What Is Really Quantum in Quantum Econophysics?

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Econophysics is a branch of economics that applies concepts and methods from physics to the financial markets. This article focuses on the approaches to quantum finance developed by Kirill Ilinski and Belal E. Baaquie to deal with the uncertainty characterizing financial time series. Allegedly, their models rest on a formal analogy between quantum mechanics and finance. In order to evaluate them, we raise the question what is really quantum in quantum econophysics. We then argue that the supposed analogy breaks in an important manner, which is relevant to explain the empirical success of the proposed models.

1. Introduction. Econophysics is a branch of economics that applies concepts and methods from physics to the financial markets (see Mantegna and Stanley's [1999] seminal book, as well as Jovanovic and Schinckus [2017] for a recent analysis). In econophysical models, one transfers the formalism of successful physical theories to the description of financial phenomena. The underlying idea is that the evolution of financial markets bears some kind of analogy to the physical phenomena described by such theories. Arguably, this is what justifies the empirical success of econophysics, just as it happens in the areas of physics from which the applied methods are borrowed. How far can one stretch the analogy, in spite of the obvious disanalogies between physical and financial phenomena, depends on the particular model one adopts. The issue is particularly outstanding when it comes to models of quantum econophysics, which rely on an analogy between quantum mechanics and finance.[1]

Received January 2020; revised August 2020.

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1. Let us note that in Mantegna and Stanley's (1999) original exposition, econophysics involved mainly statistical physics, while later on quantum physics entered into the game.

Philosophy of Science, 88 (October 2021) pp. 665–685. 0031-8248/2021/8804-0005$10.00

Such models are remarkably good at reproducing empirical data. Moreover, there exist rather elaborate, systematic formulations of quantum econophysics in the literature. Yet, it is not quite intuitive to grasp the sense in which financial quantities would be analogous to the relevant physical quantities, in particular because, for example, money, financial assets, and trade agents can hardly be regarded as quantum-mechanical objects. So, the question we wish to take up in the current article is as follows: What is really quantum in quantum econophysics?

Answering such a question helps one provide a justification to the otherwise surprising empirical success of financial markets models. For this purpose, we will explore the supposed analogy between quantum mechanics and finance in two of the main approaches to quantum econophysics, namely, those in Ilinski (2001) and Baaquie (2004), which has been further developed in Baaquie (2009, 2018). There, the dynamical evolution of prices and the uncertainty associated with it are treated with the same methods successfully applied to systems of quantum particles. The fact that the proposed models prove empirically successful in the financial markets thus seems to corroborate the analogy with quantum mechanics. Nevertheless, we contend that the latter fails, at least if intended in a strict sense, exactly when one applies the models to the financial markets. In fact, notwithstanding some formal similarity, the dynamical equations governing the evolution of prices, such as the Black-Scholes equation, have to be of a form different from the Schrödinger equation governing the evolution of quantum particles. More to the point, the imaginary unit i appearing in the dynamics and the commutation relations in quantum mechanics does not appear in the corresponding equations in finance, where one instead employs real numbers to account for prices and their variation in time. In turn, though, this rules out peculiarly quantum-mechanical phenomena, such as interference patterns, which set the quantum world apart from the classical one.

The article is organized as follows. In sections 2 and 3, we explain in which sense econophysics rests on an analogy between the relevant physical systems and the financial markets, and we present Rickles's (2007, 2011) criticism to the models elaborated by Baaquie and by Ilinski, as well as Schinckus's (2014) methodological call for quantum econophysics. In sections 4 and 5, we critically discuss the basic concepts of Ilinski's and Baaquie's approaches, respectively. For this purpose, we employ Feynman's path integral formalism, which allows a direct connection with the empirical data from the financial markets. Finally, in section 6, we compare the two models, so as to develop an answer to the question what is really quantum mechanical in quantum econophysics. The upshot of our analysis is that, perhaps ironically, what helps one justify the empirical success of quantum econophysics is just the fact that the applied models lose some of their characteristic quantum-mechanical components.

2. On the Analogy between Quantum Mechanics and Finance. Although the exact definition of econophysics is subject to debate (see Rickles 2007), a typical presupposition underlying the construction and the use of econophysical models is that formal methods from physics can be imported into finance, inasmuch as the trade market is analogous to the relevant physical systems to which these methods are already successfully applied. The formal analogy between physical and financial quantities consists of equations and mathematical relations having the same form. It is then hoped that the empirical success these formulas show in physics will carry over to finance as well. Arguably, when that happens, the analogy would appear as being further corroborated. Analogical reasoning is indeed ubiquitous in science, and there is a growing philosophical literature that attempts to refine the conditions, if any, under which a formal analogy could also be material, so as to establish a somewhat closer connection between the source and the target system of the analogy (see Bartha [2019] for a critical overview). Here, we do not aim to develop a general account of when analogical inferences are licensed in econophysics; we focus instead on understanding to what extent properties of quantum systems, intended as the source of the analogy, can be transferred to the financial market, which is the purported target system.

The supposed analogy is somehow intuitive in the case of models of econophysics based on classical statistical mechanics (see Lux and Marchesi 1999, 2000; Lux and Heitger 2001; Schinckus 2018), wherein a large number of agents involved in market transactions are supposed to behave like the molecules in a gas, and the collective result of their mutual interaction is computed with statistical methods, just as the macroscopic variables of the gas are determined by the behavior of the molecules at the microscopic level. In this respect, there is a connection, at least prima facie, between a classical gas system and the financial market understood as a complex system whose overall properties depend on how its constituents interact. For instance, according to the model proposed by Johansen, Ledoit, and Sornette (2000), real crashes occurring in the stock market are analogous to phase transitions in condensed matter physics: in both cases states of the system undergo sudden changes due to fluctuations of the relevant parameters (see Juhn, Palacios, and Weatherall [2018] for a causal analysis of this model). What is more, Rickles (2007, 2011) maintains that this approach to econophysics is rather effective in recovering the so-called stylized facts, namely, universal regularities observed in the economic data, which are common to the evolution of prices of various commodities across different times and places.[2] Indeed, these statistical generalizations

2. Lux and Heitger (2001, 123) isolate three such stylized facts: (1) prices appear to be random, (2) returns appear not be random (their time series have fat tails, i.e., exhibit a significant probability for extreme values), and (3) volatility is not uniformly distributed but clusters so that there are highly volatile and very nonvolatile periods.

can be formally captured by mathematical relations akin to those involving the universal critical exponents used to describe the similar behavior of different substances during phase transitions by means of renormalization group methods. That enforces the connection between classical statistical mechanics and the financial markets. So, one may perhaps wonder whether the same argument by analogy can ground quantum econophysics too.

