hat{x}_0)'|^\alpha = c|(z - x_0)'|^\alpha,$$

from which we get the exterior power condition for $\mathcal{O}$.

Next, we have

Lemma 4.3. *Under the assumptions of theorem 1.3, there exists $c^* > 0$ and $\bar{\varrho} > 0$ small such that for all $0 < \epsilon, |y| \leqslant \bar{\delta}$, we have*

$$-\mathcal{I}_\epsilon(\zeta, x) \geqslant c^*(\epsilon + d(x) - |y|)^{-2\sigma}, \quad \text{for all } x \in \bar{\mathcal{O}}_{\bar{\varrho}}. \tag{4.8}$$

Proof. Since $\mathcal{O}$ satisfies the assumptions that allow us to apply proposition 2.2 and noticing that $d(x) - |y| = d_{\mathcal{O}}(x)$ for all $x \in \bar{\mathcal{O}}$, we get

$$-\mathcal{I}_\epsilon(\zeta, x) \geqslant c^*(\epsilon + d(x) - |y|)^{\epsilon - 2\sigma},$$

for all x close to the boundary. Now, since the quantity $(\epsilon + d(x) - |y|)^\epsilon$ is uniformly bounded for all $\epsilon, |y| \in (0, 1)$, we conclude the result by shortening c^* if it is necessary. □

Before continuing, we prove a geometric lemma concerning the volume of the set $\mathcal{U}$.

Lemma 4.4. *Assume hypotheses of theorem 1.3 hold and let $\mathcal{U}$ be defined in (4.2). Then, there exists $C > 0$ depending only on $\partial\Omega$ such that*

$$\text{Vol}(\mathcal{U}) \leqslant C|y|.$$

Proof. Let $x \in \partial\Omega$ and consider $\phi = \phi_x : \mathbb{R}^{N-1} \to \mathbb{R}$ defined in (2.10) the mapping associated to the chart at x. Let $\xi = (\xi', \xi_N) \in B_{r_0/2} \cap \mathcal{U}$ and chose $\hat{\xi} = (\hat{\xi}', \hat{\xi}_N) \in \partial\Omega \cap B_{r_0}$ such that $d(\xi) = |\xi - \hat{\xi}|$. Define $\xi^* = (\xi', \hat{\xi}_N)$ and assume $\xi^* \neq \hat{\xi}$. Following the arguments given in lemma 4.4 of [9], we see that $|\hat{\xi}' - \xi'| \leqslant d(\xi) \leqslant r_0/2$ from which $|\hat{\xi}'| \leqslant r_0$, and using the Lipschitz continuity of ϕ, we get that

$$|\xi^* - (\xi', \phi(\xi'))| \leqslant \Lambda_0 |\hat{\xi}' - \xi'| \leqslant \Lambda_0 d(\xi),$$

and therefore, $|\xi_N - \phi(\xi')| \leqslant (1 + \Lambda_0) d(\xi)$. Notice that this inequality trivially holds when $\hat{\xi} = \xi^*$. Hence,

$$B_{r_0/2} \cap \mathcal{U} \subset \mathcal{C}_1 := \{(z', z_N) : z' \in B_{r_0/2}, |z_N - \phi(z')| \leqslant (1 + \Lambda_0)|y|\}$$

and therefore, by applying the change of variables theorem in the same fashion as in the estimate of $\tilde{I}_2$ in lemma 2.4, we conclude that

$$\mathrm{Vol}(\mathcal{U} \cap B_{r_0/2}) \leqslant C \int_{\mathcal{C}_1} \mathrm{d}z = C \int_{-(1+\Lambda_0)|y|}^{(1+\Lambda_0)|y|} \int_{B_{r_0/2}} \mathrm{d}z' \mathrm{d}s,$$

from which the result follows by the compactness of Ω. □

The above lemma is the key step to get the following

LEMMA 4.5. *Under the assumptions of theorem 1.3, where $c > 0$ such that for all $\epsilon, |y| > 0$ small enough, we have*

$$-\mathcal{I}_\epsilon(\eta, x) \geqslant -c\ m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}, \quad \text{for all } x \in \bar{\mathcal{O}}.$$

