THE JOURNAL OF SYMBOLIC LOGIC
Volume 84, Number 1, March 2019

ON A QUESTION OF KRAJEWSKI'S

FEDOR PAKHOMOV AND ALBERT VISSER

Abstract. In this paper, we study finitely axiomatizable conservative extensions of a theory U in the case where U is recursively enumerable and not finitely axiomatizable. Stanisław Krajewski posed the question whether there are minimal conservative extensions of this sort. We answer this question negatively.

Consider a finite expansion of the signature of U that contains at least one predicate symbol of arity ≥ 2. We show that, for any finite extension α of U in the expanded language that is conservative over U, there is a conservative extension β of U in the expanded language, such that $\alpha \vdash \beta$ and $\beta \nvdash \alpha$. The result is preserved when we consider either *extensions* or *model-conservative extensions* of U instead of *conservative extensions*. Moreover, the result is preserved when we replace $\dashv$ as ordering on the finitely axiomatized extensions in the expanded language by a relevant kind of interpretability, to wit *interpretability that identically translates the symbols of the U-language*.

We show that the result fails when we consider an expansion with only unary predicate symbols for conservative extensions of U ordered by interpretability that preserves the symbols of U.

Dedicated to Stanisław Krajewski.

§1. Krajewski's question. At the Workshop on Formal Truth Theories in Warsaw, September 28–30, 2017, Stanisław Krajewski asked the following question.

> Consider any theory U of finite signature and suppose that U is not finitely axiomatizable. We expand the language of U by finitely many extra predicate symbols. Can there be a finitely axiomatized α in the expanded language that conservatively extends U such that there is no finitely axiomatized β that conservatively extends U strictly $\dashv$-below α, i.e., such that $\alpha \vdash \beta$ and $\beta \nvdash \alpha$?

In this note, we prove that the answer is *no* for all U in case the expansion contains at least one symbol of arity 2 or larger. In fact, we will prove the desired result as one of several similar results, where, instead of the relation of *conservative extension*, we can also read either *extension* or *model-conservative extension* and where $\dashv$ as ordering on finitely axiomatized extensions can be replaced by some special kinds of interpretability, to wit: either *parameter-free interpretability that identically preserves the symbols of U* or *interpretability with parameters that identically preserves the symbols of U.*

For the case that we expand with only unary symbols, we provide a class of examples that illustrate that the answer may be *yes* for the interpretability orderings. See Section 5. We show, in a sense, that there is just one finite way to say that an

Received January 10, 2018.
2010 *Mathematics Subject Classification.* 03B10, 03B30, 03F25.
Key words and phrases. predicate logic, recursively enumerable theories, interpretability, linear orderings.

0022-4812/19/8401-0015
DOI:10.1017/jsl.2018.72

ordering is infinite in the language of linear orderings plus finitely many extra unary predicates. A consequence of our results is that the finite model property for finitely axiomatized theories of linear order is decidable.

Kleene [Kle52] and Craig & Vaught [CV58] show that, in case U is recursively enumerable and has no finite models, there is a finitely axiomatized α that conservatively extends U in the language expanded with at least one relation symbol of arity ≥ 2. Craig & Vaught prove an even stronger result where the finite theory α that extends U is *model-conservative* over U. For completeness, we reprove the result by Craig & Vaught below. See Section 3.

Craig and Vaught provide an example that shows that, in the case of only unary expansion, there need not be a finitely axiomatized conservative extension. This also follows by a result of Skolem that implies that there is no formula in the language with identity and only unary predicate symbols that has only infinite models—if it has any models at all.

In Appendix 7, we provide some results in the environment of our problem.

§2. Preliminaries.

We work in languages with only relational symbols. This restriction is not really a limitation since we can simulate the presence of terms using the well-known term-unwinding algorithm. We work in languages with identity as logical symbol.

Suppose U is a theory of finite signature Σ_U and let Θ be a finite signature disjoint from Σ_U. We use $A, B, \ldots$ for sentences of signature Σ_U and $\alpha, \beta, \ldots$ for sentences of signature $\Sigma_U + \Theta$. We will confuse sentences with finitely axiomatized theories. Thus, we will write, e.g., both $\alpha + A$ and $(\alpha \wedge A)$.

We write M_Θ for the maximal arity of a symbol in Θ.

2.1. Theories.

We will employ a number of specific theories in our paper.

The theory INF is the theory in the language of identity that has, for every n, an axiom that says 'there are at least n elements', or,

$$\exists x_0 \cdots \exists x_{n-1} \bigwedge_{i<j<n} x_i \neq x_j.$$

The theory S^1_2 is the weak arithmetic of p-time computability. See, e.g., [Bus86] or [HP93] for a description.

The theory AS or Adjunctive Set Theory has the following axioms.

$\mathsf{AS1}$. $\exists x\, \forall y\, y \notin x$

$\mathsf{AS2}$. $\forall x\, \forall y\, \exists u\, \forall v\, (v \in u \leftrightarrow (v \in x \vee v = y))$.

We refer the reader to [Vis13] for further information about AS.

We can interpret S^1_2 in AS. We fix one such interpretation N. We can arrange it so that assignments for formulas coded in N have desirable properties that make a definition of satisfaction meaningful. We can also arrange that, in the obvious interpretation of AS in the hereditarily finite sets, $\mathbb{HF}$, the interpretation of N is the standard numbers.

Let a signature Σ be given. We define the finitely axiomatized theory C_Σ in the Σ-language expanded with one fresh binary relation symbol R as follows: Let $x \in y :\leftrightarrow x\, R\, y \wedge \neg\, y\, R\, y$ and $\mathsf{sat}(x, y) := \exists z\, (z\, R\, z \wedge \langle x, y\rangle\, R\, z)$. Here, $\langle \cdot, \cdot \rangle$ is the Kuratowski $\in$-pairing. We take the following axioms for C_Σ.

- AS plus the version of extensionality for $\in$ that treats x such that $x\ R\ x$ as urelements:

$$\forall x, y\,((\neg\, x\ R\ x \wedge \neg\, y\ R\ y) \to (x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))).$$

- sat is a satisfaction predicate for the Σ-language with commutation conditions for Σ-formulas coded in N.

THEOREM 2.1. *Every infinite model $\mathcal{M}$ of signature Σ can be expanded to a model $\mathcal{M}^\star$ of C_Σ. Moreover, we can do this in such a way that the interpretation of* N *in $\mathcal{M}^\star$ is isomorphic to the standard numbers.*

PROOF. One can construct such an expanded model in many ways. We just sketch one such way. Consider a set of urelements X of cardinality $|\mathcal{M}|$. We consider $\mathbb{HF}(X)$. Clearly, the set Y consisting of X plus the hereditarily finite sets over X again has cardinality $|\mathcal{M}|$. Thus, we may identify, via some bijection, the domain of $\mathcal{M}$ with Y. Given this identification, the predicate sat is fixed, where we use the finite sets to code the sequences needed for the satisfaction predicate. We define:

- $m\ R\ m'$ iff (m' is a set and $m \in m'$) or (m' is an urelement and ($m = m'$ or, for some n and n', ($m = \langle n, n' \rangle$ and $\mathsf{sat}(n, n')$)))).

We now easily see that we can recover $\in$ and sat from R in the promised way. ⊣

We will write C_U for C_{Σ_U}.

REMARK 2.2. An alternative way to construct a functional equivalent of C_Σ is to expand Σ with two primitive predicates $\in$ and sat and employ a theorem of Tarski to reduce the two predicates to one in a definitionally equivalent way. See [Tar54]. This strategy is employed by Craig & Vaught in [CV58].

The theory TiS, or, *tiny set theory*, is defined as follows:

TiS1. Extensionality:
$\forall x\,\forall y\,(x = y \leftrightarrow \forall z\,(z \in x \leftrightarrow z \in y)).$

TiS2. Restricted Adjunction of Elements:
$\forall x\,\forall y\,\forall z\,(y \in z \to \exists u\,\forall v\,(v \in u \leftrightarrow (v \in x \vee v = y))).$

TiS3. Foundation for Sets:
$\forall x\,\forall y\,(y \in x \to \exists z\,(z \in x \wedge \forall v\,(v \in z \to v \notin x))).$

We have that following insights.

