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FORMAL REPRESENTATIONS OF DEPENDENCE AND GROUNDEDNESS

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Abstract. We study, in an abstract and general framework, formal representations of dependence and groundedness which occur in semantic theories of truth. Our goals are (a) to relate the different ways in which groundedness is defined according to the way dependence is represented and (b) to represent different notions of dependence as instances of a suitable generalisation of the mathematical notion of functional dependence.

§1. Introduction. The technical notions of *semantic dependence* and *semantic groundedness* crucially recur in many works on languages which contain their own truth predicates. Groundedness is intended to capture what is characteristic of sentences whose truth values seem to be determined, one way or another, by non-semantic states of affairs, whereas dependence is invoked to make sense of apparent relationships between the truth values the sentences bear. Moreover, groundedness is often explicated in terms of dependence.

We can find, in the logical literature on truth, several attempts to formalise these notions, making them operate in the context of some formal theory of self-referential truth. The main theme of the present article is to abstract some general mathematical properties shared by these formal counterparts of dependence and groundedness and to put them into a unified and general framework.

This work is intended to serve two purposes: one is helping us to separate, in the logical studies on truth involving dependence and/or groundedness, what is specific to truth from what directly relies on more general mathematical facts; the other is facilitating the comparison of different formal definitions intended to capture the informal notions of dependence and groundedness. Concerning this second aim, we can observe that, in the literature on truth, these two notions are formalised under a variety of terminology and notations that makes it difficult, *prima facie*, to distinguish them from an extensional point of view: Accordingly, this article is also conceived to serve the logico-philosophical community by collecting in one place and in a systematic way some mathematical "equivalences" between different approaches that often in the literature are assumed only implicitly.

The article is split into two parts, proceeding from the more abstract to the lesser. In the first part (§2), I will study the relationship between *groundedness*, defined in terms of a notion of dependence represented either by a set of rules or by a monotone operator, and *well-foundedness*, with respect to a notion of dependence represented by either a binary relation or a graph mapping. In the second part (§3) I will investigate the possibility of representing the abstract notion of dependence dealt with in the first part as a suitable

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generalisation of the notion of "functional dependence" usually employed in computer science and other disciplines.

1.1. General mathematical preliminaries. Throughout the article, "iff" abbreviates "if and only if." The symbol $=_{\mathrm{Df}}$ means that identity is stated as a definition. Proofs are ended by the symbol $\square$. Subproofs inside one main proof are ended by the symbol $\dashv$.

We refer to Lévy (1979) for standard set-theoretic notation. Let X, Y be sets. $\mathcal{P}(X)$ denotes the set of all subsets of X, $X \times Y$ denotes the cartesian product of X and Y, ${}^{X}Y$ denotes the set of all functions from X to Y. The *identity function on* X, denoted by id_X, is the function $f : X \to X$ such that $f(x) = x$ for every $x \in X$. The *domain* of a function f is denoted by $\mathrm{dom}(f)$, its *range* by $\mathrm{ran}(f)$. If f is a function and $X \subseteq \mathrm{dom}(f)$, $f \restriction X$ denotes the *restriction of f to X*. Ordinal numbers are denoted by the initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta$. The set of all natural numbers is denoted by ω and the class of all ordinal numbers is denoted by On.

An *inductive space* (Yablo, 1982, p. 119) is a triple (U, P, J), where U is an arbitrary set called the *universe*; P is an *inductive* family of subsets of U, namely, a family such that (a) the empty set is in P, and (b) the union of any increasing sequence of members in P is in P; and J is a *monotone* operator on (U, P), i.e., a function from P to P such that for any $S, S' \in P$, $S \subseteq S' \Rightarrow J(S) \subseteq J(S')$. A subset S of U is *J-sound* iff $S \subseteq J(S)$, and is a *fixed point* of J iff $S = J(S)$.

Yablo (1982, p. 120) proves that for every J-sound subset S of U there exists a fixed point S' of J such that $S \subseteq S'$. By the monotonicity of J, the construction of S' in the proof also shows that S' is the *least fixed point of J above* S, i.e., for any other fixed point S'' of J, if $S \subseteq S''$ then $S' \subseteq S''$. We denote the least fixed point of J above S by $\mathrm{lfp}(J, S)$. A dual proof shows that if S is a subset of U such that $J(S) \subseteq S$ then there exists the *greatest fixed point of J below* S, which we denote by $\mathrm{gfp}(J, S)$.

For each J-sound $S \in P$, define by transfinite recursion the following sequence of elements of P: $\langle J^S_\alpha \mid \alpha \in \mathrm{On}\rangle$, where $J^S_0 = S$, $J^S_{\alpha+1} = J(J^S_\alpha)$, $J^S_\delta = \bigcup\{J^S_\alpha \mid \alpha < \delta\}$ for δ limit. We call the sequence $\langle J^S_\alpha \mid \alpha \in \mathrm{On}\rangle$ the *transfinite iteration of J starting with S*. The least fixed point of J above the empty set is also the least fixed point of J, and will be denoted by $\mathrm{lfp}(J)$. Accordingly, the sequence $\langle J_\alpha \mid \alpha \in \mathrm{On}\rangle$ will denote the transfinite iteration of J starting with the empty set. It follows that $\mathrm{lfp}(J) = J_\alpha$ for some α limit. For every $x \in \mathrm{lfp}(J)$ the *J-rank* of x is the least $\beta \in \mathrm{On}$ such that $x \in J_\beta$. The rank of x is necessarily a successor ordinal.

We will work in an abstract setting in which all mathematical objects will be considered relatively to a fixed, nonempty set A (subsets of A, functions from A, etc.). In the examples taken from the literature on formal theories of truth, A will be the set of all sentences of some first-order language augmented with a truth predicate. However, in this article, we are only concerned with properties which do not depend on the internal structure of A, so we simply take A to be an arbitrary nonempty set.

We will reserve the following letters, variously decorated, to range over the corresponding classes of objects "from A":

- x, y, z, u, v, w for elements of A.
- X, Y, Z for subsets of A, i.e., elements of $\mathcal{P}(A)$.
- f, g, h for functions with domain A, i.e., elements of ${}^{A}B$ for some set B.
- p, q, r for partial functions on A, i.e., functions with domain a subset of A.
- s for sequences of elements of A, i.e., elements of ${}^{\omega}A$.
- R, Q for binary relations on A, i.e., subsets of $A \times A$.

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- D for domain functions on A, i.e., elements of ${}^{A}\mathcal{P}(A)$.
- G for (directed) graphs on A, i.e., pairs (X, R), where $X \subseteq A$ and $R \subseteq X \times X$.
- Θ for graph mappings, i.e., mappings of elements of A to graphs on A.
- Δ for operators on A, i.e., functions from $\mathcal{P}(A)$ to $\mathcal{P}(A)$.
- Φ for sets of rules, i.e., elements of $\mathcal{P}(A) \times A$.
- $\mathcal{X}$ for families of subsets of A.
- $\mathcal{R}$, $\mathcal{Q}$ for families of binary relations on A.
- $\mathcal{D}$ for families of domain functions on A.
- $\mathcal{G}$ for families of graphs on A.
- $\mathcal{F}$, $\mathcal{H}$ for families of partial functions on A.
- Λ for functions with domain a family of partial functions on A.
- Γ for operators $\Gamma : \mathcal{F} \to \mathcal{F}'$ between families of partial functions on A.

We will define several maps between subclasses of the objects listed above. Each map will be denoted by the constant symbol K indexed by a natural number. We will adopt the following convention: Given a map of the form $\mathsf{K} : B \mapsto C_B$ we will often omit the constant K, denoting the map and its result by C_B only. For any pair of classes of objects from A, there will be at most one map labelled with an indexed K, so that from the particular letters C and B, taken from the above list, it will be easy to recover the map $\mathsf{K} : B \mapsto C_B$ itself. For instance, the notation $R \mapsto \Delta_R$ will be reserved for the unique map from binary relations on A to monotone operators on A which is labelled with K_{11} in the article.

Most results will take the form either of a "natural correspondence" theorem or of a "natural reduction" theorem, in the following sense. Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are subclasses of objects from A endowed with functions $F : \mathcal{B} \to \mathcal{A}$ and $G : \mathcal{C} \to \mathcal{A}$.

In a "natural correspondence" theorem we prove that there exists a bijection $H : \mathcal{B} \to \mathcal{C}$ such that, for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$ which *correspond to each other*, namely, such that $C = H(B)$, we have $F(B) = G(C)$. Moreover, we prove the existence of such H by explicitly exhibiting two maps $K : B \mapsto C_B$ and $K' : C \mapsto B_C$ which are *inverse each other*, namely, such that if $C = C_B$ then $B_C = B$, and if $B = B_C$ then $C_B = C$.

In a "natural reduction" theorem, we show that for every $B \in \mathcal{B}$ there exists $C \in \mathcal{C}$ such that $F(B) = G(C)$ by explicitly exhibiting a map $K : B \mapsto C_B$ such that $F(B) = G(C_B)$ for every $B \in \mathcal{B}$.

In Appendix B, Table 1 lists all maps enumerated by a "K" label, and Table 2 and 3 list all correspondence and reduction theorems, respectively.

Further mathematical notation and terminology will be explained at the beginning of the section where it is used the first time.

§2. Groundedness from dependence. In the literature on truth, we can find several formalisms employed to represent an intuitive notion of dependence and to define a corresponding notion of groundedness. Let us quickly review some of these options.

Herzberger (1970a; 1970b) defines his notion of *groundlessness* (the negation of groundedness) starting with that of *domain* (function), a function which associates to every sentence ϕ the set of all sentences ϕ depends on.

Kripke (1975) defines his own notion of groundedness directly in terms of the least fixed point of a monotone operator on partial sets of sentences.

Bolander (2002) formalises semantic dependence as a binary relation on the set of all sentences, and, accordingly, takes groundedness to be well-foundedness.

Leitgeb (2005) starts with a notion of dependence expressed as a binary relation between sets of sentences and sentences and derives groundedness in terms of the least fixed point of the associated monotone operator.

Beringer and Schindler (2016) depict dependence by means of graph mappings, namely, functions which associate to every sentence a graph, and identify the groundedness of a sentence with the well-foundedness of the associated graph.

In this part of the article (§2), we study in our abstract setting the formal notions of dependence and groundedness taken from the abovementioned examples, with the exception of the Kripkean notion of grounding, which will be dealt with in the second part (§3), in the context of Kripkean valuation systems.

In §2.1, we show that the notions of groundedness defined from notions of dependence represented by binary relations, domain functions or graph mappings are in a sense all equivalent to the set-theoretic notion of well-foundedness. In §2.2, we recall from Aczel's work that the representations of dependence and groundedness in terms of "sets of rules" and of monotone operators are equivalent, and we reserve the technical term "groundedness" for these two modes of representations, with the aim, in the subsequent sections, of contrasting "(formal) groundedness" with "well-foundedness." This aim will be made more precise in §2.3, and will be achieved in two steps, in §2.4 for the restricted notion of "essential-dependence," and in §2.5 for the general case.

2.1. Binary relations, domain functions and graph mappings. In this section, I will consider three substantially equivalent ways of representing a notion of dependence between the elements of a given set: binary relations, domain functions, and graph mappings.

2.1.1. Binary relations. One possible way of formalising an intuitive notion of dependence between elements of A is by means of a binary relation $R \subseteq A \times A$, reading "$(y, x) \in R$" as something like "y contributes to determining the value of x." An example of a notion of dependence represented by a binary relation can be found in Bolander (2002, p. 45). When a notion of dependence is represented by a binary relation, the natural candidate as the formal counterpart of the informal notion of "groundedness" is "well-foundedness." Let us just recall some fundamentals about the set-theoretic notions of binary relations and well-foundedness.

Let $R \subseteq X \times X$ be any binary relation. The *restriction* of R to a subset $Y \subseteq X$, denoted by $R \restriction Y$, is the binary relation $R \cap (Y \times Y)$. R is *transitive* iff $(z, y) \in R \wedge (y, x) \in R \Rightarrow (z, x) \in R$, for all $x, y, z \in X$. R^* denotes the *transitive closure* of R, namely, the least transitive binary relation on X which extends R (Lévy, 1979, pp. 59–60). For $x \in X$, we denote by x^R the set of all *R-predecessors of x in X*, namely, $x^R = \{y \in X \mid (y, x) \in R\}$.

We say that a subset Y of X is *R-left-closed*[1] iff

$$\forall y \in Y \, \forall z \, ((z, y) \in R \Rightarrow z \in Y).$$

For any $R \subseteq A \times A$ and $Y \subseteq A$, Y is R-left-closed iff Y is R^*-left-closed. The *R-closure of x*, denoted by $\overline{x^R}$ is the smallest R-left-closed subset of X containing x as an element. For any $x \in A$, $\overline{x^R} = \{x\} \cup x^{R^*}$. For $y, x \in X$, $(y \in \overline{x^R} \Rightarrow \overline{y^R} \subseteq \overline{x^R})$.

Following Aczel (1977, p. 743), we say that $x \in X$ is *R-well-founded*[2] iff there is no infinite descending R-chain of elements of X starting with x, namely, iff there is no infinite

[1] This notion and its related properties are the obvious counterpart of those which in (Lévy, 1979, pp. 61–62) are formulated in terms of *R-right*-closure.

[2] Actually, this notion is called "R-groundedness" by many authors. We assume the Axiom of Dependent Choice throughout the article, so that R-groundedness is equivalent to the set-theoretic

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sequence $\langle x_n \mid n \in \omega \rangle$ of elements of X such that (a) $(x_{n+1}, x_n) \in R$ for every $n \in \omega$, and (b) $x_0 = x$. The *well-founded part* of X (with respect to R), denoted by $\mathsf{W}(X, R)$, is the set of all R-well-founded elements of X. A binary relation $R \subseteq X \times X$ is *well-founded on* X iff $\mathsf{W}(X, R) = X$: We abbreviate this notion by $\mathrm{Wf}(R, X)$.

2.1.2. Domain functions. A straightforwardly equivalent way of formalising dependence, between the elements of a set A, is by means of a function $D : A \to \mathcal{P}(A)$. Functions of this sort are considered, for instance, in Herzberger (1970a, p. 148) and called *domain* functions: each domain function D associates to every $x \in A$ a subset $D(x) \subseteq A$ which constitutes the "domain of x," namely, the set of all elements of A which contribute to determine the value of x. Given a domain function $D : A \to \mathcal{P}(A)$, Herzberger (1970b) defines $x \in A$ to be "groundless" iff "x is the first member of some infinite sequence of [elements], each of which belongs to the domain of its predecessor."

REMARK 2.1 (Correspondence Theorem I). *There exists a one-to-one correspondence between domain functions $D : A \to \mathcal{P}(A)$ and binary relations $R \subseteq A \times A$ given by the following maps:*[3]

$\mathsf{K}_1 : D \mapsto R_D$*, defined by*

$$(y, x) \in R_D =_{\mathsf{Df}} y \in D(x),$$

for every $x, y \in A$, and,

$\mathsf{K}_2 : R \mapsto D_R$*, defined by*

$$D_R(x) =_{\mathsf{Df}} x^R,$$

for every $x \in A$.

Moreover, whenever D and R correspond to each other, an element $x \in A$ is D-groundless in Herzberger's sense iff is not R-well-founded.

The proof of Correspondence Theorem I is straightforward.

2.1.3. Graph mappings. A third way of representing dependence between elements of A is by means of "graph mappings."

A (directed) *graph* on A can be identified with a pair $G = (\mathsf{V}(G), \mathsf{E}(G))$ where $\mathsf{V}(G)$ is a subset of A, called the set of all *vertices* of G, and $\mathsf{E}(G)$ is a binary relation on $\mathsf{V}(G)$, i.e., $\mathsf{E}(G) \subseteq \mathsf{V}(G) \times \mathsf{V}(G)$, called the set of all *edges* of G. We write, for short, $x \in G$ for $x \in \mathsf{V}(G)$ and $(y, x) \in G$ for $(y, x) \in \mathsf{E}(G)$. G^* denotes the graph $(\mathsf{V}(G), (\mathsf{E}(G))^*)$. A graph G is *well-founded*, iff $\mathsf{E}(G)$ is well-founded on $\mathsf{V}(G)$: We abbreviate this notion by $\mathrm{Wf}(G)$.

A *graph mapping* is a map $\Theta : x \mapsto \Theta(x)$, where $x \in A$ and $\Theta(x)$ is a graph *on* A, namely, $\mathsf{V}(\Theta(x))$ is a subset of A. We say that an element $x \in A$ is Θ-*well-founded* iff the graph $\Theta(x)$ is well-founded.

In what follows we will establish a natural correspondence between the class of all binary relations on A and the following subclass of the class of all graph mappings on A:

DEFINITION 2.2. *We say that a graph mapping Θ is* coherent *iff it satisfies the following conditions*:

definition of R-well-foundedness. We rest on this latter terminology in order to avoid confusion with "groundedness with respect to a monotone operator," which will be introduced in the next section.

[3] An instance of the map $\mathsf{K}_1 : D \mapsto R_D$ is in Bolander (2002, p. 45), except that Bolander actually defines the converse relation of our R_D.

1. $x \in \Theta(x)$.
2. $y \in \Theta(x) \wedge y \neq x \Rightarrow (y, x) \in (\Theta(x))^*$.
3. $\exists z\, (y, x) \in \Theta(z) \Rightarrow \forall z'\, (x \in \Theta(z') \Rightarrow (y, x) \in \Theta(z'))$.

PROPOSITION 2.3 (Correspondence Theorem II). *There exists a one-to-one correspondence between coherent graph mappings $\Theta : x \mapsto \Theta(x)$ and binary relations $R \subseteq A \times A$ given by the following maps*:

$\mathsf{K}_3 : \Theta \mapsto R_\Theta$, *defined by*

$$R_\Theta = \bigcup \{\mathsf{E}(\Theta(x)) \mid x \in A\},$$

and

$\mathsf{K}_4 : R \mapsto \Theta_R$, *defined by*

$$\Theta_R(x) = (\overline{x^R}, R \restriction \overline{x^R}),$$

for every $x \in A$.

Moreover, whenever Θ and R correspond to each other, an element $x \in A$ is R-well-founded iff is Θ-well-founded.

Since the proof of Correspondence Theorem II is a bit long but rather uninformative, we confine it in Appendix A.

An example of the use of graph mappings applied to a notion of dependence is in Beringer & Schindler (2016, Definition 3, p. 6). Actually, the two authors use an instance of the map $\mathsf{K}_5 : D \mapsto \Theta_D$ which associates to each domain function $D : A \to \mathcal{P}(A)$ a graph mapping Θ_D defined by

$$\Theta_D(x) = (D^{\mathsf{V}}(x), D^{\mathsf{E}}(x)),$$

for every $x \in A$, where (a) $D^{\mathsf{V}}(x)$ is the least subset X of A such that $x \in X$ and, for every $y \in X$, $D(y) \subseteq X$; and[4] (b) $D^{\mathsf{E}}(x) = \{(y, z) \in D^{\mathsf{V}}(x) \times D^{\mathsf{V}}(x) \mid y \in D(z)\}$.

The diagram constituted by the maps $\mathsf{K}_1 : D \mapsto R_D$, $\mathsf{K}_4 : R \mapsto \Theta_R$, and $\mathsf{K}_5 : D \mapsto \Theta_D$ commutes,[5] in the sense that for every domain function D, $\Theta_{R_D} = \Theta_D$, so that we obtain,[6] as a corollary of Correspondence Theorems I and II, the following:

COROLLARY 2.4 (Correspondence Theorem III). *There exists a one-to-one correspondence between coherent graph mappings $\Theta : x \mapsto \Theta(x)$ and domain functions $D : A \to \mathcal{P}(A)$ given by the following maps*:

$\mathsf{K}_5 : D \mapsto \Theta_D$.

