A look back at a long-forgotten trigonometric function: the versine function and its inverse

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Ask anyone who has studied mathematics to a moderate level how many trigonometric functions there are and one is likely to be presented with a range of answers depending on what the person being asked is most likely to remember. Perhaps the ‘calculator button’ three of sine, cosine, and tangent will come to mind as these are the three trigonometric functions found on any standard scientific calculator. At a stretch, perhaps the names for their respective reciprocals, cosecant, secant and cotangent, will be recalled. Beyond the modern standard six, looking at calculus or trigonometric texts published prior to 1900 one soon discovers others going by strange names such as versine, haversine, or coversine (see, for example, [1, pp. 53, 63]). There are at least six others with as many as perhaps ten to twelve having received a name at one time or another. Today all these additional trigonometric functions considered important enough to grace the pages of texts in centuries past have fallen by the wayside, to be largely forgotten in favour of the modern standard six. Of course the pedant amongst us would say there is only one trigonometric function, the sine function, which currently stands as the preferred fundamental trigonometric entity, with all others being simple variations of this function, and they would not be incorrect in asserting this. But having the current standard six seems about the right balance between the minimalistic on the one hand and convenience on the other hand.

In this paper we intend to look back at one of the forgotten trigonometric functions, the *versed sine* function, or *versine* for short, together with its associated inverse. By focusing on series expansions for the inverse versine function and other closely related functions, we shall see how this can provide a slightly different perspective on a range of interesting infinite sums containing the central binomial coefficients in its summand. Recall that a *central binomial coefficient* is a binomial coefficient given by

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2},$$

where n is a non-negative integer. They are the centrally located binomial coefficients in the even numbered rows of Pascal's triangle. Sums of such coefficients are particularly interesting and were the subject of a paper written in 1985 by the American mathematician Derrick H. Lehmer (1905-1991) which today has become a *cause célèbre* [2]. Here Lehmer defines a series to be *interesting* if its sum can be expressed in closed form in terms of well-known constants.

Until about the beginning the eighteenth century, as hard as it is to believe today, after the sine function the versine function was considered the

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next most important of the trigonometric functions. Historically, one of the primary applications of trigonometry was in the fields of astronomy and later in navigation. To help aid astronomers and navigators, many tables containing incremented functional values for various trigonometric functions were compiled. This importance meant the versine appeared in some of the earliest trigonometric tables. In fact the earliest surviving trigonometric tables are those for sine and versine where values at 3.75° intervals from 0° and 90° to an accuracy of four decimal places are given [3, p. 215]. Found in the astronomical treatise *Aryabhatiya* they were compiled around the beginning of the sixth century by the Indian mathematician and astronomer Aryabhata (476–550).

The versine function and its inverse

To understand the origin of versine, which we will denote by $\text{vers}\,x$, we would do well to remember how historically sine was defined. If a vertical chord is drawn in a unit circle (*AB* in Figure 1) the sine of the angle x was defined as half the length of this chord (the distance *AC* in the figure). The versine of the angle x was therefore defined as the length of the line segment from the centre of the chord to the centre of the arc subtended by the chord (the distance *CD* in the figure). Defining cosine of the angle x as the distance of the line segment from the centre of the circle to the centre of the chord (the distance *OC* in the figure) we see that the sum of $\cos x$ and $\text{vers}\,x$ corresponds to the radius of the circle. On the unit circle this means

$$\text{vers}\,x = 1 - \cos x.$$

Viewed geometrically in the context of the unit circle both sine and versine are distances from the centre of the chord to the edge of the circle with 'versed sine' being just sine but turned through an angle of 90°. Here the term 'versed' comes from the Latin *versus*, meaning turned. In making the turn observe that each distance remains within the confines of the unit circle. Of course today the trigonometric functions at the most elementary level are usually defined in terms of the ratio between two lengths found in a right-angled triangle instead of chords and lengths of line segments found in a circle, making sine, cosine, and tangent the more natural choice for the trigonometric functions compared to the versine function and its other long lost cousins.