Proof. By its very definition, for $x \in \bar{\mathcal{O}}$, we have

$$\mathcal{I}_\epsilon(\eta, x) = C_0(\epsilon + |y|)^{\beta_0} \int_{\mathcal{U}-x} K_\epsilon^\sigma(z)\mathrm{d}z. \tag{4.9}$$

We start considering the case for interior points $x \in \mathcal{O}$. Consider $\bar{\varrho}$ as in lemma 4.3 and $x \in \mathcal{O} \setminus \mathcal{O}_{\bar{\varrho}/2}$. Then, there exists $c_1 > 0$ depending only on $\bar{\varrho}$ such that $K_\epsilon^\sigma(z)\mathbf{1}_{\mathcal{U}-x} \leqslant c_1$. Using this and lemma 4.4, we conclude that

$$\mathcal{I}_\epsilon(\eta, x) \leqslant c_2|y| \leqslant c_3 m_f(|y|), \tag{4.10}$$

where we have used (4.6) in the last inequality. Notice that the term $(\epsilon + d(x) - |y|)^{-2\sigma}$ is uniformly bounded below when $x \in \mathcal{O} \setminus \mathcal{O}_{\bar{\varrho}/2}$ and by this observation, we get the estimate asserted in the lemma for this case.

Now we deal with the case $x \in \mathcal{O}_{\bar{\varrho}/2}$. Without loss of generality, we can assume that $d(x) > |y|$ (or equivalently $d_{\mathcal{O}}(x) > 0$) since as it can be seen in [9], we conclude the limit case by a continuity argument which is not related to the regularity of the domain.

We require to precise some elements which are going to play an important role in computing $\mathcal{I}_\epsilon(\eta, x)$ for this case. For $z \in \mathbb{R}, r \in \mathbb{R}$, we define $F(z, r) = d(z) - r$ which is a Lipschitz function in its domain of definition, and notice that $\partial\Omega = \{z \in$

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

$\mathbb{R}^N : F(z,0) = 0\}$. In view of proposition 3.1, there exists $a > 0$ such that for each $\xi = (\xi', \xi_N) \in \partial\Omega$ there exists a rotation matrix $\mathcal{R} = \mathcal{R}_\xi$ for which

$$v_N > a \quad \text{for all } v = (v', v_N) \in \mathcal{R}^{-1}\partial d(\xi).$$

Hence, assuming that $\mathcal{R}$ is the identity matrix (otherwise we argue over the 'rotated' function $\tilde{F}(z,r) = d(\mathcal{R}z) - r$, but the conclusion follows exactly the same ideas) and by definition of partial generalized gradient, we conclude that $\partial_{z_N} F(\xi, 0) \in [a, 1]$. Therefore, by the Lipschitz Implicit Function theorem (see [**5**]), there exists $\bar{r} > 0$ and a unique Lipschitz continuous function $\phi_r(z') = \phi(z', r)$ defined for $z' \in B_{\bar{r}}(\xi') \subset \mathbb{R}^{N-1}$ and $|r| \leqslant \bar{r}$, such that

$$F(z', \phi_r(z'), r) = 0, \quad \text{for all } z' \in B_{\bar{r}}(\xi'),\ |r| \leqslant \bar{r}, \tag{4.11}$$

and such that $||\phi_r||_{Lip(B_{\bar{r}}(\xi'))}$ is bounded for all $r \in (-\bar{r}, \bar{r})$, by a universal constant just depending on Λ_0. This analysis can be done for each $\xi \in \partial\Omega$ and therefore, by compactness of $\partial\Omega$, we can assume $\bar{r} > 0$ is independent of ξ. Moreover, we can assume that $\bar{r} = r_0$ with r_0 defined in (2.10).