$\mathsf{K}_6 : \Theta \mapsto D_\Theta$, *defined by*

$$D_\Theta(x) = x^{\mathsf{E}(\Theta(x))},$$

for every $x \in A$.

Moreover, whenever Θ and D correspond to each other, an element $x \in A$ is D-groundless iff it is not Θ-well-founded.

In the light of the correspondence Theorems I–III, in what follows we will loosely speak of "well-foundedness" to collectively refer to the equivalent notions introduced

[4] I only depart from Beringer & Schindler's notation in that in their article they actually define the converse relation of our $D^{\mathsf{E}}(x)$.

[5] Lemma 4.8, proved in Appendix A.

[6] A proof of Correspondence Theorem III will be given in Appendix A.

in this section, taking binary relations as our primary representatives for the notions of dependence and well-foundedness which could be equivalently defined by using either domain functions or coherent graph mappings.

2.2. Monotone operators and sets of rules. A notion of dependence on a set A can also be represented either by means of a binary relation Φ between subsets X of A and members y of A, to be read "y Φ-depends on X," or by means of an operator $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ which assigns to every subset X of A the set $\Delta(X)$ of all members of A which "Δ-depend on X." In this section, we will investigate the relationship between these two modes of representations, starting with the latter.

2.2.1. Monotone operators. In most cases, the intended interpretation of "$y \in \Delta(X)$" is that "to know the values assigned to each element in X is sufficient to determine the value of x." A desirable property of Δ, which intuitively follows from its intended interpretation, is *monotonicity*, namely, for all $X, Y \subseteq A$,

$$X \subseteq Y \Rightarrow \Delta(X) \subseteq \Delta(Y).$$

A monotone operator can be seen as an instance of the more general notion of inductive space recalled in the Introduction: Every monotone operator Δ on A can be identified with the inductive space $(A, \mathcal{P}(A), \Delta)$. The family $\mathcal{P}(A)$ of subsets of A is obviously an inductive family. Therefore we can use the same notation and terminology introduced for the inductive spaces also for the monotone operators on A: A subset $X \subseteq A$ is Δ-*sound* iff $X \subseteq \Delta(X)$, for every Δ-sound subset of A there exists the *least fixed point of* Δ *above* X denoted by $\mathrm{lfp}(\Delta, X)$, $\mathrm{lfp}(\Delta)$ denotes the least fixed point of Δ, etc.

Let us recall here some definitions about monotone operators that will be used later in the article.

DEFINITION 2.5. *We say that a monotone operator Δ on A is* surjective *iff for every $x \in A$ there exists $X \subseteq A$ such that $x \in \Delta(X)$.*

Observe that, from monotonicity, it follows that a monotone operator Δ on A is surjective iff $A \subseteq \Delta(A)$.

DEFINITION 2.6. *A monotone operator Δ on A is a* closure operator *iff Δ satisfies the following*:

1. Progressivity, *i.e., $X \subseteq \Delta(X)$, for every $X \subseteq A$, and*
2. Transitivity, *i.e., $\Delta(\Delta(X)) \subseteq \Delta(X)$, for every $X \subseteq A$.*

Observe that, by monotonicity, Transitivity is in fact strengthened to Idempotency: $\Delta(\Delta(X)) = \Delta(X)$. Moreover, by Progressivity, every closure operator is surjective.

DEFINITION 2.7. *We say that a monotone operator Δ on A*

- *has the* Binary intersection property *iff*

$$\forall X, Y \subseteq A(\Delta(X) \cap \Delta(Y) \subseteq \Delta(X \cap Y)).$$

- *has the* Generalised intersection property *iff*

$$\forall \mathcal{F} (\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(A) \Rightarrow \bigcap\{\Delta(X) \mid X \in \mathcal{F}\} \subseteq \Delta(\bigcap \mathcal{F})).$$

Observe that, by monotonicity, both inclusions in the last part of the above definitions are in fact identities: $\Delta(X) \cap \Delta(Y) = \Delta(X \cap Y)$ and $\bigcap\{\Delta(X) \mid X \in \mathcal{F}\} = \Delta(\bigcap \mathcal{F})$.

Moreover, the Generalised intersection property obviously implies the Binary intersection property.

DEFINITION 2.8. *Let* $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ *be monotone, and let* x *in* A*. We define*

- $\mathsf{S}_\Delta(x) = \{X \subseteq A \mid x \in \Delta(X)\}$.
- $\mathsf{E}_\Delta(x) = \bigcap \mathsf{S}_\Delta(x)$ *if* $\mathsf{S}_\Delta(x) \neq \emptyset$*; otherwise* $\mathsf{E}_\Delta(x) = A$.

LEMMA 2.9. *Let* Δ *be a monotone and surjective operator on* A*. Then,* Δ *has the Generalised intersection property iff every* $x \in A$ Δ*-depends on* $\mathsf{E}_\Delta(x)$*.*

Proof. Whenever Δ is surjective, the family $\mathsf{S}_\Delta(x) = \{X \subseteq A \mid x \in \Delta(X)\}$ is not empty, hence $\mathsf{E}_\Delta(x) = \bigcap \mathsf{S}_\Delta(x)$.

In one direction, suppose Δ has the Generalised intersection property and let $x \in A$. By definition of $\mathsf{S}_\Delta(x)$ and by the Generalised intersection property,

$$x \in \bigcap\{\Delta(X) \mid X \in \mathsf{S}_\Delta(x)\} \subseteq \Delta(\bigcap \mathsf{S}_\Delta(x)) = \Delta(\mathsf{E}_\Delta(x)).$$

In the other direction, assume that $x \in \Delta(\mathsf{E}_\Delta(x))$ for all $x \in A$. Let $\mathcal{F} \subseteq \mathcal{P}(A)$ be nonempty and suppose that $x \in \bigcap\{\Delta(X) \mid X \in \mathcal{F}\}$. Hence $\mathcal{F} \subseteq \mathsf{S}_\Delta(x)$. So $\mathsf{E}_\Delta(x) = \bigcap \mathsf{S}_\Delta(x) \subseteq \bigcap \mathcal{F}$. By the assumption and by monotonicity, $x \in \Delta(\mathsf{E}_\Delta(x)) \subseteq \Delta(\bigcap \mathcal{F})$. □

When a notion of dependence is represented by a monotone operator, the natural candidate as the formal counterpart of the informal notion of "groundedness" is given by the following

DEFINITION 2.10. *Let* $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ *be a monotone operator on* A*. We say that an element* $x \in A$ *is* Δ-grounded *iff* $x \in \mathrm{lfp}(\Delta)$.

2.2.2. Sets of rules. Given a monotone operator Δ on A, the relation "$x \in \Delta(X)$" is a binary relation between subsets of A and elements of A which could be taken to represent the intuitive notion of dependence as an alternative to the monotone operator Δ itself.

Aczel (1977, Definition 1.1.1, p. 741) calls any binary relation $\Phi \subseteq \mathcal{P}(A) \times A$ a *set of rules* and gives the following definitions. A set of rules Φ on A is *monotone* iff $Y \subseteq X$ and $(Y, x) \in \Phi$ implies $(X, x) \in \Phi$. A set $X \subseteq A$ is Φ*-closed* iff for every $(Y, x) \in \Phi$ if $Y \subseteq X$ then $x \in X$. The set *inductively defined by* Φ is the set

$$\mathsf{I}(\Phi) = \bigcap\{X \subseteq A \mid X \text{ is } \Phi\text{-closed}\}.$$

PROPOSITION 2.11 (Correspondence Theorem IV). (*Aczel, 1977, pp. 744–745*) *There exists a one-to-one correspondence between monotone sets of rules* $\Phi \subseteq \mathcal{P}(A) \times A$ *and monotone operators* $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ *given by the following maps*:

$\mathsf{K}_7 : \Phi \mapsto \Delta_\Phi$, *defined by*

$$\Delta_\Phi(X) =_{\mathsf{Df}} \{x \in A \mid (X, x) \in \Phi\},$$

for every $X \subseteq A$*, and*

$\mathsf{K}_8 : \Delta \mapsto \Phi_\Delta$, *defined by*

$$(X, x) \in \Phi_\Delta =_{\mathsf{Df}} x \in \Delta(X)$$

for every $x \in A$ *and* $X \subseteq A$*.*

Moreover, whenever Φ *and* Δ *correspond to each other, the set inductively defined by* Φ *and the least fixed point of* Δ *coincide, namely,* $\mathsf{I}(\Phi) = \mathrm{lfp}(\Delta)$*.*

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Proof. Let Φ be a monotone set of rules on A and let $Y \subseteq X \subseteq A$. Let $x \in \Delta_\Phi(Y)$. By definition of Δ_Φ, the rule $(Y, x) \in \Phi$. Since Φ is monotone $(X, x) \in \Phi$. Hence $x \in \Delta_\Phi(X)$. This shows that whenever Φ is monotone Δ_Φ is monotone too.

Conversely, let Δ be a monotone operator on A, and suppose $Y \subseteq X \subseteq A$ and $(Y, x) \in \Phi_\Delta$. By definition of Φ_Δ, $x \in \Delta(Y)$. Since Δ is monotone, $x \in \Delta(X)$. Hence $(X, x) \in \Phi_\Delta$. This shows that whenever Δ is monotone Φ_Δ is monotone too.

Let $\Delta = \Delta_\Phi$. Then $(X, x) \in \Phi_\Delta$ iff $x \in \Delta_\Phi(x)$ iff $(X, x) \in \Phi$. Conversely, let $\Phi = \Phi_\Delta$. Then $x \in \Delta_\Phi(X)$ iff $(X, x) \in \Phi_\Delta$ iff $x \in \Delta(X)$. Hence, the two maps $\mathsf{K}_7 : \Phi \mapsto \Delta_\Phi$ and $\mathsf{K}_8 : \Delta \mapsto \Phi_\Delta$ are inverse each other, establishing a one-to-one correspondence between monotone sets of rules on A and monotone operators on A.

The proof that, whenever Φ and Δ correspond to each other, $\mathsf{I}(\Phi) = \mathrm{lfp}(\Delta)$ is given in Aczel (1977, pp. 744–745): For, just observe that for a monotone set of rules Φ, the monotone operator associated to Φ defined by Aczel at the bottom of page 744 coincides with our Δ_Φ. □

The two maps $\mathsf{K}_7 : \Phi \mapsto \Delta_\Phi$ and $\mathsf{K}_8 : \Delta \mapsto \Phi_\Delta$ also witness a one-to-one correspondence between monotone and surjective operators and monotone and surjective (as binary relations) sets of rules. For, if Φ is surjective as a binary relation then for every $x \in A$ there exists $X \subseteq A$ such that $(X, x) \in \Phi$. Since $x \in \Delta_\Phi(X)$, X also witnesses that Δ_Φ is surjective. Conversely, if Δ is surjective then for every $x \in A$ there exists $X \subseteq A$ such that $x \in \Delta(X)$. Since $(X, x) \in \Phi_\Delta$, X also witnesses that Φ_Δ is surjective.

When we are merely interested in the set inductively defined by a set of rules we can confine ourselves to *monotone* sets of rules only, as established in the following

LEMMA 2.12 (Reduction Theorem I). *Let* $\mathsf{K}_9 : \Phi \mapsto \Phi^+$ *be the map between sets of rules on A defined by*

$$\Phi^+ =_{\mathsf{Df}} \{(X, x) \in \mathcal{P}(A) \times A \mid \exists Y\, (Y \subseteq X \wedge (Y, x) \in \Phi)\}.$$

Then

1. Φ^+ *is the* monotone closure *of* Φ, *namely, the least* (*under inclusion*) *monotone set of rules on A that extends* Φ.
2. $\mathsf{I}(\Phi^+) = \mathsf{I}(\Phi)$,

Proof. (1) Let $Y \subseteq X$ and $(Y, x) \in \Phi^+$. By definition, there exists $Z \subseteq Y$ such that $(Z, x) \in \Phi$. Since $Z \subseteq Y$ implies $Z \subseteq X$, $(X, x) \in \Phi^+$, so Φ^+ is monotone.

Let $(X, x) \in \Phi$. Since $X \subseteq X$, $(X, x) \in \Phi^+$, hence Φ^+ extends Φ.

Let Φ' be a monotone set of rules extending Φ, and let $(X, x) \in \Phi^+$. By definition, there exists $Z \subseteq X$ such that $(Z, x) \in \Phi \subseteq \Phi'$. Since Φ' is monotone and $Z \subseteq X$, also $(X, x) \in \Phi'$, hence $\Phi^+ \subseteq \Phi'$.

(2) We prove that, for every $Y \subseteq A$, Y is Φ-closed iff is Φ^+-closed, hence $\mathsf{I}(\Phi) = \mathsf{I}(\Phi^+)$ will follow from the definition of “set inductively defined from a set of rules.” In one direction, assume that Y is Φ-closed and let $(X, x) \in \Phi^+$ and $X \subseteq Y$. By definition, there exists $Z \subseteq X$ such that $(Z, x) \in \Phi$. Since $Z \subseteq Y$ and Y is Φ-closed, it follows $x \in Y$. In the other direction, assume that Y is Φ^+-closed and let $(X, x) \in \Phi$ and $X \subseteq Y$. Since $\Phi \subseteq \Phi^+$, $(X, x) \in \Phi^+$, so $x \in Y$. □

In the light of Correspondence Theorem IV, in what follows we will loosely speak of “groundedness” to collectively refer to the equivalent notions introduced in this section, taking monotone operators as our primary representatives for the notions of dependence and groundedness which could be equivalently defined by using monotone sets of rules.

An example of a notion of dependence equivalently represented as a monotone set of rules and as a monotone operator is Leitgeb's notion of semantic dependence in Leitgeb (2005, p. 161 and p. 166). This is an instance of a general method of defining a notion of dependence from a system of functions: we will see several cases of application of this method in the second part (§3) of the article.

2.3. Groundedness and well-foundedness. In the previous sections, we have introduced two notions of "groundedness" for the elements of A: one expressed in terms of *well-foundedness* with respect to a binary relation R representing dependence (or, equivalently, in terms of well-foundedness with respect to a domain function or to a coherent graph mapping), and the other expressed in terms of the least fixed point of a monotone operator Δ (or, equivalently, in terms of the set inductively defined by a monotone set of rules). Let us use the term *groundedness*, properly, only for this latter notion.

The general question we want to address concerns the possibility of representing groundedness in terms of well-foundedness and the other way round, in a uniform way. More precisely, we have two distinct goals: first, we look for a suitable subclass of monotone operators (called *essential-dependence operators*) for which it will be possible to define a one-to-one correspondence with the set of all binary relations on A such that, whenever one operator and one relation correspond to each other, groundedness defined in terms of the former correspond to well-foundedness defined in terms of the latter; second, we will consider a class of families of binary relations on A (called *saturated families*), and a suitable notion of well-foundedness with respect to these families, such that we will able to define a one-to-one correspondence with the class of all monotone operators on A and establish the expected correspondence between groundedness and well-foundedness.

I will elaborate on the first goal in §2.4, and on the second in §2.5. In this section, I will make some preliminary considerations about mapping binary relations and monotone operators on A, starting with recalling a related result on binary relations and sets of rules which we will refer to in the following as *Aczel's theorem*:[7]

PROPOSITION 2.13 (Aczel's theorem). (*Aczel, 1977, Prop. 1.2.1, p. 743*) *Let* $\mathsf{K}_{10} : R \mapsto \Phi_R$ *be the map between binary relations and sets of rules on A defined by*

$$(X, x) \in \Phi_R \Leftrightarrow X = x^R,$$

for every $x \in A$ *and* $X \subseteq A$. *Then*

$$\mathsf{W}(A, R) = \mathsf{I}(\Phi_R).$$

COROLLARY 2.14. *Let* $\mathsf{K}_{11} : R \mapsto \Delta_R$ *be the map between binary relations and monotone operators on A defined by*

$$\Delta_R(X) =_{\mathsf{Df}} \{x \in A \mid x^R \subseteq X\},$$

for every $X \in A$. *Then*

$$\text{lfp}(\Delta_R) = \mathsf{W}(A, R).$$

Proof. If $Y \subseteq X$ then $x^R \subseteq Y$ implies $x^R \subseteq X$, hence Δ_R is monotone. For every $X \subseteq A$,

$$\Delta_R(X) = \{x \in A \mid x^R \subseteq X\} = \{x \in A \mid (X, x) \in (\Phi_R)^+\} = \mathsf{K}_7((\Phi_R)^+)(X).$$

[7] As Aczel himself noticed, this result need the Axiom of Dependent Choices.

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Hence, the map $\mathsf{K}_{11} : R \mapsto \Delta_R$ is the composition of the maps $\mathsf{K}_{10} : R \mapsto \Phi_R$, $\mathsf{K}_9 : \Phi \mapsto \Phi^+$, and $\mathsf{K}_7 : \Phi \mapsto \Delta_\Phi$. Therefore, by Correspondence Theorem IV, Reduction Theorem I, and Aczel's theorem,

$$\text{lfp}(\Delta_R) = \mathsf{I}((\Phi_R)^+) = \mathsf{I}(\Phi_R) = \mathsf{W}(A, R). \qquad \square$$

Corollary 2.14 establishes a correspondence from binary relations to monotone operators on A which preserves the correspondence between R-well-foundedness and Δ-groundedness. However, we will see in the next section that not all monotone operators on A are of the form Δ_R for some $R \subseteq A \times A$.

In the other direction, in the literature on truth we can find two different ways of associating a binary relation to a monotone operator.

The first way is witnessed in Bolander (2002, p. 45). Bolander defines the set of all D_Δ-predecessors of x as follows:

$$D_\Delta(x) = \begin{cases} \text{least } X \in \mathsf{S}_\Delta(x) & \text{if such an } X \text{ does exist.} \\ \emptyset & \text{otherwise.} \end{cases}$$

As remarked in Correspondence Theorem I, Bolander's definition gives us the binary relation $R^{\text{Bol}}_\Delta = \mathsf{K}_1(D_\Delta)$ defined by

$$(y, x) \in R^{\text{Bol}}_\Delta =_{\mathsf{Df}} y \in D_\Delta(x).$$

The second way of associating a binary relation to a monotone operator is witnessed in Yablo (1982, no. 16, p. 136). The relation R^{Yab}_Δ is defined as follows:

$$(y, x) \in R^{\text{Yab}}_\Delta =_{\mathsf{Df}} \exists X \subseteq A\, (y \in X \wedge x \in \Delta(X) \wedge x \notin \Delta(X - \{y\})).$$

It follows immediately from the definitions that $R^{\text{Bol}}_\Delta \subseteq R^{\text{Yab}}_\Delta$; hence, every infinite descending R^{Bol}_Δ-chain of elements of A starting with x (witnessing that x is not R^{Bol}_Δ-well-founded) is also an infinite descending R^{Yab}_Δ-chain witnessing that x is not R^{Yab}_Δ-well-founded. Hence $\mathsf{W}(A, R^{\text{Yab}}_\Delta) \subseteq \mathsf{W}(A, R^{\text{Bol}}_\Delta)$.

For every monotone operator Δ, $\text{lfp}(\Delta) \subseteq \mathsf{W}(A, R^{\text{Bol}}_\Delta)$.[8] For his operator of semantic dependence, Leitgeb (2005, Example 15, pp. 164–165) provides an example of a Δ-ungrounded element x of A for which the set $D_\Delta(x)$ is empty, showing that in this case the inclusion $\text{lfp}(\Delta) \subset \mathsf{W}(A, R^{\text{Bol}}_\Delta)$ is proper.

