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PROJECTIVE STRUCTURES AND ρ-CONNECTIONS

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Abstract We extend T. Y. Thomas's approach to projective structures, over the complex analytic category, by involving the ρ-connections. This way, a better control of projective flatness is obtained and, consequently, we have, for example, the following application: if the twistor space of a quaternionic manifold P is endowed with a complex projective structure then P can be locally identified, through quaternionic diffeomorphisms, with the quaternionic projective space.

Keywords: complex projective structures; ρ-connections

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Introduction

The classical notion of projective line evolved, for example, into the differential geometric notion of (unparametrized) geodesic. Locally, these are defined through an equivalence class of connections (see Proposition 1.1, below; see, also, [19] for a historical background). There are several ways to give a global (coordinate free) description of the resulting 'projective structures' (see [7] for a nice review of projective structures in the smooth setting). Among these, there is [18] (see [17]) where it is, essentially, shown that any projective structure on a smooth manifold M corresponds to an invariant Ricci flat torsion free connection on $\det(TM)$. However, the extension of this approach over the complex analytic category is nontrivial as, in this case, by [2], the relevant bundles (for example, the tautological line bundle over the complex projective space) can never be endowed with a connection.

Such an extension has been carried over in [14, §7], under the assumption that the canonical line bundle admits an $(n+1)$th root, where n is the dimension of the manifold (see, also, [1] for an extension, of the T. Y. Thomas's approach, over odd dimensional complex manifolds).

In this paper, we work out this extension by involving the ρ-connections (see Definition 1.4, below). The obtained main result (Theorem 2.1) then provides a

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surprisingly simple (and improved) characterization of projective flatness (Corollary 3.1). From the applications, we mention, here, only the following: if the twistor space of a quaternionic manifold P is endowed with a complex projective structure then P can be locally identified, through quaternionic diffeomorphisms, with the quaternionic projective space.

1. Complex projective structures and ρ-connections

In this paper, we work in the category of complex manifolds. (The corresponding extensions over the smooth category are easy to be dealt with.) A good starting point for (complex) projective structures is [12]. The reader interested, also, in the 'almost complex' setting and other related facts, may consult [5].

Recall that two connections on a manifold are *projectively equivalent* if and only if they have the same geodesics (up to parametrizations). Also, any connection on a manifold is projectively equivalent to a torsion free connection, and the following result is well known. For the reader's convenience, we sketch its proof.

Proposition 1.1. *Let ∇ and $\widetilde{\nabla}$ be torsion free connections on M. Then the following assertions are equivalent:*

(i) *∇ and $\widetilde{\nabla}$ are projectively equivalent.*

(ii) *There exists a one-form α on M such that $\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X$, for any local vector fields X and Y on M.*

Proof. For this we only need the equivalence of the following facts, for a symmetric $(1, 2)$ tensor Γ on a vector space V:

(1) $\Gamma(v, v)$ is proportional to v, for any $v \in V$.

(2) There exists $\alpha \in V^*$ such that $\Gamma(u, v) = \alpha(u)v + \alpha(v)u$, for any $u, v \in V$.

Indeed, if $\dim V = 1$ then this is obvious, whilst, if $\dim V \geqslant 2$ and on assuming (1) then, for any $i_1, i_2 = 1, \ldots, \dim V$, we have $\Gamma^{i_1}_{jk} x^j x^k x^{i_2} = \Gamma^{i_2}_{jk} x^j x^k x^{i_1}$, where $(x^i)_i$ is any basis on V^* and $x^i \circ \Gamma = \Gamma^i_{jk} x^j x^k$.

Consequently, $\Gamma^i_{jk} = 0$ if $j \neq i \neq k$. Furthermore, with i fixed, the one-form $\alpha = \frac{1}{2}(\Gamma^i_{ii} x^i + 2\sum_{j \neq i} \Gamma^i_{ij} x^j)$ is well defined (that is, it does not depend on i) and satisfies (2), with $u = v$. □

The following definition is, essentially, classical.