From properties of the half angle formula for sine we see that

$$\text{vers}\,x = 1 - \cos x = 2\sin^2\frac{x}{2}. \tag{1}$$

It is immediate that the domain and codomain for $\text{vers}\,x$ are

$$\text{dom}\left(\text{vers}\,x\right) = \mathbb{R} \qquad \text{and} \qquad \text{codom}\left(\text{vers}\,x\right) = 0 \leqslant \text{vers}\,x \leqslant 2.$$

In the definition for the versine function given in (1), and in its codomain, the historical significance of the function is brought to the fore. Firstly, as the value of $\text{vers}\,x$ is always positive (when it is equal to zero, in calculating nothing further needs to be done) its logarithm could always be taken. This

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was particularly important in the days when everything had to be calculated by hand and took appreciable effort, as was the case in navigation. Secondly, versine could save one the computational expense of having to calculate the square of a sine, a particularly common task in the field of navigation.

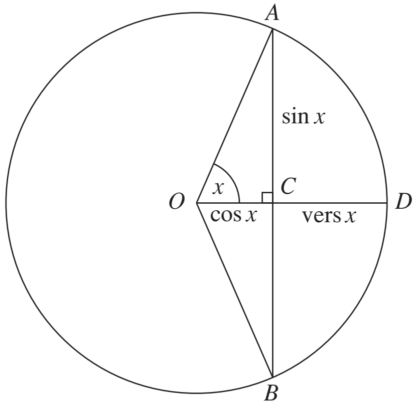

FIGURE 1: Unit circle showing sine, versine and cosine

As we have alluded to, the importance of mathematics in the task of navigation cannot be overstated. A frequent undertaking involves finding the shortest distance between two points on the surface of a sphere. It requires the latitude and longitude at each point on the surface being known. Here the so-called 'haversine' formula is used, the haversine function, $\operatorname{hav} x$, being half or 'ha' the versine function. If r is the radius of a sphere and ℓ the shortest distance between two points on its surface the central angle θ subtended between these points is given by the arc-length formula $\theta = \ell / r$. Finding the central angle θ is more difficult. Given two points (φ_1, ψ_1) and (φ_2, ψ_2) on the sphere, where φ denotes the latitude and ψ denotes the longitude of each point, the central angle can be found by applying the *haversine formula* given by [4]

$$\operatorname{hav}(\theta) = \operatorname{hav}(\varphi_2 - \varphi_1) + \cos(\varphi_1)\cos(\varphi_2)\operatorname{hav}(\psi_2 - \psi_1),$$

a result that comes from spherical trigonometry. Since $2 \operatorname{hav} x = \operatorname{vers} x$, in terms of versine the haversine formula can be written as

$$\operatorname{vers}(\theta) = \operatorname{vers}(\varphi_2 - \varphi_1) + \cos(\varphi_1)\cos(\varphi_2)\operatorname{vers}(\psi_2 - \psi_1).$$

From this result we see that working directly with versine immediately saved navigators the expense of having to calculate two sine-squared terms in each calculation made for the central angle. Not only was this a major saving in terms of time, it also made such calculations less prone to error.

Of course all the properties for the versine function follow immediately from corresponding properties of the cosine function, or the square of the half argument sine function if you prefer, and is no doubt the reason why the versine function has all but been forgotten today. For example, its derivative and Maclaurin series expansion follow directly from (1) using known properties for the cosine function. Thus

$$\frac{d}{dx}\operatorname{vers} x = \sin x \qquad \text{and} \qquad \operatorname{vers} x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}, \; x \in \mathbb{R}.$$

The occasional modern reference to the versine function and at least some of its other long lost counterparts can be found, for example, in [5, p. 78, Eq. 4.3.147], [6, p. 322, Eq. 32.13.4] or [7, pp. 167-168, Ex. 19], though these are quite atypical among present-day texts.