THEOREM 2.3. *Let $\mathcal{M}$ be a finite model of* TiS. *Then, $\mathcal{M}$ is isomorphic with $\langle \wp X, \in \rangle$, where X is a pure finite transitive set.*

PROOF. Consider a finite model $\mathcal{M}$ of TiS. Let m be a set in $\mathcal{M}$ with the maximal number of elements. By TiS2, it follows that every element of some other set is in m. Thus, either m is empty or its $\in$-minimal element, guaranteed by TiS3, is empty. Given that we have the empty set, we can use TiS2 to build any subset of m. Clearly, the relation $\in^{\mathcal{M}}$ restricted to the set of all k such that $k \in^{\mathcal{M}} m$ is well-founded and extensional. Moreover, if $n \in^{\mathcal{M}} n' \in^{\mathcal{M}} m$, then $n \in^{\mathcal{M}} m$.

Thus, the set of $k \in^{\mathcal{M}} m$ equipped with $\in^{\mathcal{M}}$ is isomorphic to X equipped with $\in$, for some finite transitive set X. Hence, the model $\mathcal{M}$ is isomorphic to the powerset of X equipped with $\in$. ⊣

THEOREM 2.4. *Suppose S is a Σ^0_1-sentence. There is a translation $S \mapsto \widetilde{S}$ of the arithmetical to the set-theoretical Σ_1-formulas, such that: S is true iff, there is a pure transitive, hereditarily finite set X, such that $\langle \wp X, \in\rangle \models \widetilde{S}$.*

PROOF. A set-theoretical Σ_1-sentence F is true, in $\langle \wp X, \in\rangle$ for a hereditarily finite set X, iff it is true in $\mathbb{HF}$. Thus, it will be sufficient to construct $\widetilde{S}$ such that $\mathbb{HF} \models \widetilde{S}$ iff S is true.

The proof is basically a careful translation of S to the set theoretical language. One way of doing that is by using the realization in Barwise's book [Bar17]. We note that $\mathbb{HF}$ is a model of KPU. Barwise embeds natural number arithmetic in KPU as the arithmetic on finite ordinals. He gives Δ-definitions for predicates $\mathsf{Nat}(x)$ and $\leq$(Section I.5). Moreover, he shows that the functions S (Section I.5), $+$, and $\times$ are Σ-functions (Section I.6). The fact that the constant 0 and the successor function are Σ-definable is trivial. We produce from S a Σ_1-sentence S' of the set-theoretic language expanded by the definitions: we replace all $\leq$-bounded quantifiers in S with $\in$-bounded, e.g., for universal quantifiers $\forall x \leq t\ A(x)$ is replaced by $\forall x \in t\ (A(x) \wedge A(t))$. He proves that, if we extend the language of KPU by Δ-predicates (Lemma I.5.2) and Σ-functions (Lemma I.5.4), then the extension will be conservative and each Σ-formula of the extended language will be equivalent to a Σ-formula of the original. He proves that any Σ-formula could be equivalently transformed to a Σ_1-formula (Theorem I.4.3). Combining these translations, we are able to transform S' to an equivalent Σ_1-sentence $\widetilde{S}$ of the pure set-theoretic language. ⊣

The finite models of TiS can only have cardinalities that are powers of 2. This is too restrictive for our purposes. We will need that, if our small set theory plus $\widetilde{S}$ has a finite model, then it has a model of any greater cardinality and that the construction of this larger model is sufficiently uniform. There are many ways of achieving this. We sketch two of them. The first is to allow elements with loops in the domain. We replace TiS by its relativization to the elements a such that $a \notin a$. Alternatively, we replace identity by extensional equivalence in the axioms. We adopt this second strategy and describe it in a bit more detail.

Let us write $a \approx b$ for $\forall x\,(x \in a \leftrightarrow x \in b)$. We now define the theory:

TiS*1. Congruence:
$\forall x\, \forall x'\, \forall y\, \forall y'\, ((x \approx x' \wedge y \approx y') \to (x \in y \leftrightarrow x' \in y'))$.

TiS*2. Restricted Adjunction of Elements:
$\forall x\, \forall y\, \forall z\, (y \in z \to \exists u\, \forall v\, (v \in u \leftrightarrow (v \in x \vee v \approx y)))$.

TiS*3. Foundation for Sets:
$\forall x\, \forall y\, (y \in x \to \exists z\, (z \in x \wedge \forall v\, (v \in z \to v \notin x)))$.

We modify our mapping $S \mapsto \widetilde{S}$ by replacing $=$ by $\approx$. Say the resulting formula is $\breve{S}$. We write $[S]$ for TiS$^\star + \breve{S}$. It is clear that any model of $[S]$ can be modified into an $[S]$-model of greater cardinality simply by adding extra elements to a $\approx$-equivalence class. We will return to this idea in the proof of Theorem 2.7.

REMARK 2.5. Our use of TiS and TiS* was inspired by Harvey Friedman's use of theories of a number in [Fri07].

2.2. Relations between theories. Suppose Σ and Θ are two disjoint finite signatures. A Σ,Θ-translation τ is given by a number p and a mapping that sends the predicate symbols P of Θ of arity n to formulas $\tau(P)$ of the form $\pi(\vec{v},\vec{w})$, where the $\vec{v},\vec{w}$ are pairwise-disjoint designated variables and $\vec{v}$ has length n and $\vec{w}$ has length p. The translation τ can be lifted to all Σ,Θ-formulas in which the $\vec{w}$ do not occur as free variables as follows:

- If Q is a symbol from Σ: $Q^\tau(\vec{x}\,) :\leftrightarrow Q(\vec{x}\,)$,
- If P is a symbol from Θ: $P^\tau(\vec{x}\,) :\leftrightarrow \tau(P)[\vec{v} := \vec{x}\,]$, where we rename bound variables in $\tau(P)$ if in case they obstruct substitutability of the $\vec{x}$.
- $(\cdot)^\tau$ commutes with the logical connectives where we rename variables of quantifiers if they would bind any of the $\vec{w}$.

Let V and W be Σ,Θ-theories. A Σ,Θ-interpretation $\mathfrak{t} : V \to W$, is given by a Σ,Θ-translation τ and a parameter domain $\mathsf{par}_\mathfrak{t}(\vec{w})$ in the Σ,Θ-language. We demand that $W \vdash \exists\vec{w}\,\mathsf{par}_\mathfrak{t}(\vec{w})$ and that, for all Σ,Θ-sentences α, if $V \vdash \alpha$, then $W \vdash \forall\vec{w}\,(\mathsf{par}_\mathfrak{t}(\vec{w}) \to \alpha^\tau)$. An interpretation is *parameter-free* if the dimension p of the parameter domain associated to its translation τ is zero. In case of parameter-free interpretations, we will simply omit the parameter domain—since it is $\top$ modulo W-provable equivalence.

We write $W \rhd V$ (or, more officially, $W \rhd_{\Sigma,\Theta} V$) if there is an Σ,Θ-interpretation $\mathfrak{t} : V \to W$. We will always suppress the Σ,Θ-subscript since the relevant pair of signatures is, in all cases, contextually given.

We write $W \rhd_{\mathsf{pf}} V$ if there is a parameter-free interpretation $\mathfrak{t} : V \to W$.

We write $\alpha^\mathfrak{t}$ for $\alpha^{\tau_\mathfrak{t}}$, etcetera.

REMARK 2.6. Kentaro Fujimoto in his paper [Fuj10] introduced the notion of *relative truth definability*. This notion is our notion $\rhd_{\mathsf{pf}}$ restricted to expansions with a truth predicate.

We have the following small insight.

THEOREM 2.7. *Consider any finitely axiomatized theory β. Then, $W \rhd \beta$ iff, for some translation τ, we have $W \vdash \exists\vec{w}\,\beta^\tau$.*

PROOF. From left to right is immediate. From right to left, we take as parameter-domain β^τ. ⊣

As a consequence it suffices, in case the target theory of an interpretability claim is finitely axiomatized, to just specify the translation.