However, Rickles also contrasts the approach based on classical statistical mechanics with the approaches to quantum econophysics elaborated by Ilinski (2001) and Baaquie (2004), which are based on gauge theory and quantum field theory, respectively, and contends: "The proposed connection to gauge theory and quantum field theory exists at a far more abstract level. Certainly, at this level there is no suggestion that markets correspond to the models in any *realistic* sense: we have here a pair of purely *phenomenological* approaches to market dynamics and pricing" (2007, 967).

Allegedly, these models do not seem to retain a direct correspondence with the financial quantities they aim to account for. Indeed, Rickles then goes as far as claiming that they even fall outside the domain of econophysics, in that they fail to recover some crucial stylized fact, such as the randomness of price changes and returns. Later on, in his 2011 article, he reiterates his criticism toward Ilinski and Baaquie, but he admits that they display remarkable empirical success: as he puts it, "One doesn't get the same intuitive connection with models based on quantum field theory and gauge theory—though, it has to be said, they do surprisingly well at *reproducing* economic data" (Rickles 2011, 2). That is what, due to the asserted lack of a concrete connection with the relevant financial quantities, would make the models of quantum econophysics purely phenomenological rather than realistic. Unfortunately, though, Rickles does not offer a detailed enough discussion of the content of Ilinski's and Baaquie's proposals. Therefore, a more systematic analysis is needed in order to adequately assess their status.

Surely, in quantum econophysics the underlying analogy between quantum systems and the financial market is not quite intuitive, since one could not possibly idealize traders and investors as quantum objects exhibiting nonclassical properties, like superposition or long-distance entanglement. Yet, numerical computations show pretty good matches between the models and real economic data, at least for short-term dynamics. Empirical success thus justifies the practical use of these models. Furthermore, the formalism of quantum mechanics provides powerful methods to deal with uncertainty, which motivates one to apply them to financial transactions wherein traders and investors operate in conditions of incomplete information. According to Schinckus (2014), the treatment of economic uncertainty is actually what offers a methodological justification for the development of quantum econophysics. As he explains, in contrast with the classical account of randomness adopted in statistical and agent-based econophysics, "randomness observed in quantum processes . . . is considered as a fundamental feature of nature which is independent of our ignorance. This way of dealing with randomness paves the way to another characterization of emerging phenomena which could be very interesting in finance or economics. Despite the existence of a 'natural indeterminism,' individual quantum events are constrained by statistical laws which make them interesting for analogy with financial phenomena" (311). In fact, unlike in the classical case, uncertainty appears as deeply entrenched in the quantum world, rather than just reflecting our ignorance about the phenomena. In this respect, somewhat differently from Rickles's critical perspective, Schinckus calls for a quantum econophysics that could complement classical econophysics in the understanding of complex economic systems characterized by randomness. However, he does not directly discuss the models of quantum finance proposed by Ilinski and by Baaquie.[3] Here we wish to fill this gap in his analysis.

In our view, unless one establishes a compelling correspondence between physical and financial quantities, the ability to treat economic uncertainty does not in itself explain why quantum methods can prove so effective to describe the evolution of prices. For the sake of elucidating this issue, we suggest that one ought to identify what is really quantum mechanical in quantum econophysics. As we argue in the rest of the article, although Ilinski's and Baaquie's approaches hinge on different analogies between quantum mechanics and finance, they both share a common component that actually renders them more classical than it may appear at first. Before reviewing each proposal in section 4, it is worth highlighting some basic aspects of quantum theory, which mark its departure from classical mechanics.

3. What Is Quantum Mechanical? In standard quantum mechanics, the state of a microscopic system is defined by a unitary vector $|\psi\rangle$ belonging to a (separable) Hilbert space $\mathcal{H}$, intended as a complex vector space that is also complete with respect to the metric induced by the inner product. In addition, the observables of the theory, namely, the physical quantities of interest, are represented by linear operators acting over the underlying Hilbert space, so that the possible values of an observable A are obtained as the eigenvalues of the corresponding operator $\hat{A}$, that is, by means of the equation $\hat{A}|\psi\rangle = a|\psi\rangle$,

3. Instead, he refers to other proposals, such as those by Maslov (2002), Busemeyer, Wang, and Townsend (2006), Bagarello (2007), Guevara (2007), and Yukalov and Sornette (2008), which mainly combine quantum methods with game theory and classical decision theory. Arguably, these proposals share the three main characteristics of quantum econophysics identified by Schinckus (2014, 312), namely, the use of the quantum formalism to model economic and financial processes, the application of quantum-mechanical analogies, and the application of what he calls "quantum mechanical ideology," which actually paves the way for a new treatment of randomness with respect to the classical approaches to econophysics.

so that the possible value of the observable A is an eigenvalue a of $\hat{A}$ relative to the eigenvector $|\psi\rangle$. In quantum mechanics the operators are Hermitian (more precisely, essentially self-adjoint); therefore, their spectrum consists of real numbers. Clearly, the eigenvalue equation is not satisfied for an arbitrary wave function $|\psi\rangle$. For example, if the state $|\psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle$ is a superposition of two eigenvectors $|\psi_1\rangle$ and $|\psi_2\rangle$ of observable A with different eigenvalues a_1 and a_2, then $|\psi\rangle$ is not an eigenvector of $\hat{A}$. In general, one can compute the probability that a measurement of the physical quantity associated with an operator $\hat{A}$ yields a certain eigenvalue a_i of $\hat{A}$ when the system is in state $|\psi\rangle$ by the formula $P(a_i) = |\langle\psi|\psi_i\rangle|^2$. Since $P(a_i) < 1$ for all eigenvalues a_i of $\hat{A}$, unless $|\psi\rangle$ is an eigenvector of $\hat{A}$, the existence of superposition states betrays an intrinsic uncertainty arising at the atomic level, and the formalism of quantum mechanics is designed to deal with it. Indeed, the superposition of wave functions gives rise to phenomena that have no classical analogue, for example, the interference patterns that threw classical mechanics into crisis at the beginning of the twentieth century. While probability is built into the structure of quantum mechanics, the time evolution of a quantum system is deterministic. For, let the state $|\psi_t\rangle$ denote the wave function at each instant t: then, the dynamics is dictated by the Schrödinger equation, namely, a linear partial differential equation of the form

