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CONTINUOUS NILPOTENTS ON TOPOLOGICAL SPACES

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Abstract

K. D. Magill has investigated the semigroup generated by the idempotent continuous mappings of a topological space into itself and examined whether this semigroup determines the space to within homeomorphism. By analogy with this (and related work of Bridget Bos Baird) we now consider the semigroup generated by nilpotent continuous partial mappings of a space into itself.

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1. Introduction

In [2] Howie investigated the semigroup generated by all the idempotents in $\mathscr{T}(X)$, the full transformation semigroup on a set X, and subsequently in [4] Magill considered the corresponding subsemigroup of $S(X)$, the semigroup of all continuous self-maps of a topological space X.

Certain transformation semigroups contain a "zero" and hence "nilpotents": for example, the semigroup $\mathscr{I}(X)$ of all partial one-to-one transformations of a set X contains the "empty mapping" $\square$ as a zero as well as maps f for which $f^m = \square$ for some $m \geqslant 2$, and in [7] we characterised the elements of $\mathscr{I}(X)$ that can be written as a product (under composition) of such "nilpotents". In this paper we commence by investigating the extent to which the semigroup $N_c(X)$ generated by all continuous nilpotents whose domains are closed subsets of a topological space X determines the underlying space to within a homeomorphism; we also consider the same question for various subsemigroups of $N_c(X)$

(this work is analogous to that of Baird [1] and Magill [4]) and remark that the problem of characterising the elements of $N_c(X)$ appears insurmountable. However, when we specialise to the closed unit interval I in Section 3 and define a nilpotent to be any continuous map $f\colon I \to I$ such that $f^m = 0$, the zero constant map, we obtain a more complete answer: namely that nilpotents in this sense can be characterised in terms of an order property (using the natural order on I), and the product of any two nilpotents is again nilpotent.

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2. Partial nilpotents

If X is a set and $f, g \in \mathscr{P}(X)$, the semigroup (under composition) of all partial transformations of X, then $\operatorname{dom}(fg) = \operatorname{dom} g \cap g^{-1}(\operatorname{dom} f)$. This simple fact will enable us to define various semigroups S of continuous *partial* maps by demanding that the domain of each element of S possess some topological property. We say $f \in \mathscr{P}(X)$ is *nilpotent* if $f^m = \square$ for some $m \geqslant 1$ and that f has *index m* if $f^m = \square$ and $f^{m-1} \neq \square$. Clearly, if $f \in \mathscr{P}(X)$ is nilpotent in this sense, then $\operatorname{dom} f$ and $\operatorname{ran} f$ are proper subsets of X.

Our first aim is to show that if $N_c(X)$ denotes the semigroup generated (under composition) by all continuous nilpotents whose domains are closed subsets of a topological space X, then $N_c(X)$ determines all regular T_1 spaces to within homeomorphism (recall that a space X is *regular* if for each closed $Q \subseteq X$ and $a \notin Q$, there exist disjoint open $U, V \subseteq X$ such that $Q \subseteq U$ and $a \in V$; and X is T_1 if and only if the points of X are closed).

We shall need the following result: it is analogous to Lemma 2.5 in [4].

LEMMA 1. *If X is a regular T_1 space, then $\{f^{-1}(x)\colon x \in X,\ f \in N_c(X)\}$ is a basis for the closed subsets of X.*

PROOF. Suppose Q is a non-empty closed subset of X, and let $a \notin Q$. Since X is regular, there exist disjoint open subsets U, V of X such that $Q \subseteq U$ and $a \in V$. Define a partial map f_a by putting $\operatorname{dom} f_a = X \setminus V$ (a closed subset of X containing U) and $f_a(x) = a$ for all $x \notin V$. Then f_a is a continuous nilpotent of index 2, and it is easy to check that $Q = \bigcap\{f_a^{-1}(a)\colon a \in X \setminus Q\}$.

We now recall a result from [6]: if X is an arbitrary set, any subsemigroup S of $\mathscr{P}(X)$ is briefly referred to as a *transformation semigroup* and S is said to *cover* X if for all $x \in X$, there exists an idempotent constant in S with range equal to $\{x\}$.

THEOREM 1. *If S is a transformation semigroup covering an arbitrary set X, and if ϕ is an automorphism of S, then there exists a permutation g: $X \to X$ such that $\phi(f) = gfg^{-1}$ for all $f \in S$.*

The following theorem can be proven in a manner identical with that of Theorem 1 in [6], so we omit the details.