Definition 1.2. A *projective covering* on a manifold M is a family $\{\nabla^U\}_{U \in \mathcal{U}}$, where:

(a) $\mathcal{U}$ is an open covering of M,

(b) ∇^U is a torsion free connection on U, for any $U \in \mathcal{U}$,

(c) ∇^U and ∇^V are projectively equivalent on $U \cap V$, for any $U, V \in \mathcal{U}$.

Two projective coverings are *equivalent* if their union is a projective covering. A *projective structure* is an equivalence class of projective coverings.

For any manifold M, endowed with a projective structure, there exists a representative of it $\{\nabla^U\}_{U\in\mathcal{U}}$ such that, for any $U\in\mathcal{U}$, the connection induced by ∇^U on $\det(TU)$ is flat; such a representative will be called *special*. The existence of special representatives (an essentially known fact) is proved as follows. Let $\{\widetilde{\nabla}^U\}_{U\in\mathcal{U}}$ be any representative of the projective structure. By passing to a refinement of $\mathcal{U}$, if necessary, we may suppose that each $U\in\mathcal{U}$ is the domain of a frame field $(u^1_U,\dots,u^n_U)$ on M, over U, where $\dim M=n$. Let α_U be the local connection form, with respect to $u^1_U\wedge\ldots\wedge u^n_U$, of the connection induced by $\widetilde{\nabla}^U$ on $\det(TU)$. Let $\beta_U=-\frac{1}{n+1}\alpha_U$ and ∇^U be given by $\nabla^U_XY=\widetilde{\nabla}^U_XY+\beta_U(X)Y+\beta_U(Y)X$, for any $U\in\mathcal{U}$ and any local vector fields X and Y on U. Then $\{\nabla^U\}_{U\in\mathcal{U}}$ is as required.

Let $\{\nabla^U\}_{U\in\mathcal{U}}$ be a representative of a projective structure on M. For any overlapping $U,V\in\mathcal{U}$, denote by α_{UV} the one-form on $U\cap V$ which gives $\nabla^V-\nabla^U$, through Proposition 1.1. Then $(\alpha_{UV})_{(U,V)\in\mathcal{U}^*}$ is a cocycle representing, up to a nonzero factor, the obstruction [2] to the existence of a principal connection on $\det(TM)$, where $\mathcal{U}^*=\{(U,V)\in\mathcal{U}\times\mathcal{U}\,|\,U\cap V\neq\emptyset\}$. Recall that this can be defined as the obstruction to the splitting of the following exact sequence of vector bundles

$$0\longrightarrow M\times\mathbb{C}\longrightarrow E\overset{\rho}{\longrightarrow}TM\longrightarrow 0,$$

where $E=\frac{T(\det(TM))}{\mathbb{C}\setminus\{0\}}$ and $\rho:E\to TM$ is the projection induced by the differential of the projection $\det(TM)\to M$.

Let L be a line bundle on M. Denote $E=\frac{T(L^*\setminus 0)}{\mathbb{C}\setminus\{0\}}$, and $\rho:E\to TM$ the projection. Recall that the sheaf of sections of E is given by the sheaf of vector fields on $L^*\setminus 0$ which are invariant under the action of $\mathbb{C}\setminus\{0\}$. Therefore to any local sections s and t of E (defined over the same open set of M) we can associate their bracket $[s,t]$. Then $[\cdot,\cdot]$ is skew-symmetric, satisfies the Jacobi identity and ρ intertwines it and the usual Lie bracket on local vector fields on M; that is, $(E,\rho,[\cdot,\cdot])$ is a Lie algebroid (see [13]).

Remark 1.3. Let L be a line bundle on M and denote $E=\frac{T(L^*\setminus 0)}{\mathbb{C}\setminus\{0\}}$. If we replace L by L^n, for some $n\in\mathbb{Z}\setminus\{0\}$, then in the exact sequence $0\longrightarrow M\times\mathbb{C}\overset{\iota}{\longrightarrow}E\overset{\rho}{\longrightarrow}TM\longrightarrow 0$, we just need to replace ι by $(1/n)\,\iota$.