Now we address $\mathcal{I}_\epsilon(\eta, x)$. We chose $\hat{x} \in \partial\Omega$ a projection of x to the boundary and without loss of generality, we can shorten $\bar{\varrho}$ to get $d(x) \leqslant \bar{r}/4$. Then, we split the integral term as

$$\mathcal{I}_\epsilon(\eta, x) = C_0(\epsilon + |y|)^{\beta_0}\Big(I_1(x) + I_2(x)\Big)$$

where

$$I_1(x) := \int_{\mathcal{U}\setminus B_{\bar{r}/2}(\hat{x})} K_\epsilon^\sigma(z - x)\mathrm{d}z; \quad \text{and} \quad I_2(x) := \int_{\mathcal{U}\cap B_{\bar{r}/2}(\hat{x})} K_\epsilon^\sigma(z - x)\mathrm{d}z.$$

Notice that $x \in B_{\bar{r}/2}(\hat{x})$ by the above choice of $\bar{\varrho}$.

The same analysis leading us to (4.10) drives us to the estimate

$$I_1(x) \leqslant c_4(\epsilon + |y|)^{\beta_0}|y|, \tag{4.12}$$

where $c_4 > 0$ is a universal constant (depending on $\bar{r}$ and Γ).

Now we continue with the estimate of $I_2(x)$. We apply the coarea formula (see [**8**]) to arrive at

$$I_2(x) = C_0(\epsilon + |y|)^{\beta_0}\int_{-|y|}^{|y|}\int_{S_r} K_\epsilon^\sigma(z - x)\mathrm{d}S(z)\mathrm{d}r,$$

where $S_r = \{z \in B_{\bar{r}/2}(\hat{x}) \ : \ d(z) = r\}$ and dS denotes the $N-1$ dimensional Haussdorf measure.

We assume $\bar{R} = \bar{R}_{\hat{x}}$ given by proposition 3.1 is the identity matrix. Thus, the Implicit Function expression (4.11) is valid in this case and we can cast ϕ_r as a parametrization of the set S_r. Then, using ϕ_r as a change of variables, there exists a constant $C > 0$ depending only on Λ_0 such that

$$I_2(x) \leqslant C\ \Gamma\ C_0(\epsilon + |y|)^{\beta_0} \int_{-|y|}^{|y|} \int_{B_{\bar{r}/2}(\hat{x}')} \frac{\mathrm{d}z'}{\epsilon^{N+2\sigma} + |(z', \phi_r(z')) - x|^{N+2\sigma}} \mathrm{d}r. \quad (4.13)$$

Now we estimate the integral in z' dividing it in two parts. First, we consider

$$I_{21} := \int_{B_{\bar{r}}(\hat{x}') \cap B_{d(x)-r}(x')} \frac{\mathrm{d}z'}{\epsilon^{N+2\sigma} + |(z', \phi_r(z')) - x|^{N+2\sigma}}.$$

Noting that for each z such that $d(z) = r$, we have $|x - z| \geqslant d(x) - r$, we can write

$$\begin{aligned} I_{21} &\leqslant \int_{B_{d(x)-r}(x')} \frac{\mathrm{d}z'}{\epsilon^{N+2\sigma} + (d(x) - r)^{N+2\sigma}} \\ &\leqslant C(\epsilon + d(x) - r)^{-(N+2\sigma)} \int_{B_{d(x)-r}(x')} \mathrm{d}z', \end{aligned}$$

that is

$$I_{21} \leqslant C(\epsilon + d(x) - r)^{-(1+2\sigma)}. \quad (4.14)$$

On the contrary, we denote

$$I_{22} := \int_{B_{\bar{r}/2}(\hat{x}') \setminus B_{d(x)-r}(x')} \frac{\mathrm{d}z'}{\epsilon^{N+2\sigma} + |(z', \phi_r(z')) - x|^{N+2\sigma}}.$$

Then, applying the direct inequality $|(z', \phi_r(z')) - x| \geqslant |z' - x'|$, followed by the change of variables $\xi' = x' - z'$, we can write

$$I_{22} \leqslant \int_{B_{\bar{r}} \setminus B_{d(x)-r}} \frac{\mathrm{d}z'}{\epsilon^{N+2\sigma} + |\xi'|^{N+2\sigma}}.$$