On the interval $[0, \pi]$ an inverse to the versine function can be defined. Taken as the principal branch for the inverse function it is denoted by $\operatorname{vers}^{-1} x$. The versine function and its inverse are plotted in Figure 2. Some special values for the versine function and its inverse are immediate. These are

$$\begin{aligned} \operatorname{vers}(0) &= 0, & \operatorname{vers}^{-1}(0) &= 0, \\ \operatorname{vers}\left(\frac{\pi}{3}\right) &= \frac{1}{2}, & \operatorname{vers}^{-1}\left(\frac{1}{2}\right) &= \frac{\pi}{3}, \\ \operatorname{vers}\left(\frac{\pi}{2}\right) &= 1, & \operatorname{vers}^{-1}(1) &= \frac{\pi}{2}, \\ \operatorname{vers}\left(\frac{2\pi}{3}\right) &= \frac{3}{2}, & \operatorname{vers}^{-1}\left(\frac{3}{2}\right) &= \frac{2\pi}{3}, \\ \operatorname{vers}(\pi) &= 2, & \operatorname{vers}^{-1}(2) &= \pi. \end{aligned}$$

These values for the inverse will be used extensively when it comes to finding values for various infinite sums.

To find the derivative for the inverse versine function we start by setting $y = \operatorname{vers}^{-1} x$. Since $x = \operatorname{vers} y$, differentiating implicitly with respect to x gives

$$\frac{dy}{dx} = \frac{1}{\sin y}.$$

As $\operatorname{vers} y = x = 1 - \cos y$, we see that $\cos y = 1 - x$ giving $\sin y = \sqrt{2x - x^2}$. Here the positive square root is selected since $\operatorname{vers}^{-1} x \geqslant 0$ for $x \in [0, \pi]$. Thus

$$\frac{d}{dx}\left(\operatorname{vers}^{-1} x\right) = \frac{1}{\sqrt{2x - x^2}}. \tag{2}$$

An integral representation for the inverse versine function follows immediately from (2). It is

$$\operatorname{vers}^{-1} x = \int_0^x \frac{dt}{\sqrt{2t - t^2}}. \tag{3}$$

Alternatively, the integral found in (3) can be evaluated by completing the square of the quadratic found in the denominator of the integrand before

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integrating. Doing so we find

$$\operatorname{vers}^{-1} x = \int_0^x \frac{dt}{\sqrt{1 - (1 - t)^2}} = \int_{1-x}^1 \frac{du}{\sqrt{1 - u^2}} = \cos^{-1}(1 - x),$$

after making the substitution $u = 1 - t$. Finally, directly from (1) one can find an expression for the inverse of the versine function in terms of the inverse for the sine function. Putting this all together we have

$$\operatorname{vers}^{-1} x = \cos^{-1}(1 - x) = 2 \sin^{1}\left(\sqrt{\frac{x}{2}}\right). \tag{4}$$

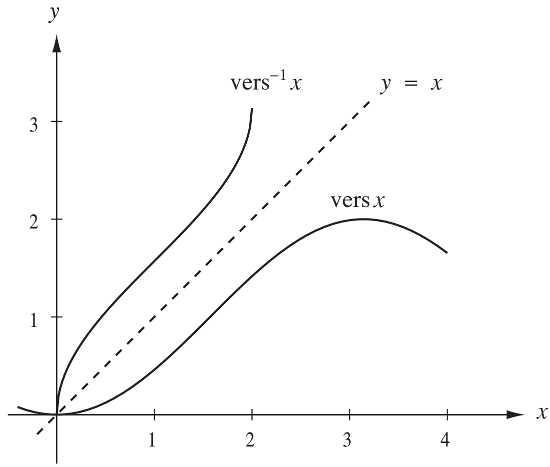

FIGURE 2: Plot of vers x and its inverse $\operatorname{vers}^{-1} x$

It is the inverse of the versine function and series expansions for it and other closely related functions that we shall focus on for the remainder of the paper. In view of (4) it should come as no surprise to see how the inverse versine function parallels closely with the inverse sine function. By taking a look back at this forgotten function, what it provides is an interesting alternative perspective to the standard approach encountered when the inverse sine function is used. At times it may be slightly simpler compared to the standard approach using the inverse sine function. At other times it is just different, but at all times we hope to reveal the hidden beauty of this long-forgotten function.