If F is a vector bundle over M we denote by $\Gamma(F)$ the corresponding sheaf of sections; that is, $\Gamma(U,F)$ is the space of sections of F over U, for any open set $U\subseteq M$.

The following definition is taken from [16] (cf. [6]).

Definition 1.4. (1) Let M be endowed with a vector bundle E, over it, and a morphism of vector bundles $\rho:E\to TM$.

If F is a vector bundle over M a *ρ-connection* on F is a linear sheaf morphism $\nabla:\Gamma(F)\to\Gamma(\mathrm{Hom}(E,F))$ such that $\nabla_s(ft)=\rho(s)(f)\,t+f\nabla_st$, for any local function f on M, and any local sections s of E and t of F.

(2) Suppose (for simplicity) that $\rho:E\to TM$ is the projection, with $E=\frac{T(L^*\setminus 0)}{\mathbb{C}\setminus\{0\}}$ and L a line bundle over M. Then the *curvature form* of a ρ-connection ∇ on F is the section R of $\mathrm{End}(F)\otimes\Lambda^2E^*$ given by $R(s_1,s_2)\,t=[\nabla_{s_1},\nabla_{s_2}]\,t-\nabla_{[s_1,s_2]}\,t$, for any local sections s_1,s_2 of E and t of F.

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If ∇ is a ρ-connection on E then its *torsion* is the section T of $E \otimes \Lambda^2 E^*$ given by $T(s_1, s_2) = \nabla_{s_1} s_2 - \nabla_{s_2} s_1 - [s_1, s_2]$, for any local sections s_1, s_2 of E.

Remark 1.5. (1) With the same notations as in Definition 1.4(1), if L admits a (classical) connection then any ρ-connection on F corresponds to a (non-unique) pair formed of a connection on F and a section of $\mathrm{End}(F) \otimes E^*$.

(2) Let E be a vector bundle over M, let $\rho : E \to M$ be a morphism of vector bundles, and let ∇ be a ρ-connection on E. On defining $\{s_1, s_2\} = \nabla_{s_1} s_2 - \nabla_{s_2} s_1$, for any local sections s_1 and s_2 of E, we obtain that $\{\cdot, \cdot\}$ is bilinear skew-symmetric and $\{s_1, f s_2\} = \rho(s_1)(f) s_2 + f\{s_1, s_2\}$, for any local sections s_1 and s_2 of E, and any local function f on M. Obviously, ρ intertwines $\{\cdot, \cdot\}$ and the usual bracket on $\Gamma(TM)$ if and only if the section [16] (cf. [15]) of $TM \otimes \Lambda^2 E^*$ given by $\mathcal{T}(s_1, s_2) = \rho \circ (\nabla_{s_1} s_2 - \nabla_{s_2} s_1) - [\rho \circ s_1, \rho \circ s_2]$, for any local sections s_1 and s_2 of E, is zero.

Suppose, now, that we are given a bracket $[\cdot, \cdot]$ on E such that $(E, \rho, [\cdot, \cdot])$ is a Lie algebroid. Then we may define T just like in Definition 1.4(2), thus, obtaining $\mathcal{T} = \rho \circ T$. Furthermore, $T = 0$ if and only if $\{\cdot, \cdot\} = [\cdot, \cdot]$, and each one of these two relations implies $\mathcal{T} = 0$.

Any (classical) connection ∇ on F defines a ρ-connection $\widetilde{\nabla}$ given by $\widetilde{\nabla}_s t = \nabla_{\rho(s)} t$, for any local sections s of E and t of F.

However, not all ρ-connections are obtained this way. For example, if M is compact Kähler then a line bundle over M admits a connection if and only if its (first) Chern class with complex coefficients is zero [2].