REMARK 2.8. Mycielski, Pudlák, and Stern, in their fundamental paper [MPS90], *define* interpretability with parameters for finitely axiomatized theories in the style of Theorem 2.7 without a parameter-domain.

We will need the following insight.

THEOREM 2.9. *Consider any true Σ^0_1-sentence S. Suppose $\mathcal{M}$ is a model of $[S] := \mathsf{TiS}^\star + \check{S}$ of cardinality n. Then, $\exists x_0,\ldots,x_{n-1} \bigwedge_{i<j<n} x_i \neq x_j \rhd [S]$.*

PROOF. Suppose the domain of $\mathcal{M}$ is $m_0,\ldots,m_{n-1}$. We take as parameter-domain $\bigwedge_{i<j<n} w_i \neq w_j$. We take $\in_{\vec{w}}$ as translation of $\in$:

- $x \sim y :\leftrightarrow \bigwedge_{0<i<n}(x \neq w_i \wedge y \neq w_i) \vee \bigvee_{0<i<n}(x = w_i \wedge y = w_i)$.
- $v_0 \in_{\vec{w}} v_1 := \bigvee\{(v_0 \sim w_i \wedge v_1 \sim w_j) \mid i,j < n \text{ and } m_i \in^{\mathcal{M}} m_j\}$.

https://doi.org/10.1017/jsl.2018.72 Published online by Cambridge University Press

It is easy to see that the resulting parameter-domain plus translation witness $\exists x_0, \ldots, x_{n-1} \bigwedge_{i<j<n} x_i \neq x_j \blacktriangleright [S]$. ⊣

Here are some further notions.

- $U \dashv_{\mathsf{c}} \alpha$ iff $\alpha \vdash U$ and, for all A, if $\alpha \vdash A$, then $U \vdash A$. In case $U \dashv_{\mathsf{c}} \alpha$, we say that α is *a conservative extension of* U.
- $U \dashv_{\mathsf{mc}} \alpha$ iff $\alpha \vdash U$ and every model of U can be expanded to a model of α. In case $\alpha \dashv_{\mathsf{mc}} U$, we say that α is *a model-conservative extension of* U.
- $\alpha \vdash\!\!\!\!\not\,\,\, \beta$ iff $\alpha \vdash \beta$ and $\beta \nvdash \alpha$.
- $\alpha \blacktriangleright\!\!\!\!\not\,_{\mathsf{pf}} \beta$ iff $\alpha \blacktriangleright_{\mathsf{pf}} \beta$ and $\beta \not\blacktriangleright_{\mathsf{pf}} \alpha$.
- $\alpha \blacktriangleright\!\!\!\!\not\,\, \beta$ iff $\alpha \blacktriangleright \beta$ and $\beta \not\blacktriangleright \alpha$.

We note the following useful fact.

FACT 2.10. *Let* $\sqsubseteq$ *be one of* $\dashv$, $\dashv_{\mathsf{c}}$, *or* $\dashv_{\mathsf{mc}}$. *Let* $\preceq$ *be one of* $\dashv$, $\blacktriangleleft_{\mathsf{pf}}$, *or* $\blacktriangleleft$. *Suppose* $U \sqsubseteq \alpha$ *and* $U \dashv \beta \preceq \alpha$. *Then,* $U \sqsubseteq \beta$.

§3. Existence.

In this section, we provide a proof of the Kleene-Craig-Vaught result. We also provide some insights in its direct neighborhood.

THEOREM 3.1 (Kleene-Craig-Vaught). *Suppose* U *is a recursively enumerable theory without finite models. We expand the signature of* U *with* Θ *with* $\mathsf{M}_\Theta \geq 2$. *Then, there is an* α *such that* $U \dashv_{\mathsf{mc}} \alpha$.

We note that if U has a finitely axiomatized conservative extension α, then U must be recursively enumerable. So, the existence theorem is best possible as far as the complexity of U is concerned.

PROOF. We first treat the case where we expand U with a binary relation symbol R. Let $S(x)$ be a Σ^0_1-formula that represents the theorems of U. Let true be the truth-predicate that is based on the satisfaction predicate sat of the theory C_U. Let $\alpha := (\mathsf{C}_U \wedge \forall A \in \mathsf{sent}^N_U\, (S^N(A) \to \mathsf{true}(A)))$. Clearly, $\alpha \vdash U$.

Consider any model $\mathcal{M}$ of U. By Theorem 2.1 we can expand U to a model of C_U in which the numbers given by N are standard. It follows that the expansion also satisfies α.

We extend the result to any Θ with $\mathsf{M}_\Theta \geq 2$, by using any predicate $P(\vec{x})$ of Θ of arity ≥ 2 as replacement of R, using the first two argument places to mimic the argument places of R and treating the remaining ones as dummies. We also treat the remaining predicate symbols of Θ as don't care. ⊣

We show how to extend Theorem 3.1 to the case where U has only finitely many finite models modulo isomorphism.

LEMMA 3.2. *Let* U *be a recursively enumerable theory. We expand the signature of* U *with* Θ *with* $\mathsf{M}_\Theta \geq 2$. *Suppose that, for some* A, *we have* $U \dashv A$ *and* $U + \neg A$ *has no finite models. Then,* U *has a finitely axiomatized model-conservative extension* α *in the expanded language.*

PROOF. We apply Theorem 3.1 to $U + \neg A$. This gives us a β with $U + \neg A \dashv_{\mathsf{mc}} \beta$.

We show that $U \dashv_{\mathsf{mc}} (A \vee \beta) =: \alpha$. Clearly, $A \vee \beta \vdash U$. Let $\mathcal{M}$ be a model of U. In case $\mathcal{M} \models A$, we are done. In case $\mathcal{M} \not\models A$, we have $\mathcal{M} \models U + \neg A$. So, we can expand $\mathcal{M}$ to a model of β, and we are, again, done. ⊣

THEOREM 3.3. *Suppose U is a recursively enumerable theory and suppose U has only finitely many finite models modulo isomorphism. We expand the signature of U with Θ with $\mathsf{M}_\Theta \geq 2$. Then, for some α, we have $U \dashv_{\mathsf{mc}} \alpha$.*

PROOF. We take A the disjunction of the model descriptions of the finite models of U. It is easily seen that A satisfies the conditions of Lemma 3.2. ⊣

We note that the set of finite models of a recursively enumerable theory might be complete Π^0_1. On the other hand, whenever $U \vdash_{\mathsf{c}} \alpha$, the set of finite models for U is NP, by Fagin's theorem. See e.g., [Imm12]. So, not all recursively enumerable U can have a finite extension α in an expanded language such that $U \dashv_{\mathsf{c}} \alpha$.

OPEN QUESTION 3.4. *Is there a recursively enumerable theory U with an NP set of finite models (modulo isomorphism) such that there is no α with $U \dashv_{\mathsf{c}} \alpha$?*

OPEN QUESTION 3.5. *Can we find a recursively enumerable U and an α in an expanded language, such that $U \dashv_{\mathsf{c}} \alpha$, where there is no β such that $U \dashv_{\mathsf{mc}} \beta$?*

In case we work with languages without identity symbol, the situation changes, since we could have $U \dashv_{\mathsf{c}} \alpha$, where U has finite models and α has not. In fact, by slightly modifying the proof of Theorem 3.1, we can, in the identity-free case, find an α such that $U \dashv_{\mathsf{c}} \alpha$, for any recursively enumerable U. We note that this last observation does not hold for the case of $\dashv_{\mathsf{mc}}$. In this case the application of Fagin's Theorem still obtains.

In case we expand the language of U only with unary predicates, we need not be able to find an α such that $U \dashv_{\mathsf{c}} \alpha$, as shown in [CV58].

Finally, we consider what happens when we consider the relation $\dashv$ instead of $\dashv_{\mathsf{c}}$ and $\dashv_{\mathsf{mc}}$. It is clear that we can always find an α such that $U \dashv \alpha$, to wit $\alpha := \bot$. As a consolation, for those who find this example too trifling, we have the following result.