LEMMA 2.15. *Let Δ be a monotone operator on A having the Binary intersection property. Then*

$$\text{lfp}(\Delta) \subseteq \mathsf{W}(A, R^{Yab}_\Delta).$$

Proof. Let $R = R^{\text{Yab}}_\Delta$. Mimicking the proof of (Leitgeb, 2005, Lemma 13, p. 169), we will prove the contrapositive: If $x \in A$ is not R-well-founded then $x \notin \text{lfp}(\Delta)$. Assume that $s \in {}^\omega A$ is an infinite descending R-chain of elements of A starting with $s(0) = x$ and suppose, towards a contradiction, $x \in \text{lfp}(\Delta)$.

CLAIM. $s(n) \in \text{lfp}(\Delta)$, for every $n \in \omega$.

Proof of the Claim. By induction on n. Let $n = k+1$. By the inductive hypothesis, $s(k) \in \text{lfp}(\Delta)$. Let $\langle \Delta_\alpha \mid \alpha \in \text{On} \rangle$ be the transfinite iteration of Δ starting with the empty set, and

[8] For a proof, the reader can check that the proof of the analogous statement (Leitgeb, 2005, Lemma 13, p. 169) about Leitgeb's operator of semantic dependence actually works for any monotone operator Δ and binary relation R^{Bol}_Δ defined as above.

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let $\alpha = \beta + 1$ be the rank of $s(k)$. Hence $s(k) \in \Delta_{\beta+1} = \Delta(\Delta_\beta)$. Since $(s(k+1), s(k)) \in R^{\mathrm{Yab}}_\Delta$, there exists $Y \subseteq A$ such that $s(k+1) \in Y$, $s(k) \in \Delta(Y)$, and $s(k) \notin \Delta(Y - \{s(k+1)\})$. By the Binary intersection property, $s(k) \in \Delta(\Delta_\beta) \cap \Delta(Y) \subseteq \Delta(\Delta_\beta \cap Y)$. Suppose, towards a contradiction, $s(k+1) \notin \Delta_\beta$. Hence $\Delta_\beta \cap Y \subseteq Y - \{s(k+1)\}$ and, by monotonicity of Δ, $s(k) \in \Delta(Y - \{s(k+1)\})$: Contradiction. Thus $s(k+1) \in \Delta_\beta \subseteq \mathrm{lfp}(\Delta)$. ⊣

By the claim, we can define a sequence of ordinals $\langle \alpha_n \mid n \in \omega \rangle$ as follows: α_n is the rank of $s(n)$ for every $n \in \omega$. The proof of the claim shows, by induction on n, that if $\alpha_n = \beta + 1$ is the rank of $s(n)$ then $s(n+1) \in \Delta_\beta$, hence its rank α_{n+1} is strictly lesser than α_n. It follows $\alpha_{n+1} < \alpha_n$ for every $n \in \omega$, contradicting the fact that every set of ordinals is well-ordered. Therefore, $x \notin \mathrm{lfp}(\Delta)$. □

We can show[9] that Leitgeb's operator of semantic dependence provides an example of a monotone operator Δ having the Binary intersection property, yet for which all inclusions $\mathrm{lfp}(\Delta) \subset \mathsf{W}(A, R^{\mathrm{Yab}}_\Delta) \subset \mathsf{W}(A, R^{\mathrm{Bol}}_\Delta)$ are proper.

2.4. Aczel's theorem and essential-dependence. In this section, we will prove a correspondence theorem linking all binary relations on A with the subclass of monotone operators on A given by the following

DEFINITION 2.16. *A monotone operator Δ on A is an* essential-dependence operator *iff Δ satisfies Surjectivity and the Generalised intersection property.*

Observe that, by Lemma 2.9, whenever Δ is an essential-dependence operator,

$$\mathsf{S}_\Delta(x) = \{X \subseteq A \mid \mathsf{E}_\Delta(x) \subseteq X\},$$

for every $x \in A$.

LEMMA 2.17. *If Δ is an essential-dependence operator, then*

$$R^{Yab}_\Delta = R^{Bol}_\Delta.$$

.

Proof. We already remarked that $R^{\mathrm{Bol}}_\Delta \subseteq R^{\mathrm{Yab}}_\Delta$ holds for every monotone operator Δ. It remains to show that, whenever Δ is an essential-dependence operator, $R^{\mathrm{Yab}}_\Delta \subseteq R^{\mathrm{Bol}}_\Delta$ holds too.

Assume $(y, x) \in R^{\mathrm{Yab}}_\Delta$, namely, there exists $Y \subseteq A$ such that $y \in Y$, $x \in \Delta(Y)$, and $x \notin \Delta(Y - \{y\})$. As remarked above, $\mathsf{S}_\Delta(x) = \{X \subseteq A \mid \mathsf{E}_\Delta(x) \subseteq X\}$, hence $\mathsf{E}_\Delta(x) \subseteq Y$. Suppose, towards a contradiction, $y \notin \mathsf{E}_\Delta(x)$. Hence $\mathsf{E}_\Delta(x) \subseteq Y - \{y\}$ and, by monotonicity, $x \in \Delta(\mathsf{E}_\Delta(x))$ implies $x \in \Delta(Y - \{y\})$: Contradiction. Therefore, $y \in \mathsf{E}_\Delta(x)$. Since $\mathsf{E}_\Delta(x)$ is least in $\mathsf{S}_\Delta(x)$, by definition of R^{Bol}_Δ, $(y, x) \in R^{\mathrm{Bol}}_\Delta$. □

In the light of Lemma 2.17, let us denote by $\mathsf{K}_{12} : \Delta \mapsto R_\Delta$ the map that, when Δ is an essential-dependence operator, can equivalently be defined either by $R_\Delta = R^{\mathrm{Bol}}_\Delta$ or by $R_\Delta = R^{\mathrm{Yab}}_\Delta$.

We are now ready to state the main proposition of this section:

PROPOSITION 2.18 (Correspondence Theorem V). *There exists a one-to-one correspondence between binary relations $R \subseteq A \times A$ and essential-dependence operators $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ given by the maps*:

[9] See Rivello (Forthcoming) for a proof.

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$\mathsf{K}_{11} : R \mapsto \Delta_R$, *and*
$\mathsf{K}_{12} : \Delta \mapsto R_\Delta$.
Moreover, whenever R and Δ correspond to each other,

$$\mathsf{W}(A, R) = \mathrm{lfp}(\Delta).$$

To prove Correspondence Theorem V we will first prove two related Correspondence theorems about binary relations and sets of rules.

DEFINITION 2.19. (*Aczel, 1977, Definition 1.2.2, p. 744*) *A set of rules Φ is* deterministic *iff Φ is injective as a binary relation between $\mathcal{P}(A)$ and A.*

Observe that, whenever Φ is both surjective and deterministic, the converse relation of Φ is the domain function $D_\Phi : A \rightarrow \mathcal{P}(A)$ which associates to each $x \in A$ the unique subset X of A such that $(X, x) \in \Phi$. We denote by $\mathsf{K}_{13} : \Phi \mapsto D_\Phi$ the bijection between the surjective and deterministic sets of rules and their converse relations.

LEMMA 2.20 (Correspondence Theorem VI). *There exists a one-to-one correspondence between binary relations $R \subseteq A \times A$ and surjective and deterministic sets of rules $\Phi \subseteq \mathcal{P}(A) \times A$ given by the following maps*:
$\mathsf{K}_{10} : R \mapsto \Phi_R$, *where*

$$(X, x) \in \Phi_R =_{\mathsf{Df}} X = x^R$$

for every $x \in A$ and $X \subseteq A$, and
$\mathsf{K}_{14} : \Phi \mapsto R_\Phi$, *where* (*Aczel, 1977, p. 744*)

$$(y, x) \in R_\Phi =_{\mathsf{Df}} \exists X \subseteq A\, (y \in X \wedge (X, x) \in \Phi).$$

Moreover, whenever Φ and R correspond to each other,

$$\mathsf{I}(\Phi) = \mathsf{W}(A, R).$$

Proof. We already know by Aczel's theorem (Proposition 2.13) that whenever R and Φ correspond to each other in the map $\mathsf{K}_{10} : R \mapsto \Phi_R$, $\mathsf{I}(\Phi) = \mathsf{W}(A, R)$. So it only remains to prove that the two maps K_{10} and K_{14} are inverse bijections between the set of all binary relations on A and the set of all surjective and deterministic sets of rules on A.

We observed above that the map $\mathsf{K}_{13} : \Phi \mapsto D_\Phi$ is a bijection between surjective and deterministic sets of rules and domain functions on A. The map K_{14} is clearly the composition of the bijection $\mathsf{K}_{13} : \Phi \mapsto D_\Phi$ with the bijection $\mathsf{K}_1 : D \mapsto R_D$, hence K_{14} is a bijection between surjective and deterministic sets of rules and binary relations on A.

Let $R = R_\Phi$. Then $(X, x) \in \Phi_R$ iff $X = x^{R_\Phi}$ iff $X = D_\Phi(x)$ iff $(X, x) \in \Phi$. Hence, the map $\mathsf{K}_{10} : R \mapsto \Phi_R$ is the inverse map of $\mathsf{K}_{14} : \Phi \mapsto R_\Phi$. □

DEFINITION 2.21. *A set of rules Φ is* essential *iff*

$$\forall x \in A\, \exists Z \subseteq A\, \forall X \subseteq A\, ((X, x) \in \Phi \Leftrightarrow Z \subseteq X).$$

Observe that an essential set of rules is both monotonic and surjective. Furthermore, for each $x \in A$, the witnessing Z is unique and is given by

$$\mathsf{E}_\Phi(x) = \bigcap \{X \subseteq A \mid (X, x) \in \Phi\}.$$

For, by definition $Z \subseteq \mathsf{E}_\Phi(x)$. Conversely, since $Z \subseteq Z$ implies $(Z, x) \in \Phi$, $\mathsf{E}_\Phi(x) \subseteq Z$.

LEMMA 2.22 (Correspondence Theorem VII). *There exists a one-to-one correspondence between surjective and deterministic sets of rules and essential sets of rules on A given by the following maps*:
$\mathsf{K}_9 : \Phi \mapsto \Phi^+$ *(defined in Lemma 2.12), and*
$\mathsf{K}_{15} : \Phi \mapsto \Phi^-$, *where* Φ *is essential and*

$$\Phi^- = \{(\mathsf{E}_\Phi(x), x) \mid x \in A\}.$$

Moreover, if Φ *is surjective and deterministic,* Φ' *is essential and* Φ, Φ' *correspond to each other,*

$$\mathsf{I}(\Phi') = \mathsf{I}(\Phi).$$

Proof. We show that whenever Φ is surjective and deterministic, Φ^+ is essential. Let $x \in A$. Since Φ is surjective and deterministic there exists a unique $Z \subseteq A$ such that $(Z, x) \in \Phi$, the set $Z = D_\Phi(x)$. Let $X \subseteq A$. If $(X, x) \in \Phi^+$ then there exists $Y \subseteq A$ such that $(Y, x) \in \Phi$ and $Y \subseteq X$. Hence, $Z = Y \subseteq X$. Conversely, if $Z \subseteq X$ then, by monotonicity of Φ^+, $(X, x) \in \Phi^+$. Therefore, Φ^+ is essential.

On the other hand, for an essential Φ the set of rules Φ^- is precisely defined as the converse relation of the domain function $x \mapsto \mathsf{E}_\Phi(x)$, so is both surjective and deterministic (injective).

Let Φ be surjective and deterministic. Then,

$$\mathsf{E}_{\Phi^+}(x) = \bigcap\{Y \subseteq A \mid (Y, x) \in \Phi^+\} = \bigcap\{Y \subseteq A \mid \exists Z\,(Z \subseteq Y \wedge (Z, x) \in \Phi)\} = \bigcap\{Y \subseteq A \mid D_\Phi(x) \subseteq Y\} = D_\Phi(x).$$

Therefore, $(X, x) \in (\Phi^+)^-$ iff $X = \mathsf{E}_{\Phi^+}(x) = D_\Phi(x)$ iff $(X, x) \in \Phi$.

Conversely, let Φ be essential. Then $(X, x) \in (\Phi^-)^+$ iff $\exists Y\,(Y \subseteq X \wedge (Y, x) \in \Phi^-)$ iff $\mathsf{E}_\Phi(x) \subseteq X$ iff $(X, x) \in \Phi$.

Hence, the two maps $\mathsf{K}_9 : \Phi \mapsto \Phi^+$ and $\mathsf{K}_{15} : \Phi \mapsto \Phi^-$ are inverse each other, when restricted to surjective and deterministic sets of rules and to essential sets of rules, respectively.

Finally, whenever Φ and Φ' correspond to each other, $\Phi' = \Phi^+$, hence $\mathsf{I}(\Phi') = \mathsf{I}(\Phi)$ by Reduction Theorem I. □

Proof of Correspondence Theorem V (*Proposition 2.18*).

CLAIM. The two maps $\mathsf{K}_7 : \Phi \mapsto \Delta_\Phi$ and $\mathsf{K}_8 : \Delta \mapsto \Phi_\Delta$ also witness a one-to-one correspondence between essential sets of rules and essential-dependence operators.

Proof of the Claim. In one direction, let Δ be an essential-dependence operator and let $\Phi = \Phi_\Delta$. By Lemma 2.9, for every $x \in A$ and $X \subseteq A$,

$$(X, x) \in \Phi_\Delta \Leftrightarrow x \in \Delta(X) \Leftrightarrow \mathsf{E}_\Delta(x) \subseteq X.$$

Hence Φ_Δ is essential. In the other direction, let Φ be essential and let $\Delta = \Delta_\Phi$. Any essential set of rules is monotone and surjective, and we already observed in §2.1 that whenever Φ is monotone and surjective Δ_Φ is monotone and surjective too. So we only have to check that Δ satisfies the Generalised intersection property. By Lemma 2.9, this is equivalent to checking that every $x \in A$ belongs to $\Delta(\mathsf{E}_\Delta(x))$. For, observe that

$$\mathsf{E}_\Delta(x) = \bigcap \mathsf{S}_\Delta(x) = \bigcap\{X \subseteq A \mid x \in \Delta_\Phi(X)\} = \bigcap\{X \subseteq A \mid (X, x) \in \Phi\} = \bigcap\{X \subseteq A \mid \mathsf{E}_\Phi \subseteq X\} = \mathsf{E}_\Phi(x).$$

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Hence, $x \in \Delta(\mathsf{E}_\Delta(x))$ iff $(\mathsf{E}_\Phi(x), x) \in \Phi$, which is true for every $x \in A$. Hence, Δ_Φ satisfies the Generalised intersection property. ⊣

Let X be any subset of A.

$$\Delta_{(\Phi_R)^+}(X) = \{x \in A \mid (X, x) \in (\Phi_R)^+\} = \{x \in A \mid \exists Y \subseteq X\, (Y, x) \in \Phi_R\} = \{x \in A \mid \exists Y \subseteq X\, (Y = x^R)\} = \{x \in A \mid x^R \subseteq X\} = \Delta_R(X).$$

Hence, the map $\mathsf{K}_{11} : R \mapsto \Delta_R$ can be constructed as the composition of the bijections $\mathsf{K}_{10} : R \mapsto \Phi_R$, $\mathsf{K}_9 : \Phi_R \mapsto (\Phi_R)^+$, and $\mathsf{K}_7 : (\Phi_R)^+ \mapsto \Delta_{(\Phi_R)^+}$. Therefore, the map $\mathsf{K}_{11} : R \mapsto \Delta_R$ is a bijection between binary relations and essential-dependence operators.

Let $\Delta = \Delta_R$. By Lemma 2.17,

$$(y, x) \in R_\Delta \Leftrightarrow y \in \mathsf{E}_\Delta(x) \Leftrightarrow y \in \bigcap\{X \subseteq A \mid x \in \Delta_R(X)\} \Leftrightarrow y \in \bigcap\{X \subseteq A \mid x^R \subseteq X\} \Leftrightarrow y \in x^R \Leftrightarrow (y, x) \in R.$$

Therefore, the map $\mathsf{K}_{12} : \Delta \mapsto R_\Delta$ is the inverse map of $\mathsf{K}_{11} : R \mapsto \Delta_R$.

Finally, from Aczel's theorem, Reduction Theorem I, and Correspondence Theorem IV it follows

$$\mathsf{W}(A, R) = \mathsf{I}(\Phi_R) = \mathsf{I}((\Phi_R)^+) = \mathrm{lfp}(\Delta_R).$$ □

2.5. Yablo's theorem and saturated families of binary relations. Correspondence Theorem V establishes that every binary relation can be represented by a unique monotone operator in such a way that the well-founded part of the relation coincides with the least fixed point of the operator.

The particular correspondence established by Correspondence Theorem V represents binary relations by essential-dependence operators. However, not all monotone operators are essential-dependence operators: For instance, the operator defined in (Leitgeb, 2005, p. 166) provides an example of a monotone and surjective operator that satisfies the Binary intersection property but not the Generalised intersection property. Hence, we cannot use the converse of the map $\mathsf{K}_{11} : R \mapsto \Delta_R$ to represent every monotone operator by a binary relation preserving the identity of groundedness and well-foundedness.

A technique for representing Δ-groundedness in terms of R-well-foundedness which works in the general case of Δ monotone on A (and even in more general situations) is offered by Yablo's analysis of Kripke's notion of groundedness in Yablo (1982).

Yablo deals with groundedness and well-foundedness, considered in an abstract setting, in the first part of his article, where he formalises the "inheritance" aspects of groundedness (with respect to a monotone operator) in terms of the generation of its least fixed point, and the "dependence" aspect of the same intuitive notion of groundedness in terms of well-foundedness with respect to a suitable family of binary relations associated with the monotone operator (I recall that in the present article we chose to reserve the term "groundedness" to refer to the least fixed point construction only).

Let us briefly recall Yablo's theorem, reformulated in our current notation.

Let $(A, \mathcal{X}, J)$ be any inductive space with universe A. For each $x \in A$, define $\mathsf{S}_J(x) = \{X \in \mathcal{X} \mid x \in J(X)\}$. To each element X of $\mathcal{X}$, Yablo (1982, Definition 5, p. 121) associates a family $\mathcal{R}_{J,X}$ of binary relations on A (called *X-dependence relations*) defined as follows:[10] $R \in \mathcal{R}_{J,X}$ iff $R \subseteq A \times A$ and satisfies the following condition for every $x \in A$:

- If $x \in X$ then $x^R = \emptyset$.
- If $x \notin X$ and $\mathsf{S}_J(x) \neq \emptyset$ then $x^R \in \mathsf{S}_J(x)$.
- If $x \notin X$ and $\mathsf{S}_J(x) = \emptyset$ then $x^R = \{x\}$.

To establish the required correspondence between groundedness and well-foundedness, we need a reasonable notion of "well-foundedness" with respect to a family of binary relations, still due to Yablo:

DEFINITION 2.23. (*Yablo, 1982, p. 122*)[11] *Let $\mathcal{R}$ be a family of binary relations on A and let $x \in A$. We say that x is* $\mathcal{R}$-well-founded *iff there exists $R \in \mathcal{R}$ such that x is R-well-founded.*

The well-founded part *of A with respect to $\mathcal{R}$, denoted by* $\mathsf{W}(A, \mathcal{R})$ *is the set of all $\mathcal{R}$-well-founded elements of A.*

Clearly, for any family $\mathcal{R}$ of binary relations on A, $\mathsf{W}(A, \mathcal{R}) = \bigcup\{\mathsf{W}(A, R) \mid R \in \mathcal{R}\}$.