Nevertheless, any line bundle L over a manifold M is endowed with a *canonical* flat ρ-connection ∇, where $\rho : E \to TM$ is the projection, with $E = \frac{T(L^* \setminus 0)}{\mathbb{C} \setminus \{0\}}$. This can be defined as follows. First, recall that any local section s of E over an open set $U \subseteq M$ can be seen as a $\mathbb{C} \setminus \{0\}$ invariant vector field on $L^* \setminus 0$, whilst any section t of L over U corresponds to a function f_t on $\pi^{-1}(U)$, where $\pi : L^* \setminus 0 \to M$ is the projection. Then, by definition, $\nabla_s t = s(f_t)$.

For another example, let V be a vector space and let L be the dual of the tautological line bundle over the projective space PV. From $L^* \setminus 0 = V \setminus \{0\}$, it follows that $\frac{T(L^* \setminus 0)}{\mathbb{C} \setminus \{0\}} = L \otimes (PV \times V)$. Thus, although PV does not admit a connection, we can associate to it the *canonical* flat ρ-connection given by the tensor product of the canonical ρ-connection of L and the canonical flat connection on $PV \times V$. Note that, the canonical ρ-connection of the projective space is torsion free.

The following fact will be used later on.

Remark 1.6. Let L be a line bundle over M and let V be a finite dimensional subspace of the space of sections of L. Then V induces a section s_V of $L \otimes V^* (= \mathrm{Hom}(M \times V, L))$ given by $s_V(x, s) = s_x$, for any $x \in M$ and $s \in V$. Obviously, the base point set S_V of V is equal to the zero set of s_V. Assume that $S_V = \emptyset$.

Then the differential of the corresponding map $\varphi : M \to PV^*$ is induced by $\nabla s_V : E \to L \otimes V^*$, where $E = \frac{T(L^* \setminus 0)}{\mathbb{C} \setminus \{0\}}$ and ∇ is the tensor product of the canonical ρ-connection of L and the canonical flat connection on $M \times V^*$. This means that, if we, also, denote by $\mathrm{d}\varphi$

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the morphism $TM \to \varphi^*(T(PV^*))$ corresponding to the differential of φ, then $\mathrm{d}\varphi \circ \rho = \rho_V \circ (\nabla s_V)$, where $\rho : E \to TM$ and $\rho_V : L \otimes V^* \to \varphi^*(T(PV^*))$, are the projections.

2. The main result on projective structures

In this section, we prove the following result (cf. [14, 17, 18]).

Theorem 2.1. *Let M be a manifold, $\dim M = n \geqslant 2$, denote $E = \frac{T(\det(TM))}{\mathbb{C}\setminus\{0\}}$ and let $\rho : E \to TM$ be the projection. There exists a natural correspondence between the following:*

(i) *Projective structures on M.*

(ii) *Torsion free ρ-connections ∇ on E satisfying:*

(ii1) $\nabla_{\mathbb{1}} s = -\frac{1}{n+1}\, s$, *for any local section s of E, where $\mathbb{1}$ is the section of E given by $x \mapsto (x, 1) \in M \times \mathbb{C} \subseteq E$;*

(ii2) *The ρ-connection induced by ∇ on $\Lambda^{n+1}E$ corresponds, under the isomorphism $\Lambda^{n+1}E = \Lambda^n(TM)$, with the canonical ρ-connection of $\Lambda^n(TM)$;*

(ii3) $\mathrm{Ric} = 0$, *where* $\mathrm{Ric}(s_1, s_2) = \mathrm{trace}(t \mapsto R(t, s_2)s_1)$, *for any $s_1, s_2 \in E$, with R the curvature form of ∇.*

Proof. Suppose that E is endowed with a torsion free ρ-connection ∇ such that, for any local section s of E, we have $\nabla_{\mathbb{1}} s = -\frac{1}{n+1}\, s$. Then, also, $\nabla_s \mathbb{1} = -\frac{1}{n+1}\, s$, as ∇ is torsion free and $[\mathbb{1}, s] = 0$, for any local section s of E.