Using spherical coordinates and defining $\tau = \epsilon/(\epsilon + d(x) - r)$, we get

$$\begin{aligned} I_{22} &\leqslant C \int_{d(x)-r}^{\bar{r}} \frac{t^{N-2}\mathrm{d}t}{\epsilon^{N+2\sigma} + t^{N+2\sigma}} \\ &\leqslant C(\epsilon + d(x) - r)^{-(1+2\sigma)} \int_{1-\tau}^{+\infty} \frac{t^{N-2}\mathrm{d}t}{\tau^{N+2\sigma} + t^{N+2\sigma}}, \end{aligned}$$

and since the last integral is uniformly bounded, independent of τ, we conclude the same estimate for I_{22} as in (4.14). Using this in (4.13), we conclude

$$I_2(x) \leqslant C(\epsilon + |y|)^{\beta_0} \int_{-|y|}^{|y|} (\epsilon + d(x) - r)^{-(1+2\sigma)} \mathrm{d}r. \quad (4.15)$$

At this point, we consider two cases: if $\epsilon \leqslant |y|$, by integration and assumption (4.6) we get

$$I_2(x) \leqslant C|y|^{\beta_0}(\epsilon + d(x) - |y|)^{-2\sigma} \leqslant Cm(|y|)(\epsilon + d(x) - |y|)^{-2\sigma},$$

and when $\epsilon > |y|$, we note that since $d(x) > |y|$, we have $\epsilon + |y| \leqslant 2(\epsilon + d(x) - |y|)$ and therefore, from (4.15), we see that

$$\begin{aligned} I_2(x) &\leqslant C(\epsilon + d(x) - |y|)^{\beta_0} \int_{-|y|}^{|y|} (\epsilon + d(x) - |y|)^{-(1+2\sigma)} \mathrm{d}r \\ &\leqslant C(\epsilon + d(x) - |y|)^{-1-2\sigma+\beta_0} |y| \\ &\leqslant C(\epsilon + d(x) - |y|)^{-2\sigma+\beta_0-\alpha} |y|^{\alpha} \\ &\leqslant Cm(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}, \end{aligned}$$

where we used again (4.6). From here and (4.12), we conclude the proof. □

Proof of Proposition 4.2*.* By linearity of $\mathcal{I}_\epsilon$ and using lemmas 4.3 and 4.5, we see that

$$-\mathcal{I}_\epsilon(W, x) \geqslant (-c + Ac^*)m(|y|)(\epsilon + d(x) - |y|)^{-2\sigma}, \quad \text{for all } x \in \mathcal{O}.$$

Taking $A > 0$ large enough in terms of the data (but independent of ϵ or y), we conclude the result. □

Now we are ready to provide the proof of the compactness of the family of solutions of the approximating problems.

Proof of Theorem 4.1*.* Noticing that the quantity $(\epsilon + \varrho/2)^{\epsilon}$ is uniformly bounded below by a strictly positive constant as $\epsilon \to 0$, in view of the previous proposition, arguing as in the proof of existence of theorem 1.3, there exist $C_1, C_2 > 0$ not depending on $\epsilon, |y|$ such that the function

$$C_1 \min\{W(x), C_2 m_f(|y|)\}, \quad x \in \mathbb{R}^N$$

is a supersolution to (4.4). Hence, by comparison principle, we get that $w \leqslant W$ in $\bar{\mathcal{O}}$ and therefore, we get the inequality

$$u_\epsilon(x + y) - u(x) \leqslant c_1 A m(y) \quad \text{for all } x \in \bar{\mathcal{O}},$$

which establishes an upper bound for the modulus of continuity of u_ϵ. A similar lower bound can be given. This concludes the proof. □

5. The non-linear problem

Here we briefly describe the main directions to get the results for the non-linear case stated in theorem 1.5. It is going to be convenient for the forthcoming analysis to introduce the *extremal Pucci operators* associated witth the family $\mathcal{K}$ satisfying (1.5). For $\epsilon \geqslant 0$, we define

$$\mathcal{M}_\epsilon^+(u, x) := \sup_{K \in \mathcal{K}} \mathcal{I}_{\epsilon,K}(u, x), \quad \text{and} \quad \mathcal{M}_\epsilon^-(u, x) := \inf_{K \in \mathcal{K}} \mathcal{I}_{\epsilon,K}(u, x),$$

where the linear nonlocal operator $\mathcal{I}_{\epsilon,K}$ is given in (1.7) and (1.8).