PROPOSITION 2.24 (Yablo's theorem). (*Yablo, 1982, p. 126*) *Let $(A, \mathcal{X}, \Delta)$ be an inductive space and let X be a J-sound element of $\mathcal{X}$. Then*

$$\text{lfp}(J, X) = \mathsf{W}(A, \mathcal{R}_{J,X}).$$

We already observed in §2.2 that any monotone operator Δ on A can be identified with the inductive space $(A, \mathcal{P}(A), \Delta)$. By applying Yablo's theorem to the special case $X = \emptyset$ (which is Δ-sound for any Δ) we obtain

$$\text{lfp}(\Delta) = \text{lfp}(\Delta, \emptyset) = \mathsf{W}(A, \mathcal{R}_{\Delta,\emptyset}).$$

Write $\mathcal{R}_\Delta$ for $\mathcal{R}_{\Delta,\emptyset}$. Then the map $\mathsf{K}_{16} : \Delta \mapsto \mathcal{R}_\Delta$ is a map from the class of all monotone operators on A into the class of all families of binary relations on A, and the last equation provides, for every monotone operator Δ on A, a representation of its least fixed point as the well-founded part of the corresponding family $\mathcal{R}_\Delta$. To turn Yablo's theorem in a Correspondence theorem in our sense, we need to define:

- A subclass $\mathcal{B}$ of the class of all monotone operators on A.
- A subclass $\mathcal{C}$ of the class of all families of binary relations on A.
- A function $K : \mathcal{C} \to \mathcal{B}$

such that, whenever $\mathsf{K}_{16} : \Delta \mapsto \mathcal{R}_\Delta$ is restricted to $\mathcal{B}$, its image is exactly $\mathcal{C}$ and K_{16} and K are inverse each other, so witnessing a bijection between $\mathcal{B}$ and $\mathcal{C}$ which preserves, by Yablo's theorem, the identity of Δ-groundedness and $\mathcal{R}_\Delta$-well-foundedness.

The subclass of monotone operators on A will be the class of all monotone *and surjective* operators on A. Observe that, for a surjective monotone operator $(A, \mathcal{P}(A), \Delta)$, the definition of the map $\mathsf{K}_{16} : \Delta \mapsto \mathcal{R}_\Delta$ can be simplified in:

$$R \in \mathcal{R}_\Delta =_{\mathsf{Df}} \forall x \in A\, (x \in \Delta(x^R)).$$

Moreover, Δ surjective implies that $\mathcal{R}_\Delta$ is not empty, since the trivial relation $A \times A$ belongs to $\mathcal{R}_\Delta$.

[10] I only depart from Yablo's definition in that each Yablo's X-dependence relation is actually the converse relation of a member of our $\mathcal{R}_{J,X}$.

[11] Actually, Yablo uses "$\mathcal{R}$-grounded" for our "$\mathcal{R}$-well-founded."

The corresponding subclass of families of binary relations on A will be proved to be the class of all families which are "saturated" according to the following

DEFINITION 2.25. *Let $\mathcal{R}$, $\mathcal{Q}$ be two families of binary relations on A. We write $\mathcal{R} \sqsubseteq \mathcal{Q}$ iff*

$$\mathcal{R} \subseteq \mathcal{Q} \wedge \forall Q \in \mathcal{Q}\, \forall x \in A\, \exists R \in \mathcal{R}\, (x^R \subseteq x^Q).$$

We say that a family $\mathcal{R}$ is saturated *iff $\mathcal{R}$ is nonempty and $\sqsubseteq$-maximal, namely, for all families $\mathcal{Q}$:*

$$\mathcal{R} \sqsubseteq \mathcal{Q} \Rightarrow \mathcal{R} = \mathcal{Q}.$$

Finally, let $\mathsf{K}_{17} : \mathcal{R} \mapsto \Delta_{\mathcal{R}}$ be the map between families of binary relations on A and operators on A defined as follows:

$$\Delta_{\mathcal{R}}(X) =_{\mathsf{Df}} \{x \in A \mid \exists R \in \mathcal{R}\, (x^R \subseteq X)\},$$

for every $X \subseteq A$.

THEOREM 2.26 (Correspondence Theorem VIII). *There exists a one-to-one correspondence between saturated families of binary relations on A and surjective monotone operators on A given by the maps*:

$\mathsf{K}_{16} : \Delta \mapsto \mathcal{R}_\Delta$, *and*

$\mathsf{K}_{17} : \mathcal{R} \mapsto \Delta_{\mathcal{R}}$.

Moreover, whenever $\mathcal{R}$ and Δ correspond to each other,

$$\mathsf{W}(A, \mathcal{R}) = \mathrm{lfp}(\Delta).$$

Proof. CLAIM I. For every surjective monotone operator Δ, $\mathcal{R}_\Delta$ is saturated.

Proof of Claim I. Let $\mathcal{Q}$ be a family such that $\mathcal{R}_\Delta \sqsubseteq \mathcal{Q}$, and let $Q \in \mathcal{Q}$ and $x \in A$. By definition of $\sqsubseteq$, there exists $R \in \mathcal{R}_\Delta$ such that $x^R \subseteq x^Q$. By definition of $\mathcal{R}_\Delta$ and by monotonicity of Δ, $x \in \Delta(x^R) \subseteq \Delta(x^Q)$, so $Q \in \mathcal{R}_\Delta$. ⊣

CLAIM II. For every family $\mathcal{R}$, $\Delta_{\mathcal{R}}$ is monotone and surjective.

Proof of Claim II. For monotonicity, let $Y \subseteq X$ and let $x \in \Delta_{\mathcal{R}}(Y)$. Hence there exists $R \in \mathcal{R}$ such that $x^R \subseteq Y \subseteq X$, so $x \in \Delta_{\mathcal{R}}(X)$.

For surjectivity, let $x \in A$. Pick any $R \in \mathcal{R}$. Since $x^R \subseteq x^R$ it follows that $x \in \Delta_{\mathcal{R}}(x^R)$. ⊣

CLAIM III. Let $\Delta = \Delta_{\mathcal{R}}$. Then $\mathcal{R} \subseteq \mathcal{R}_\Delta$. Moreover, if $\mathcal{R}$ is saturated then $\mathcal{R} = \mathcal{R}_\Delta$.

Proof of Claim III. By Claim II, $\Delta = \Delta_{\mathcal{R}}$ is monotone and surjective, so $\mathcal{R}_\Delta$ is well defined. Let $R \in \mathcal{R}$ and $x \in A$. Since $x^R \subseteq x^R$, $x \in \Delta_{\mathcal{R}}(x^R)$, so $R \in \mathcal{R}_\Delta$.

Suppose, further, that $\mathcal{R}$ is saturated. Since $\mathcal{R} \subseteq \mathcal{R}_\Delta$ and $\mathcal{R}$ is $\sqsubseteq$-maximal, it is enough to show that $\forall Q \in \mathcal{R}_\Delta\, \forall x \in A\, \exists R \in \mathcal{R}\, (x^R \subseteq x^Q)$, hence $\mathcal{R} \sqsubseteq \mathcal{R}_\Delta$, so $\mathcal{R} = \mathcal{R}_\Delta$. For, let $Q \in \mathcal{R}_\Delta$. By definition of $\mathcal{R}_\Delta$, for all $x \in A$, $x \in \Delta_{\mathcal{R}}(x^Q)$. By definition of $\Delta_{\mathcal{R}}$, $\Delta_{\mathcal{R}}(x^Q) = \{y \in A \mid \exists R \in \mathcal{R}\, (y^R \subseteq x^Q)\}$. Thus, $\exists R \in \mathcal{R}\, (x^R \subseteq x^Q)$. ⊣

CLAIM IV. Let Δ be monotone and surjective and let $\mathcal{R} = \mathcal{R}_\Delta$. Then $\Delta_{\mathcal{R}} = \Delta$.

Proof of Claim IV. Let $X \subseteq A$ and $x \in \Delta_{\mathcal{R}}(X)$. By definition of $\Delta_{\mathcal{R}}$, there exists $R \in \mathcal{R}_\Delta$ such that $x^R \subseteq X$. By definition of $\mathcal{R}_\Delta$, $x \in \Delta(x^R) \subseteq \Delta(X)$.

Conversely, let $x \in \Delta(X)$. Pick any $R \in \mathcal{R}$ and define R' as follows: $(y, z) \in R'$ iff $(z = x \wedge y \in X)$ or $(z \neq x \wedge y \in z^R)$. Clearly, $R' \in \mathcal{R} = \mathcal{R}_\Delta$. Moreover, $x^{R'} = X$, hence $x \in \Delta_{\mathcal{R}}(X)$. ⊣

Finally, whenever Δ and $\mathcal{R}$ correspond to each other, $\mathcal{R} = \mathcal{R}_\Delta$, hence $\mathsf{W}(A, \mathcal{R}) = \mathrm{lfp}(\Delta)$ by Yablo's theorem. □

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2.5.1. Reference graphs. An example of characterisation of groundedness in terms of well-foundedness, which is covered by Correspondence Theorem VIII, is in Beringer & Schindler (2017). Among other results, Beringer & Schindler (2017, Corollary 3.3, p. 459) obtain that "a sentence is grounded [in Leitgeb's sense] iff it has a well-founded reference graph."

We can recast Beringer & Schindler's result in our abstract setting and show that the characterisation in terms of reference graphs provided by the two authors for the least fixed point of Leitgeb's operator actually works for *any surjective monotone operator*, and can be obtained as a consequence of Correspondence Theorem VIII.

First, we state the abstract version of Beringer & Schindler' definition of "reference graph."[12]

DEFINITION 2.27. (*Beringer & Schindler, 2017, Definition 2.8, p. 453*) *Let* Δ *be an operator on* A*. For each* $x \in A$*, define a family* $\mathcal{G}_\Delta(x)$ *of graphs on* A *(the "reference graphs" of* x*) as follows. For* $x \in A$ *and* G *a graph on* A*,* $G \in \mathcal{G}_\Delta(x)$ *holds iff*

1. $x \in G$,
2. $\forall y \in G\,(y \neq x \Rightarrow (y, x) \in G^*)$, *and*
3. $\forall y \in G\,(y \in \Delta(y^{\mathsf{E}(G)}))$.

Then, the abstract version of Beringer & Schindler's theorem is:

PROPOSITION 2.28. *Let* Δ *be a surjective monotone operator on* A*. Then*

$$\mathrm{lfp}(\Delta) = \{x \in A \mid \exists G\,(G \in \mathcal{G}_\Delta(x) \ \wedge\ \mathrm{Wf}(G))\}.$$

Observe that, given any family $\mathcal{R}$ of binary relations on A, we can use the map $\mathsf{K}_4 : R \mapsto \Theta_R$, defined in §2.1, to define, for each $x \in A$, a family $\mathcal{G}_\mathcal{R}(x)$ of graphs on A as follows:

$$\mathcal{G}_\mathcal{R}(x) = \{\Theta_R(x) \mid R \in \mathcal{R}\}.$$

To obtain Proposition 2.28 as a corollary of Correspondence Theorem VIII, we only need the following

LEMMA 2.29. *Let* Δ *be a surjective monotone operator on* A *and let* $\mathcal{R} = \mathcal{R}_\Delta$*. Then, for all* $x \in A$*,*

$$\mathcal{G}_\mathcal{R}(x) = \mathcal{G}_\Delta(x).$$

Proof. In one direction, let $G \in \mathcal{G}_\mathcal{R}(x)$, hence $G = \Theta_R(x)$ for some $R \in \mathcal{R}_\Delta$, and check that G verifies Condition 1–3 of Definition 2.27. By Lemma 4.4 (Appendix A), Θ_R is coherent, so Conditions 1 and 2 are satisfied.

CLAIM. For any relation $R \subseteq A \times A$ and for every $y \in \overline{x^R}$, $y^{\mathsf{E}(\Theta_R(x))} = y^R$.

Proof of the Claim. By definition, $(z, y) \in \mathsf{E}(\Theta_R(x))$ iff $z \in \overline{x^R} \wedge (z, y) \in R$ iff $(z, y) \in R$, since $(z \in \overline{y^R} \Rightarrow \overline{z^R} \subseteq \overline{y^R})$.

Since $R \in \mathcal{R}_\Delta$, by the claim $y \in \Delta(y^R) = \Delta(y^{\mathsf{E}(\Theta_R(x))})$. Hence Condition 3 is satisfied. ⊣

[12] I only depart from Beringer & Schindler definition in that in their article they actually define the converse graph of our "reference graph." Moreover, observe that Beringer & Schindler's two definitions of "reference graph" in Beringer & Schindler (2016) and in Beringer & Schindler (2017), are extensionally equivalent.

In the other direction, let $G \in \mathcal{G}_\Delta(x)$ and pick any $R \in \mathcal{R}_\Delta$. Then define

$$R_G = \{(z, y) \in A \times A \mid (y \in G \wedge (z, y) \in G) \vee (y \notin G \wedge (z, y) \in R)\}.$$

By Condition 3 (of Definition 2.27), $\forall y \in A\,(y \in \Delta(\overline{y^{R_G}}))$. Hence $R_G \in \mathcal{R}_\Delta$. We have to check that $G = \Theta_{R_G} = (\overline{x^{R_G}}, R_G \restriction \overline{x^{R_G}})$. Let $y \in G$ and $(z, y) \in R_G$. By definition of R_G, $(z, y) \in G$, hence $z \in G$. This shows that $\mathsf{V}(G)$ is R_G-left closed. By Condition 1, $x \in \mathsf{V}(G)$, hence $\overline{x^{R_G}} \subseteq \mathsf{V}(G)$. Conversely, let $y \in \overline{x^{R_G}}$ and $(z, y) \in G$. Since $\overline{x^{R_G}} \subseteq \mathsf{V}(G)$, $y \in G$. Hence, by definition of R_G, $(z, y) \in R_G$, so $z \in \overline{x^{R_G}}$. This shows that $\overline{x^{R_G}}$ is $\mathsf{E}(G)$-left closed. Since $x \in \overline{x^{R_G}}$, $\overline{x^{\mathsf{E}(G)}} \subseteq \overline{x^{R_G}}$. By Conditions 1 and 2, $\mathsf{V}(\mathsf{G}) \subseteq \overline{x^{\mathsf{E}(G)}}$. Therefore $\mathsf{V}(\mathsf{G}) \subseteq \overline{x^{R_G}}$. This shows that $\mathsf{V}(\mathsf{G}) = \mathsf{V}(\Theta_{R_G})$. From this, it follows by the definitions of R_G and Θ_{R_G} that $\mathsf{E}(G) = \mathsf{E}(\Theta_{R_G})$. □

Proof of Proposition 2.28. Let $\mathcal{R} = \mathcal{R}_\Delta$. By Correspondence Theorem VIII,

$$\mathrm{lfp}(\Delta) = \mathsf{W}(A, \mathcal{R}) = \{x \in A \mid \exists R \in \mathcal{R}\,(x \in \mathsf{W}(A, R))\}.$$

For every $x \in A$, let $\mathcal{G}_\mathcal{R}(x) = \{\Theta_R(x) \mid R \in \mathcal{R}\}$. If $G \in \mathcal{G}_\mathcal{R}(x)$ then $G = \Theta_R(x)$ for some $R \in \mathcal{R}$. Hence, by Correspondence Theorem II, $x \in \mathsf{W}(A, R) \Leftrightarrow \mathrm{Wf}(G)$. Hence $\mathrm{lfp}(\Delta) = \{x \in A \mid \exists G\,(G \in \mathcal{G}_\mathcal{R}(x) \wedge \mathrm{Wf}(G))\}$. By Lemma 2.29, for every $x \in A$, $\mathcal{G}_\mathcal{R}(x) = \mathcal{G}_\Delta(x)$. Therefore, $\mathrm{lfp}(\Delta) = \{x \in A \mid \exists G\,(G \in \mathcal{G}_\Delta(x) \wedge \mathrm{Wf}(G))\}$. □

2.5.2. Reducing inductive spaces to monotone operators. As noticed above, Yablo's theorem has a wider scope than Correspondence Theorem VIII, in that it applies to inductive spaces of the form $(A, \mathcal{X}, J)$ for which it is not necessary that $\mathcal{X}$ is the power set of A nor that J is surjective as a monotone operator on $\mathcal{X}$. Yablo's theorem is also more informative, since it preserves the identity between the notions of groundedness and well-foundedness even when their are considered *relatively* to a J-sound *nonempty* subset of A.

Nonetheless, in the remaining of this section we want to show that the class of the surjective monotone operators and the map $\mathsf{K}_{16} : \Delta \mapsto \mathcal{R}_\Delta$ are, in some sense, all we need to extend Correspondence Theorem VIII to cover the full generality of Yablo's theorem.

Let us start with extending Correspondence Theorem VIII to handle nonempty Δ-sound subsets of A when $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ is surjective. Yablo obtains his result by defining a distinct family $\mathcal{R}_{\Delta,X}$ for every subset $X \subseteq A$. We can uniformly recover each $\mathcal{R}_{\Delta,X}$ from $\mathcal{R}_\Delta = \mathcal{R}_{\Delta,\emptyset}$ as follows.

DEFINITION 2.30. *Let R be any binary relation on A and let $X \subseteq A$. The* truncation of R at X, *denoted by $R \setminus X$ is the binary relation on A defined by*

$$R \setminus X = \{(y, x) \in R \mid x \notin X\}.$$

It follows immediately from the definitions that, for $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ surjective, for every $X \subseteq A$, $\mathcal{R}_{\Delta,X} = \{R \setminus X \mid R \in \mathcal{R}_\Delta\}$. Hence, for surjective monotone operators on A, we can recast Yablo's theorem as an extension of Correspondence Theorem VIII as follows:

COROLLARY 2.31. *Let Δ be a surjective monotone operator on A and let $\mathcal{R}$ be a saturated family of binary relations on A. Whenever Δ and $\mathcal{R}$ correspond to each other in the maps $\mathsf{K}_{16} : \Delta \mapsto \mathcal{R}_\Delta$ and $\mathsf{K}_{17} : \mathcal{R} \mapsto \Delta_\mathcal{R}$, for every Δ-sound $X \subseteq A$,*

$$\mathrm{lfp}(\Delta, X) = \bigcup\{\mathsf{W}(A, R \setminus X) \mid R \in \mathcal{R}\}.$$

The next step is to show that, in order to compute $\mathrm{lfp}(J, X)$ for all J-sound $X \in \mathcal{X}$, we can reduce any inductive space $(A, \mathcal{X}, J)$ to a surjective and monotone operator Δ, as established by the following

THEOREM 2.32 (Reduction Theorem II). *Let $(A, \mathcal{X}, J)$ be an inductive space on A. Then there exist a subset $A^\#$ of A and a surjective monotone operator $\Delta : \mathcal{P}(A^\#) \to \mathcal{P}(A^\#)$ such that*

1. *$\mathcal{P}(A^\#) \cap \mathcal{X}$ contains all J-sound members of $\mathcal{X}$.*
2. *Δ and J agree on the set $\mathcal{P}(A^\#) \cap \mathcal{X}$.*
3. *For every J-sound member X of $\mathcal{X}$, X is Δ-sound and* $\mathrm{lfp}(\Delta, X) = \mathrm{lfp}(J, X)$.

The proof of Reduction Theorem II will follow from two lemmata.

(I) First, we reduce any inductive space $(A, \mathcal{X}, J)$ on A to a *surjective* inductive space on a subset of A, namely, an inductive space of the form $(A^\#, \mathcal{X}^\#, J')$ for which $A^\# \subseteq A$ and J' is surjective as a monotone operator on $\mathcal{X}^\#$, i.e., for every $x \in A^\#$ there exists $X \in \mathcal{X}^\#$ such that $x \in J'(X)$.