We define the geodesics of ∇ to be those immersed curves c in M for which, locally, up to a parametrization, there exists a section s of E, over c, such that $\rho \circ s = \dot{c}$ and $\nabla_s s = 0$ (compare [15, Remark 1.1]). Note that, then, if t is another section of E, over c, such that $\rho \circ t = \dot{c}$ then $t = s + f\mathbb{1}$ for some function f, on the domain of c, and, consequently, $\nabla_t t = 0$ if and only if $f = 0$; that is, $s = t$.

We shall show that for any $x \in M$ and any $X \in T_xM \setminus \{0\}$ there exist a curve c on M and a section s of E, over c, such that $\dot{c}(0) = X$, $\rho \circ s = \dot{c}$, and $\nabla_s s = 0$; in particular, c is a geodesic (in a neighbourhood of x).

For this, let V be the typical fibre of E and let $(P, M, \mathrm{GL}(V))$ be the frame bundle of E; denote by $\pi : P \to M$ the projection. Then ∇ corresponds [16] to a map $C : P \times V \to TP$ satisfying:

$$\begin{aligned} \mathrm{d}\pi(C(u,\xi)) &= \rho(u\xi), \\ C(ua, a^{-1}\xi) &= \mathrm{d}R_a(C(u,\xi)), \end{aligned} \tag{2.1}$$

for any $u \in P$, $a \in \mathrm{GL}(V)$ and $\xi \in V$, and where R_a is the '(right) translation' on P defined by a. Note that, similarly to the classical case, we have

$$\nabla_{u\xi} s = u\, C(u,\xi)(f_s), \tag{2.2}$$

for any local section s of E, any $u \in P$ such that $\pi(u)$ is in the domain of s, and any $\xi \in V$, and where f_s is the equivariant (V-valued) function on P corresponding to s.

For $\xi \in V$, we denote [16] by $C(\xi)$ the vector field on P given by $u \mapsto C(u, \xi)$.

Now, let $x \in M$ and $X \in T_xM \setminus \{0\}$. Choose $u_0 \in P$ and $\xi \in V$ such that $\rho(u_0\xi) = X$ and let c be the projection, through π, of the integral curve u of $C(\xi)$ through u_0. Thus, if we denote $s = u\xi$, then the first relation of (2.1) implies $\rho \circ s = \mathrm{d}\pi(\dot{u}) = \dot{c}$; in particular, $\dot{c}(0) = X$. Furthermore, by (2.2), we have $\nabla_s s = u\,C(u,\xi)(\xi) = 0$, where the second ξ denotes the corresponding constant function along u.

To show that we have constructed, indeed, a projective structure, let $c_U : TU \to E|_U$ be the local section of ρ corresponding to a connection on $\det(TU)$, for some open set $U \subseteq M$ (note that, we may cover M with such open sets U). Then $E|_U = TU \oplus (U \times \mathbb{C})$, where we have identified TU and the image of c_U; in particular, $\rho|_U$ is just the projection from $E|_U$ onto TU. Let ∇^U be the (torsion free) connection on U given by $\nabla^U_X Y = \rho(\nabla_X Y)$, for any local vector fields X and Y on U. Then if we intersect with U any geodesic of ∇ we obtain a geodesic of the projective structure on U, determined by ∇^U.

We have, thus, proved that any torsion free ρ-connection ∇ on E, satisfying the condition $\nabla_{\mathbb{1}} s = -\frac{1}{n+1}\,s$, for any local section s of E, determines a projective structure on M.

Conversely, suppose that M is endowed with a projective structure given by the special projective covering $\{\nabla^U\}_{U\in\mathcal{U}}$.

As ∇^U induces a flat connection on $\det(TU)$, it corresponds to a section c_U, over U, of ρ; furthermore, $c_U \circ [X,Y] = [c_U \circ X, c_U \circ Y]$ for any local vector fields X and Y on U. Therefore there exists a unique $\beta_U \in \Gamma(E^*|_U)$ such that, for any $t \in E|_U$, we have $t = c_U(\rho(t)) + \beta_U(t)\mathbb{1}$.