THEOREM 2. *If S, T are transformation semigroups covering arbitrary sets X, Y, respectively, and if ϕ: $S \to T$ is an isomorphism, then there exists a unique bijection h: $X \to Y$ such that $\phi(f) = hfh^{-1}$ for all $f \in S$. Moreover, if ϕ: $S \to T$ is an isomorphism, and if $\iota_A \in S$ for some $A \subseteq X$, then $\phi(\iota_A) = \iota_{h(A)}$, where h: $X \to Y$ is the bijection associated with ϕ.*

As in [6] we let A_x denote the constant map with domain A and range $\{x\}$; moreover, if $A = \{a\}$, we abbreviate A_x to a_x. With this notation, we now observe that if X is a T_1 space, then $N_c(X)$ contains all the maps x_x for $x \in X$ (since $x_x = y_x \circ x_y$ for $y \neq x$), and so $N_c(X)$ covers X. Thus we may conclude from Theorem 2 that if X, Y are T_1 spaces and ϕ: $N_c(X) \to N_c(Y)$ is an isomorphism, then there exists a bijection h: $X \to Y$ such that $\phi(f) = hfh^{-1}$ for all $f \in N_c(X)$. This establishes part (i) of the next result. The proof of parts (ii) and (iii) is akin to that of Lemma 2.8 [4], so we omit the details (in fact the only difference lies in the use of the injective constants x_x rather than the total constants X_x).

LEMMA 2. *If X, Y are T_1 spaces and ϕ an isomorphism from $N_c(X)$ onto $N_c(Y)$, then there exists a bijection h: $X \to Y$ such that*

(i) *$\phi(f) = hfh^{-1}$ for each $f \in N_c(X)$,*
(ii) *$h(f^{-1}(x)) = \phi(f)^{-1}(h(x))$ for each $f \in N_c(X)$ and $x \in X$, and*
(iii) *$h^{-1}(g^{-1}(y)) = (\phi^{-1}(g))^{-1}(h^{-1}(y))$ for each $g \in N_c(Y)$ and $y \in Y$.*

From Lemmas 1 and 2, we readily deduce

THEOREM 3. *If X, Y are regular T_1 spaces, then ϕ: $N_c(X) \to N_c(Y)$ is an isomorphism if and only if there exists a homeomorphism h: $X \to Y$ such that $\phi(f) = hfh^{-1}$ for all $f \in N_c(X)$.*

We now consider the subsemigroup $N_{co}(X)$ of $N_c(X)$ generated by all continuous maps whose domains are closed-and-open (clopen) subsets of a topological space X, and ask whether $N_{co}(X)$ determines X to within a homeomorphism. In fact, it is easy to show that if X is 0-dimensional (that is, has a basis consisting of clopen subsets of X) and also T_1, then $\{f^{-1}(x)\colon x \in X,\ f \in N_{co}(X)\}$ is a basis for the closed subsets of X (compare Lemma 2.5 [4] and Lemma 1 above). Moreover, if $a \in X$ and $b, c \in X \setminus a$ (an open subset of X), then there exist clopen subsets H_1 and H_2 of $X \setminus a$ such that $b \in H_1$ and $c \in H_2$. Thus, $E = H_1 \cup H_2$ is a proper clopen subset of X containing b and c, and if $d \in X \setminus E$, then $E_b = (X \setminus E)_b \circ E_d$, a product of nilpotents each with index 2. In other words, if X is a 0-dimensional T_1 space, then, for each $x, y \in X$, $N_{co}(X)$ contains an idempotent constant whose domain contains $\{x, y\}$, and whose range equals $\{x\}$.

The last remark guarantees that if X, Y are 0-dimensional T_1 spaces, then $N_{co}(X)$ and $N_{co}(Y)$ cover X and Y, respectively, and so, by Theorem 2, if ϕ: $N_{co}(X) \to N_{co}(Y)$ is an isomorphism, then there exists a bijection $h\colon X \to Y$ such that $\phi(f) = hfh^{-1}$ for all $f \in N_{co}(X)$. To prove that in this situation we have

$$h\big(f^{-1}(x)\big) = \phi(f)^{-1}\big(h(x)\big)$$

for all $x \in X$, we use the discussion of the last paragraph in showing the equivalence of the following statements (compare with [4], page 239):

$y \in h(f^{-1}(x))$.
$y = h(z)$ and $f(z) = x$.
$y = h(z)$ and $f \circ A_z = A_x$ for some A containing $\{x, z\}$.
$y = h(z)$ and $\phi(f) \circ \phi(A_z) = \phi(A_x)$.
$y = h(z)$ and $\phi(f) \circ B_{h(z)} = B_{h(x)}$ for some B containing $\{h(x), h(z)\}$.
$y = h(z)$ and $\phi(f)(h(z)) = h(x)$.
$y \in \phi(f)^{-1}(h(x))$.