The following properties concerning the extremal operators are useful in what follows: recalling F_ϵ defined in (1.9), for each $\epsilon \geqslant 0$, we see that

$$\mathcal{M}_\epsilon^-(u - v) \leqslant F_\epsilon(u) - F_\epsilon(v) \leqslant \mathcal{M}_\epsilon^+(u - v). \tag{5.1}$$

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

5.1. Well-posedness

Using (5.1) with $v = 0$, it is easy to see that a supersolution to the problem

$$-\mathcal{M}_\epsilon^+(u) = f \quad \text{in } \Omega; \qquad u = 0 \quad \text{in } \Omega^c, \tag{5.2}$$

is a supersolution to the problem (1.11). Hence, it is enough to construct an upper barrier for (5.2) to get an upper barrier for (1.11), and for this task the key computations are related to the corresponding nonlinear estimate given by proposition 2.2. In order to handle the supremum in $\mathcal{M}_\epsilon^+$, we use the inequality $K \leqslant \Gamma$ in the estimates I_0, I_1 and I_3, and the inequality $\gamma \leqslant K$ in the estimate of I_2 in proposition 2.2. It is important to notice that the choice of the power profile β_0 of the barrier in this proposition is determined in the estimates concerning I_3. Since our analysis of the linear problem is carried out for a general kernel K, these estimates are also valid for the non-linear case and therefore, the same β_0 is obtained for the non-linear case.

Then, in the proof of Theorem 1.3, we replace $\tilde{C}$ by $\tilde{C}/\lambda$ and fixing $C_1, C_2 > 0$ adequate to the mentioned changes, we get the upper barrier. An analogous procedure can be used to get lower barriers for problem (1.11) by addressing a problem involving $\mathcal{M}_\epsilon^-$.

Once we get the barriers, Perron's method applies to get the result. Standard viscosity comparison principle drives to the uniqueness.

5.2. Regularity

Concerning interior regularity, the nonlinear nature of F can be managed in the proof of Theorem 2.5 through the following fact: For each $a > 0$, u and x, there exists K^* in the family of kernels $\mathcal{K}$ such that

$$F(u, x) \leqslant \mathcal{I}_{K^*}(u, x) + a.$$

Hence, considering u the viscosity solution to (1.10) given by the previous procedure, and assuming that $\mathrm{osc}_{B_1}(u) > 0$ (otherwise the result follows), the contrast of the viscosity evaluations of u at $\bar{x}$ (as subsolution) and of u at $\bar{y}$ (as supersolution) represented by formula (2.14) in the proof of Theorem 2.5 can be formally written this time as

$$-\mathcal{I}_{K^*}(u, \bar{x}) + \mathcal{I}_{K^*}(u, \bar{y}) \leqslant f(\bar{x}) - f(\bar{y}) + \mathrm{osc}_{B_1}(u),$$

for some K^* depending on $\mathrm{osc}_{B_1}(u), \bar{x}$ and u. From this, we control the positive and negative contributions of the different parts of the splitting of the integral terms arising in the left-hand side of the above inequality with quantities depending on the ratio $\Gamma/\gamma \in (0, +\infty)$ to get the interior regularity. The regularity up to the boundary is basically accomplished by the positive homogeneity of the operator F.