THEOREM 3.6. *Suppose U is a consistent recursively enumerable theory. We expand the signature of U with a binary relation symbol R. Then, there is a consistent α in the expanded language such that $U \dashv \alpha$.*

PROOF. Our theorem is a direct consequence of Theorem 6.1 applied to the relations $\dashv$ of $\sqsubseteq$ and $\dashv$ for $\preceq$ and $\bot$ as initial example of $U \dashv \bot$.

However, we can also reason as follows: Suppose $U + \mathsf{INF}$ is consistent. In that case we can apply Theorem 3.1 to $U + \mathsf{INF}$ to obtain the desired consistent α. If $U + \mathsf{INF}$ is inconsistent, U clearly only has models of size $\leq n$, for some n. In this case, U can be axiomatized by the disjunction of the model descriptions of its finite models. We can now take this disjunction as our α. ⊣

§4. The main theorem.

We formulate our main theorem.

THEOREM 4.1. *Consider any recursively enumerable theory U that is not finitely axiomatizable and any finite expansion Θ of the signature of U with $\mathsf{M}_\Theta \geq 2$. Then, for all $U \dashv \alpha$, there is a β with $U \dashv \beta \ntrianglelefteq \alpha$.*

We note that Theorem 4.1 is not a direct answer to Krajewski's question. However, we will show, in Corollary 4.2, that the negative answer to Krajewski's question is an immediate consequence of Theorem 4.1.

https://doi.org/10.1017/jsl.2018.72 Published online by Cambridge University Press

Proof of Theorem 4.1. We first treat the case where Θ consists of a single binary relation symbol R. At the end of the proof we will describe how to adapt the argument to the more general case.

Suppose U is recursively enumerable and not finitely axiomatizable. We split the proof in two cases:

A. There is no finitely axiomatizable subtheory U_0 of U such that $U_0 + \mathsf{INF} \vdash U$.
B. There is a finitely axiomatizable subtheory U_0 of U such that $U_0 + \mathsf{INF} \vdash U$.

We treat case (A). By Craig's trick, we can find a Σ^b_1 formula $\mathsf{ax}_U(x)$ defining some axiomatization of U. Let true be the truth predicate derived from sat of C_U.

Suppose $U \dashv \alpha$. Note that the arithmetizations of the statements $\varphi \rhd \psi$ can be made to be $\exists\Sigma^b_1$ formulas. Thus, we can find as a fixed point a $\exists\Sigma^b_1$ sentence k such that, in the standard model, we have k iff $\gamma \rhd \alpha$, where $\mathsf{k} = \exists x\, \mathsf{k}_0(x)$ for some $\mathsf{k}_0 \in \Sigma^{\mathsf{b}}_1$ and

$$\gamma :\leftrightarrow \mathsf{C}_U \wedge \exists x \in \mathsf{N}\,(\mathsf{k}^{\mathsf{N}}_0(x) \wedge \forall y <^{\mathsf{N}} x\,(\mathsf{ax}^{\mathsf{N}}_U(y) \to \mathsf{true}(y))).$$

Our first order of business is to prove that k is false.

Suppose k were true. Let it be witnessed by k. Then, we have $\mathsf{C}_U \vdash (\mathsf{k}_0(\underline{k}))^{\mathsf{N}}$. Using the commutation conditions it follows that:

$$U + \mathsf{C}_U \vdash \forall y <^{\mathsf{N}} \underline{k}\,(\mathsf{ax}^{\mathsf{N}}_U(y) \to \mathsf{true}(y)).$$

Thus, $U + \mathsf{C}_U \vdash \gamma$. By compactness, for some finite subtheory U_0 of U, we have $U_0 + \mathsf{C}_U \vdash \gamma$.

Since, by assumption, k is true we have $\gamma \rhd \alpha$. Suppose we have $U \vdash B$, for any Σ_U-sentence B. Then, $\alpha \vdash B$, and thus, $\gamma \rhd B$. Since B is in the U-language, we find $\gamma \vdash B$. Since B was an arbitrary consequence of U, we find $\gamma \vdash U$ and, hence, $U_0 + \mathsf{C}_U \vdash U$. Since, every model of $U_0 + \mathsf{INF}$ can be expanded to a model of C_U, we have $U_0 + \mathsf{INF} \vdash U$, *quod non*, by assumption (A).

We have shown that $\gamma \not\rhd \alpha$. The falsity of k also implies that $\gamma \vdash U$, since γ knows of every standard number that it is not a witness of k. It follows that $U \dashv (\alpha \vee \gamma) \lhd \alpha$ and $(\alpha \vee \gamma) \not\rhd \alpha$. So, $U \dashv (\alpha \vee \gamma) \mathrel{\underset{\neq}{\lhd}} \alpha$. Thus, we can take $\beta := (\alpha \vee \gamma)$.

We treat case (B). We suppose, in order to obtain a contradiction, that $\alpha^\star$ is $\lhd$-minimal such that $U \dashv \alpha^\star$.

Let A be a sentence axiomatizing a finitely axiomatizable subtheory U_0 of U such that $U_0 + \mathsf{INF} \vdash U$. We find, using the Gödel Fixed Point Lemma, a Σ^0_1-sentence k such that, in the standard model, k iff $(A \wedge [\mathsf{k}]) \rhd \alpha^\star$. Here we use R in the role of $\in$.

We claim that k is false. Suppose, to obtain a contradiction, that k is true. Then, we have a finite model of $[\mathsf{k}] = \mathsf{TiS}^\star + \breve{\mathsf{k}}$, say, of size n. By Theorem 2.9, we have:

$$(A + \exists x_0 \cdots \exists x_{n-1} \bigwedge_{i<j<n} x_i \neq x_j) \rhd (A + [\mathsf{k}]).$$

Since $(A \wedge [\mathsf{k}]) \rhd \alpha^\star$, we find:

$$(\dagger) \quad (A + \exists x_0 \cdots \exists x_{n-1} \bigwedge_{i<j<n} x_i \neq x_j) \rhd \alpha^\star.$$

https://doi.org/10.1017/jsl.2018.72 Published online by Cambridge University Press

For finite Σ_U-models $\mathcal{M}$, we fix Σ_U-sentences $\mathsf{D}_{\mathcal{M}}$ that are true in a Σ_U-model $\mathcal{N}$ iff $\mathcal{N}$ is isomorphic to $\mathcal{M}$. We have:

$$U + \forall x_0 \cdots \forall x_{n-1} \bigvee_{i<j<n} x_i = x_j \vdash \bigvee \{\mathsf{D}_{\mathcal{M}} \mid \mathcal{M} \text{ is a } U\text{-model of cardinality} < n\}.$$

Since, whenever $\mathcal{M}$ is finite and $\mathcal{M} \models U$, we have $\mathsf{D}_{\mathcal{M}} \vdash U$, it follows, by the $\blacktriangleleft$-minimality of $\alpha^\star$, that $\mathsf{D}_{\mathcal{M}} \blacktriangleright \alpha^\star$. Ergo,

$$(\ddagger) \quad (U + \forall x_0 \cdots \forall x_{n-1} \bigvee_{i<j<n} x_i = x_j) \blacktriangleright \alpha^\star.$$

We may conclude from ($\dagger$) and ($\ddagger$) that $U \blacktriangleright \alpha^\star$.

We find that, for some finitely axiomatized $U_1 \subseteq U$, we have $U_1 \blacktriangleright \alpha^\star \vdash U$. It follows that $U_1 \vdash U$, in contradiction to the fact that U is not finitely axiomatizable.

Thus, k is false.

Since k is false, we have $(A \wedge [\mathsf{k}]) \not\blacktriangleright \alpha^\star$. Moreover, $A + [\mathsf{k}] \vdash A + \mathsf{INF} \vdash U$. It follows that $\beta := (\alpha^\star \vee (A \wedge [\mathsf{k}]))$ is $\blacktriangleleft\!\!\!\!\!\neq$-below $\alpha^\star$ and $\dashv$-above U. A contradiction.

We can easily extend our proof to the general case where Θ is finite and $\mathsf{M}_\Theta \geq 2$. We simply choose one predicate symbol P with arity ≥ 2. We use the first two argument places of P to simulate the argument places of R. The remaining argument places are treated as dummy variables. All other predicate symbols in Θ are don't care. $\dashv$

We can now answer Krajewski's question.