That is, when X, Y are 0-dimensional T_1 spaces, we can establish a result for isomorphisms between $N_{co}(X)$ and $N_{co}(Y)$ that is entirely similar to Lemma 2 above. We put all this together in

THEOREM 4. *If X, Y are 0-dimensional T_1 spaces, then ϕ: $N_{co}(X) \to N_{co}(Y)$ is an isomorphism if and only if there exists a homeomorphism $h\colon X \to Y$ such that $\phi(f) = hfh^{-1}$ for all $f \in N_{co}(X)$.*

Our next aim is to provide conditions under which $NI_F(X)$, the semigroup generated by all nilpotent *homeomorphisms* whose domains are closed subsets of a topological space X, determines X to within a homeomorphism of X. However, we first note that in Theorem 2, when we restrict our attention to inverse subsemigroups of $\mathscr{I}(X)$, the semigroup of all injective partial transformations of an arbitrary set X, we obtain Theorem 3.1 of [1]. Since a particular example of an inverse semigroup covering X is the semigroup $I_F(X)$ of all homeomorphisms whose domains are closed subsets of a T_1 space X, we obtain the next result directly from Theorem 2 (compare with Corollary 3.5 in [1]).

COROLLARY 1. *If X, Y are T_1 spaces, then ϕ: $I_F(X) \to I_F(Y)$ is an isomorphism if and only if there exists a homeomorphism h: $X \to Y$ such that $\phi(f) = hfh^{-1}$ for all $f \in I_F(X)$.*

Since $NI_F(X)$ is also an inverse semigroup and since, when X is T_1, $NI_F(X)$ covers X, we readily obtain (using results analogous to Lemmas 1 and 2 to show that h is a homeomorphism)

COROLLARY 2. *If X, Y are regular T_1 spaces, then ϕ: $NI_F(X) \to NI_F(Y)$ is an isomorphism if and only if there exists a homeomorphism h: $X \to Y$ such that $\phi(f) = hfh^{-1}$ for all $f \in NI_F(X)$.*

We now prove a result similar to Corollary 2.9 in [4].

THEOREM 5. *If X, Y are regular T_1 spaces, then any isomorphism from $NI_F(X)$ onto $NI_F(Y)$ has a unique extension to an isomorphism from $I_F(X)$ onto $I_F(Y)$.*

PROOF. Suppose ϕ is an isomorphism from $NI_F(X)$ onto $NI_F(Y)$. By Corollary 2 there is a homeomorphism h: $X \to Y$ such that $\phi(f) = hfh^{-1}$ for every $f \in NI_F(X)$. We extend ϕ to an isomorphism $\bar{\phi}$: $I_F(X) \to I_F(Y)$ by defining $\bar{\phi}(f) = hfh^{-1}$ for each $f \in I_F(X)$; the extension is unique since any isomorphism ψ from $I_F(X)$ onto $I_F(Y)$ that agrees with ϕ on $NI_F(X)$ must also agree with $\bar{\phi}$ on $\{x_x: x \in X\}$, and this suffices to prove that $\psi = \bar{\phi}$.

REMARK 1. As above, it is possible to show that the semigroup generated by all nilpotent homeomorphisms whose domains are *clopen* subsets of a 0-dimensional T_1 space determines that space to within a homeomorphism.

REMARK 2. In general, if X is an arbitrary set and $f \in \mathscr{P}(X)$ is nilpotent, then $hfh^{-1} \in \mathscr{P}(X)$ is nilpotent for every permutation h: $X \to Y$. With this in mind, it readily follows that if X is a regular T_1 space, then the automorphism group of $N_c(X)$ is isomorphic to the group of all homeomorphisms from X into X (compare with Corollary 2.10 [4]). Similar statements could also be made about the automorphism groups of $NI_F(X)$ and $N_{co}(X)$ for suitable spaces X.