$$i\hbar \frac{d}{dt}|\psi_t\rangle = \hat{H}|\psi_t\rangle, \tag{1}$$

where $\hbar$ is the reduced Planck constant and $\hat{H}$ is the Hamiltonian operator representing the total energy H of the system. Accordingly, if the latter is independent of time, the evolution of the state from the initial time $t = 0$ is uniquely given by $|\psi_t\rangle = e^{i\hbar\hat{H}}t|\psi_0\rangle$. In particular, for a particle of mass m immersed in a real potential field V, the Hamiltonian can be written as

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \hat{V}, \tag{2}$$

where $\hat{V}$ is the multiplication operator corresponding to the potential V, and the kinetic term being added to it in the right-hand side is a differential operator $\hat{K} = -(\hbar^2/2m)\nabla^2$. As it turns out, $\hat{H}$ is an Hermitian operator acting on $\mathcal{H}$, and hence it has real spectrum. We will return to this point later on when comparing the Schrödinger equation with the Black-Scholes equation describing the evolution of prices in the stock market. Here, let us stress that the equation depends on the reduced Planck constant $\hbar$ and includes the imaginary unit i. Such quantities also appear in the right-hand side of the Heisenberg commutation relation for position $\hat{X}$ and momentum $\hat{P}$:

$$[\hat{X}, \hat{P}] = i\hbar I, \tag{3}$$

https://doi.org/10.1086/713921 Published online by Cambridge University Press

which is often regarded as a way to express the intrinsic uncertainty characterizing the quantum world, in that it entails that one cannot simultaneously determine position and momentum. Noncommutativity marks another peculiar quantum-mechanical fact, since classical mechanics corresponds to the limit where $\hbar$ goes to 0, and hence the position and momentum operators commute.

Finally, let us recall that, beside Schrödinger's picture placing emphasis on the wave function and Heisenberg's picture placing emphasis on operators (which are provably equivalent within the Hilbert space formulation of the theory), there is a third description of quantum particles one can adopt, namely, Feynman path integral formalism (see Feynman 1948; Feynman and Hibbs 2010). Within this picture, the position x_t of a particle at each instant t plays the role of the independent random variable, to which quantum probabilities are assigned. Specifically, one computes the probability amplitude that the particle evolves from the initial position x_i at time t_i and the final position x_f at time t_f as $T(x_i, t_i, x_f, t_f) = |\langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)}|x_i\rangle|^2$. That is the square of the absolute value of the *transition amplitude*, which is obtained by integrating over all possible virtual paths the particle can take between the initial and final position, so as to obtain the Feynman path integral $\prod_{t_i<t<t_f}\int_{-\infty}^{+\infty}dx(t)e^S$, where the action S depends on the Lagrangian. For instance, in the case of a particle with mass m immersed in the field potential V, the quantum-mechanical action specialized to one degree of freedom is given by

$$S(x, \dot{x}) = \frac{i}{\hbar}\int_{t_i}^{t_f}\left[\frac{1}{2}m\dot{x}^2 - V(x)\right]dt, \tag{4}$$

and thus it depends on the reduced Planck constant $\hbar$ and takes imaginary values. Since in Feynman's formulation one appeals to functional integration, the peculiar quantum-mechanical elements of Schrödinger's and Heisenberg's pictures, namely, the Hilbert space structure and noncommutativity, are implicitly incorporated in the path integral. What is more, this approach enables one to cope with cases in which the system presents a mathematically intractable Hamiltonian $H = K + V$, since the path integral formulation requires the Lagrangian $L = K - V$, wherein the potential term V is subtracted from instead of being added to the kinetic term K. Specifically, Feynman suggested a procedure whereby time is discretized. That is, the total time required to complete a trajectory is divided into N steps, so that the position of the particle at each step is denoted by $x(t_j)$ with $j = 1, \ldots, N$, and hence its velocity $\dot{x}(t_j)$ is approximately given by $[x(t_{j+1}) - x(t_j)]/\Delta t$, where $\Delta t = t_{j+1} - t_j$. The Lagrangian of the system thus becomes

$$L \simeq \frac{1}{2}m\frac{|x(t_{j+1}) - x(t_j)|^2}{\Delta t} + V(x(t_j)). \tag{5}$$

Then, at the limit for continuous time, the amplitude obtained by integrating the quantity $e^{\frac{i}{\hbar}L}$ over all paths corresponds to the Schrödinger wave function.

The path integral formulation has, among others, the advantage of making clear that in the semiclassical limit $\hbar \to 0$ the only trajectory with positive probability is the classical one. As we will see later on, in quantum econophysics working with the Lagrangian for discretized time enables one to depict more realistically the evolution of prices in the stock market.

The above-described aspects of quantum mechanics, namely, the probabilistic structure of Hilbert space, noncommutativity, and the Feynman path integral, are peculiar to the theory in the sense that they are not shared by classical mechanics. Indeed, they are introduced in order to deal with the intrinsic uncertainty of the quantum world. As such, one may hope to employ them in finance, so as to account for the uncertainty characterizing the financial markets. Thus, the issue to be addressed here is whether, and in what sense, these distinct quantum-mechanical features are built into quantum econophysics as well.

4. Baaquie's Approach: A Quantum-Mechanical Black-Scholes Equation? Baaquie's approach to econophysics aims to transfer formal methods adopted in quantum field theory to the account of financial data. Quantum field theory describes systems with infinite degrees of freedom, and it is used by Baaquie to describe "infinite dimensional" financial problems such as the evolution of interest rates. However, for the purpose of dealing with his purported analogy between quantum mechanics and finance, we restrict our discussion to the standard treatment of a single quantum particle with one degree of freedom, corresponding to the evolution of the price of a single asset. Indeed, Baaquie puts forward his analogy already at the beginning of chapter 4 of his 2004 book on quantum finance, in which he first introduces the Hamiltonian formulation of stock options in the financial market. As he puts it, "In contrast, in quantum mechanics, the particle's evolution is random, analogous to the case of the evolution of a stock price having non-zero volatility ($\sigma \neq 0$)" (Baaquie 2004, 45).