LEMMA 2.33. *Let $(A, \mathcal{X}, J)$ be an inductive space. Then there exists a subset $A^\# \subseteq A$ and a subfamily $\mathcal{X}^\# \subseteq \mathcal{X}$ of subsets of $A^\#$ such that*

1. *$(A^\#, \mathcal{X}^\#, J \restriction \mathcal{X}^\#)$ is a surjective inductive space.*
2. *$\mathcal{X}^\#$ contains all J-sound elements of $\mathcal{X}$.*

Proof.

(1) Define, by transfinite induction, $A_0 = \emptyset$, $A_{\alpha+1} = \{x \in A \mid \forall X \in \mathcal{X}\,(x \in J(X) \Rightarrow X \cap A_\alpha \neq \emptyset)\}$, $A_\delta = \bigcup\{A_\alpha \mid \alpha < \delta\}$, for δ limit. (Observe that $A_1 = \{x \in A \mid \forall X \in \mathcal{X}\,(x \notin J(X))\}$).

CLAIM I. $\langle A_\alpha \mid \alpha \in \mathrm{On}\rangle$ is a hierarchy, namely, $\beta < \alpha \Rightarrow A_\beta \subseteq A_\alpha$.

Proof of Claim I. It is enough to show, by induction on α, that $A_\alpha \subseteq A_{\alpha+1}$, for every $\alpha \in \mathrm{On}$. For $\alpha = 0$, $A_0 = \emptyset$, so $A_0 \subseteq A_1$. Let $\alpha = \beta + 1$ and let $x \in A_{\beta+1}$. Hence, $\forall X \in \mathcal{X}\,(x \in J(X) \Rightarrow X \cap A_\beta \neq \emptyset)$. By the inductive hypothesis, $A_\beta \subseteq A_{\beta+1}$, so $X \cap A_{\beta+1} \neq \emptyset$. Thus, $x \in A_{\beta+2}$. Finally, let α be limit. Let $x \in A_\alpha$ and $X \in \mathcal{X}$ be such that $x \in J(X)$. By definition, there exists $\beta < \alpha$ such that $x \in A_\beta$. By the inductive hypothesis, $A_\beta \subseteq A_{\beta+1}$, so $x \in A_{\beta+1}$. Thus, $X \cap A_\beta \neq \emptyset$. Since α is limit and $\beta < \alpha$, $A_\beta \subseteq A_\alpha$, hence $X \cap A_\alpha \neq \emptyset$. Thus, $x \in A_{\alpha+1}$. ⊣

Let $A_\infty = \bigcup\{A_\alpha \mid \alpha \in \mathrm{On}\}$ and define:

- $A^\# = A - A_\infty$.
- $\mathcal{X}^\# = \{X \in \mathcal{X} \mid X \cap A_\infty = \emptyset\}$.

CLAIM II. $\mathcal{X}^\#$ is an inductive family of subsets of $A^\#$.

Proof of Claim II. Let $X \in \mathcal{X}^\#$ and $x \in X$. Since $X \cap A_\infty = \emptyset$, $x \notin A_\infty$, so $x \in A^\#$. Hence, $\mathcal{X}^\#$ is a family of subsets of $A^\#$. We check that $\mathcal{X}^\#$ is inductive.

$\emptyset \in \mathcal{X}^\#$ is trivial. Let $\mathcal{C} \subseteq \mathcal{X}^\#$ be a chain, and let $Z = \bigcup \mathcal{C}$. Since $\mathcal{C} \subseteq \mathcal{X}^\# \subseteq \mathcal{X}$, and $\mathcal{X}$ is inductive, it follows that $Z \in \mathcal{X}$. Suppose, towards a contradiction, that there exists $z \in Z \cap A_\infty$. Hence, there exists $X \in \mathcal{C} \subseteq \mathcal{X}^\#$ such that $z \in X \cap A_\infty$: Contradiction. So, $Z \cap A_\infty = \emptyset$, namely, $Z \in \mathcal{X}^\#$. ⊣

CLAIM III. $J \restriction \mathcal{X}^\#$ is a surjective monotone operator from $\mathcal{X}^\#$ to $\mathcal{X}^\#$.

Proof of Claim III. Monotonicity of $J \restriction \mathcal{X}^\#$ immediately follows from the fact that J is monotone.

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We want to show that $\forall X \in \mathcal{X}^{\#}\,(J(X) \in \mathcal{X}^{\#})$. Let $X \in \mathcal{X}^{\#}$. Suppose, towards a contradiction, that $J(X) \cap A_{\infty} \neq \emptyset$ and let $x \in J(X) \cap A_{\infty}$. Hence, there exists α such that $x \in J(X) \cap A_{\alpha}$. Let α be the first such. Since $A_0 = \emptyset$, α cannot be zero. Since α is the first one, it cannot be limit. Let $\alpha = \beta + 1$. Then, $x \in J(X) \Rightarrow X \cap A_{\beta} \neq \emptyset$. Thus, $X \cap A_{\infty} \neq \emptyset$, namely $X \notin \mathcal{X}^{\#}$: Contradiction.

Finally, surjectivity means that $\forall x \in A\, \exists X \in \mathcal{X}^{\#}\,(x \in J(X))$. Let $x \in A$ and suppose, towards a contradiction, that $\forall X \in \mathcal{X}^{\#}\,(x \notin J(X)$ or, equivalently, that $\forall X \in \mathcal{X}\,(x \in J(X) \Rightarrow X \cap A_{\infty} \neq \emptyset)$. By the definition of A_{∞} and by first-order logic, this is equivalent to saying that $\forall X \in \mathcal{X}\, \exists \alpha\, (x \in J(X) \Rightarrow X \cap A_{\alpha} \neq \emptyset)$. Let $\alpha_X = \min\{\alpha \in \mathrm{On} \mid x \in J(X) \Rightarrow X \cap A_{\alpha} \neq \emptyset\}$, and let $\overline{\alpha} = \sup\{\alpha_X \mid X \in \mathcal{X}\}$. It follows that $\forall X \in \mathcal{X}\,(x \in \Delta(X) \Rightarrow X \cap A_{\overline{\alpha}} \neq \emptyset)$. For, let $X \in \mathcal{X}$ be such that $x \in J(X)$. By definition, $X \cap A_{\alpha_X} \neq \emptyset$. Since $\alpha_X \leq \overline{\alpha} \Rightarrow A_{\alpha_X} \subseteq A_{\overline{\alpha}}$, we have that $X \cap A_{\overline{\alpha}} \neq \emptyset$. Therefore, $x \in A_{\overline{\alpha}+1} \subseteq A_{\infty}$: Contradiction. Thus, $\exists X \in \mathcal{X}\,(x \in J(X) \wedge X \cap A_{\infty} = \emptyset)$, namely, $\exists X \in \mathcal{X}^{\#}\,(x \in J(X))$. ⊣

CLAIM IV. $A^{\#} = \bigcup \mathcal{X}^{\#}$.

Proof of Claim IV. By Claim II, $\mathcal{X}^{\#}$ is a family of subsets of $A^{\#}$, so $\bigcup \mathcal{X}^{\#} \subseteq A^{\#}$. Conversely, let $x \in A^{\#}$. By Claim III, there exists $X \in \mathcal{X}^{\#}$ such that $x \in J(X)$. By Claim III again, $J(X) \in \mathcal{X}^{\#}$. Thus, $x \in \bigcup \mathcal{X}^{\#}$. ⊣

Claims I–III show that $(A^{\#}, \mathcal{X}^{\#}, J \restriction \mathcal{X}^{\#})$ is a surjective inductive space.

(2) Let $X \in \mathcal{X}$ be J-sound, i.e., $X \subseteq J(X)$. We will show, by transfinite induction, that $X \cap A_{\alpha} = \emptyset$ for every $\alpha \in \mathrm{On}$. In particular, $X \cap A_{\infty} = \emptyset$, so it will follow that $X \in \mathcal{X}^{\#}$. If $\alpha = 0$, then $A_0 = \emptyset$, so $X \cap \emptyset = \emptyset$. Let $\alpha = \beta + 1$. Suppose, towards a contradiction, that $x \in X \cap A_{\alpha}$. By the hypothesis, $x \in X$ implies $x \in J(X)$. On the other hand, $x \in A_{\beta+1}$ and $x \in J(X)$ implies $X \cap A_{\beta} \neq \emptyset$. However, by the inductive hypothesis, $X \cap A_{\beta} = \emptyset$: Contradiction. If α is limit, by the inductive hypothesis $X \cap A_{\beta} = \emptyset$ for all $\beta < \alpha$, hence $X \cap A_{\alpha} = X \cap \bigcup\{A_{\beta} \mid \beta < \alpha\} = \emptyset$. □

(II) Secondly, we reduce any surjective inductive space to a surjective monotone operator.

LEMMA 2.34. *Let (U, P, J) be a surjective inductive space. Then there exists a surjective monotone operator $\Delta : \mathcal{P}(U) \to \mathcal{P}(U)$ such that $\Delta \restriction P = J$.*

Proof. Define, for every $S \subseteq U$:

$$\Delta(S) = \bigcup\{J(S') \mid S' \subseteq S \wedge S' \in P\}.$$

It is immediate to see that Δ is monotone for every J and that Δ is surjective whenever J is. Moreover, for $S \in P$, $\Delta(S) = J(S)$, by the monotonicity of Δ. □

Proof of Reduction Theorem II. Let $(A, \mathcal{X}, J)$ be an inductive space on A. By applying Lemma 2.33 we obtain a surjective inductive space $(A^{\#}, \mathcal{X}^{\#}, J \restriction \mathcal{X}^{\#})$ such that $\mathcal{X}^{\#}$ contains all J-sound members of $\mathcal{X}$. By applying Lemma 2.34 to the inductive space $(A^{\#}, \mathcal{X}^{\#}, J \restriction \mathcal{X}^{\#})$ we obtain a surjective monotone operator Δ on $A^{\#}$ such that $\Delta \restriction \mathcal{X}^{\#} = J \restriction \mathcal{X}^{\#}$. By the proof of Lemma 2.33, $A^{\#} = A - A_{\infty}$ and $\mathcal{X}^{\#} = \{X \in \mathcal{X} \mid X \cap A_{\infty} = \emptyset\}$. Hence $X \in \mathcal{P}(A^{\#}) \cap \mathcal{X}$ implies $X \cap A_{\infty} = \emptyset$, so $X \in \mathcal{X}^{\#}$. By Lemma 2.33, $\mathcal{X}^{\#} \subseteq \mathcal{P}(A)^{\#} \cap \mathcal{X}$, so $\mathcal{X}^{\#} = \mathcal{P}(A^{\#}) \cap \mathcal{X}$. Therefore, $\mathcal{P}(A^{\#}) \cap \mathcal{X}$ contains all J-sound members of $\mathcal{X}$ and Δ and J agree on $\mathcal{P}(A^{\#}) \cap \mathcal{X}$.

Finally, by transfinite induction, we can show that for each J-sound member X of $\mathcal{X}$ the transfinite iterations Δ^X_{α} and J^X_{α} built from X applying Δ and J, respectively, coincide. So $\mathrm{lfp}(\Delta, X) = \Delta^X_{\infty} = J^X_{\infty} = \mathrm{lfp}(J, X)$. □

Combining Yablo's theorem with Correspondence Theorem VIII and Reduction Theorem II, we get the following representation of J-groundedness in terms of R-well-foundedness for generic inductive spaces:

COROLLARY 2.35. *Let $(A, \mathcal{X}, J)$ be an inductive space, let Δ be the surjective monotone operator on $A^{\#} \subseteq A$ given by Reduction Theorem II and let $\mathcal{R}$ be the saturated family of binary relations on $A^{\#}$ given by the map $\mathsf{K}_{16} : \Delta \mapsto \mathcal{R}_{\Delta}$. Then, for all J-sound $X \in \mathcal{X}$,*

$$\mathrm{lfp}(J, X) = \bigcup\{\mathsf{W}(A^{\#}, R \setminus X) \mid R \in \mathcal{R}\}.$$

§3. Dependence from a valuation system. In the first part of the article (§2) we studied how the notion of "groundedness" can be defined from a *given* "dependence relation," and we contrasted different ways of representing dependence in the mathematical language. In this section, we take monotone operators as our primary representatives of dependence relations and study different ways of *defining* monotone operators intended to represent dependence.

Our goal is to subsume all kinds of definitions of dependence we will deal with under one umbrella notion of Γ*-dependence* (a generalisation of "functional dependence," as we will see) and to show that this notion is comprehensive enough to (a) represent all monotone operators, and (b) capture most definitions of dependence we find in the literature on semantic theories of truth and beyond.

In this section, I will formalise the notion of Γ-dependence, and I will prove Reduction Theorem III, stating that every monotone operator can be seen as the notion of Γ-dependence induced by a suitable "valuation system" Γ. In the subsequent sections, I will recast in the framework of Γ-dependence three usual ways of formalising relations of dependence: Kripkean jump operators, functional dependence, and Leitgeb's operator of semantic dependence.

The general notion of dependence I have in mind can be roughly described as follows. We assume that the "data" (or the "independent variables") are given by several assignments of values to a collection of objects, while the "unknown" (or the "dependent variables") are represented by the values which are correspondingly assigned to some other objects. We think that an object y "depends on" a collection of objects X whenever to know the values of the objects in X under some assignment is sufficient in order to determine the value of y under the corresponding assignment. This intuition entails that, in order to define dependence, we first need a given "system of valuation" linking the assignments of values to the independent variables to the assignments of values to the dependent ones. We make precise this requirement in the following

DEFINITION 3.1. *A* valuation system *is a triple $(\mathcal{F}, \mathcal{F}', \Gamma)$, where both $\mathcal{F}$ and $\mathcal{F}'$ are nonempty sets of functions and Γ is an operator $\Gamma : \mathcal{F} \to \mathcal{F}'$.*

We put no constraint either on the domains or on the co-domains of the functions in $\mathcal{F}$ and in $\mathcal{F}'$. However, it will prove more convenient to assume that domains and co-domains are subsets of fixed sets that we can reconstruct from $\mathcal{F}$ and $\mathcal{F}'$ as follows. The *common domain* of the valuation system $(\mathcal{F}, \mathcal{F}', \Gamma)$ is the set $A = \mathrm{dom}(\mathcal{F} \cup \mathcal{F}') = \bigcup\{\mathrm{dom}(p) \mid p \in \mathcal{F} \cup \mathcal{F}'\}$. Therefore, every function $p \in \mathcal{F} \cup \mathcal{F}'$ can be thought as a *partial function* from A. Further, the *common co-domain* of the valuation system $(\mathcal{F}, \mathcal{F}', \Gamma)$ is the set $B = \mathrm{ran}(\mathcal{F} \cup \mathcal{F}') = \bigcup\{\mathrm{ran}(p) \mid p \in \mathcal{F} \cup \mathcal{F}'\}$. Then, the union $\mathcal{F} \cup \mathcal{F}'$ can be seen as a space of partial functions from A to B, namely, as a subset of the set ${}^{[A]}B$ (partially ordered by inclusion) of all partial functions from A to B.

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In the subsequent treatment of valuation systems what actually matter is not the full knowledge of $\mathcal{F}'$, rather only the image of $\mathcal{F}$ under the operator Γ. Hence the relevant part of a valuation system can be recovered from its operator alone. For this reason, sometimes we will sloppy and we denote the valuation system $(\mathcal{F}, \mathcal{F}', \Gamma)$ simply by Γ. We allow in the definition of valuation system $\mathcal{F}$ and $\mathcal{F}'$ to be the same set of functions, and indeed this will be the case in most applications.

Let $y \in A$ and $q \in \mathcal{F}$ and suppose we know that $y \in \mathrm{dom}(\Gamma(q))$. Then we can say that "$y$ depends on X" iff $X \subseteq \mathrm{dom}(q)$ and to know the values of q on the elements of X is enough to determine the value of $\Gamma(q)$ at y. This leads to our official definition of dependence that we will state after introducing some convenient notation: Let $X \subseteq A$, $q, q' \in \mathcal{F}$, and $y \in A$. We write

$$q \equiv_X q'$$

as short for "$X \subseteq \mathrm{dom}(q) \cap \mathrm{dom}(q') \ \wedge \ q \upharpoonright X = q' \upharpoonright X$," and

$$\Gamma(q)(y) \equiv \Gamma(q')(y)$$

as short for "$y \in \mathrm{dom}(\Gamma(q)) \cap \mathrm{dom}(\Gamma(q')) \ \wedge \ \Gamma(q)(y) = \Gamma(q')(y)$."

DEFINITION 3.2. *Let* $(\mathcal{F}, \mathcal{F}', \Gamma)$ *be a valuation system and let* $A = \mathrm{dom}(\mathcal{F} \cup \mathcal{F}')$. *Let* $X \subseteq A$ *and* $y \in A$. *We say that* y Γ-depends on X *iff*

$$\forall q, q' \in \mathcal{F}\, (q \equiv_X q' \Rightarrow \Gamma(q)(y) \equiv \Gamma(q')(y)).$$

Observe that if y Γ-depends on X then for every $q \in \mathcal{F}$ if $X \subseteq \mathrm{dom}(q)$ then $y \in \mathrm{dom}(\Gamma(q))$.

The claim that Definition 3.2 captures the intuitive idea of a "functional" notion of dependence is supported by our first proposition that provides an alternative equivalent way of defining Γ-dependence:

PROPOSITION 3.3. *Let* $(\mathcal{F}, \mathcal{F}', \Gamma)$ *be a valuation system and let* $A = \mathrm{dom}(\mathcal{F} \cup \mathcal{F}')$ *and* $B = \mathrm{ran}(\mathcal{F} \cup \mathcal{F}')$. *Let* $X \subseteq A$ *and* $y \in A$. *Then the following are equivalent*:

1. y Γ*-depends on* X.
2. *There exists a function* $\Lambda : {}^X B \to B$ *such that, for every* $q \in \mathcal{F}$, *if* $X \subseteq \mathrm{dom}(q)$ *then* $y \in \mathrm{dom}(\Gamma(q))$ *and*

$$\Gamma(q)(y) = \Lambda(q \upharpoonright X). \qquad (*)$$

Proof. In one direction, suppose that y Γ-depends on X. Let $p \in {}^X B$. If there exists $q \in \mathcal{F}$ such that $p \subseteq q$ and $y \in \mathrm{dom}(\Gamma(q))$, define $\Lambda(p) = \Gamma(q)(y)$; otherwise, let $\Lambda(p) = b$, where b is an arbitrary element of B. Λ is well defined. For, if $q' \in \mathcal{F}$ is such that $p \subseteq q'$, then $q \equiv_X q'$. Therefore, since y Γ-depends on X, $\Gamma(q')(y) \equiv \Gamma(q)(y)$. To check that Λ satisfies (*), let $q \in \mathcal{F}$ and suppose $X \subseteq \mathrm{dom}(q)$. Let $p = q \upharpoonright X$. Hence $p \in {}^X B$ and $p \subseteq q$, so $\Lambda(q \upharpoonright X) = \Lambda(p) = \Gamma(q)(y)$.