In [7] we characterised products of nilpotents in $\mathscr{I}(X)$ as those $f \in \mathscr{I}(X)$ such that $|X \setminus \operatorname{ran} f| \geqslant \operatorname{rank} f$ and $|X \setminus \operatorname{dom} f| \geqslant \operatorname{rank} f$, where $\operatorname{rank} f$ is defined to be $|\operatorname{ran} f|$. Unfortunately, this latter condition does not suffice to ensure that if $f \in I_F(X)$, and f satisfies the condition, then $f \in NI_F(X)$. One problem entails deciding when, for $f \in I_F(X)$, there exists a closed set $Q \subseteq X \setminus \operatorname{dom} f$ and a (nilpotent) homeomorphism $g\colon \operatorname{dom} f \to Q$ (compare the proof of Theorem 3 in [7]). For example, suppose that $X = A \cup B$ is a disjoint union of two sets with cardinal equal to $|X| \geqslant \aleph_0$. If we take X, A and $\square$ as the open subsets of X, then $NI_F(X) = \{\square\}$ since $I_F(X) = \{\iota_X, \iota_B, \square\}$, and, although ι_B satisfies the above-mentioned condition, it cannot be written as a product of non-zero nilpotents in $I_F(X)$ (since there are none!). For a more interesting example, let X be an infinte set and fix a proper subset Y of X for which $|Y| = |X|$. We topologise X with the smallest (T_1) topology under which Y and all finite subsets of X are closed. In this event, $Q \subseteq X$ is closed if and only if Q is finite or $Q = Y \cup F$ for some finite $F \subseteq X$. Suppose $\iota_Y = f_1 \cdots f_n$, a product of nilpotents in $I_F(X)$. Then $Y \subseteq \operatorname{dom} f_n$, and since $f_n \in I_F(X)$, we deduce that $\operatorname{dom} f_n = Y \cup E$ and $\operatorname{ran} f_n = Y \cup F$ for some finite sets $E, F \subseteq X$. However, since f_n can be regarded as a nilpotent element of $\mathscr{I}(Z)$, where $Z = Y \cup E \cup F$, we must have $|Z \setminus \operatorname{dom} f_n| = |F \setminus E| \geqslant \operatorname{rank} f_n = |X|$, which is plainly a contradiction.

Of course if we sufficiently narrow our vision, then everything will work. Let us say that a space X is *bonded* if for every proper closed $A \subseteq X$, there exists some closed $B \subseteq X \setminus A$ such that $|B| = |A|$. Rather surprisingly, such spaces exist in abundance. For example, suppose X, Y are sets such that $|X| > |Y| \geqslant \aleph_0$, and let us topologise X by saying that $A \subseteq X$ is closed if and only if $|A| \leqslant |Y|$. Clearly, X with this topology is a bonded space, and for such X we shall write $X = X_k$ where $k = |Y|$ and call X a *bonded k-space*. (We thank Dr. Peter Jupp for giving us this example.) However, not every bonded space is an X_k for some cardinal k: for instance, any countably infinite set X, topologised by saying that $A \subseteq X$ is closed if and only if A is finite, is a bonded space and does not equal any X_k by definition.

Every bonded k-space is T_1 (obviously) and normal: if $A, B \subseteq X_k$ are disjoint closed sets then $|X \setminus (A \cup B)| = |X|$, and we can write $X \setminus (A \cup B) = P \dot\cup Q$, where $|P| = |Q| = |X|$. In this event, $P \cup A$ and $Q \cup B$ are both open in X_k and contain A and B, respectively. However, not every normal space is a bonded k-space: any set X with the discrete topology is normal and cannot be an X_k for any cardinal $k < |X|$.

Bonded k-spaces also have the property: if $A, B \subseteq X_k$ are closed with the same cardinal, and $f\colon A \to B$ is any bijection, then f is a homeomorphism. For, if $C \subseteq X_k$ is closed, then $f(A \cap C) = B \cap f(C)$ and, by virtue of the cardinality condition, both $f(C)$ and $B \cap f(C)$ are closed in X_k: that is, f is a closed map. Likewise, if $D \subseteq X_k$ is closed, then $f^{-1}(B \cap D) = A \cap f^{-1}(D)$, where $f^{-1}(D)$ is closed. (Note that this is true because f is injective; thus, f is a continuous map.) Finally, observe that since both $|\operatorname{ran} f|$ and $|\operatorname{dom} f|$ are less than k and $k < |X|$, we have $|X \setminus \operatorname{ran} f| = |X \setminus \operatorname{dom} f| = |X|$: that is, by [7] Theorem 3, f is a product of nilpotents in $I_F(X_k)$; in fact, since $(X \setminus \operatorname{ran} f) \cap (X \setminus \operatorname{dom} f)$ has cardinal equal to $|X|$, it is easy to see that f is actually a product of just two nilpotents each with index 2.