In the above quotation, the declared contrast must be understood with respect to classical mechanics, where the time evolution of particles is described by Newton's law of motion. Here the relevant difference is the presence of the Planck constant $\hbar$ in quantum theory that plays the same role as the nonzero volatility σ for stock prices, whereas the case analogous to classical theory is just that with zero volatility ($\sigma = 0$). On this basis, Baaquie proceeds to recast the Black-Scholes equation with $\sigma \neq 0$ for the evolution of prices in the quantum formalism. As we will see in the current section, though, while that enables one to draw an interesting mathematical parallelism between quantum mechanics and finance, it highlights deep formal disanalogies that betray substantial differences between the two domains.

The parallelism established by Baaquie hinges on two basic ingredients of the quantum formalism, namely, the fact that quantum state functions are elements of a linear vector space and the fact that observables are linear operators acting on such a space. In the same vein, in finance states are given by vectors $|\mathcal{C}\rangle$ representing the possible prices taken on by a given stock option, which belong to a linear vector space $\mathcal{V}$. In order to keep the notation simple, in the formulas below we refer to the vector $|\mathcal{C}\rangle$ simply as $\mathcal{C}$. The Black-Scholes equation describes the way in which option prices evolve in the course of time,[4] and, when the volatility is assumed to be constant, it takes the form

$$\frac{\partial \mathcal{C}}{\partial t} = -\frac{1}{2}\sigma^2 \mathcal{S}^2 \frac{\partial^2 \mathcal{C}}{\partial \mathcal{S}^2} - r\mathcal{S}\frac{\partial \mathcal{C}}{\partial \mathcal{S}} + r\mathcal{C}. \tag{6}$$

Here $\mathcal{S}$ denotes the price of the underlying asset, which is assumed to follow a geometric Brownian motion. The derivation of the Black-Scholes equation is based on the assumption that the price of the underlying asset follows a geometric Brownian motion. A hypothetical dealer builds an auxiliary portfolio, called delta hedge, by selling one option and buying $\partial\mathcal{C}/\partial\mathcal{S}$ stocks. One can show that such portfolio is well balanced (i.e., delta hedged) and that its value increases at the risk-free rate of the market r, which is a known quantity. The Brownian motion assumption, together with Itô's lemma and some algebra, leads to the Black-Scholes equation. In this derivation the no-arbitrage assumption is crucial. Let us point out here that the analysis of such a condition is central in Ilinski's approach, and thus we will discuss it in greater detail in the next section. Here, following Baaquie, we enforce the analogy with the Schrödinger equation (1) by enacting the change of variable $\mathcal{S} = e^x$ in equation (6), so that x becomes the only degree of freedom, thereby yielding an equation of the form $\partial\mathcal{C}/\partial t = H_{\mathrm{BS}}\mathcal{C}$, where the Black-Scholes Hamiltonian reads

$$H_{\mathrm{BS}} = -\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r. \tag{7}$$

Compared with the quantum-mechanical Hamiltonian (2) specialized to one degree of freedom, the square of the volatility σ in equation (7) plays the same role as the Planck constant $\hbar$ over the mass m of the particle, whereas the drift term represents a potential depending on the rate of change of the option price. So, similarly to the observables in quantum mechanics, the thus-defined Black-Scholes Hamiltonian is a linear operator acting on the vector $|\mathcal{C}\rangle$ in the state-space $\mathcal{V}$ for option prices.

4. More precisely, this evolution equation estimates the price of European put and call options. Let us recall that a European put option gives a holder the right to sell the underlying asset at a specified price at a specified date. Similarly, a European call option gives a holder the right to buy the underlying asset at a specified price at a specified date.

The apparent formal analogy with quantum mechanics can be extended even to the Heisenberg picture, in that one can construct a commutation relation displaying the same structure as (3). For, let $x = \ln \mathcal{S}$ be the logarithm of the stock price $\mathcal{S}(t)$ at any given time t, as defined above for the security in the Black-Scholes equation (6). Then, if one knows its value, one cannot simultaneously determine the velocity at which it changes in time and vice versa. In fact, by representing the relevant quantities in terms of operators on the state space, for instance, by choosing the Hermitian operator $X \coloneqq x$ and the anti-Hermitian operator $P \coloneqq -\sigma^2 \frac{\partial}{\partial x}$, respectively, one obtains the following commutation relation:

$$[X(t), P(t)] = \sigma^2 I, \tag{8}$$

where the time dependence is made explicit by adding the variable t, so that the equation is understood as holding at each instant. Accordingly, inasmuch as one has nonzero volatility $\sigma \neq 0$, the operators X and P do not commute with each other, just like position and momentum in the quantum-mechanical commutation relation (3) with $\hbar \neq 0$. One can then obtain a financial analogue of Heisenberg's uncertainty principle for option prices.[5]

Baaquie's proposal thus establishes a prima facie formal analogy between quantum mechanics and finance. In both cases, states are represented by elements of a vector space, and the observable quantities correspond to linear operators acting on such a space. One can then construct the Black-Scholes equation describing the time evolution of option prices in the same way in which the Schrödinger equation dictates the dynamics of a quantum particle, as well as derive a commutation relation between the operators representing prices at a given time and their rate of change with the same form as Heisenberg's quantum-mechanical commutation relation between position and momentum. Nevertheless, in spite of this parallelism, there are also profound differences breaking the purported analogy, which prove essential in order to guarantee that, when being applied to finance, the formalism correctly recovers economical data. In fact, as Baaquie himself notices (see 2004, 76–77), the Schrödinger equation (1) is a time-reversible partial differential equation given as an initial value problem, whereas the Black-Scholes equation is time irreversible, and it is given as a final value problem, in that the final option price is given in advance. This corresponds to the fact that the price of an option at maturity as a function of the underlying asset is known, since it corresponds to the price of the underlying asset. Furthermore, the former implies that the wave function $|\psi\rangle$ is complex valued, while the latter must yield a real-valued formula for the price of an option $|\mathcal{C}\rangle$: as Baaquie puts

5. Specifically, the uncertainty of the value of x for any given vector state f is defined as $\Delta x = \sqrt{\langle f|x^2|f\rangle - \langle f|x|f\rangle^2}$, and similarly for its derivative $\dot{x}$. It follows that the uncertainty principle for finance reads $\Delta x \Delta \dot{x} \geq \sigma^2/2$ (see Baaquie 2004, 101).

it, "One can think of the Black-Scholes equation as the Schrödinger equation for imaginary time" (76). We submit that this entails that what is peculiar to quantum mechanics in the proposed formalism is effectively removed in its application to finance. In order to see why that is the case, let us dwell on the differences with the distinctive quantum-mechanical features described in section 3.