Conversely, suppose $\Lambda : {}^X B \to B$ is a function satisfying the condition stated in the proposition. Let $q, q' \in \mathcal{F}$ be such that $q \equiv_X q'$. In particular $X \subseteq \mathrm{dom}(q) \cap \mathrm{dom}(q')$, so $y \in \mathrm{dom}(\Gamma(q)) \cap \mathrm{dom}(\Gamma(q'))$ and $\Gamma(q)(y) = \Lambda(q \upharpoonright X) = \Lambda(q' \upharpoonright X) = \Gamma(q')(y)$. Hence, $\Gamma(q)(y) \equiv \Gamma(q')(y)$. □

Clearly, the function Λ whose existence is granted by Proposition 3.3 is not unique. However, if we fix one element $b \in B$, as in the proof of Proposition 3.3, then for every pair (X, y) there exists exactly one function $\Lambda_{X,y}$ satisfying the condition stated in the proposition.

We think the dependence relation defined by the two equivalent conditions of Proposition 3.3 as a sort of "generalised functional dependence" in that the usual notion of "functional dependence" is just one special case, as we will see below in §2.4

The following remark allows us to apply to all instances of generalised functional dependence the abstract treatment of dependence and groundedness developed in the first part of the article:

REMARK 3.4. *The operator* $\Delta_\Gamma : \mathcal{P}(A) \to \mathcal{P}(A)$ *defined by*

$$\Delta_\Gamma(X) = \{y \in A \mid y \ \Gamma\text{-depends on } X\},$$

is monotone.

Proof. Let $X \subseteq Y \subseteq A$, $y \in \Delta_\Gamma(X)$ and let $q, q' \in \mathcal{F}$ be such that $q \equiv_Y q'$ holds. Clearly, $q =_X q'$ holds too. Since y Γ-depends on X, $\Gamma(q)(y) \equiv \Gamma(q')(y)$ follows. Hence, y Γ-depends on Y. So, $y \in \Delta_\Gamma(Y)$. □

Observe that to get the monotonicity of Δ_Γ we do not need to assume Γ itself to be monotone.

We conclude this section by showing that every monotone operator $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ can be reconstructed as the notion of dependence defined from a suitable valuation system. For this purpose, we do not even need the full class of valuation systems given by Definition 3.1: a special class of valuation systems, that we call *total* valuation systems, will be enough to represent all monotone operators.

DEFINITION 3.5. *A* valuation system $(\mathcal{F}, \mathcal{F}', \Gamma)$ *is* total *whenever* $\mathcal{F} \subseteq {}^A B$*, namely,* $\mathcal{F}$ *is a set of total functions on A.*

Observe that the operator Γ of a total valuation system is trivially monotone, because any two distinct total functions are incomparable. When f, f' are total on A, $X \subseteq \mathrm{dom}(f) \cap \mathrm{dom}(f')$ obviously holds for every subset X of A: In this case, we write $f =_X f'$ as short for $f \restriction X = f' \restriction X$.

PROPOSITION 3.6 (Reduction Theorem III). *Let* $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ *be a monotone operator on A. Then, there exists a total valuation system* $(\mathcal{F}, \mathcal{F}', \Gamma)$ *such that* $\mathrm{dom}(\mathcal{F} \cup \mathcal{F}') = A$ *and*

$$\Delta_\Gamma = \Delta.$$

Proof. First,[13] we handle the trivial case in which $\Delta(\emptyset) = A$. By monotonicity of Δ, $\Delta(\emptyset) = A$ iff $\Delta(X) = A$ for every $X \subseteq A$. In this case, take $B = \{0\}$, $f^0 : A \to B$ be the constant function whose image is 0 for every $x \in A$, $\mathcal{F} = \mathcal{F}' = \{f^0\} = {}^A B$, and $\Gamma(f^0) = f^0$. Hence, for all $X \subseteq A$, $\Delta_\Gamma(X) = A = \Delta(X)$, as required.

If $\Delta(\emptyset) \neq A$, fix an element z_0 of A such that $z_0 \notin \Delta(\emptyset)$, and let $B = \{0, 1\} \times \mathcal{P}(A) \times A$. For each $(X, y) \in \mathcal{P}(A) \times A$ define $\mathcal{F}_{X,y} = \{f^0_{X,y}, f^1_{X,y}\}$ and $\mathcal{G}_{X,y} = \{g^0_{X,y}, g^1_{X,y}\}$ as follows. Both $f^0_{X,y}$ and $f^1_{X,y}$ are functions from A to B. For every $x \in A$,

$$f^0_{X,y}(x) = (0, X, y),$$

and

[13] Our proof will mimic that given by Väänänen (2016, p. 7) for the special case in which Δ is a *closure operator* on A.

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$$f^1_{X,y}(x) = \begin{cases} (0, X, y) & \text{if } x \in X \\ (1, X, y) & \text{otherwise.} \end{cases}$$

Observe that, for $X \neq A$, $f^0_{X,y}$ and $f^1_{X,y}$ are distinct.

Next we define a total function $g^0_{X,y} \in {}^{A}B$ and a partial function $g^1_{X,y} \in {}^{[A]}B$. For every $z \in A$,

$$g^0_{X,y}(z) = \begin{cases} (0, \emptyset, z_0) & \text{if } z \neq y \text{ or } z \in \Delta(\emptyset) \\ (0, X, y) & \text{otherwise.} \end{cases}$$

and

$$g^1_{X,y}(z) = \begin{cases} (0, \emptyset, z_0) & \text{if } z \neq y \text{ or } z \in \Delta(\emptyset) \\ \textit{undefined} & \text{otherwise.} \end{cases}$$

Let $\mathcal{F} = \bigcup\{\mathcal{F}_{X,y} \mid y \notin \Delta(X)\}$ and $\mathcal{F}' = \bigcup\{\mathcal{G}_{X,y} \mid y \notin \Delta(X)\}$. By our assumption, $z_0 \notin \Delta(\emptyset)$, so both $\mathcal{F}$ and $\mathcal{F}'$ are not empty. Moreover, since both $f^0_{\emptyset,z_0}$ and $g^0_{\emptyset,z_0}$ are total functions, it follows that $\mathrm{dom}(\mathcal{F} \cup \mathcal{F}') = A$. Every $f \in \mathcal{F}$ is of the form $f^i_{X,y}$ $(i = 0, 1)$ for some index (i, X, y). If $X \neq A$ then the index is unique, otherwise there are exactly two indices such that $f = f^0_{A,y} = f^1_{A,y}$. Every $g \in \mathcal{G}$ is of the form $g^{i'}_{X',y'}$ for a unique index (i', X', y'). Define $\Gamma : \mathcal{F} \to \mathcal{F}'$ as follows. For every $f = f^i_{X,y} \in \mathcal{F}$, if $X = A$ put $\Gamma(f) = g^0_{A,y}$, if $y \in \Delta(A)$, and $\Gamma(f) = g^1_{A,y}$ if $y \notin \Delta(A)$. If $X \neq A$, put $\Gamma(f) = g^i_{X,y}$. We want to show that, for every $X \subseteq A$, $\Delta_\Gamma(X) = \Delta(X)$.

In one direction, suppose $y \in \Delta(X)$ and let $f, f' \in \mathcal{F}$, $f = f^i_{Y,z}$, $f' = f^j_{Y',z'}$ be such that $f =_X f'$. Observe that, for $g, g' \in \mathcal{F}'$, $g(z) = (0, \emptyset, z_0) = g'(z)$ for every $z \in A$ such that $z \in \Delta(\emptyset)$. Hence, if $X = \emptyset$ then $\Gamma(f)(y) = \Gamma(f')(y)$ by definition of Γ. Otherwise, there exists $x \in X$ such that $f^i_{Y,z}(x) = f^j_{Y',z'}(x)$, hence it must be $Y = Y'$ and $z = z'$. If $X = A$ then $f = f'$. By the hypothesis, $y \in \Delta(A)$ implies that $\Gamma(f) = g^0_{A,z}$ is total, hence $\Gamma(f)(y) = \Gamma(f)(y)$ holds. If $X \neq A$ suppose, without loss of generality, $f = f^0_{Y,z}$ and $f' = f^1_{Y,z}$, hence $\Gamma(f) = g^0_{Y,z}$ and $\Gamma(f') = g^1_{Y,z}$. Since, for all $x \in X$, $f^0_{Y,z}(x) = f^1_{Y,z}(x)$, it must be $X \subseteq Y$. By monotonicity of Δ, $y \in \Delta(Y)$. Since $f \in \mathcal{F}$, $z \notin \Delta(Y)$, so $y \neq z$, hence $y \in \mathrm{dom}(\Gamma(f')(y))$ and $\Gamma(f)(y) = g^0_{Y,z}(y) = (0, \emptyset, z_0) = g^1_{Y,z}(y) = \Gamma(f')(y)$, namely $\Gamma(f)(y) = \Gamma(f')(y)$. Therefore, $y \in \Delta_\Gamma(X)$.

In the other direction, assume $y \notin \Delta(X)$. Hence $f^1_{X,y} \in \mathcal{F}$ and, whatever is X, $\Gamma(f^1_{X,y}) = g^1_{X,y}$. By monotonicity of Δ, $y \notin \Delta(X)$ implies $y \notin \Delta(\emptyset)$, hence $g^1_{X,y}$ is not defined at y. Thus, we have found a function $f = f^1_{X,y}$ such that $X \subseteq \mathrm{dom}(f)$ but $y \notin \mathrm{dom}(\Gamma(f))$, witnessing $y \notin \Delta_\Gamma(X)$. □

By Remark 3.4 every operator Δ_Γ induced by a valuation system Γ is monotone so, by Reduction Theorem III, for every valuation system Γ we can find a total valuation system Γ' inducing the same dependence relation as Γ.

We say that a valuation system $(\mathcal{F}, \mathcal{F}', \Gamma)$ is *regular* iff for every total $f \in \mathcal{F}$ its image $\Gamma(f) \in \mathcal{F}'$ is total too. The monotone operator Δ_Γ associated to a total and regular valuation system is surjective. Moreover, by adapting the proof of Reduction Theorem III, we can see that every monotone and surjective operator $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ can be reconstructed as the notion of dependence Δ_Γ induced by a total and regular valuation system Γ.

3.1. Kripkean valuation systems. The "jump" operators studied in Kripke (1975) can be seen as particular cases of valuation systems. Kripke defines several monotone functions taking as arguments "partial sets," namely, pairs (X, Y) of disjoint subsets of the set A of all sentences of a suitable first-order language. Disregarding the internal structure of A as a set of sentences, Kripke's construction fits our present setting as follows. There is an obvious one-to-one correspondence between the set of all partial sets on A and the set of all *partial characteristic functions* on A, namely, functions of the form $p : Z \to \{\mathbf{t}, \mathbf{f}\}$, where Z is a subset of A and $\mathbf{t}$ and $\mathbf{f}$ are two distinct objects. For, it suffices to take the map $(X, Y) \mapsto \mathsf{p}_{X,Y}$, where $\mathsf{p}_{X,Y}$ is defined as follows: $\text{dom}(\mathsf{p}_{X,Y}) = X \cup Y$, and, for every $x \in \text{dom}(\mathsf{p}_{X,Y})$, $\mathsf{p}_{X,Y}(x) = \mathbf{t}$ if $x \in X$, and $\mathsf{p}_{X,Y}(x) = \mathbf{f}$ if $x \in Y$. Disjointness of X and Y ensures that $\mathsf{p}_{X,Y}$ is indeed well defined as a partial function from A into $\{\mathbf{t}, \mathbf{f}\}$. Moreover, it is not difficult to check that (a) the map $(X, Y) \mapsto \mathsf{p}_{X,Y}$ witnesses a one-to-one correspondence between partial sets and partial characteristic functions, and (b) one partial set (X', Y') extends (in Kripke's sense) another partial set (X, Y) iff $\mathsf{p}_{X,Y} \subseteq \mathsf{p}_{X',Y'}$. Therefore, we can faithfully reconstruct in our present setting every jump operator considered by Kripke in his article as the monotone operator Γ of a valuation system $(\mathcal{F}, \mathcal{F}', \Gamma)$, where $\mathcal{F}$ and $\mathcal{F}'$ are suitable sets of partial characteristic functions from A.

Kripke considers five monotone jumps, usually denoted[14] by the letters $\mu, \kappa, \sigma, \sigma_1$, and σ_2. All five Kripkean jumps are examples of the following class of valuation systems:

DEFINITION 3.7. *A* Kripkean valuation system *is a valuation system* $(\mathcal{F}, \mathcal{F}', \Gamma)$ *satisfying*

1. $\mathcal{F}' = \mathcal{F}$.
2. $\mathcal{F}$ *is an* inductive[15] *family of partial functions from A.*
3. $\Gamma : \mathcal{F} \to \mathcal{F}$ *is monotone.*
4. $\mathcal{F}$ *is* downward closed*, namely, for all partial functions* p, q *on A, if* $q \in \mathcal{F}$ *and* $p \subseteq q$ *then* $p \in \mathcal{F}$.
5. *For all* $q \in \mathcal{F}$ *there exists a total* $f \in \mathcal{F}$ *such that* $q \subseteq f$.

Condition (1) of Definition 3.7 ensures the iterability of the jump operator Γ. Conditions (2) and (3) grant the existence of the least fixed point of Γ. Observe, further, that a jump operator Γ is also an instance of an *inductive space*,[16] and indeed the main application of inductive spaces in Yablo (1982).

The dependence relation induced on A by a Kripkean valuation system Γ can be expressed in terms of the *domains* (of the assignments in $\mathcal{F}$) only, as shown by the following[17]

PROPOSITION 3.8. *Let* $(\mathcal{F}, \mathcal{F}, \Gamma)$ *be a Kripkean valuation system. Then the following are equivalent*:

1. $y \in \Delta_\Gamma(X)$.
2. $y \in \bigcap\{\text{dom}(\Gamma(q)) \mid q \in \mathcal{F} \wedge X = \text{dom}(q)\}$.

Proof. First, observe that by Condition (5) of Definition 3.7, there exists at least one total function f in $\mathcal{F}$. By Condition (4), for every $X \subseteq A$, the function $q = f \restriction X$ belongs to $\mathcal{F}$, hence the set $\{\text{dom}(\Gamma(q)) \mid q \in \mathcal{F} \wedge X = \text{dom}(q)\}$ is not empty and its intersection is well defined.

[14] See (Kremer, 2009, pp. 365–366 and 369) for definitions of the five jumps.

[15] A family $\mathcal{F}$ of partial functions is *inductive* iff the empty function belongs to $\mathcal{F}$ and the union of any increasing (under inclusion) sequence of members of $\mathcal{F}$ is also in $\mathcal{F}$.

[16] For, consider each element in $\mathcal{F}$ as a subset of the set $U = A \times B$, where $B = \text{ran}(\mathcal{F})$. Then the triple $(U, \mathcal{F}, \Gamma)$ is an inductive space in the sense of Yablo.

[17] Actually, Proposition 3.8 has a wider scope, in that the Conditions (1) and (2) of the definition of "Kripkean valuation system" are not used in the proof.

In one direction, if $y \in \Delta_\Gamma(X)$ we already observed, immediately after Definition 3.2, that for every $q \in \mathcal{F}$ if $X = \mathrm{dom}(q)$ then $y \in \mathrm{dom}(\Gamma(q))$.

Conversely, assume that $y \in \bigcap\{\mathrm{dom}(\Gamma(q)) \mid q \in \mathcal{F} \wedge X = \mathrm{dom}(q)\}$ and let $q, q' \in \mathcal{F}$ be such that $q \equiv_X q'$. Let $p = q \restriction X = q' \restriction X$. Since $\mathcal{F}$ is downward closed, $p \in \mathcal{F}$. By the hypothesis, $X = \mathrm{dom}(p)$ implies $y \in \mathrm{dom}(\Gamma(p))$. Since $p \subseteq q, q'$, by monotonicity $\Gamma(p) \subseteq \Gamma(q), \Gamma(q')$, hence $\Gamma(q)(y) \equiv \Gamma(q')(y)$, namely, $y \in \Delta_\Gamma(X)$. □

The kinds of "functionality" captured by the notions of Γ-dependence associated to (a) a Kripkean valuation system and (b) a total and regular valuation system, respectively, are complementary. Proposition 3.3 shows that, whenever y Γ-depends on X, for every assignment of values to X of the form $q \restriction X$ for $q \in \mathcal{F}$ we can prove existence and uniqueness of a value b for y satisfying $\Gamma(q)(y) = b$, whichever $q \in \mathcal{F}$ we choose. In the case of a total and regular valuation system, whenever an assignment of values to X is given in the form $f \restriction X$ for some total function $f \in \mathcal{F}$, the *existence* of a value $\Gamma(f)(x)$ is already granted by the regularity of Γ, namely, by the fact that $\Gamma(f)$ is *total* too, so that y necessarily belongs to its domain. What the notion of Γ-dependence adds is the *uniqueness* of such value, namely, the fact that for every other $f' \in \mathcal{F}$ such that $f' \restriction X = f \restriction X$ the values $\Gamma(f)(y)$ and $\Gamma(f')(y)$ coincide. By contrast, in a Kripkean valuation system, whenever an assignment of values to X is given in the form $q \restriction X$ for some $q \in \mathcal{F}$, the *uniqueness* of the value of x is already granted by the downward closure of $\mathcal{F}$, namely, by the fact that the function $p = q \restriction X$ belongs to $\mathcal{F}$, so that we can directly apply the operator Γ to p, i.e., to a partial function whose domain is exactly X. However, in general, if p is *partial* so is $\Gamma(p)$, and the possibility of applying $\Gamma(p)$ to y, namely, the *existence* of a value for y, is exactly what the Γ-dependence of y from X states, as shown by Proposition 3.8

Actually Kripke (1975) does not define a notion of dependence associated to a jump operator: Such an investigation is conducted, for instance, in Yablo (1982) and in Bolander (2002). However, Kripke (1975, p. 706) directly defines a notion of *groundedness* with respect to a jump operator Γ: An element $x \in A$ is Γ-*grounded* iff $x \in \mathrm{dom}(\mathrm{lfp}(\Gamma))$.

By contrast, observe that for *any* valuation system $(\mathcal{F}, \mathcal{F}', \Gamma)$ we can consider the notion of Δ-*groundedness* (Definition 2.10) as applied to the monotone operator Δ_Γ, obtaining that, for every valuation system $(\mathcal{F}, \mathcal{F}, \Gamma)$, an element $x \in A = \mathrm{dom}(\mathcal{F} \cup \mathcal{F}')$ is Δ_Γ-*grounded* iff $x \in \mathrm{dom}(\mathrm{lfp}(\Delta_\Gamma))$.

Therefore, for Kripkean valuation systems we have two notions of groundedness at work, and we can show that Δ_Γ-groundedness implies Γ-groundedness,[18] as it results from the following

THEOREM 3.9. *Let* $(\mathcal{F}, \mathcal{F}, \Gamma)$ *be a Kripkean valuation system. Define an auxiliary jump operator* $\Gamma^{\mathsf{aux}} : \mathcal{F} \to \mathcal{F}$ *by putting, for every* $q \in \mathcal{F}$,

$$\Gamma^{\mathsf{aux}}(q) = \Gamma(q) \restriction \Delta_\Gamma(\mathrm{dom}(q)).$$

Then

1. *There exists a monotone Galois connection*[19] *between the set of all fixed points of* Γ *and the set of all fixed points of* Γ^{aux}.

[18] In general, the converse is not true. See Rivello (Forthcoming).

[19] A pair (d, e) of functions $d : P \to P'$ and $e : P' \to P$ between two partially ordered sets $(P, \preceq_P)$ and $(P', \preceq_{P'})$ is a *monotone* Galois connection iff (1) both d and e are monotone maps; and (2) $p \preceq_P e(d(p))$ and $d(e(q)) \preceq_{P'} q$, for every $p \in P$ and $q \in P'$.