Let $U, V \in \mathcal{U}$, be such that $U \cap V \neq \emptyset$, and let α_{UV} be the one-form on $U \cap V$ given by Proposition 1.1 applied to $\nabla^U|_{U\cap V}$ and $\nabla^V|_{U\cap V}$. Then, on $U \cap V$, we have $c_V = c_U - (n+1)\alpha_{UV}\mathbb{1}$; equivalently, $(n+1)\alpha_{UV}(\rho(t)) = \beta_V(t) - \beta_U(t)$, for any $t \in E|_{U\cap V}$.

For any $U \in \mathcal{U}$, we define a ρ-connection $\widetilde{\nabla}^U$ on $E|_U$ by

$$\widetilde{\nabla}^U_s t = c_U(\nabla^U_{\rho(s)}(\rho(t)) - \tfrac{1}{n+1}\beta_U(s)\rho(t) - \tfrac{1}{n+1}\beta_U(t)\rho(s)) + (b_U(s,t) + \rho(s)(\beta_U(t)))\mathbb{1},$$

for any local sections s and t of $E|_U$, where b_U is some section of $\odot^2 E^*|_U$; consequently,

$$\begin{aligned}\widetilde{\nabla}^U_s t - \widetilde{\nabla}^U_t s &= c_U([\rho(s),\rho(t)]) + (\rho(s)(\beta_U(t)) - \rho(t)(\beta_U(s)))\mathbb{1}\\ &= [c_U(\rho(s)), c_U(\rho(t))] + (\rho(s)(\beta_U(t)) - \rho(t)(\beta_U(s)))\mathbb{1} = [s,t],\end{aligned}$$

that is, $\widetilde{\nabla}^U$ is torsion free.

Let $U \in \mathcal{U}$, and denote by Ric^U the Ricci tensor of ∇^U defined by $\mathrm{Ric}^U(X,Y) = \mathrm{trace}(Z \mapsto R^U(Z,Y)X)$, for any $X, Y \in TM$, where R^U is the curvature form of ∇. For $s, t \in E|_U$, we define

$$b_U(s,t) = \tfrac{n+1}{n-1}\,\mathrm{Ric}^U(\rho(s),\rho(t)) - \tfrac{1}{n+1}\beta_U(s)\beta_U(t).$$

Then a straightforward computation shows that $\widetilde{\nabla}^U|_{U\cap V} = \widetilde{\nabla}^V|_{U\cap V}$, for any $U, V \in \mathcal{U}$, with $U \cap V \neq \emptyset$. We have, thus, obtained a torsion free ρ-connection ∇ on E which is easy to prove that it satisfies (ii1).

Further, we may suppose that, for any $U \in \mathcal{U}$, there exists an n-form ω_U on U such that $\nabla^U\omega_U = 0$. Consequently, for any $t \in E|_U$, we have $\nabla_t(\rho^*\omega_U) = \frac{n}{n+1}\beta_U(t)\,\rho^*\omega_U$, on $\mathrm{im}\,c_U$.

Now, the isomorphism $\Lambda^n(T^*U) = \Lambda^{n+1}(E^*|_U)$ is expressed by $\omega_U \mapsto \beta_U \wedge \rho^*\omega_U$. Also, (ii1) implies that, for any $t \in E|_U$, we have $\nabla_t(\beta_U \wedge \rho^*\omega_U) = \beta_U(t)\,\beta_U \wedge \rho^*\omega_U$.

On the other hand, the relation $t = c_U(\rho(t)) + \beta_U(t)\mathbb{1}$, for any $t \in E|_U$, means that β_U is the 'difference' between the connection induced by ∇^U on $\Lambda^n(TU)$ and the canonical ρ-connection $\overset{\mathrm{can}}{\nabla}$ on $\Lambda^n(TU)$; equivalently, $\overset{\mathrm{can}}{\nabla}_t \omega_U = \beta_U(t)\,\omega_U$, for any $t \in E|_U$. Thus, ∇ satisfies (ii2).