COROLLARY 4.2. *Consider any theory U that is not finitely axiomatizable and any finite expansion Θ of the signature of U with $\mathsf{M}_\Theta \geq 2$. Then, for all α with $U \dashv_{\mathsf{c}} \alpha$, there is a β with $U \dashv_{\mathsf{c}} \beta \underset{\neq}{\dashv} \alpha$.*

We note that, unlike in the case of Theorem 4.1, we do not need to demand that U is recursively enumerable. If the theory has a finite conservative extension at all, then the theory is automatically recursively enumerable.

PROOF. Suppose U that is not finitely axiomatizable. Let Θ be an expansion of the signature of U with $\mathsf{M}_\Theta \geq 2$. Suppose $U \dashv_{\mathsf{c}} \alpha$. It follows that U is recursively enumerable and $U \dashv \alpha$, and, hence, by Theorem 4.1, there is a γ with $U \dashv \gamma \underset{\neq}{\blacktriangleleft} \alpha$. Let $\beta := (\alpha \vee \gamma)$. Clearly, $U \dashv \beta \dashv \alpha$. Suppose that we would have $\beta \vdash \alpha$. Then, it would follow that $\gamma \vdash \alpha$. From this, we get $\gamma \blacktriangleright \alpha$. Quod non. Hence, $\beta \underset{\neq}{\dashv} \alpha$. Finally, since a subtheory of a conservative theory is conservative, we have $U \dashv_{\mathsf{c}} \beta$. $\dashv$

In fact we can do more than Corollary 4.2. We can replace $\dashv$ in Theorem 4.1 by any of $\dashv_{\mathsf{c}}$ or $\dashv_{\mathsf{mc}}$ and we can replace $\blacktriangleleft$ by any of $\dashv$ or $\blacktriangleleft_{\mathsf{pf}}$. In all these cases, we obtain a valid theorem. The reasoning for the seven further cases is fully analogous to the reasoning for Theorem 4.2. We spell this out in Appendix 6.

We end this Section with some questions.

OPEN QUESTION 4.3. *Consider any theory U that is not finitely axiomatizable and any finite expansion Θ of the signature of U with $\mathsf{M}_\Theta \geq 2$.*

Suppose α is not interpretable (in the full sense of interpretability) in U. Is there an extension β of U, such that α is not interpretable in β?

Open Question 4.4. *Consider any theory U that is not finitely axiomatizable and any finite expansion Θ of the signature of U with $\mathsf{M}_\Theta \geq 2$.*

We take as the length of a proof the number of symbols in the proof written in a fixed finite alphabet. We define $\beta \preceq_{\mathsf{sp}} \alpha$ iff there is a polynomial $P(x)$ such that, for every A (of the language of U), if A is provable from β by a proof of the length n, then A is provable from α by a proof of length $\leq P(n)$.

Can there be a $\preceq_{\mathsf{sp}}$-minimal conservative extension $\alpha^\star$ of U?

§5. The case of unary predicates.

In this section, we provide an example of a class of theories U such that, in the language expanded with a nonempty finite signature of unary predicate symbols, we do have a $\blacktriangleleft_{\mathsf{pf}}$-minimal $\alpha^\star$ such that $U \dashv_{\mathsf{mc}} \alpha^\star$.

Let LIN be the theory of linear order. Suppose Θ is a finite signature consisting of unary predicate symbols and let P be a designated symbol in Θ. We take Λ to be the signature of the theory of order extended with Θ. Let $\mathsf{UB}(P)$ be the property of P that P defines a nonempty set that either does not have a minimal element or does not have a maximal element. More formally:

- $\mathsf{UB}(P) :\leftrightarrow \exists x\, P(x) \;\wedge$
  $(\forall y\, (P(y) \to \exists z\, (P(z) \wedge z < y)) \vee \forall y\, (P(y) \to \exists z\, (P(z) \wedge y < z)))$.

The main theorem of this section is as follows:

Theorem 5.1. *Suppose Θ is a nonempty finite signature consisting of unary predicate symbols and let P be in Θ. Suppose A is a finite extension of LIN and let $U := A + \mathsf{INF}$. We take $\alpha^\star := (A \wedge \mathsf{UB}(P))$. We have:*

a. $U \dashv_{\mathsf{mc}} \alpha^\star$.
b. For all α such that $U \dashv \alpha$, we have $\alpha^\star \blacktriangleleft_{\mathsf{pf}} \alpha$.

So, in a sense, the theorem tells us that, in the extended language, there is only one finite way to say that we exclude finite models of A. We note that Theorem 5.1 with $\blacktriangleleft_{\mathsf{pf}}$ replaced by $\blacktriangleleft$ follows from Theorem 5.1. However the following is open.

Open Question 5.2. *Is there an example of a theory U and an α in the language of U extended with a nonempty finite signature of unary predicate symbols, such that α is a $\dashv$-minimal conservative extension of U?*

We prove (a) of Theorem 5.1 now and postpone the proof of (b) until we have done some preparatory work.

Proof of Theorem 5.1(a). Consider any infinite model $\mathcal{N}$ of A. Since $\mathcal{N}$ cannot be both well-founded and converse well-founded, we can find the desired interpretation of P. $\dashv$

We start our prepratory work with a theorem on linear orderings. Consider the theory LIN of linear order. We extend the signature of LIN. with a signature Θ consisting of finitely many unary predicate symbols. Say the resulting signature is Λ. As usual, we let $\alpha, \beta, \ldots$ range over Λ-language We add two two unary predicates $\triangle_0$ and $\triangle_1$ to the Λ-language. We write $\alpha^{\triangle_i}$ for the result of relativizing all quantifiers in α to $\triangle_i$. We add the following axioms to LIN:

- $\forall x\, (\triangle_0(x) \leftrightarrow \neg\, \triangle_1(x))$.
- $\forall x\, \forall y\, ((\triangle_0(x) \wedge y < x) \to \triangle_0(y))$.

https://doi.org/10.1017/jsl.2018.72 Published online by Cambridge University Press

Note that we allow the $\triangle_i$ to be empty. Relativization to the empty domain is as expected: an existential sentence relativized to the empty domain is false and a universal one is true.

Say the resulting theory (in the language of signature $\Lambda(\triangle_0, \triangle_1)$) is $\mathsf{LIN}^{\mathsf{split}}$. Consider a formula α. Let $\vec{x}; \vec{y}$ be a partition in two parts of a finite set of variables. Let the context $\mathsf{C}(\vec{x}; \vec{y})$ be a conjunction of all formulas $\triangle_0(x_i)$ and $\triangle_1(y_j)$. By a classical result of Rubin the elementary theories of two linear orders with unary predicates determine the elementary theory of their sum [Rub74]. The following theorem is a more explicit form of the reduction.

THEOREM 5.3. *Consider a formula $\alpha(\vec{x}, \vec{y})$ with all free variables shown. Then, over $\mathsf{LIN}^{\mathsf{split}} + \mathsf{C}(\vec{x}; \vec{y})$, the formula $\alpha(\vec{x}, \vec{y})$ is equivalent to a Boolean combination of formulas of the form $\eta^{\triangle_0}(\vec{x})$ (all free variables shown) and $\theta^{\triangle_1}(\vec{y})$ (all free variables shown).*

PROOF. The proof is by induction on α. For the atomic case, we have:

- $\mathsf{LIN}^{\mathsf{split}} + \triangle_0(x) \wedge \triangle_1(y) \vdash x < y \leftrightarrow \top$.
- $\mathsf{LIN}^{\mathsf{split}} + \triangle_0(x) \wedge \triangle_1(y) \vdash y < x \leftrightarrow \bot$.
- $\mathsf{LIN}^{\mathsf{split}} + \triangle_0(x_0) \wedge \triangle_0(x_1) \vdash x_0 < x_1 \leftrightarrow x_0 < x_1$.
- $\mathsf{LIN}^{\mathsf{split}} + \triangle_1(y_0) \wedge \triangle_1(y_1) \vdash y_0 < y_1 \leftrightarrow y_0 < y_1$.
- $\mathsf{LIN}^{\mathsf{split}} + \triangle_0(x) \vdash P(x) \leftrightarrow P(x)$.
- $\mathsf{LIN}^{\mathsf{split}} + \triangle_1(y) \vdash P(y) \leftrightarrow P(y)$.
- Similary, for further unary predicates.