To begin with, in quantum mechanics states are normalized vectors, that is, $\langle \psi|\psi \rangle = 1$, which assures that probabilities are computed by means of the inner product. By contrast, in finance, even though option prices are elements of the underlying vector space $\mathcal{V}$, they are such that the inner product $\mathcal{C}|\mathcal{C}$ can take in principle any positive arbitrary value. In fact, here the price $\mathcal{C}$ is supposed to be directly observable, and hence it should not be interpreted in a probabilistic sense. That raises the question how probabilities are introduced in Baaquie's approach. The answer rests on the notion of a *price kernel*, which enables one to describe the backward random evolution of securities as a conditional probability. Specifically, let T be the final time of a given path, so that $\tau = T - t$ fixes the time interval with respect to any earlier instant t. The price kernel is defined as $p(x, \tau, \dot{x}) = \langle x|e^{-\tau H}|\dot{x} \rangle$, where the notation stands to indicate $x = e^S$ for an arbitrary security S, as above, and $\dot{x}$ is its time derivative. In general, the matrix element $e^{-\tau H}$ cannot be exactly evaluated since, if the kinetic term K and the potential term V of the Hamiltonian H do not commute with each other, one has $e^{-\tau H} \neq e^{-\tau K}e^{-\tau V}$. However, when the interval of time is extremely small, say of the order of an infinitesimal $\epsilon \rightarrow 0$, one can approximate the matrix element as $e^{-\epsilon H} \simeq e^{-\epsilon K}e^{-\epsilon V}$, which motivates one to adopt Feynman path integral formulation. For, if one subdivides time into N steps, so that $\epsilon = \frac{\tau}{N}$, the variable $x(t)$ takes on the form $x(t_j) = j\epsilon$ for each step $j = 1, \ldots, N$. The formula for the kernel price can thus be rewritten in terms of the option-pricing Lagrangian $L[x(t_j), x(t_{j-1}); \epsilon]$ for discretized time instead of the intractable Hamiltonian H, just like in the case of formula (5) in quantum mechanics, so that the quantity $\langle x(t_j|e^{-\epsilon H}|x(t_{j-1}) \rangle$ becomes proportional to $e^{\varepsilon L}$. Then, if one takes the overall action $\epsilon\Sigma_{j=1}^{N}L[x(t_j), x(t_{j-1}); \epsilon]$ for the N time steps and integrates over all the possible virtual paths, one obtains a Feynman path integral for option prices. Accordingly, the price kernel is constructed as the financial analogue of the transition amplitude in Feynman's formulation of quantum mechanics. On this point, we would like to add also that in finance working with discretized time appears to be a more realistic procedure since it mimics what actually happens in the stock market, where information about prices is not provided continuously but only at different intervals of time.

Be that as it may, the fact that, differently from quantum mechanics, the vector state $|\mathcal{C}\rangle$ representing an option price is not normalized, thereby lacking probabilistic significance, has consequences even on how one defines the linear operators acting on it. Indeed, that the quantum-mechanical wave functions are complex valued is related with the fact that the observables are Hermitian operators, which admit only real eigenvalues. In particular, the Hamiltonian operator H appearing in the Schrödinger equation remains invariant under the adjoint operation '†', as one can explicitly see in the case of the expression (2) for which $H^{\dagger} = H$. To the contrary, the Hamiltonian appearing in the Black-Scholes equation is non-Hermitian: for, if one takes the adjoint of expression (7), one obtains

$$H_{\mathrm{BS}}^{\dagger} = -\frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} - \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r, \tag{9}$$

which is clearly different from H_{BS}. This fact generalizes to all the linear operators representing financial quantities, and hence their eigenvalues can be complex numbers. The point becomes even more evident by contrasting the commutation relations for quantum mechanics and finance. Indeed, although there is a formal analogy between them in that in both cases the right-hand side is nonzero, in (3) one has a real coefficient (i.e., σ^2), whereas in (8) one has a complex coefficient (i.e., $i(\hbar/m)$). Therefore, while it is true that according to Baaquie's approach the volatility σ plays the same role as the Planck constant (the more so because the limit $\hbar \rightarrow 0$ yields a classical description, similarly to the case of vanishing volatility), the presence of the imaginary unit i in quantum mechanics betrays a substantial disanalogy with respect to finance.[6] In fact, it is responsible for some peculiar quantum-mechanical phenomena such as interference, which do not have any analogue in the financial markets.

5. Ilinski's Approach: Minimizing Arbitrage. Ilinski's approach to econophysics aims to transfer methods adopted in quantum field theory to the study of financial series, just as in Baaquie's approach, but it is based on a more 'concrete' analogy between physics and finance than Baaquie's. In Ilinski's 2001 book, the similarities between the description of quantum phenomena and the treatment of random prices are made explicit in section 3.3, which is titled "Uncertainty and Quantization." There, he emphasizes that, due to the fact that one cannot predict the exact future values of exchange rates, one is required to appeal to probabilities in order to describe the random

6. It should be stressed that this point did not escape Baaquie's attention, since in n. 12 of chap. 5 he comments: "in finance one is working with Euclidean time and hence there is no factor i, and $\sigma^2 = \hbar/m$" (2004, 100). A clarification is in order, though. The lack of the imaginary unit marks a disanalogy between quantum theory and finance just with respect to the expression of the commutation relations. Instead, that it is also connected with the use of Euclidean time is admittedly less relevant: as Baaquie (2018) explains when extending his analysis to quantum field theory, "to make the path integral a rigorously defined expression, one analytically continues from Minkowski time to Euclidean time" (181). We thank an anonymous referee for pointing out this fact.

path that prices follow in the course of time, thereby enforcing an analogy with quantum mechanics. As he puts it, "Why 'quantum'? . . . Quantum mechanics is a probabilistic theory by its very nature. This is why many convenient and powerful methods have been developed in the framework of quantum field theory, one of which is the calculus of random trajectories. In this language, the theory of the random financial market and quantum electrodynamics in imaginary time almost coincide" (Ilinski 2001, 39).