Series expansions for $\operatorname{vers}^{-1} x$ *and related functions*

In this section we find a number of series expansions related to the inverse versine function. We start by recalling the well-known result for the generating function for the central binomial coefficients:

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^{2n} = \frac{1}{\sqrt{1 - 4x}}, \qquad |x| < \frac{1}{4}.$$

This result follows directly from the generalised binomial theorem. Replacing x with $\frac{1}{8}x$ gives

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{2^{3n}} = \frac{1}{\sqrt{1-\frac{1}{2}x}}, \qquad |x| < 2. \tag{5}$$

From the integral representation for the inverse versine function given in (3), after substituting (5) for the reciprocal of the square root term and interchanging the order of the integration with the summation, which is permissible due to Tonelli's theorem [8, p. 138] as all term involved are positive, one has

$$\begin{aligned}\text{vers}^{-1}x &= \int_0^x \frac{dt}{\sqrt{2t-t^2}} = \int_0^x \frac{dt}{\sqrt{2t}\sqrt{1-\frac{1}{2}t}} = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{3n}}\int_0^x t^{n-1/2}dt \\ &= \sqrt{2x}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{2^{3n}(2n+1)},\end{aligned}$$

yielding

$$\frac{\text{vers}^{-1}x}{\sqrt{2x}} = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{2^{3n}(2n+1)}, \qquad 0 \leqslant x \leqslant 2. \tag{6}$$

The series found in (6) is thought to have been given first by Isaac Newton in a letter sent by his colleague the English mathematician John Collins (1625-1683) on behalf of Newton to the Scottish mathematician and astronomer James Gregory (1638-1675) in December 1670 [9, p. 43]. Here Newton gave it as a corollary to the series expansion he had just found for the inverse sine function. It is important to note that (6) represents a Maclaurin series expansion for the function $\text{vers}^{-1}x/\sqrt{2x}$ rather than a Maclaurin series expansion for the inverse versine function itself. Indeed, no Maclaurin series expansion for the inverse versine function exists. Moving the $\sqrt{2x}$ term to the right-hand side of (6) we see the series expansion that results contains half-order powers of x and is what is technically referred to as a *Puiseux* series [10, p. 2407]. Setting $x = \frac{1}{2}, 1, \frac{3}{2}$, and 2 in (6) we immediately obtain the following sums involving the central binomial coefficients

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{4n}(2n+1)} = \frac{\pi}{3}, \qquad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{3n}(2n+1)} = \frac{\pi}{2\sqrt{2}},$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{3^n}{2^{4n}(2n+1)} = \frac{2\pi}{3\sqrt{3}}, \qquad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n}(2n+1)} = \frac{\pi}{2}.$$

Sums of this type can be found in [2].

Other series involving the inverse versine function can now be readily found from (6). Replacing x with t in (6) before integrating with respect to t from 0 to x one finds

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+1}}{2^{3n}(n+1)(2n+1)} = \int_0^x \frac{\text{vers}^{-1}t}{\sqrt{2t}}\,dt.$$

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The integral that has appeared can be found by integrating by parts. Doing so we find

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+1}}{2^{3n}(n+1)(2n+1)} = \sqrt{2x}\,\text{vers}^{-1}x - \sqrt{2}\int_0^x \frac{dt}{\sqrt{2-t}},$$

or

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+1}}{2^{3n}(n+1)(2n+1)} = \sqrt{2x}\,\text{vers}^{-1}x + 2\sqrt{4-x} - 4,$$

and is valid for $0 \leqslant x \leqslant 2$. Setting $x = \frac{1}{2}, 1, \frac{3}{2}$, and 2 in (7) yields the rather interesting sums

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{4n}(n+1)(2n+1)} = \frac{2\pi}{3} + 4\sqrt{3} - 8,$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{3n}(n+1)(2n+1)} = \frac{\pi}{\sqrt{2}} + 2\sqrt{2} - 4,$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n}(n+1)(2n+1)} = \pi - 2,$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{3^n}{2^{4n}(n+1)(2n+1)} = \frac{4\pi}{3\sqrt{3}} - \frac{4}{3}.$$

It is worth noting the four sums just given are in fact related to the Catalan numbers. Recall the nth Catalan number C_n is defined recursively by