Finally, let R be the curvature form of ∇. Then a straightforward calculation shows that, on each $U \in \mathcal{U}$, we have

$$R = c_U(\rho^* W^U) + \tfrac{n+1}{n-1}(\rho^* C^U)\,\mathbb{1}, \tag{2.3}$$

where W^U and C^U are the projective Weyl and Cotton–York tensors of ∇^U, respectively, given by

$$W^U(X,Y)Z = R^U(X,Y)Z + \tfrac{1}{n-1}(\mathrm{Ric}^U(X,Z)Y - \mathrm{Ric}^U(Y,Z)X),$$
$$C^U(X,Y,Z) = (\nabla^U_X \mathrm{Ric}^U)(Y,Z) - (\nabla^U_Y \mathrm{Ric}^U)(X,Z),$$

for any $X, Y, Z \in TU$.

Note that, (2.3) implies that ∇ satisfies (ii3). Moreover, condition (ii2) fixes the 'horizontal' part $\rho \circ \nabla$ of ∇ (among the torsion free ρ-connections satisfying (ii1)), whilst (ii3) fixes the 'vertical' part $\beta_U \circ \nabla$, for $U \in \mathcal{U}$. □

Remark 2.2. (1) Suppose that, in Theorem 2.1, there exists a line bundle L such that $L^{n+1} = \Lambda^n(TM)$. Then we may replace $\det(TM)$ by $L^* \setminus 0$, and, by Remark 1.3, condition (ii1) becomes $\nabla_{\mathbb{1}} s = s$, for any local section s of E, as satisfied by the canonical ρ-connection of the projective space. Furthermore, the canonical ρ-connection of the projective space, also, satisfies (ii2) and (ii3), and the corresponding geodesics are the projective lines (as the 'the second fundamental form', with respect to the canonical ρ-connection, of any projective subspace is zero).

(2) Let $\lambda \in \mathbb{C} \setminus \{0\}$ and let M be a manifold. Denote $E = \frac{T(\det(TM))}{\mathbb{C}\setminus\{0\}}$ and let $\rho : E \to TM$ be the projection. Then, similarly to the proof of Theorem 2.1, any torsion free ρ-connection ∇ on E satisfying $\nabla_s \mathbb{1} = \lambda\, s$, for any $s \in E$, defines a projective structure on M.

3. Applications

In this section, first, we explain how the well known characterization of 'projective flatness' can be improved by using our approach.

Corollary 3.1. *Let M be endowed with a projective structure, given by the torsion free ρ-connection ∇, and suppose that there exists a line bundle L over M such that $L^{n+1} = \Lambda^n(TM)$, where $\dim M = n \geqslant 2$.*

Then ∇ is flat if and only if there exists a (globally defined) local diffeomorphism from a covering space of M to $\mathbb{C}P^n$ mapping the geodesics to projective lines.

Proof. Assume, for simplicity, M is simply connected. Also, by Remark 2.2(1), we may suppose that $E = \frac{T(L^*\setminus 0)}{\mathbb{C}\setminus\{0\}}$ so that $\nabla_{\mathbb{1}} s = s$, for any local section s of E. Then, on denoting by V the typical fibre of E, we have that ∇ is flat if and only if $L \setminus 0$ is a reduction to

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$\mathbb{C}\setminus\{0\}$ of the frame bundle of E, where $\mathbb{C}\setminus\{0\}\subseteq \mathrm{GL}(V)$ through $\lambda\mapsto\lambda\,\mathrm{Id}_V$. Equivalently, ∇ is flat if and only if there exists an isomorphism of vector bundles $\alpha: E\to L\otimes V$, preserving the ρ-connections. In particular, if we define $s=\alpha(\mathbb{1})$ then s is a section of $L\otimes V$ which is nowhere zero; note, also, that $\nabla s=\alpha$. Therefore s induces a section of $P(L\otimes V)=M\times PV$ given by $x\mapsto(x,\varphi(x))$, for any $x\in M$, for some map $\varphi: M\to PV$. Moreover, φ is as required, as, by Remark 1.6, its differential is induced by α. The proof is complete. □

Denote by $\mathcal{O}(n)$ the line bundle of Chern number $n\in\mathbb{Z}$ over the projective line.