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2. $\mathrm{lfp}(\Gamma, \mathrm{lfp}(\Gamma^{\mathsf{aux}})) = \mathrm{lfp}(\Gamma)$.
3. $\mathrm{lfp}(\Delta_\Gamma) = \mathrm{dom}(\mathrm{lfp}(\Gamma^{\mathsf{aux}})) \subseteq \mathrm{dom}(\mathrm{lfp}(\Gamma))$.

Proof. First, we check that $\Gamma^{\mathsf{aux}} : \mathcal{F} \to \mathcal{F}$ is well defined. We already observed that, for $y \in \Delta_\Gamma(X)$ and $q \in \mathcal{F}$, if $X \subseteq \mathrm{dom}(q)$ then $y \in \mathrm{dom}(\Gamma(q))$. In particular, $\Delta_\Gamma(\mathrm{dom}(q)) \subseteq \mathrm{dom}(\Gamma(q))$, hence $\Gamma^{\mathsf{aux}}(q)$ is well defined. Moreover, being $\mathcal{F}$ downward closed, $\Gamma^{\mathsf{aux}}(q) \in \mathcal{F}$.

(1) Monotonicity of Γ and Δ_Γ directly implies the monotonicity of Γ^{aux}. Immediately from the definition, we have that $\Gamma^{\mathsf{aux}}(q) \subseteq \Gamma(q)$ for every q, hence q is Γ-sound whenever q is Γ^{aux}-sound and $\Gamma^{\mathsf{aux}}(q) \subseteq q$ whenever $\Gamma(q) \subseteq q$. In particular, $\mathrm{lfp}(\Gamma, q)$ exists for every fixed point q of Γ^{aux} and $\mathrm{gfp}(\Gamma^{\mathsf{aux}}, q)$ exists for every fixed point q of Γ. Let $d : \mathrm{Fix}(\Gamma^{\mathsf{aux}}) \to \mathrm{Fix}(\Gamma)$ be defined by $d : q \mapsto \mathrm{lfp}(\Gamma, q)$, and let $e : \mathrm{Fix}(\Gamma) \to \mathrm{Fix}(\Gamma^{\mathsf{aux}})$ be defined by $e : q \mapsto \mathrm{gfp}(\Gamma^{\mathsf{aux}}, q)$. Clearly, both d and e are monotone maps. By definition, $e(d(q))$ is the greatest fixed point of Γ^{aux} below $d(q)$: Since $q \in \mathrm{Fix}(\Gamma^{\mathsf{aux}})$ and $q \subseteq d(q)$, $q \subseteq e(d(q))$. Dually, $d(e(q))$ is the least fixed point of Γ above $e(q)$: since $q \in \mathrm{Fix}(\Gamma)$ and $e(q) \subseteq q$, $d(e(q)) \subseteq q$. Hence (d, e) is a monotone Galois connection between $\mathrm{Fix}(\Gamma^{\mathsf{aux}})$ and $\mathrm{Fix}(\Gamma)$.

(2) Let $\bar{q}^{\mathsf{aux}} = \mathrm{lfp}(\Gamma^{\mathsf{aux}})$ and $\bar{q} = \mathrm{lfp}(\Gamma)$. By definition of $\bar{q}$ and e, $\bar{q}^{\mathsf{aux}} \subseteq e(\bar{q}) \subseteq \bar{q}$. By definition of $\bar{q}^{\mathsf{aux}}$ and d and by (1), $\bar{q} \subseteq d(\bar{q}^{\mathsf{aux}}) \subseteq d(e(\bar{q})) \subseteq \bar{q}$. Hence

$$\mathrm{lfp}(\Gamma) = \bar{q} = d(\bar{q}^{\mathsf{aux}}) = \mathrm{lfp}(\Gamma, \mathrm{lfp}(\Gamma^{\mathsf{aux}})).$$

(3) By definition of Γ^{aux} the domain of any fixed point of Γ^{aux} is also a fixed point of Δ_Γ, so $\mathrm{lfp}(\Delta_\Gamma) \subseteq \mathrm{dom}(\mathrm{lfp}(\Gamma^{\mathsf{aux}}))$. Let $q = \mathrm{lfp}(\Gamma^{\mathsf{aux}})$, $X = \mathrm{lfp}(\Delta_\Gamma)$, and $q' = q \restriction X$. By definition, $\Gamma^{\mathsf{aux}}(q) = \Gamma(q) \restriction \Delta_\Gamma(X) = \Gamma(q) \restriction X$. Since $q' \subseteq q$ and Γ^{aux} is monotone, $\Gamma^{\mathsf{aux}}(q') \subseteq \Gamma^{\mathsf{aux}}(q) = q$, hence $\Gamma^{\mathsf{aux}}(q') = q \restriction X = q'$. Since q' is a fixed point of Γ^{aux} it follows that $q \subseteq q'$, hence $q' = q$, so $\mathrm{dom}(\mathrm{lfp}(\Gamma^{\mathsf{aux}})) = \mathrm{dom}(q') = X = \mathrm{lfp}(\Delta_\Gamma)$. Moreover, by (2), $\mathrm{dom}(\mathrm{lfp}(\Gamma^{\mathsf{aux}}) \subseteq \mathrm{dom}(\mathrm{lfp}(\Gamma))$. □

3.2. Functional dependence. The notion of "functional dependence" is ubiquitous in mathematics, logic, computer science, economics, ... We refer to Väänänen (2016) for a comprehensive illustration of variants and applications of this notion. In this section, we only recall some features that characterise functional dependence as a special case of Γ-dependence (Definition 3.2).

Väänänen calls a *team* any set of functions $\mathcal{F}$ on a same domain[20] A, and define an operator $\mathsf{Cl}_\mathcal{F}$ on A as follows: for any $X \subseteq A$,

$$\mathsf{Cl}_\mathcal{F}(X) = \{y \in A \mid \forall f, f' \in \mathcal{F}\,(f \restriction X = f \restriction X' \Rightarrow f(y) = f(y'))\}.$$

In our setting, a team $\mathcal{F}$ is clearly identifiable with a *total* valuation system $(\mathcal{F}, \mathcal{F}, \Gamma)$ in which Γ is the *identity function* on $\mathcal{F}$, i.e., for every $f \in \mathcal{F}$, $\Gamma(f) = f$. For, since every $f \in \mathcal{F}$ is total, $f \equiv_X f'$ is equivalent to $f \restriction X = f \restriction X'$, and $f(y) \equiv f'(y)$ is equivalent to $f(y) = f'(y)$. Hence our notion of Γ-dependence and Väänänen's definition of functional dependence coincide, i.e., $\mathsf{Cl}_\mathcal{F} = \Delta_\Gamma$.

With a slight abuse of language we will use the term "team" both for the family $\mathcal{F}$ and for the valuation system $(\mathcal{F}, \mathcal{F}, \mathrm{id}_\mathcal{F})$. A team is a total and regular valuation system, so we already know that $\mathsf{Cl}_\mathcal{F}$ is a monotone and surjective operator on A. Actually, as

[20] We only drop from Väänänen's definition of "team" the requirement that the domain A has to be finite.

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Väänänen observes, $\mathsf{Cl}_{\mathcal{F}}$ is a *closure operator* on A, namely (a) surjectivity is strengthened in *progressivity*, i.e., $X \subseteq \mathsf{Cl}_{\mathcal{F}}(X)$, and (b) the operator also satisfies *idempotency*, i.e., $\mathsf{Cl}_{\mathcal{F}}(\mathsf{Cl}_{\mathcal{F}}(X)) = \mathsf{Cl}_{\mathcal{F}}(X)$. Moreover, Väänänen (2016, Theorem 1, p. 7) shows that for every closure operator Δ on A there exists a team $\mathcal{F}$ such that $\Delta = \mathsf{Cl}_{\mathcal{F}}$.

The notion of functional dependence, as noticed by Väänänen (2016), traces back to the work of Grelling and Oppenheim. Grelling's functional setting can be identified with a team, in Väänänen's sense. From a given team, Grelling defines several notions of dependence. Among them, the notion labelled "Equidep" by Grelling shows an intimate connection with the modern notion of functional dependence. However, at the first glance, Equidep and $\mathsf{Cl}_{\mathcal{F}}$ are operators on different objects.

Indeed, Grelling (1939, p. 217) insists on the idea that the entities between which we can predicate a notion of dependence have to be *functions*: "The analysis ... leads to the following statements concerning the logical form of the propositions involved:

1. Anything said to depend upon something else is—or at least can be described as—a *function*.
2. What something is said to depend upon is a *class* generally consisting *of* several *functions*. In special cases this class may have only one element.
3. All the functions involved in the same statement of dependence must have the *same argument*, i.e., it must be possible to use the same letter, say 'z', as the argument for all the functions occurring in one formula."

Grelling describes the notion of functional dependence in the following terms: "(E) If, for some argument x_1, every function belonging to F, i.e., every function upon which f depends, takes the same values as for the argument x_2, then f itself must take *equal* values for x_1 and x_2 as well." (Grelling, 1939, p. 218). Grelling proposes formalising the notion expressed by the statement (E) by means of a binary relation Equidep between a function f and a set of functions $\mathcal{H}$ defined on the same arguments. In the present setting

DEFINITION 3.10 (Grelling's Equidep). *Let $\mathcal{F}$ be a team and let $A = \mathrm{dom}(\mathcal{F})$, $f \in \mathcal{F}$ and $\mathcal{H} \subseteq \mathcal{F}$. Then*

$$\mathrm{Equidep}(f, \mathcal{H}) =_{\mathsf{Df}} \forall x, y \in A\ (\forall h \in \mathcal{H}\ h(x) = h(y) \Rightarrow f(x) = f(y)).$$

Actually, it is not so difficult to turn Grelling's relation Equidep into a relation between arguments, rather than between functions. There is a sort of intuitive duality between the fact that the value of a function depends on the values of some other functions and the fact that the values assigned to some argument depends on the values assigned to some other arguments, as noticed in (Väänänen, 2016, p. 5).

The duality between Väänänen's $\mathsf{Cl}_{\mathcal{F}}$ and Grelling's Equidep can be made explicit as follows. In a team, the functions are the elements of $\mathcal{F}$ while the arguments are the elements of $A = \mathrm{dom}(\mathcal{F})$. We can reverse the roles of function and argument in the pair $(\mathcal{F}, A)$ by associating, to each team $\mathcal{F}$, a team $\mathcal{F}^{\mathsf{d}}$ in such a way that Grelling's notion of Equidep in $\mathcal{F}$ corresponds to Väänänen's notion of $\mathsf{Cl}_{\mathcal{F}^{\mathsf{d}}}$, and the other way round. Formally, for each $x \in A$, let F_x denote the function defined by $\mathsf{F}_x(f) = f(x)$ for all $f \in \mathcal{F}$. Accordingly, for $X \subseteq A$ let $\mathsf{F}_X = \{\mathsf{F}_x \mid x \in X\}$.

DEFINITION 3.11. *Let $\mathcal{F}$ be a team with domain A. The* dual team *of $\mathcal{F}$ is the team $\mathcal{F}^{\mathsf{d}} = \mathsf{F}_A = \{\mathsf{F}_x \mid x \in A\}$ with domain $\mathcal{F}$.*

PROPOSITION 3.12. *Let $\mathcal{F}$ be a team with domain A and let $\mathcal{F}^{\mathsf{d}}$ be its dual team. Then, for every $X \subseteq A$, $x \in A$, $\mathcal{H} \subseteq \mathcal{F}$, $f \in \mathcal{F}$:*

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1. $x \in \mathsf{Cl}_{\mathcal{F}}(X) \Leftrightarrow \text{Equidep}(\mathsf{F}_x, \mathsf{F}_X)$.
2. $\text{Equidep}(f, \mathcal{H}) \Leftrightarrow f \in \mathsf{Cl}_{\mathcal{F}^{\mathsf{d}}}(\mathcal{H})$.

Proof. In one direction, assume $x \in \mathsf{Cl}_{\mathcal{F}}(X)$, let $f, f' \in \mathcal{F}$ and suppose that, for all $g \in \mathsf{F}_X$, $g(f) = g(f')$. Let $y \in X$. By definition, $\mathsf{F}_y \in \mathsf{F}_X$, so $f(y) = \mathsf{F}_y(f) = \mathsf{F}_y(f') = f'(y)$. By the arbitrariness of y, this means $f =_X f'$ thus, by the assumption, $\mathsf{F}_x(f) = f(x) = f'(x) = \mathsf{F}_x(f')$.

In the other direction, assume $\text{Equidep}(\mathsf{F}_x, \mathsf{F}_X)$ and let $f, f' \in \mathcal{F}$ be such that $f =_X f'$. Let $g \in \mathsf{F}_X$. By definition, $g = \mathsf{F}_y$ for some $y \in X$. Hence, $g(f) = \mathsf{F}_y(f) = f(y) = f'(y) = \mathsf{F}_y(f') = g(f')$. So, by the assumption, $f(x) = \mathsf{F}_x(f) = \mathsf{F}_x(f') = f'(x)$. Therefore, $x \in \mathsf{Cl}_{\mathcal{F}}(X)$.

(2) In one direction, assume $\text{Equidep}(f, \mathcal{H})$ and let $\mathsf{F}_y, \mathsf{F}_z \in \mathcal{F}^{\mathsf{d}}$ be such that $\mathsf{F}_y =_{\mathcal{H}} \mathsf{F}_z$. Hence, for all $h \in \mathcal{H}$, $h(y) = \mathsf{F}_y(h) = \mathsf{F}_z(h) = h(z)$. By the assumption, $\mathsf{F}_y(f) = f(y) = f(z) = \mathsf{F}_z(f)$. So, $f \in \mathsf{Cl}_{\mathcal{F}^{\mathsf{d}}}(\mathcal{H})$.

In the other direction, assume $f \in \mathsf{Cl}_{\mathcal{F}^{\mathsf{d}}}(\mathcal{H})$, let $y, z \in A$ and suppose that, for all $h \in \mathcal{H}$, $h(y) = h(z)$. It follows that, for all $h \in \mathcal{H}$, $\mathsf{F}_y(h) = h(y) = h(z) = \mathsf{F}_z(h)$, namely $\mathsf{F}_y =_{\mathcal{H}} \mathsf{F}_z$. By the assumption, $f(y) = \mathsf{F}_y(f) = \mathsf{F}_z(f) = f(z)$. Therefore, $\text{Equidep}(f, \mathcal{H})$ holds. □

3.3. Leitgeb-style valuation systems. We conclude this section by recasting in our framework Leitgeb's notion of dependence.

(Leitgeb, 2005, p. 166) defines a monotone operator[21] Δ^{L} (on the set A of all sentences of a suitable first-order language) intended to capture an informal notion of *semantic dependence*. As in the Kripkean case, we can reconstruct Leitgeb's definition in our abstract setting, disregarding the internal structure of A. To every subset $Y \subseteq A$, Leitgeb associates a function $\mathsf{Val}_Y : A \to \{\mathbf{t}, \mathbf{f}\}$, and give the following definition of dependence: For $y \in A$ and $X \subseteq A$, "y Δ^{L}-depends on X," writing $y \in \Delta^{\mathsf{L}}(X)$, iff for all $Y_1, Y_2 \subseteq A$, if $Y_1 \cap X = Y_2 \cap X$ then $\mathsf{Val}_{Y_1}(y) = \mathsf{Val}_{Y_2}(y)$.

Every subset $Y \subseteq A$ can be identified with its *characteristic function*, namely with the function $\mathsf{h}_Y : A \to \{\mathbf{t}, \mathbf{f}\}$ such that $\mathsf{h}_Y(x) = \mathbf{t} \Leftrightarrow x \in Y$. Moreover, for every $Y \subseteq A$, Val_Y itself is a characteristic function. Hence we can identify the map $Y \mapsto \mathsf{Val}_Y$ with an operator $\tau : {}^A\{\mathbf{t}, \mathbf{f}\} \to {}^A\{\mathbf{t}, \mathbf{f}\}$ from the set of all characteristic functions on A into itself. Finally, if h_1 and h_2 are the characteristic functions of the subsets Y_1, Y_2, respectively, then $Y_1 \cap X = Y_2 \cap X$ holds iff $h_1 =_X h_2$ holds. Hence, Leitgeb's operator Δ^{L} can be equivalently defined as follows:

$$y \in \Delta^{\mathsf{L}}(X) \Leftrightarrow \forall h_1, h_2 \in {}^A\{\mathbf{t}, \mathbf{f}\} \, (h_1 =_X h_2 \Rightarrow \tau(h_1)(y) = \tau(h_2)(y)).$$

Consequently, Leitgeb's operator of dependence is an example of Γ-dependence, for Γ falling in the following class of valuation systems

DEFINITION 3.13. *A* Leitgeb-style valuation system *is a valuation system* $(\mathcal{F}, \mathcal{F}', \Gamma)$ *such that* $\mathcal{F}' = \mathcal{F} = {}^A B$*, where* $A = \text{dom}(\mathcal{F})$ *and* $B = \text{ran}(\mathcal{F})$.

Both teams and Leitgeb-style valuation systems are total and regular valuation systems for which $\mathcal{F}' = \mathcal{F}$. A special case falling under both classes is that of a valuation system Γ of the form $({}^A B, {}^A B, \Gamma)$, where Γ is the identity on ${}^A B$. In this case, provided the set of values B has at least two elements, the induced dependence operator Δ_Γ is the identity on

[21] Leitgeb's notation is D^{-1}. We use Δ^{L} to avoid possible confusion with the inverse map of a domain function D.

$\mathcal{P}(A)$, namely, $\Delta_\Gamma(X) = X$ for every $X \subseteq A$. Teams are characterised by the condition $\Gamma = \mathrm{id}_{\mathcal{F}}$, however they admit a nontrivial dependence operator Δ_Γ by allowing $\mathcal{F}$ to be a proper subset of ${}^AB$. By contrast, Leitgeb-style valuation systems are characterised by the condition $\mathcal{F} = {}^AB$, and admit a nontrivial dependence operator Δ_Γ by allowing Γ not to be the identity $\mathrm{id}_{\mathcal{F}}$.

Since a Leitgeb-style valuation system Γ is total and regular, we already know that Δ_Γ is a monotone and surjective operator on A. Moreover, as noticed by Leitgeb for Δ^{L}, Δ_Γ has the *Binary intersection property*, namely, for all $X, Y \subseteq A$

$$\Delta_\Gamma(X) \cap \Delta_\Gamma(Y) \subseteq \Delta_\Gamma(X \cap Y).$$

For, let $y \in \Delta_\Gamma(X) \cap \Delta_\Gamma(Y)$ and let $f, f' \in {}^AB$ be such that $f =_{X\cap Y} f'$. Let $g = f \restriction X \cup f' \restriction A - X$. Then $g =_X f$ and $g =_Y f'$ hold, hence $f(y) = g(y) = f'(y)$, namely, $y \in \Delta_\Gamma(X \cap Y)$.

In §2.4 we saw that every monotone and surjective operator on A of the form Δ_R for some binary relation $R \subseteq A \times A$ satisfies the Generalised intersection property. The operator Δ^{L} originally defined in (Leitgeb, 2005, p. 166) provides an example of a monotone and surjective operator which satisfies the Binary intersection property but not the Generalised intersection property.

§4. Appendix A. In this Appendix we will prove Correspondence Theorem II (Proposition 2.3) and Correspondence Theorem III (Corollary 2.4).

4.1. Correspondence Theorem II. There exists a one-to-one correspondence between coherent graph mappings $\Theta : x \mapsto \Theta(x)$ and binary relations $R \subseteq A \times A$ given by the following maps:

$\mathsf{K}_3 : \Theta \mapsto R_\Theta$, defined by

$$R_\Theta = \bigcup\{\mathsf{E}(\Theta(x)) \mid x \in A\},$$

and

$\mathsf{K}_4 : R \mapsto \Theta_R$, defined by

$$\Theta_R(x) = (\overline{x^R}, R \restriction \overline{x^R}),$$

for every $x \in A$.