Preservation of the desired property under the propositional connectives is trivial. We treat the case of the existential quantifier. Suppose $\alpha = \exists u\, \alpha_0(u, \vec{x}, \vec{y})$. After some rewriting we have:

- $\mathsf{LIN}^{\mathsf{split}} + \mathsf{C}(u, \vec{x}; \vec{y}) \vdash \alpha_0(u, \vec{x}, \vec{y}) \leftrightarrow \bigvee_{i<n}(\eta_i^{\triangle_0}(u, \vec{x}) \wedge \theta_i^{\triangle_1}(\vec{y}))$.
- $\mathsf{LIN}^{\mathsf{split}} + \mathsf{C}(\vec{x}; u, \vec{y}) \vdash \alpha_0(u, \vec{x}, \vec{y}) \leftrightarrow \bigvee_{j<m}(\kappa_i^{\triangle_0}(\vec{x}) \wedge \nu_i^{\triangle_1}(u, \vec{y}))$.

So, we have:

$$
\begin{aligned}
\mathsf{LIN}^{\mathsf{split}} + \mathsf{C}(\vec{x}; \vec{y}) \vdash \exists u\, \alpha_0(u, \vec{x}, \vec{y}) &\leftrightarrow \exists u\, (\triangle_0(u) \wedge \alpha_0(u, \vec{x}, \vec{y})) \;\vee \\
&\qquad \exists u\, (\triangle_1(u) \wedge \alpha_0(u, \vec{x}, \vec{y})) \\
&\leftrightarrow \exists u\, (\triangle_0(u) \wedge \bigvee_{i<n}(\eta_i^{\triangle_0}(u, \vec{x}) \wedge \theta_i^{\triangle_1}(\vec{y}))) \vee \\
&\qquad \exists u\, (\triangle_1(u) \wedge \bigvee_{j<m} (\kappa_i^{\triangle_0}(\vec{x}) \wedge \nu_i^{\triangle_1}(u, \vec{y}))) \\
&\leftrightarrow \bigvee_{i<n}(\exists u\, (\triangle_0(u) \wedge \eta_i^{\triangle_0}(u, \vec{x})) \wedge \theta_i^{\triangle_1}(\vec{y})) \;\vee \\
&\qquad \bigvee_{j<m} (\kappa_i^{\triangle_0}(\vec{x}) \wedge \exists u\, (\triangle_1(u) \wedge \nu_i^{\triangle_1}(u, \vec{y}))) \\
&\leftrightarrow \bigvee_{i<n}((\exists u\, \eta_i(u, \vec{x}))^{\triangle_0} \wedge \theta_i^{\triangle_1}(\vec{y})) \;\vee \\
&\qquad \bigvee_{j<m} (\kappa_i^{\triangle_0}(\vec{x}) \wedge (\exists u\, \nu_i(u, \vec{y}))^{\triangle_1}).
\end{aligned}
$$

So, we are done. (We note that the calculation also works when one of the domains is empty.) ⊣

Let α be a sentence of signature Λ with $\mathsf{LIN} \dashv \alpha$. We write $y \in [0,x]$ for $y \leq x$ and $y \in (x,\infty)$ for $x < y$. Let η_i, for $i < n$ and θ_j, for $j < \ell$, be the sentences produced for α in Theorem 5.3, when we take Δ_0 to be $[0,x]$ and Δ_1 to be (x,∞). Let s be a 0, 1-sequence of of length ℓ. We define:

- $\beta^s(x)$ is the conjunction of the sentences $\theta_j^{(x,\infty)}$ when $s_j = 1$ and $\neg\,\theta_j^{(x,\infty)}$ if $s_j = 0$.

We will say that x *witnesses* s for $\beta_s(x)$. We note that each x witnesses a unique s. We define the theory F_α as follows:

F_α1. α
F_α2. $\exists x\, \forall y\; x \leq y$ (Zero)
F_α3. $\forall x\, \forall y\, (x < y \to \exists z\, (x < z \wedge \forall u\, (x < u \to z \leq u)))$ (Restricted Successor)
F_α4. $\exists x\, \beta^s(x) \to \exists x\, (\beta^s(x) \wedge \forall y\, (x < y \to \neg\, \beta^s(y)))$, for each $s : \ell \to 2$.
(If s has a witness at all, it has a largest witness.)

THEOREM 5.4. *Let α be a sentence in the language of linear orderings expanded with finitely many unary predicate symbols. Suppose* $\mathsf{LIN} \dashv \alpha$. *Then, α has a finite model iff F_α is consistent.*

PROOF. The left-to-right direction holds since any finite model LIN is also model of F_α.

We prove right-to-left. Suppose F_α is consistent. Then, F_α has a model $\mathcal{N}$. We write $\mathfrak{n}(\mathcal{N})$ for the number of s that are witnessed infinitely often in $\mathcal{N}$. Let $\mathcal{M}$ be a model of F_α such that $\mathfrak{n}(\mathcal{M})$ is minimal. Suppose $\mathcal{M}$ is infinite.

The model $\mathcal{M}$ begins with a copy of ω. By the pigeon-hole principle, there is an s^* that is witnessed infinitely often in this copy of ω. Clearly, the initial copy of ω contains a smallest witness a of s^*. Let b be the maximal witness of s^*.

We now remove the interval $(a,b]$ from $\mathcal{M}$, thus obtaining a new model $\mathcal{M}'$.

We claim that $\mathcal{M}'$ again satisfies F_α.

To prove $(\mathsf{F}_\alpha 1)$, we note that: $\mathcal{M} \models \eta_i^{[0,a]}$ iff $\mathcal{M}' \models \eta_i^{[0,a]}$. Moreover, we have $\mathcal{M} \models \theta_j^{(a,\infty)}$ iff $\mathcal{M} \models \theta_j^{(b,\infty)}$, and $\mathcal{M} \models \theta_j^{(b,\infty)}$ iff $\mathcal{M}' \models \theta_j^{(a,\infty)}$. So, $\mathcal{M} \models \theta_j^{(a,\infty)}$ iff $\mathcal{M}' \models \theta_j^{(a,\infty)}$. Hence, $\mathcal{M}' \models \alpha$.

The preservation of $(\mathsf{F}_\alpha 2)$ and $(\mathsf{F}_\alpha 3)$ is immediate.

Finally, consider any s that is witnessed in $\mathcal{M}'$. In case s only has $\mathcal{M}'$-witnesses in $[0,a]$ we are done, since $[0,a]$ is finite. In case s has an $\mathcal{M}'$-witness in (a,∞), then it has an $\mathcal{M}$-witness in (a,∞), since the question whether c is a witness only depends on what happens above c. It follows that s has a maximal $\mathcal{M}$-witness in (a,∞), and, hence, s has a maximal $\mathcal{M}'$-witness in (a,∞). This gives us $(\mathsf{F}_\alpha 4)$.

We note that if s is witnessed infinitely often in $\mathcal{M}'$, then it is witnessed infinitely often in $\mathcal{M}$. On the other hand, s^* is not witnessed in (b,∞), so it is only witnessed finitely often in $\mathcal{M}'$. Thus, $\mathfrak{n}(\mathcal{M}') < \mathfrak{n}(\mathcal{M})$. This contradicts the minimality of $\mathfrak{n}(\mathcal{M})$.

We may conclude that $\mathcal{M}$ must be finite. ⊣

Since, Theorem 5.4 tells us that the property of α having a finite model is both recursively enumerable and corecursively enumerable, we have the following corollary:

COROLLARY 5.5. *Suppose α is a sentence in the language of* LIN *expanded with finitely many unary predicate symbols and* $\mathsf{LIN} \dashv \alpha$. *Then, it is decidable whether α has a finite model.*

OPEN QUESTION 5.6. *Suppose α is a sentence in the language of* LIN *expanded with finitely many unary predicate symbols and* $\mathsf{LIN} \dashv \alpha$. *Is there a better algorithm than the one suggested for Corollary* 5.5 *to determine whether α has a finite model*?