3. Total nilpotents

As noted by Kuratowski [3, pp. 285–286], "every 0-dimensional space is homeomorphic to a subspace of the Cantor discontinuum", and "every topological space with cardinal less than $\mathbf{c}$ (the continuum) is 0-dimensional". Since therefore we hae to some extent been working in Section 2 with spaces of cardinal less than $\mathbf{c}$, it seems natural now to focus the light on spaces with cardinal $\mathbf{c}$. In particular, we here study nilpotents defined on $I = [0, 1]$, the closed unit interval. For this context it seems inappropriate to consider "partial" nilpotents. Instead, we let $S_0(I)$ denote the semigroup (under composition) of all continuous self-maps of I that fix 0, and observe that $S_0(I)$ contains as "zero" the mapping 0: $[0, 1] \to 0$. In this section we characterise the "nilpotents" of $S_0(I)$ and show that the product of any two nilpotents is again nilpotent.

An $f \in S_0(I)$ will be called *nilpotent* if $f^m = f \circ \cdots \circ f = 0$ for some $m \geqslant 1$; we say f has *index* m if $f^m = 0$ but $f^{m-1} \neq 0$, and we let $N(I)$ denote the set of all nilpotents in $S_0(I)$.

Clearly nilpotents in this sense are never surjective; to say more about them, we shall use (without mention) the fact that real-valued continuous functions map closed intervals to (possibly degenerate) closed intervals and possess the Intermediate Value Property.

Lemma 3. *If $f \in N(I)$ and $f \neq 0$, then there exist $a > 0$ and $b < 1$ such that $f([0, a]) = 0$ and $f(I) = [0, b]$. Moreover, if $a_f = \sup\{a \in I\colon f([0, a]) = 0\}$, then $f(x) < x$ for all $x > a_f$.*

Proof. Suppose $f^{m+1} = 0$ and $f^m \neq 0$. Then f^m: $[0, 1] \to [u, v]$ for some $u, v \in I$ and, since $f^m(f(0)) = 0$, we deduce that $0 \in [u, v]$. Hence $u \neq v$, and in fact, $u = 0$; since $f^{m+1} = 0$, we obtain $f([0, v]) = 0$ for $v > 0$, and consequently f: $[0, 1] \to [0, b]$ for some $b < 1$.

Now, the set $\{a \in I\colon f([0, a]) = 0\}$ is non-empty and bounded above by 1; hence, a_f exists and it equals 1 if and only if $f = 0$. If $x > a_f > 0$, then $f([0, x]) = [0, c_1]$ for some $c_1 \in I$. If $x \geqslant c_1$, then $f(x) \leqslant c_1 \leqslant x$, and the result

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follows (note that $f(x) \neq x$ since f is nilpotent and $x \neq 0$); on the other hand, if $x < c_1$, then $f([0, c_1]) = [0, c_2]$, and again, if $x \geqslant c_2$, the result follows, while if $x < c_2$, we repeat the argument. This process must stop, since otherwise we obtain a sequence $\{c_n\}$ for which $0 < x < c_n$ and $f^n([0, x]) = [0, c_n]$ for all n, contradicting the nilpotency of f.

We note for future reference that the above proof actually shows a little more: namely, if $f \in N(I)$ and $x > a_f$, then $f([0, x]) = [0, c]$, where $c \leqslant x$.

LEMMA 4. *If $f \in S_0(I)$, and if there exists $a > 0$ such that $f([0, a]) = 0$ and $f(x) < x$ for all $x > a$, then f is nilpotent.*

PROOF. If $a = 1$, then $f = 0$, so we suppose that $a < 1$ and let $f(x_1)$ be a global maximum of f on $[a, 1]$. Then $a < x_1$ (otherwise $f = 0$, contrary to supposition), and $f(x_1) < x_1$. If $f(x_1) \leqslant a$, then $f^2 = 0$ since f: $[0, 1] \to [0, x_1]$. So we suppose that $a < f(x_1)$ and let $f(x_2)$ be a global maximum of f on $[a, f(x_1)]$. Then $f([a, f(x_1)]) = [0, f(x_2)]$, and $f(x_2) < x_2 \leqslant f(x_1)$. If $f(x_2) \leqslant a$, then we obtain $f^3 = 0$. So we suppose $a < f(x_2)$ and repeat the argument. In this fashion we can generate a strictly descending sequence $\{x_n\}$ bounded below by a:

$$a < \cdots \leqslant f(x_n) < x_n \leqslant f(x_{n-1}) < x_{n-1} \leqslant \cdots < x_1.$$