More to the point, the alleged connection with physics is established in two steps. For one, the value of prices can be equivalently expressed in different currencies, and thus the financial theory ought to retain a sort of gauge symmetry akin to the invariance under the relevant transformations that the electromagnetic field exhibits in electrodynamics already at the classical level. Moreover, given that information affecting prices cannot be known in advance, and hence there is risk associated with financial operations, that calls for the use of quantum methods designed to cope with the resulting uncertainty. For purposes of evaluating what is really quantum in Ilinski's models of econophysics, it is sufficient to focus on the second step, whereas we will limit our analysis of the first step to the treatment of nonzero arbitrage without elaborating on the geometry of fiber bundles adopted by Ilinski. As the end of above quotation emphasizes, the difference between quantum electrodynamics and the theory of random financial markets rests on the use of the imaginary unit, in a somewhat similar fashion as in Baaquie's proposal. However, since Ilinski does not expand on this remark, we need to fill in the missing details.

Let us begin by stressing that a parallelism between finance and quantum mechanics was already mentioned by Ilinski earlier on in the introductory pages of his book, when discussing George Soros's (1987) original proposal to connect Heisenberg's uncertainty principle with the decision-making strategy that investors ought to implement in the face of imperfect knowledge. Soros actually argued that this analogy is only superficial, since in physics the uncertainty of experimental outcomes results from an act of measurement that is external to the objects being measured, while in economics uncertainty is due to the interactions between the market participants, and hence it is somehow internal to human thinking. In response, Ilinski insists that the indeterminacy encoded in the quantum-mechanical commutation relations (3) cannot be removed or even reduced by simply improving the experimental devices; it is a fundamental law of nature, and "thus the uncertainties in the social and natural sciences can both be considered as endogenous" (2001, 10). In light of this, just as physicists developed a new formalism based on the noncommutativity of quantum operators, he submits that in the social sciences one should look for "new mathematical tools to be applied to accommodate the peculiarities of thinking systems" (10). Thus, he sets himself the goal to construct the sought-after formalism for the economical theory of the financial market. Yet, as we will explain below, in his proposed framework for quantum econophysics the key ingredient is not the machinery of non-commuting operators but rather Feynman's path integrals.

Indeed, Ilinski starts from an analogy between the Lagrangian functional of classical mechanics and the concept of arbitrage in finance, which is the opportunity to make a risk-free profit with a rate of return higher than the risk-free interest rate. To fix the idea, the evolution of a classical dynamical system with N degrees of freedom is determined by the Lagrangian function $L : \mathbb{R}^{2N} \to \mathbb{R}$. More precisely, if $q : [a, b] \to \mathbb{R}^N$ is the trajectory of the system from point $q(a)$ to point $q(b)$, then q minimizes the action

$$S = \int_a^b L(\dot{q}, q)dt,$$

among all H^1 functions with the same endpoints of q. So, if the integral is computed by discretizing time, one obtains

$$S \simeq \sum_i L(q_{i+1} - q_i, q_i).$$

Then, recall that a basic assumption of classical finance is just the impossibility of arbitrage. More precisely, it is supposed that all actors in the market have access to complete information; therefore, any chance of arbitrage is immediately erased by the traders. In other words, the market itself minimizes the arbitrage possibilities, where the minimum is set to zero. Ilinski showed that arbitrage can be evaluated with a suitable "action." To see this, consider a market where one single asset is traded. Suppose that cash guarantees a certain return r_1 (fixed and known), while the asset S is a share whose price evolves at a rate r_2 variable and unknown. Let $P(t)$ be the price of the share at time t. For discretized time, one can take different instants $t_i = i\Delta$, with $i = 0, \ldots , N$ and $\Delta > 0$ being some time step. If an investor with one unit of cash at time t_i wants to have one unit of share at time t_{i+1}, there are two possible options: that is, to keep the cash until time t_{i+1} and then buy the share or, alternatively, to buy the share at time t_i and then keep it until time t_{i+1}. In the first case the investor will own $e^{r_1}P^{-1}(t_{i+1})$ units of shares, whereas in the second case she will own $P^{-1}(t_i)e^{r_2}$ units. Of course, if these two numbers are not the same, there is a possibility for arbitrage. A simple computation (see, e.g., Ilinski 1997) shows that the excess return between time t_i and time t_{i+1} is given by

$$\mathcal{S}_i = \frac{1}{2\Delta}(P^{-1}(t_i)e^{r_2\Delta}P(t_{i+1})e^{-r_1\Delta} + P(t_i)e^{r_1\Delta}P(t_{i+1})^{-1}e^{-r_2\Delta} - 2). \qquad (10)$$

Such a quantity can thus be used as a measure of arbitrage. It follows that the no-arbitrage condition assumed in classical finance, by which one must have $\mathcal{S}_i = 0$ at each time step, becomes equivalent to the equality

$$P^{-1}(t_i)e^{r_2\Delta}P(t_{i+1})e^{-r_1\Delta} = P(t_i)e^{r_1\Delta}P(t_{i+1})^{-1}e^{-r_2\Delta} = 1$$

since it guarantees that the rate of return is balanced out by the rate of risk-free interest, provided that the investor's strategy is optimal in the sense that she has complete information about the market.