We write:

- $\gamma\langle\delta\rangle\theta$ iff $(\gamma \wedge \delta) \vee (\theta \wedge \neg\delta)$.

Thus, $\gamma\langle\delta\rangle\theta$ means: γ if δ, else θ.

We are now ready and set to prove Theorem 5.1(b).

PROOF OF THEOREM 5.1(b). Let $\mathsf{LIN} \dashv A$, $U := A + \mathsf{INF}$ and $\alpha^\star := (A + \mathsf{UB}(P))$. Suppose $\alpha \vdash U$ and, suppose, to get a contradiction, that (‡) $\alpha \not\blacktriangleright_{\mathsf{pf}} \alpha^\star$. We note that this is equivalent to $\alpha \not\blacktriangleright_{\mathsf{pf}} \mathsf{UB}(P)$, since A is in the $<$-language.

We show that the theory $W := \alpha + \{\neg\,\mathsf{UB}(\phi) \mid \phi \in \mathsf{Form}^1_\Lambda\}$ is consistent. Here Form^1_Λ is the set of Λ-formulas with at most the variable v free. Suppose W were inconsistent. Then, by compactness, we would have $\alpha \vdash \bigvee_{i<k} \mathsf{UB}(\phi_i)$, for some k and for some choice of the ϕ_i.

Let $\psi := \phi_0\langle\mathsf{UB}(\phi_0)\rangle(\phi_1\langle\mathsf{UB}(\phi_1)\rangle(\ldots))$. Clearly, it would follow that $\alpha \vdash \mathsf{UB}(\psi)$ and, hence, $\alpha \blacktriangleright_{\mathsf{pf}} \mathsf{UB}(P)$. *Quod non*, by Assumption (‡).

Let $\mathcal{M}$ be a model of W. In W every definable nonempty set has both a maximum and a minimum. We verify that $\mathcal{M}$ satisfies F_α. Clearly, $\mathcal{M}$ satisfies ($\mathsf{F}_\alpha 1$) and ($\mathsf{F}_\alpha 2$). Suppose a is not maximal and a has no direct successor. Then, there is a minimal such element $a^\star$. Since $a^\star$ is definable, there is a least $b > a^\star$. But this b must be the direct successor of $a^\star$. A contradiction. So, every nonmaximal a has a direct successor. This gives us ($\mathsf{F}_\alpha 3$). Finally, ($\mathsf{F}_\alpha 4$) is again immediate.

We have shown that F_α is consistent and, hence, by Theorem 5.4, has a finite model. A contradiction. We may conclude that Assumption (‡) is false. ⊣

§6. Appendix A: Proof of nine cases.

We formulate the generalized version of our main theorem.

THEOREM 6.1. *Consider any recursively enumerable theory U that is not finitely axiomatizable and any finite expansion Θ of the signature of U with $\mathsf{M}_\Theta \geq 2$. Let $\sqsubseteq$ be one of $\dashv$, $\dashv_{\mathsf{c}}$, $\dashv_{\mathsf{mc}}$, and let $\preceq$ be one of $\dashv$, $\blacktriangleleft_{\mathsf{pf}}$, or $\blacktriangleleft$.*

Then, for all $\alpha \sqsupseteq U$, there is a β with $\alpha \succ \beta \sqsupseteq U$.

We note that Theorem 6.1 tells us that, if U is not finitely axiomatizable, then $\{\alpha \mid U \sqsubseteq \alpha\}$ has no $\preceq$-minimal element, if $\sqsubseteq$ is one of $\dashv$, $\dashv_{\mathsf{c}}$, and $\dashv_{\mathsf{mc}}$ and if $\preceq$ is one of $\dashv$, $\blacktriangleleft_{\mathsf{pf}}$, and $\blacktriangleleft$. Since $\{\alpha \mid U \sqsubseteq \alpha\}$ is closed under $\preceq$-infima, an element is $\preceq$-minimal in $\{\alpha \mid U \sqsubseteq \alpha\}$ iff it is a $\preceq$-minimum. Thus, Theorem 6.1 tells us that, if U is not finitely axiomatizable, then $\{\alpha \mid U \sqsubseteq \alpha\}$ has no $\preceq$-minimum.

The following two lemmas allow us to derive the nine cases of Theorem 6.1 from Theorem 4.1.

https://doi.org/10.1017/jsl.2018.72 Published online by Cambridge University Press

LEMMA 6.2. *Suppose, for some* $\sqsubseteq_0$ *and* $\sqsubseteq_1$, *we have*:

i. *For all* α, *we have*: *if* $U \sqsubseteq_0 \alpha$, *then* $U \dashv \alpha$.
ii. *For all* α, *we have*: *if* $U \sqsubseteq_1 \alpha$, *then* $U \sqsubseteq_0 \alpha$.
iii. *For all* α, β, *we have*: *if* $U \sqsubseteq_1 \alpha$ *and* $U \dashv \beta \preceq \alpha$, *then* $U \sqsubseteq_1 \beta$.

Suppose (a): *for all* α *such that* $U \sqsubseteq_0 \alpha$, *there is a* β *with* $U \sqsubseteq_0 \beta \prec \alpha$. *Then, we have* (b): *for all* α *such that* $U \sqsubseteq_1 \alpha$, *there is a* β *with* $U \sqsubseteq_1 \beta \prec \alpha$.

PROOF. Suppose we have (a) and $U \sqsubseteq_1 \alpha$. Then, by (ii), $U \sqsubseteq_0 \alpha$. Thus, by (a), there is a β with $U \sqsubseteq_0 \beta \prec \alpha$. By (i), we find $U \dashv \beta \prec \alpha$. Since, $U \sqsubseteq_1 \alpha$, we may conclude, by (iii), that $U \sqsubseteq_1 \beta \prec \alpha$. ⊣

LEMMA 6.3. *Suppose, for some* $\preceq_0$ *and* $\preceq_1$, *we have*:

i. *For all* α *and* β, *we have*: *if* $\beta \dashv \alpha$, *then* $\beta \preceq_1 \alpha$.
ii. *For all* α *and* β, *we have*: *if* $\beta \preceq_1 \alpha$, *then* $\beta \preceq_0 \alpha$.
iii. $\preceq_1$ *is transitive.*
iv. *For all* α, *if* $U \sqsubseteq \alpha$, *then* $U \dashv \alpha$.
v. *For all* α *and* β, *if* $U \sqsubseteq \alpha$ *and* $U \dashv \beta \dashv \alpha$, *then* $U \sqsubseteq \beta$.

Suppose (a): *for all* α *such that* $U \sqsubseteq \alpha$, *there is a* β *with* $U \sqsubseteq \beta \prec_0 \alpha$. *Then, we have* (b): *for all* α *such that* $U \sqsubseteq \alpha$, *there is a* β *with* $U \sqsubseteq \beta \prec_1 \alpha$.

PROOF. Suppose we have (a) and $U \sqsubseteq \alpha$. Then, by (a), there is γ with $U \sqsubseteq \gamma \prec_0 \alpha$. We find $\beta := (\alpha \vee \gamma) \dashv \alpha$. Hence, by (i), $\beta \preceq_1 \alpha$.

Suppose we would have $\alpha \preceq_1 \beta$. We have, $\beta \dashv \gamma$, and, hence, by (i), $\beta \preceq_1 \gamma$. So, by (iii), we would have $\alpha \preceq_1 \gamma$. But then, by (ii), we would have $\alpha \preceq_0 \gamma$. *Quod non*. We may conclude that $\beta \prec_1 \alpha$.

Finally, we have $U \sqsubseteq \alpha$ and $U \sqsubseteq \gamma$. Hence, by (iv), $U \dashv \alpha$ and $U \dashv \gamma$. It follows that $U \dashv \beta \dashv \alpha$. But then, by (v), we have $U \sqsubseteq \beta$. ⊣

PROOF. Proof of Theorem 6.1 We note that, if we interpret $\preceq$ as any of $\dashv$, $\lhd_{\mathsf{pf}}$, or $\lhd$, then any of the pairs $\dashv, \dashv_{\mathsf{c}}$ and $\dashv_{\mathsf{c}}, \dashv_{\mathsf{mc}}$ satisfies the assumptions (i)-(iii) of Lemma 6.2. By applying Lemma 6.2, we find that in order to prove Theorem 6.1, it suffices to prove the cases where $\sqsubseteq$ is $\dashv$.