Moreover, whenever Θ and R correspond to each other, an element $x \in A$ is R-well-founded iff is Θ-well-founded.

The proof of both statements in Correspondence Theorem II will follow from a series of lemmata.

LEMMA 4.1. *Assume that Θ satisfies Condition 3 in the Definition 2.2 of* coherent *graph mapping. Then for every $y, x \in A$,*

$$\exists z \in A\, (y, x) \in (\Theta(z))^* \Rightarrow \forall u \in A\, (x \in \Theta(u) \Rightarrow (y, x) \in (\Theta(u))^*).$$

Proof. For every $u \in A$, Let $H(u) = \{(y, x) \mid x \in \Theta(u) \Rightarrow (y, x) \in (\Theta(u))^*\}$.

CLAIM. $\forall z, u\, ((\Theta(z))^* \subseteq H(u))$.

Proof of the Claim. By definition of transitive closure, it is enough to show that $H(u)$ extends $\Theta(z)$ and that $H(u)$ is transitive.

Let $(y, x) \in \Theta(z)$. Suppose $x \in \Theta(u)$. Hence, by Condition 3, $(y, x) \in \Theta(u) \subseteq (\Theta(u))^*$. Hence $(y, x) \in H(u)$.

For transitivity, let $(y, w) \in H(u)$ and $(w, x) \in H(u)$. Suppose $x \in \Theta(u)$. Since $(w, x) \in H(u)$, it follows that $(w, x) \in (\Theta(u))^*$. Thus, $w \in \Theta(u)$. From $(y, w) \in H(u)$ and $w \in \Theta(u)$ it follows $(y, w) \in (\Theta(u))^*$. Since $(\Theta(u))^*$ is transitive, $(y, x) \in (\Theta(u))^*$. We have

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shown that $x \in \Theta(u)$ implies $(y, x) \in (\Theta(u))^*$, therefore $(y, x) \in H(u)$. Thus, $H(u)$ is transitive. □

Let $z \in A$ be such that $(y, x) \in (\Theta(z))^*$. Let $u \in A$. By the claim, $(\Theta(z))^* \subseteq H(u)$, hence $x \in \Theta(u) \Rightarrow (y, x) \in (\Theta(u))^*$. □

LEMMA 4.2. *Let $\Theta : x \mapsto \Theta(x)$ be a coherent graph mapping, and let $R = R_\Theta$. Then*

$$R^*_\Theta = \bigcup\{(\mathsf{E}(\Theta(x)))^* \mid x \in A\}.$$

Proof. Let $Q = \bigcup\{(\mathsf{E}(\Theta(x)))^* \mid x \in A\}$. To prove that $R^*_\Theta \subseteq Q$ we show that Q is a transitive binary relation on A which extends R_Θ. $R_\Theta \subseteq Q$ since for every $x \in A$, $\mathsf{E}(\Theta(x)) \subseteq (\mathsf{E}(\Theta(x)))^*$. Let $(y, u) \in Q$ and $(u, z) \in Q$. By definition of Q there exists $x, w \in A$ such that $(y, u) \in (\mathsf{E}(\Theta(x)))^*$ and $(u, z) \in (\mathsf{E}(\Theta(w)))^*$. The latter condition implies $u \in \Theta(w)$ hence, by Lemma 4.1, $(y, u) \in (\mathsf{E}(\Theta(w)))^*$. By transitivity, $(y, z) \in (\mathsf{E}(\Theta(w)))^*$ follows. Hence $(y, z) \in Q$. Therefore, Q is transitive.

For the converse, we have to show that for every $x \in A$, $(\mathsf{E}(\Theta(w)))^* \subseteq R^*_\Theta$. This immediately follows from the fact that R^*_Θ is transitive and extends $\mathsf{E}(\Theta(x))$, since $\mathsf{E}(\Theta(x)) \subseteq R_\Theta \subseteq R^*_\Theta$. □

LEMMA 4.3. *For every $x \in A$, $(\Theta_R(x))^* = \Theta_{R^*}(x)$.*

Proof. By definition, $(\Theta_R(x))^* = (\overline{x^R}, (R \restriction \overline{x^R})^*)$, and $\Theta_{R^*}(x) = (\overline{x^{R^*}}, R^* \restriction \overline{x^{R^*}})$.

$\overline{x^R}$ is R-left-closed, so it is also R^*-left-closed. Since $x \in \overline{x^R}$ it follows $\overline{x^{R^*}} \subseteq \overline{x^R}$. Symmetrically, $\overline{x^{R^*}}$ is R^*-left-closed, so it is also R-left-closed. Since $x \in \overline{x^{R^*}}$ it follows $\overline{x^R} \subseteq \overline{x^{R^*}}$. Hence $\overline{x^{R^*}} = \overline{x^R}$. So, it only remains to prove that $(R \restriction \overline{x^R})^* = R^* \restriction \overline{x^R}$.

Since $R \subseteq R^*$, it follows $R \restriction \overline{x^R} \subseteq R^* \restriction \overline{x^R}$. Since R^* is transitive so is $R^* \restriction \overline{x^R}$. Since $R^* \restriction \overline{x^R}$ is transitive and extends $R \restriction \overline{x^R}$ it follows $(R \restriction \overline{x^R})^* \subseteq R^* \restriction \overline{x^R}$.

For the converse, suppose $(y, z) \in R^* \restriction \overline{x^R}$. In particular, $(y, z) \in R^*$, so there exists an R-chain from y to z. Let (y_i, y_{i+1}) be a member of this chain. Since both $(y_i, z) \in R^*$ and $(y_{i+1}, z) \in R^*$ hold, and since $z \in \overline{x^R}$, it follows that also $(y_i, x) \in R^*$ and $(y_{i+1}, x) \in R^*$ hold. Thus, y_i and y_{i+1} belong to $\overline{x^R}$. This means that z is reachable from y by means of an R-chain all whose members belong to $\overline{x^R}$, namely, that z is reachable from y by means of an $(R \restriction \overline{x^R})$-chain. Thus, $(y, z) \in (R \restriction \overline{x^R})^*$. □

LEMMA 4.4. *For any relation R, Θ_R is a coherent graph mapping* (*Definition 2.2*).

Proof. (Condition 1). Trivial.

(Condition 2). Suppose $y \in \Theta_R(x)$ and $y \neq x$. So $y \in x^{R^*}$, namely $(y, x) \in R^*$ and both y and x are in $\Theta_R(x)$. So, by Lemma 4.3, $(y, x) \in \Theta_{R^*}(x) = (\Theta_R(x))^*$.

(Condition 3). Suppose $\exists z\, (y, x) \in \Theta_R(z)$, let $u \in A$ and suppose $x \in \Theta_R(u)$. Hence $y, x \in \{z\} \cup z^{R^*}$, $(y, x) \in R$ and $x \in \{u\} \cup u^{R^*}$. We have to show that $(y, x) \in \Theta_R(u)$, namely that $y \in \{u\} \cup u^{R^*}$. If $x = u$, then $(y, x) \in R$ implies $y \in u^R \subseteq u^{R^*}$. If $x \in u^{R^*}$, then $(y, x) \in R$ and $(x, u) \in R^*$ implies $(y, u) \in R^*$, so $y \in u^{R^*}$. □

LEMMA 4.5. *Let $\Theta : x \mapsto \Theta(x)$ be a coherent graph mapping, and let $R = R_\Theta$. Then $\Theta_R(x) = \Theta(x)$, for every $x \in A$.*

Proof. First we show that $\mathsf{V}(\Theta_R(x)) = \mathsf{V}(\Theta(x))$.

$\mathsf{V}(\Theta_R(x)) = \overline{x^R} \subseteq \mathsf{V}(\Theta(x))$ since $\mathsf{V}(\Theta(x))$ contains x (by Condition 1 of the definition of coherent graph mapping) and is R-left-closed, i.e., $\forall y, z \in A$, if $y \in \Theta(x)$ and $(z, y) \in R$, then $z \in \Theta(x)$. For, $(z, y) \in R = R_\Theta$ implies that there exists $u \in A$ such that $(z, y) \in \Theta(u)$. Hence, by Condition 3, $y \in \Theta(x)$ implies $(z, y) \in \Theta(x)$, so $z \in \Theta(x)$.

Conversely, we want to show that $\mathsf{V}(\Theta(x)) \subseteq \mathsf{V}(\Theta_R(x))$. Suppose $y \in \Theta(x)$. If $y = x$ then $y \in \Theta_R(x)$ since, by Lemma 4.4, Θ_R satisfies Condition 1. Otherwise, by Condition 2, $(y, x) \in (\Theta(x))^*$ hence, by Lemma 4.2, $(y, x) \in R^*_\Theta = R^*$. Hence $y \in \Theta_{R^*}(x)$. By Lemma 4.3, $y \in (\Theta_R(x))^*$, so $y \in \mathsf{V}(\Theta_R(x))$.

We have shown that the sets of the vertices of $\Theta(x)$ and $\Theta_R(x)$ coincide. Now we have to show that also the sets of the edges of the two graphs coincide. Let $(y, z) \in \Theta_R(x)$. Hence $(y, z) \in R_\Theta$, namely there exists u such that $(y, z) \in \Theta(u)$. Since $z \in R_\Theta(x)$ and $\mathsf{V}(\Theta_R(x)) = \mathsf{V}(\Theta(x))$, it follows that $z \in \Theta(x)$ so, by Condition 3, $(y, z) \in \Theta(x)$. Conversely, let $(y, z) \in \Theta(x)$. Hence $(y, z) \in R_\Theta$. Since $y, z \in \mathsf{V}(\Theta(x)) = \mathsf{V}(\Theta_R(x)) = \overline{x^R}$, and $(y, z) \in R_\Theta = R$, it follows $(y, z) \in R \restriction \overline{x^R} = \mathsf{E}(\Theta_R(x))$. □

LEMMA 4.6. *Let R be any binary relation on A and let $\Theta = \Theta_R$. Then* $(y, x) \in R_\Theta \Leftrightarrow$ $(y, x) \in R$, *for every* $x, y \in A$.

Proof. Let $(y, x) \in R_\Theta$. Hence, there exists $z \in A$ such that $(y, x) \in \Theta_R(z)$. Thus $(y, x) \in R$. Conversely, let $(y, x) \in R$. By definition, $y, x \in \mathsf{V}(\Theta_R(x))$, hence $(y, x) \in \mathsf{E}(\Theta_R(x))$. Thus $(y, x) \in R_\Theta$. □

By the Lemmatas 4.5 and 4.6, the two maps $\Theta \mapsto R_\Theta$ and $R \mapsto \Theta_R$ between coherent graph mappings and binary relations on A are inverse each other.

LEMMA 4.7. *Let R be any binary relation on A, and let $\Theta_R : x \mapsto \Theta_R(x)$ be its corresponding graph mapping. Then, for every $x \in A$,*

$$x \in \mathsf{W}(A, R) \Leftrightarrow \Theta_R(x) \textit{ is well-founded}.$$

Proof. From the definitions it follows that $x \in \mathsf{W}(A, R) \Leftrightarrow \mathrm{Wf}(x^{R^*}, R \restriction x^{R^*})$, and that $\mathrm{Wf}(\Theta_R(x)) \Leftrightarrow \mathrm{Wf}(\overline{x^R}, R \restriction \overline{x^R})$. From $x^{R^*} \subseteq \overline{x^R}$, it follows $R \restriction x^{R^*} \subseteq R \restriction \overline{x^R}$. Hence $\mathrm{Wf}(\overline{x^R}, R \restriction \overline{x^R})$ implies $\mathrm{Wf}(\overline{x^R}, R \restriction x^{R^*})$ (Lévy, 1979, p. 63), and $\mathrm{Wf}(\overline{x^R}, R \restriction x^{R^*})$ implies $\mathrm{Wf}(x^{R^*}, R \restriction x^{R^*})$ (Lévy, 1979, p. 67). Therefore $\mathrm{Wf}(\Theta_R(x)) \Rightarrow x \in \mathsf{W}(A, R)$.

Conversely, let $x \in \mathsf{W}(A, R)$. Suppose, towards a contradiction, that $\Theta_R(x)$ is not well-founded: hence, there exists $Z \subseteq \overline{x^R}$ such that $\emptyset \neq Z$ and $\forall z \in Z \exists y \in Z\, (y, z) \in R \restriction \overline{x^R}$. Let $Z' = Z \cap x^{R^*}$. If $Z' = \emptyset$, then $Z = \{x\}$, therefore $(x, x) \in R \restriction \overline{x^R}$. But, $(x, x) \in R \Rightarrow x \in x^{R^*}$, so $x \notin \mathsf{W}(A, R)$: Contradiction. If $Z' \neq \emptyset$, from $x \in \mathsf{W}(A, R)$ it follows that there exists $z' \in Z'$ such that $\forall u \in Z'\, (u, z') \notin R \restriction x^{R^*}$. Since $Z' \subseteq Z$, we have $z' \in Z$, so by the hypothesis that $\Theta_R(x)$ is not well-founded, there exists $y' \in Z$ such that $(y', z') \in R \restriction \overline{x^R}$. From $x \in \mathsf{W}(A, R)$ it follows that $x \notin x^{R^*}$, hence $z' \neq x$. Suppose $y' \in Z'$, hence both $y', z' \in x^{R^*}$ and $(y', z') \in R$, so $(y', z') \in R \restriction x^{R^*}$, contradicting $x \in \mathsf{W}(A, R)$. Thus $y' \notin Z'$, namely, $y' = x$. $(x, z') \in R \restriction \overline{x^R}$ implies $(x, z') \in R$, and $z' \in x^{R^*}$ implies $(z', x) \in R^*$. By transitivity of R^*, $(x, x) \in R^*$, contradicting $x \in \mathsf{W}(A, R)$. □

4.2. Correspondence Theorem III. There exists a one-to-one correspondence between coherent graph mappings $\Theta : x \mapsto \Theta(x)$ and domain functions $D : A \to \mathcal{P}(A)$ given by the following maps:

$\mathsf{K}_5 : D \mapsto \Theta_D$.

$\mathsf{K}_6 : \Theta \mapsto D_\Theta$, defined by

$$D_\Theta(x) = x^{\mathsf{E}(\Theta(x))},$$

for every $x \in A$.

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Moreover, whenever Θ and D correspond to each other, an element $x \in A$ is D-groundless iff it is not Θ-well-founded.

LEMMA 4.8. *Let $D : A \to \mathcal{P}(A)$ be any domain function and let $R = R_D$ be the image of D in the map K_1. Then*

$$\Theta_D = \Theta_R.$$

Proof. By definition of the map $\mathsf{K}_1 : D \mapsto R_D$, for any $y \in A$, $y^R = y^{R_D} = D(y)$. Thus, by definition of the map $\mathsf{K}_5 : D \mapsto \Theta_D$, $D^{\mathsf{V}}(x) = \overline{x^R}$ and $D^{\mathsf{E}}(x) = R \restriction \overline{x^R}$. Hence, by definition of the map $\mathsf{K}_4 : R \mapsto \Theta_R$,

$$\Theta_D(x) = (D^{\mathsf{V}}(x), D^{\mathsf{E}}(x)) = (\overline{x^R}, R \restriction \overline{x^R}) = \Theta_R(x). \qquad \square$$

Proof of Correspondence Theorem III. Lemma 4.8 shows that the map $\mathsf{K}_5 : D \mapsto \Theta_D$ defined by Beringer & Schindler (2016, Definition 3, p. 6) can be expressed as the composition of the two bijections $\mathsf{K}_1 : D \mapsto R_D$ and $\mathsf{K}_4 : R \mapsto \Theta_R$. This proves that K_5 is a one-to-one correspondence between the class of all domain functions and the class of all coherent graph mappings on A. The map $\mathsf{K}_6 : \Theta \mapsto D_\Theta$ is explicitly defined as the composition of the inverse map K_3 of K_4 with the inverse map K_2 of K_1: Hence K_6 coincides with the inverse map of K_5. Finally, if Θ and D correspond to each other, then $\Theta = \Theta_D = \Theta_{R_D}$. Therefore, by Correspondence Theorems I and II, an element $x \in A$ is D-groundless iff is not R_D-well-founded iff is not Θ_{R_D}-well-founded iff is not Θ-well-founded. $\square$

§5. Appendix B.

In this Appendix we list all enumerated maps, correspondence theorems and reduction theorems which we dealt with in the article.

Table 1. *List of enumerated maps*

Map	Domain	Co-domain	Page
$\mathsf{K}_1 : D \mapsto R_D$	Domain functions	Binary relations	p. 109
$\mathsf{K}_2 : R \mapsto D_R$	Binary relations	Domain functions	p. 109
$\mathsf{K}_3 : \Theta \mapsto R_\Theta$	Graph mappings	Binary relations	p. 110
$\mathsf{K}_4 : R \mapsto \Theta_R$	Binary relations	Graph mappings	p. 110
$\mathsf{K}_5 : D \mapsto \Theta_D$	Domain functions	Graph mappings	p. 110
$\mathsf{K}_6 : \Theta \mapsto D_\Theta$	Graph mappings	Domain functions	p. 110
$\mathsf{K}_7 : \Phi \mapsto \Delta_\Phi$	Sets of rules	Set operators	p. 112
$\mathsf{K}_8 : \Delta \mapsto \Phi_\Delta$	Set operators	Sets of rules	p. 112
$\mathsf{K}_9 : \Phi \mapsto \Phi^+$	Sets of rules	Sets of rules	p. 113
$\mathsf{K}_{10} : R \mapsto \Phi_R$	Binary relations	Sets of rules	p. 114
$\mathsf{K}_{11} : R \mapsto \Delta_R$	Binary relations	Set operators	p. 114
$\mathsf{K}_{12} : \Delta \mapsto R_\Delta$	Set operators	Binary relations	p. 116
$\mathsf{K}_{13} : \Phi \mapsto D_\Phi$	Surjective and deterministic sets of rules	Domain functions	p. 117
$\mathsf{K}_{14} : \Phi \mapsto R_\Phi$	Surjective and deterministic sets of rules	Binary relations	p. 117
$\mathsf{K}_{14} : \Phi \mapsto R_\Phi$	Sets of rules	Binary relations	p. 117
$\mathsf{K}_{15} : \Phi \mapsto \Phi^-$	Essential sets of rules	Sets of rules	p. 118
$\mathsf{K}_{16} : \Delta \mapsto \mathcal{R}_\Delta$	Monotone operators	Families of binary relations	p. 120
$\mathsf{K}_{17} : \mathcal{R} \mapsto \Delta_\mathcal{R}$	Families of binary relations	Set operators	p. 121

Table 2. *List of correspondence theorems*

Theorem	Objects	Objects	Page
I	Domain functions	Binary relations	p. 109
II	Coherent graph mappings	Binary relations	p. 110
III	Coherent graph mappings	Domain functions	p. 110
IV	Monotone sets of rules	Monotone operators	p. 112
V	Binary relations	Essential-dependence operators	p. 116
VI	Binary relations	Surjective-deterministic sets of rules	p. 117
VII	Surjective-deterministic sets of rules	Essential sets of rules	p. 117
VIII	Saturated families of binary relations	Surjective monotone operators	p. 121

Table 3. *List of reduction theorems*

Theorem	Objects	Objects	Page
I	Sets of rules	Monotone sets of rules	p. 113
II	Inductive spaces	Surjective monotone operators	p. 124
III	Monotone operators	Total valuation systems	p. 128

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