In order to formulate his theory of quantum econophysics, Ilinski then postulates five conditions, which establish the purported analogy between the random financial market and quantum electrodynamics, that is, (1) gauge invariant dynamics, (2) locality, (3) free field theory–correspondence principle, (4) extremal action principle, and (5) limited rationality and uncertainty. The first postulate entails that the dynamics for the exchange and discount factors has to be constructed from gauge invariant quantities: in fact, it states that all the observable properties of the financial market are independent from the specific choice of currency. This is a very natural assumption since one expects rates and prices to remain the same if one changes currency. In this respect, gauge invariance is a condition akin to that governing the time evolution in physical theories, such as electrodynamics, in which fields do not change under symmetry transformations of the Lagrangian. The second condition postulates that the dynamics of an asset is local, in the sense that it is influenced only by connected (in the sense of Γ connectivity on the base graph L) assets only. Hence, the stock prices can be functions of the prices of other stocks quoted in the same currency but do not depend directly on the prices of stocks quoted in different currencies. In particular, this leads to the fact that the action (i.e., the arbitrage) is additive with respect to the excess returns at each time and each asset. The next postulate guarantees that, when there is no money flow, the theory is equivalent to classical finance, thereby drawing a correspondence with the free field theory in physics. Indeed, just like in the physical case in which matter fields are absent, with the continuous time limit one obtains the Brownian motion of the returns. Here, we will not elaborate further on these first three conditions, since they are not directly related to our main thesis.

Instead, the remaining two postulates are essential in the analogy between quantum mechanics and financial markets. For the extremal action condition asserts that, if the economic environment is fully rational and certain, then the excess rate return, namely, the rate of return above the risk-free interest rate, must take the smallest possible value. In other words, one cannot get something from nothing (no free lunch principle). That is tantamount to minimizing the action, that is, the quantity (10) for the allowed arbitrage. Yet, according to the last postulate, if the economic environment is not fully rational and certain, there exist nonzero probabilities to get other excess rates of return. In fact, in the real market, exchange rates, prices, and interest rates do fluctuate, and hence local arbitrage opportunities are actually possible.

These conditions thus lead one to the main assumption underlying Ilinski's approach, namely, that it is possible to obtain a riskless excess return, but the probability of this happening at each time step is given by

$$P = e^{-\beta \mathcal{S}_i}, \tag{11}$$

where $\beta > 0$ is a parameter that "measures" the rationality of the market. More to the point, when all time intervals $t_0, \ldots, t_N$ are taken into account, one can compute the probability P_N of the stock price at the final time t_N, provided that the price at the initial time t_0 is

$$P(0, S_0, T_N, S_N) = \int_0^{+\infty} \frac{dS_1}{S_1} \int_0^{+\infty} \frac{dS_2}{S_2} \ldots \int_0^{+\infty} \frac{dS_N}{S_N} e^{-\beta \sum_i \mathcal{S}_i}. \tag{12}$$

Note that the measure dS_i/S_i is chosen because it is gauge invariant, in accordance with the first postulate. One can also show that (12) induces a Brownian motion with volatility $\sigma = \beta^{-1/2}$.

It is important to stress that the derivation of the probability formula (12) follows closely the derivation of the path integral for the quantum mechanics of a particle introduced by Feynman. In particular, Planck's constant is replaced with the volatility squared. Thus, just as the trajectory of a particle becomes classical in the limit $\hbar \to 0$, the discretized path integral (12) shows that the trajectory of prices satisfies $\mathcal{S} = 0$ when $\beta \to +\infty$. It is the presence of uncertainty in the financial market being due to the random walk of the share price and its risk that motivates the appeal to the quantum formalism. As Ilinski observes, "if the randomness of the price is similar to quantization, the rate of return on an arbitrage operation is now a 'quantum' variable that does not have a well-defined value and cannot be taken as a real number. This exactly resembles the situation with the electro-magnetic field which, after quantization, is not a number but a quantum variable, an operator. . . . In the same way, we understand the arbitrage rate of return in the financial setting. It causes money flows, it is virtual, it fluctuates" (2001, 39–40). Accordingly, just like in quantum electrodynamics positive and negative charges interact through the electromagnetic field, in the financial market securities and debts interact with each other through the gauge field of arbitrage.

Nevertheless, we wish to point out that there is an important difference, which breaks the proposed analogy between quantum mechanics and finance. Indeed, contrary to Feynman's path integral description of the dynamics of quantum particles, in Ilinski's derivation of equation (12) the imaginary constant i in front of the Lagrangian functional is missing. So, while in quantum mechanics the path integral represents a phase and the probability of the trajectory is given by the modulus square of such a phase, in the theory of the random financial market formulated by Ilinski the path integral has no complex values and it provides directly the transition probability. This difference explains the passage put forward in the quotation at the beginning of the section, where he submits that his theory almost coincides with quantum electrodynamics in imaginary time. Yet, as we claimed in the context of Baaquie's approach, the lack of the imaginary unit i marks a profound disanalogy with quantum theory, thereby depriving quantum econophysics of a distinctive quantum-mechanical component.

6. What Is Needed for Quantum Methods to Treat Random Financial Processes? Both Baaquie's and Ilinski's approaches develop a systematic formulation of quantum econophysics that rests on a supposed analogy between quantum mechanics and the financial market. However, they hinge on quite different assumptions. The former begins with well-accepted financial models, such as the Black-Scholes equation, and then shows that such models can be seen as the Schrödinger equation with a suitable choice of Hamiltonian, although without the imaginary constant in front of the derivative with respect to time. The latter, instead, takes a somewhat more basic road, in that it builds a model based on the assumption of not perfect efficiency of the financial markets, that is, allowing for some residual arbitrage, and from that model Ilinski obtains a probability formula based on Feynman's path integral, again without the imaginary unit in front of the Lagrangian functional. What appears as rather surprising is that such different starting points lead them to essentially the same expression for the transition probabilities, once the Schrödinger equation is transformed into the corresponding path integral. The thus-obtained transition probabilities prove extremely useful to account for the dynamics of the financial processes they apply to. Indeed, the proposed models of quantum finance can be quite accurate.