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}, \qquad C_0 = 1.$$

An explicit expression for C_n is known. From [11, p. 4, Eq. 1.6] it is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Thus rewritten in terms of a summand containing the Catalan numbers the above four sums become

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{C_n}{2^{4n}(2n+1)} = \frac{2\pi}{3} + 4\sqrt{3} - 8, \quad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{C_n}{2^{3n}(2n+1)} = \frac{\pi}{\sqrt{2}} + 2\sqrt{2} - 4,$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{C_n}{2^{2n}(2n+1)} = \pi - 2, \qquad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{3^n C_n}{2^{4n}(2n+1)} = \frac{4\pi}{3\sqrt{3}} - \frac{4}{3}.$$

Another related series expansion from (7) can be found by employing a partial fraction decomposition. As

$$\frac{1}{(2n+1)(n+1)} = \frac{2}{2n+1} - \frac{1}{n+1},$$

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the series in (7) can be expressed as

$$\sqrt{2x}\,\mathrm{vers}^{-1}x + 2\sqrt{4-2x} - 4 = 2x\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{2^{3n}(2n+1)} - x\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{2^{3n}(n+1)},$$

or

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{2^{3n}(n+1)} = \frac{4-2\sqrt{4-2x}}{x}, \qquad 0 \leqslant x \leqslant 2, \tag{8}$$

after the series expansion given in (6) has been used. It is a series expansion that does not involve the inverse versine function. Setting $x = \frac{1}{2}, 1, \frac{3}{2}$, and 2 into (8) yields the interesting sums

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{4n}(n+1)} = 8 - 4\sqrt{3}, \qquad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{3n}(n+1)} = 4 - 2\sqrt{2},$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{3^n}{2^{4n}(2n+1)} = \frac{4}{3}, \qquad \sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n}(2n+1)} = 2.$$

Finding still other series expansions, dividing both sides of (7) by x before replacing x with t then integrating again with respect to t from 0 to x gives

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+1}}{2^{3n}(n+1)^2(2n+1)} = 2\int_0^x \frac{\mathrm{vers}^{-1}t}{\sqrt{2t}}\,dt + 2\int_0^x \frac{\sqrt{4-2t}-2}{t}\,dt,$$

or

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+1}}{2^{3n}(n+1)^2(2n+1)} = 2\sqrt{2x}\,\mathrm{vers}^{-1}x + 8\sqrt{4-2x} + 16\log 2$$
$$-8\log\left(\sqrt{4-2x}+2\right) - 16, \tag{9}$$

and is valid for $0 \leqslant x \leqslant 2$. Here the first of the integrals was found by parts when the expression for the series in (7) was obtained while the second of the integrals is elementary. Setting once more $x = \frac{1}{2}, 1, \frac{3}{2}$, and 2 in (9) yields the interesting sums

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{4n}(n+1)^2(2n+1)} = \frac{4\pi}{3} + 16\sqrt{3} - 32 + 16\log\left(8-4\sqrt{3}\right),$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{3n}(n+1)^2(2n+1)} = \pi\sqrt{2} + 8\sqrt{2} - 16 + 8\log\left(4-2\sqrt{2}\right),$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n}(n+1)^2(2n+1)} = 2\pi - 8 + 4\log 2,$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{3^n}{2^{4n}(n+1)^2(2n+1)} = \frac{8\pi}{3\sqrt{3}} - \frac{16}{3} + \frac{16}{3}\log\frac{4}{3}.$$

If instead we multiply both sides of (6) by x before replacing x with t, then integrating with respect to t from 0 to x we obtain

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+2}}{2^{3n}(n+2)(2n+1)} = \frac{1}{\sqrt{2}}\int_0^x \sqrt{t}\,\mathrm{vers}^{-1}\,t\,dt,$$

or after integrating by parts

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+2}}{2^{3n}(n+2)(2n+1)} = \frac{2}{3}x\sqrt{x}\,\mathrm{vers}^{-1}\,x - \frac{2}{3}\int_0^x \frac{t}{\sqrt{2-t}}\,dt.$$