Suppose $y = \lim x_n$. Then $y \geqslant a$: if $y = a$, we have $a \leqslant \lim f(x_n) = f(y) = 0$, contradicting $a > 0$; while if $y > a$, we have $y = \lim f(x_n) = f(y)$, contradicting the fact that $f(x) < x$ whenever $x > a$. Hence, the foregoing process must stop in a finite number of steps and when it does we deduce that f is nilpotent.

We now use the characterisation of nilpotents presented in Lemmas 3 and 4 to show that $N(I)$ is a subsemigroup of $S_0(I)$.

THEOREM 6. *If $f, g \in S_0(I)$ are nilpotent, then fg is also.*

PROOF. If either f or g equals 0 then $fg = 0$. So, we suppose that $f \neq 0$ and $g \neq 0$, in which case $0 < a_g < 1$ and $fg([0, a_g]) = 0$. Let $d = \sup\{z \in I$: $fg([0, z]) = 0\}$. Then $0 < a_g \leqslant d \leqslant 1$, and $d = 1$ if and only if $fg = 0$. If $d < 1$ and $d < x$ then $a_g < x$ and $g(x) < x$ by Lemma 3; if $g(x) \leqslant a_f$ then $fg(x) = 0$, contradicting the choice of d. Hence $a_f < g(x)$ and we obtain $fg(x) < g(x) < x$, as required.

The automorphisms of various subsemigroups of $S(I)$ have been determined in [5] and [8]. We now provide evidence to suggest that every automorphism ϕ of $N(I)$ is "inner": that is, there exists a homeomorphism k: $I \to I$ such that $\phi(f) = kfk^{-1}$ for all $f \in N(I)$ (compare Remark 2 above). For this purpose we let $E(a, b)$ denote the set of all $f \in N(I)$ for which $a_f = a$ and $f([0,1]) = [0, b]$. The next result provides a useful characterisation of the sets $E(a, a)$ that readily shows they are preserved by automorphisms of $N(I)$.

LEMMA 5. *If $f \in N(I)$ then $f \in E(a, a)$ for some $a \in I$ if and only if $fgf = 0$ for all $g \in N(I)$.*

PROOF. Suppose $f \in E(a, a)$ and let $g \in E(b, c)$. If $a \leqslant b$, then $gf = 0$ and if $b < a$ then $g([0, a]) = [0, d]$ where $d \leqslant a$ and so $fgf = 0$. Conversely, suppose $fgf = 0$ for all $g \in N(I)$ where $f \in E(a, b)$ and $f \neq 0$. If $a < b$ then $f(t) \neq 0$ for some t where $a < t < b$; we then let $0 < s < a$ and define a map g via: $g(x) = 0$ for $0 \leqslant x \leqslant a$, $g(x) = s(x - a)/(t - a)$ for $a \leqslant x \leqslant t$, $g(x) = t$ for $b \leqslant x \leqslant 1$, and

$$g(x) = \frac{t(x-t) + s(b-x)}{b-t} \quad \text{for } t \leqslant x \leqslant b.$$

Then $g^3 = 0$ and if $f(w) = b$ then $fgf(w) \neq 0$; a dual argument for the case $b < a$ concludes the proof.

To prove the next result we shall use the obvious fact that whenever $0 < a, b, c < 1$, we have: $a \leqslant b$ if and only if $E(b, c) \cdot E(a, a) = 0$.

LEMMA 6. *Suppose ϕ is an automorphism of $N(I)$. Then there is a strictly increasing function k: $I \to I$ such that $\phi(E(a, b)) = E(k(a), k(b))$ for all a, b with $0 < a, b < 1$.*

PROOF. For $0 < a, b < 1$ we define $k(a) = b$ if and only if $\phi(E(a, a)) = E(b, b)$. From the above remark, $x = y$ if and only if $E(x, x) \cdot E(y, y) = E(y, y) \cdot E(x, x) = 0$: this and Lemma 5 can be used to show that k is a well-defined permutation of $I \setminus \{0, 1\}$. We put $k(0) = 0$ and $k(1) = 1$, and again use the remark to deduce that k is strictly increasing.

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