Corollary 3.2. *Let M be a manifold endowed with a projective structure and an immersion $t:\mathbb{C}P^1\to M$ with normal bundle $k\mathcal{O}(1)\oplus(n-k-1)\mathcal{O}$, where* $\dim M=n\geqslant 2$ *and $k\in\{0,\ldots,n-1\}$.*

Then t is a geodesic, $k=n-1$ and the projective structure of M is flat.

Proof. Denote $E=\frac{T(\det(TM))}{\mathbb{C}\setminus\{0\}}$ and let $\rho: E\to TM$ be the projection. For simplicity, we denote by $F|_t$ the pull back by t of any vector bundle F over M, and, accordingly, by $\rho|_t$ the induced morphism from $E|_t$ to $TM|_t$. We have an exact sequence

$$0\longrightarrow\mathcal{O}\longrightarrow E|_t\xrightarrow{\rho|_t}\mathcal{O}(2)\oplus k\mathcal{O}(1)\oplus(n-k-1)\mathcal{O}\longrightarrow 0.$$

This exact sequence corresponds to $k+2\in\mathbb{C}=H^1(t,\mathcal{O}(-2)\oplus k\mathcal{O}(-1)\oplus(n-k-1)\mathcal{O})$ (the Chern number of $\mathcal{O}(k+2)=\det(TM)|_t$), and, consequently, we must have $E|_t=(k+2)\mathcal{O}(1)\oplus(n-k-1)\mathcal{O}$.

Let ∇ be the ρ-connection on E giving the projective structure of M. The second fundamental form of t, with respect to ∇, is a section of

$$(2\mathcal{O}(1))^*\otimes(2\mathcal{O}(1))^*\otimes(k\mathcal{O}(1)\oplus(n-k-1)\mathcal{O})=4k\mathcal{O}(-1)\oplus 4(n-k-1)\mathcal{O}(-2)$$

and therefore it is zero. Thus, t is a geodesic, and, then, similarly to the proof of [3, Proposition 4], we may assume that t is an embedding.

By using [11], we deduce that M contains a locally complete $(n+k-1)$-dimensional family of embedded projective lines whose normal bundles are $k\mathcal{O}(1)\oplus(n-k-1)\mathcal{O}$. As the space of geodesics sufficiently close to t has dimension $2n-2$, it follows that $k=n-1$.

Let R be the curvature form of ∇ and note that we can see it as a section of $E\otimes\bigotimes^3 E^*$. Then the restriction of R to any embedded projective line, with normal bundle $(n-1)\mathcal{O}(1)$, is a section of $(n+1)\mathcal{O}(1)\otimes\bigotimes^3((n+1)\mathcal{O}(-1))=(n+1)^4\mathcal{O}(-2)$ and therefore it is zero. Consequently, $R=0$ and the proof is complete. □

The first application of Corollaries 3.1 and 3.2 is that if the twistor space of a quaternionic manifold P is endowed with a complex projective structure then P can be locally identified, through quaternionic diffeomorphisms, with the quaternionic projective space.

Also (compare [3]), any projective structure that admits a geodesic which is an immersed projective line must be flat.

Finally, as any Fano manifold is compact simply connected and satisfies the hypothesis of Corollary 3.2 (see [9] and the references therein) from Corollary 3.1 we obtain the following fact [4] : the projective space is the only Fano manifold which admits a projective structure (compare [10, (5.3)], [8], [9]).

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