5.3. Approximation Scheme

We start recalling that the properties coming from the Lipschitz regularity of $\partial\Omega$ given in § 3 are independent of the nature of the equations we address. Thus, the analysis of the nonlinear problem is circumscribed to § 4.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

In this case, given $u_\epsilon \in C(\bar{\Omega})$ solution to (1.11) and defining w exactly as in (4.3), the invariance under translation of the operator F together with (5.1), we see that w solves

$$-\mathcal{M}_\epsilon^+(w) \leqslant m_f(|y|) \quad \text{in } \bar{\mathcal{O}}; \quad w \leqslant \tilde{C}_0(\epsilon + |y|)^{\tilde{\beta}_0} \quad \text{in } \bar{\mathcal{O}}^c, \tag{5.3}$$

for some $\tilde{C}_0, \tilde{\beta}_0 > 0$ depending on the data (as we discussed above, we can take $\tilde{\beta}_0 = \beta_0$ given in proposition 2.2).

From here, we follow exactly the arguments given in § 4 to construct an upper barrier to (5.3) using W defined in (4.7) with $A > 0$ depending on the data and the ratio Γ/γ.

Acknowledgements

Part of this work was developed while the second author visited the department of mathematics of Jianxi Normal University of China and Shanghai New York University of China. He would like to express a deep gratitude to Huyuan Chen and Ying Wang for the hospitality.

P.F. was partially supported by BASAL-CMM projects PFB 03. E. T. was partially supported by Conicyt PIA Grant No. 79150056 and Millennium Nucleus Center for Analysis of PDE NC130017.

References

1 G. Barles, E. Chasseigne and C. Imbert. On the Dirichlet problem for second order elliptic integro-differential equations. *Indiana Univ. Math. J.* **57**(1) (2008), 213–246.

2 G. Barles, E. Chasseigne and C. Imbert. Hölder continuity of solutions of second-order nonLinear elliptic integro-differential equations. *J. Eur. Math. Soc. (JEMS)* **13**(1) (2011), 1–26.

3 J. C. Bellido and C. Mora–Corral. Existence for nonlocal variational problems in peridynamics. *SIAM J. Math. Anal.* **46**(1) (2014), 890–916.

4 L. Caffarelli and L. Silvestre. Regularity theory for nonlocal integro-differential equations. *Comm. Pure Appl. Math* **62**(5) (2009), 597–638.

5 F. H. Clarke. Optimization and nonsmooth analysis. *Classics in Appl. Math.* **5** (1990).

6 A. Di Castro, T. Kuusi and G. Palatucci. Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33**(5) (2016), 1279–1299.

7 E. Di Neza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional sobolev spaces. *Bull. Sci. Math.* **136**(5) (2012), 521–573.

8 L. C. Evans and R. Gariepy. *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics, (Boca Raton, Fl: CRC Press, 1992).

9 P. Felmer and E. Topp. Uniform equicontinuity for a family of zero order operators approaching the fractional Laplacian. *Comm. Partial Diff. Eq.* **40**(9) (2015), 1591–1618.

10 M. Felsinger, M. Kassmann and P. Voigt. The Dirichlet problem for nonlocal operators. *Math. Z.* **279** (2015), 779–809.

11 H. Ishii and G. Nakamura. A class of integral equations and approximation of p-Laplace equations. *Calc. Var. PDE* **37** (2010), 485–522.

12 R. T. Rockafellar. Clarke's tangent cones and the boundaries of closed sets in $\mathbb{R}^n$. *Nonlinear. Anal. T.M.A.* **3** (1979), 145–154.

13 R. T. Rockafellar and R. J. B. Wets. *Variational Analysis* (Springer-Verlag, 1998).

14 X. Ros–Oton and J. Serra. The Dirichlet Problem for the fractional laplacian: regularity up to the boundary. *Journal de Mathématiques Pures et Appliquées* **101**(3) (2014), 275–302.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press

15 R. Servadei and E. Valdinoci. Weak and viscosity solutions of the fractional Laplace equation. *Publ. mat.* **58**(133) (2014), 154.

16 R. Servadei and E. Valdinoci. The Brezis–Niremberg result for the fractional Laplacian. *Trans. Am. Math. Soc.* **367**(1) (2015), 67–102.

17 L. Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana U. Math. J.* **55**(3) (2006), 1155–1174.

https://doi.org/10.1017/prm.2018.38 Published online by Cambridge University Press