Performing the final integration, which is elementary, one is left with

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+2}}{2^{3n}(n+2)(2n+1)} = \frac{1}{3}x\sqrt{2x}\,\mathrm{vers}^{-1}\,x + \frac{2}{9}\sqrt{4-2x}(x+4) - \frac{16}{9}, \quad (10)$$

a result valid for $0 \leqslant x \leqslant 2$. Setting once more $x = \frac{1}{2}, 1, \frac{3}{2}$, and 2, this time in (10), yields the interesting sums

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{4n}(n+2)(2n+1)} = \frac{2\pi}{9} + 4\sqrt{3} - \frac{64}{9},$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{3n}(n+2)(2n+1)} = \frac{\pi}{3\sqrt{2}} + \frac{20}{9\sqrt{2}} - \frac{16}{9},$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2^{2n}(n+2)(2n+1)} = \frac{\pi}{3} - \frac{4}{9},$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{3^n}{2^{4n}(n+2)(2n+1)} = \frac{4\pi}{9\sqrt{3}} - \frac{20}{81}.$$

One can obviously keep continuing in this manner, provided the integrals that result can be found, generating new and interesting sums containing the central binomial coefficients but we shall stop here and move on to some series expansions containing central binomial coefficients and the Fibonacci numbers.

Some series containing the Fibonacci numbers

A number of series containing products of Fibonacci numbers and central binomial coefficients can be found from the various series expansions involving the inverse versine function that have been given. Recall that the nth Fibonacci number F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$ with $F_0 = 0$ and $F_1 = 1$.

Note that when $x = \frac{1}{2}\varphi^2$ and $x = 1/(2\varphi^2)$, where φ denotes the golden ratio $(1 + \sqrt{5})/2$, we have

$$\mathrm{vers}^{-1}\left(\frac{\varphi^2}{2}\right) = \cos^{-1}\left(1 - \frac{\varphi^2}{2}\right) = \cos^{-1}\left(\frac{1-\sqrt{5}}{4}\right) = \frac{3\pi}{5},$$

and

$$\mathrm{vers}^{-1}\left(\frac{1}{2\varphi^2}\right) = \cos^{-1}\left(1 - \frac{1}{2\varphi^2}\right) = \cos^{-1}\left(\frac{1+\sqrt{5}}{4}\right) = \frac{\pi}{5}.$$

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These two special values for $\text{vers}^{-1} x$ are key to finding some interesting series containing products between the central binomial coefficients and the Fibonacci numbers.

From Binet's formula [12, p. 90], namely

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \frac{(-1)^n}{\varphi^n}\right),$$

replacing n with $2n$ yields

$$F_{2n} = \frac{1}{\sqrt{5}}\left(\varphi^{2n} - \frac{1}{\varphi^{2n}}\right). \tag{11}$$

Setting $x = \frac{1}{2}\varphi^2$ followed by $x = 1/\left(2\varphi^2\right)$ in the various series expansions that have been found, taking their difference before dividing throughout by $\sqrt{5}$, from (11) series containing the term F_{2n} can be found. Doing so for series (6), (7), and (8) containing the central binomial coefficient we obtain the intriguing results of

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{F_{2n}}{2^{4n}(2n+1)} = \frac{\pi}{25}\left(5 - 2\sqrt{5}\right),$$

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{F_{2n}}{2^{4n}(n+1)(2n+1)} = \frac{2\pi}{25}\left(5 - 2\sqrt{5}\right) - \frac{4}{5}\sqrt{125 - 10\sqrt{5}} + 8,$$

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{F_{2n}}{2^{4n}(n+1)} = \frac{4}{5}\sqrt{125 - 10\sqrt{5}} - 8.$$

Conclusion

By taking a look back at one particular long-lost trigonometric function we have seen how it allows one to retrace and navigate what is familiar mathematical terrain from a slightly different point of view. This simulacrum of moving between the strangely recognisable yet unfamiliar world of the versine was shown to provide alternative pathways to some classic results concerning sums containing central binomial coefficients. Perhaps at times simpler, perhaps at other times just different, we hope to have shown while versine may be lost to the modern reader it is by no means forgotten.

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