In spite of their empirical success, the approaches by Baaquie and by Ilinski have been subject to some criticism. As we noted in the quotations presented in section 2, Rickles (2007, 2011) blames them for lacking a realistic connection with financial markets, which would make them purely phenomenological. More to the point, he claims that they fail to account for some of the stylized facts mentioned by Lux and Heitger (2001), such as the nonrandomness of prices and the fat tails phenomenon (see n. 2). However, it seems to us that this presumptive shortcoming can be readily remedied. For there are ways to readjust the proposed models so as to cope with the required stylized facts: for instance, Paolinelli and Arioli (2018) implemented a modified version of Ilinski's model whose computer simulations display a remarkable match with actual financial data. Thus, the issue of explaining why these approaches to quantum econophysics are so empirically successful demands a somewhat orthogonal assessment. Granted, we partially agree with Rickles that the analogy between quantum theory and finance is established at a more abstract level than the connection with the financial markets underlying other approaches to econophysics, such as those based on classical statistical physics. Yet, differently from his analysis, we submit that for the sake of addressing the issue at stake one needs to determine what is really quantum in Baaquie and Ilinski's formulations. For this purpose, we first clarify the extent to which they enable one to deal with economic uncertainty, and then we pinpoint where and how they breach the alleged analogy when it comes to its application to financial phenomena.

To begin with, let us recall from sections 2 and 3 that Schinckus's (2014) methodological call for quantum econophysics rests on the fact that the quantum formalism offers an effective way to treat randomness, which is regarded as an intrinsic aspect of the quantum world, contrary to what happens in the classical context where it merely reflects our ignorance of the phenomena. How exactly the quantum-like treatment of randomness applies to the relevant processes in the financial markets, for instance, the unpredictable evolution of stock prices, depends on the specific approach one employs. For one, Baaquie derives a financial analogue of Heisenberg commutation relations, namely, equation (8), whereby option prices and their rate of change in time cannot be simultaneously determined in cases of nonzero volatility. Economic uncertainty thus arises as an inherent feature of financial transactions that is due to the dispersion of returns for a given security. As such, it ought not to be traced back to any lack of knowledge of the agents operating in the market: rather, once the value of the volatility $\sigma \neq 0$ is fixed, then a function of its square imposes a lower bound to the uncertainty concerning prices and their velocity of change at any instant. Regarding Ilinski, although he does not construct a formal analogue of the Heisenberg uncertainty principle, by taking the move from Soros's (1987) discussion of the quantum-mechanical commutation relations in connection with the social sciences, he contends that uncertainty is also an ineliminable feature of the financial markets, in that it is intrinsically due to the interactions between the agents. Indeed, it is a basic assumption of his theory of arbitrage that the real economic environment is not fully rational and certain, where the measure of rationality is captured by the parameter β in equation (11). Just as in Baaquie, albeit in a less radical sense, even in this case the uncertainty does not follow from the ignorance of the agents themselves: instead, it is a consequence of the random trajectories of shared prices and their risks, which allows for the possibility of excess rates of return. Ilinski then goes on to show that financial randomness is analogous to quantization in Feynman's path integral formalism. In such a respect, we argue, the models developed by Baaquie and by Ilinski constitute distinct yet equally noteworthy ways to fulfill Schinckus's methodological call for the treatment of economic uncertainty in quantum econophysics. In addition, it is worth emphasizing that within both frameworks one can recover the scenarios described by classical econophysics, that is, the case of zero volatility (i.e., $\sigma = 0$) in Baaquie as well as the case of a fully rational and certain market (i.e., $\beta \rightarrow +\infty$) in Ilinski.

Next, in order to fully understand the sense in which the approaches elaborated by Baaquie and by Ilinski effectively deal with economic uncertainty, thereby successfully recovering the empirical data from the financial markets, it must be stressed that both approaches breach the analogy between quantum theory and finance in an important way. In fact, as we explained in several passages, their proposed econophysical models lack the peculiar quantum component related to the imaginary unit i. For instance, contrary to Heisenberg's commutation relations, in Baaquie's equation (8) the commutator between the operators representing an option price and its derivative in time is a real multiple of the identity operator I. Likewise, contrary to Feynman's path integral description of quantum particles, in Ilinski the Lagrangian functional used in the derivation of the probabilistic equation (12) does not have any complex coefficient. The lack of the imaginary unit i therefore reveals a formal disanalogy between quantum theory and finance. However, this point has implications that go beyond a mere mathematical fact, since it marks a profound difference in the way in which the formalism is applied to describe physical and financial processes, respectively. Indeed, what matters here is not quite the imaginary unit per se, which could even be discarded, for example, by rotating the time coordinate in the complex plane, but rather the dynamics that the relevant choice of mathematical structure implies, which in turn enables one to make empirical predictions matching one's observations in physics and in finance.

For the significant structural difference is that in quantum mechanics the state variable is a phase (namely, a vector of norm 1), while in quantum finance the state variable is directly the probability function. Since the Schrödinger equation is linear, phases can be added so as to give rise to interference patterns. However, as a matter of fact, no similar phenomenon appears in finance. To be sure, the proposed econophysical models are also linear, and hence the superposition principle holds, that is, any linear combination of solutions is also a solution, just like in quantum physics. This fact in itself is not surprising, since one may well expect that the sum of the results of two independent investments will be equal to the result of the joint investment. The crucial point, instead, is that the linearity of these models is intended with respect to the probabilities. By contrast, in quantum mechanics the linearity is intended with respect to the wave function, whose squared modulus yields the relevant probability: it is this fact that, courtesy of the properties of complex numbers, implies that in the quantum world inference patterns can arise, as in the famous double-slit experiment with a beam of electrons. Yet, such a typical quantum-mechanical phenomenon has no analogue in quantum finance. So, if one were to retain the full analogy with quantum theory, one would predict empirical effects that are not observed in the financial markets.

7. Conclusion. In this article we raised the question what is really quantum in quantum econophysics within the context of the approaches formulated by Baaquie and Ilinski in quantum finance. After discussing the alleged similarities between quantum mechanics and finance that lie underneath the relevant models, we surveyed the salient aspects of the two proposals under investigation and then we critically compared them. The upshot of our analysis is that the outstanding question what is really quantum in quantum econophysics has a twofold answer. On the one hand, the ability of the quantum methods to treat uncertainty provides a powerful tool to deal with the intrinsic randomness of financial processes. On the other hand, the formal disanalogy between quantum theory and finance, which manifests itself in the lack of the imaginary unit in the models proposed by Baaquie and Ilinski, allows one to apply these methods in such a way as to effectively account for financial data without entailing unobserved empirical predictions that are just peculiar to the quantum world.

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