We also note that, if we interpret $\sqsubseteq$ as any of $\dashv$, $\dashv_{\mathsf{c}}$, or $\dashv_{\mathsf{mc}}$, then any of the pairs $\lhd_{\mathsf{pf}}, \dashv$ and $\lhd, \lhd_{\mathsf{pf}}$ satisfies the assumptions (i)-(v) of Lemma 6.3. By applying Lemma 6.3, we find that in order to prove Theorem 6.1, it suffices to prove the cases where $\preceq$ is $\lhd$.

Thus, we may conclude that we have Theorem 6.1. ⊣

§7. Appendix B: Some results in the environment of our problem. The following result is Theorem 5.3 of [Vis17a].

THEOREM 7.1. *Let* A *and* U *be theories, where* A *is finitely axiomatized and* U *is recursively enumerable and sequential. Suppose* $A \ntrianglerighteq U$. *Then, there is a finitely axiomatized theory* B *such that* $A \ntrianglerighteq B \ntrianglerighteq U$. *Moreover, if* A *is sequential,* B *is sequential too.*

OPEN QUESTION 7.2. *Can we extend Theorem* 7.1 *to a wider class of theories* U?

The following result is Theorem 2 of [Vis17b]. The theory R is the Tarski-Mostowski-Robinson theory R from [TMR53].

THEOREM 7.3. *Suppose* $\mathsf{R} \subseteq A$, *where* A *is finitely axiomatized and consistent. Then, there is a finitely axiomatized* B *such that* $\mathsf{R} \subseteq B \subseteq A$ *and* $B \ntriangleright A$.

https://doi.org/10.1017/jsl.2018.72 Published online by Cambridge University Press

EXAMPLE 7.4. It is very well possible that a nonfinitely axiomatizable theory has a minimal finite extension in the same language w.r.t. $\dashv$. An example is Peano Arithmetic that has the inconsistent theory as its only finite extension in the same language.

If the reader objects to having the inconsistent theory as an example, let e.g., A be the conjunction of the axioms of EA plus $\Box_{\mathsf{PA}}\bot$. Let U be the theory axiomatized by axioms $B \vee A$, where B is an axiom of PA. Clearly, A is a finite consistent extension of U. Suppose C is another such extension. We note that $C \wedge \neg A$ extends PA, so $C \wedge \neg A \vdash \bot$ and, hence, $C \vdash A$.

§8. Appendix C: List of questions.

Q1. It there a recursively enumerable theory U with an NP set of finite models (modulo isomorphism) such that there is no α with $U \dashv_{\mathsf{c}} \alpha$? (This is Question 3.4.)

Q2. Can we find a recursively enumerable U and an α in an expanded language, such that $U \dashv_{\mathsf{c}} \alpha$, where there is no β such that $U \dashv_{\mathsf{mc}} \beta$? (This is Question 3.5.)

Q3. Consider any theory U that is not finitely axiomatizable and any finite expansion Θ of the signature of U with $\mathsf{M}_\Theta \geq 2$. Suppose α is not interpretable (in the full sense of interpretability) in U. Is there an extension β of U, such that α is not interpretable in β? (This is Question 4.3.)

Q4. Consider any theory U that is not finitely axiomatizable and any finite expansion Θ of the signature of U with $\mathsf{M}_\Theta \geq 2$. We take as the length of a proof the number of symbols in the proof written in a fixed finite alphabet. We define $\beta \preceq_{\mathsf{sp}} \alpha$ iff there is a polynomial $P(x)$ such that, for every A (of the language of U), if A is provable from β by a proof of the length n, then A is provable from α by a proof of length $\leq P(n)$. Can there be a $\preceq_{\mathsf{sp}}$-minimal conservative extension $\alpha^\star$ of U? (This is Question 4.4.)

Q5. Is there an example of a theory U and an α in the language of U extended with a nonempty finite signature of unary predicate symbols, such that α is a $\dashv$-minimal conservative extension of U? (This is Question 5.2.)

Q6. Suppose α is a sentence in the language of LIN expanded with finitely many unary predicate symbols and $\mathsf{LIN} \dashv \alpha$. Is there a better algorithm than the one suggested for Corollary 5.5 to determine whether α has a finite model? (This is Question 5.6.)

Q7. Can we extend Theorem 7.1 to a wider class of theories U? (This is Question 7.2.)

Acknowledgments. We are grateful to Leszek Kołodziejczyk for an enlightening discussion of the question answered in Section 5. Fedor Pakhomov's research is supported in part by the Young Russian Mathematics award.

REFERENCES

[Bar17] J. BARWISE, ***Admissible Sets and Structures***, Perspectives in Mathematical Logic, vol. 7, Cambridge University Press, Cambridge, 2017.

[Bus86] S. R. BUSS, ***Bounded Arithmetic***, Bibliopolis, Napoli, 1986.

[CV58] W. Craig and R. L. Vaught, *Finite axiomatizability using additional predicates*, this Journal, vol. 23 (1958), no. 3, pp. 289–308.

[Fri07] H. Friedman, Interpretations according to Tarski. This is one of the 2007 Tarski Lectures at Berkeley, 2007. The lecture is available at `http://www.math.osu.edu/~friedman.8/pdf/Tarski1,052407.pdf`.

[Fuj10] K. Fujimoto, *Relative truth definability of axiomatic truth theories*. ***Bulletin of Symbolic Logic***, vol. 16 (2010), no. 3, pp. 305–344.

[HP93] P. Hájek and P. Pudlák, ***Metamathematics of First-Order Arithmetic***, Perspectives in Mathematical Logic, Springer, Berlin, 1993.

[Imm12] N. Immerman, ***Descriptive Complexity***, Springer Science & Business Media, New York, 2012.

[Kle52] S. C. Kleene, *Finite axiomatizability of theories in the predicate calculus using additional predicate symbols*, ***Two Papers on the Predicate Calculus***, Memoirs of the American Mathematical Society, vol. 10, American Mathematical Society, Providence, RI, 1952, pp. 27–68.

[MPS90] J. Mycielski, P. Pudlák, and A. S. Stern, ***A Lattice of Chapters of Mathematics (Interpretations Between Theorems)***, Memoirs of the American Mathematical Society, vol. 84, American Mathematical Society, Providence, RI, 1990.

[Rub74] M. Rubin, *Theories of linear order*. ***Israel Journal of Mathematics***, vol. 17 (1974), no. 4, pp. 392–443.

[Tar54] A. Tarski, *A general theorem concerning the reduction of primitive notions*, this Journal, vol. 19 (1954), pp. 158–159.

[TMR53] A. Tarski, A. Mostowski, and R. M. Robinson, ***Undecidable Theories***, North–Holland, Amsterdam, 1953.

[Vis13] A. Visser, *What is sequentiality*? ***New Studies in Weak Arithmetics*** (P. Cégielski, C. Cornaros, and C. Dimitracopoulos, editors), CSLI Lecture Notes, vol. 211, CSLI Publications and Presses Universitaires du Pôle de Recherche et d'Enseingement Supérieur Paris-est, Stanford, 2013, pp. 229–269.

[Vis17a] ———, *The interpretation existence lemma*, ***Feferman on Foundations*** (G. Jäger and W. Sieg, editors), Outstanding Contributions to Logic, vol. 13, Springer, Cham, 2017, pp. 101–144.

[Vis17b] ———, *On* Q. ***Soft Computing***, vol. 21 (2017), no. 1, pp. 39–56.

DEPARTMENT OF MATHEMATICAL LOGIC
STEKLOV MATHEMATICAL INSTITUTE
8 GUBKINA STREET, MOSCOW 119991
RUSSIA
E-mail: pakhfn@mi.ras.ru

PHILOSOPHY, FACULTY OF HUMANITIES
UTRECHT UNIVERSITY, JANSKERKHOF 13
3512BL UTRECHT, THE NETHERLANDS
E-mail: a.visser@uu.nl