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Maximum area with Minkowski measures of perimeter

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The oldest competition for an optimal (area-maximizing) shape was won by the circle. But if the fixed perimeter is measured by the line integral of $|\mathrm{d}x| + |\mathrm{d}y|$, a square would win. Or if the boundary integral of $\max(|\mathrm{d}x|, |\mathrm{d}y|)$ is given, a diamond has maximum area. For any norm in $\mathbb{R}^2$, we show that when the integral of $\|(\mathrm{d}x, \mathrm{d}y)\|$ around the boundary is prescribed, the area inside is maximized by a ball in the dual norm (rotated by $\pi/2$).

This 'isoperimetrix' was found by Busemann. For polyhedra it was described by Wulff in the theory of crystals. In our approach, the Euler–Lagrange equation for the support function of S has a particularly nice form. This has application to computing minimum cuts and maximum flows in a plane domain.

1. Introduction

We cannot really claim that Queen Dido of Carthage was a mathematician. But according to Virgil [38] she guessed the correct solution to an isoperimetric problem. Since the area to be maximized was partly bounded by water, she even went beyond the single constraint 'perimeter = L' that leads to a circle. The queen's solution was a semicircle bounded by the water (because the perimeter along that flat side did not count).

The authenticity of this example might be questioned, but it illustrates two themes of this small paper:

(i) the definition of perimeter is crucial;

(ii) a constraint $S \subset \Omega$ will affect the optimal shape S.

For the perimeter, part of the question is whether to count a boundary piece where ∂S meets $\partial\Omega$. (Dido did not and we will.) But answering a different question is the main purpose of this note. Suppose the *measure of arclength is changed* from the ℓ^2 norm $\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$. We might use the ℓ^1 norm $\mathrm{d}s = |\mathrm{d}x| + |\mathrm{d}y|$ or any other norm on $\mathbb{R}^2$. Then the perimeter constraint $\int \|s'(t)\| \,\mathrm{d}t = L$ is changed, and a new shape S provides maximum area.

We prove that *the optimal S is a rotated ball* $\|(x, y)^{\perp}\|_{\mathrm{D}} \leqslant R$, in the norm dual to the norm that defines arclength. Since ℓ^∞ is dual to ℓ^1, the square $\max(|x|, |y|) \leqslant L/8$ provides maximum area when the perimeter is $\int |\mathrm{d}x| + |\mathrm{d}y| = L$. The diamond $|x| + |y| \leqslant L/4$ (a ball in ℓ^1) is optimal when the perimeter is measured by

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$\int \max(|\mathrm{d}x|, |\mathrm{d}y|) = L$. Since ℓ^2 is dual to itself, the classical isoperimetric problem produces a circle.

All these optimal shapes are known! Section 4 gives a brief history of this isoperimetric problem in the *Minkowski plane*—maximizing area when the perimeter is specified in a prescribed norm. The 'isoperimetrix' S was discovered by Busemann [7], applying the Brunn–Minkowski inequality [16]. For polyhedra, the optimal shape was described much earlier, in Wulff's theory of crystals. Many authors have contributed, and our purpose is to give a direct and elementary argument using the calculus of variations.

Our approach computes perimeter and area from the support function $p(\phi)$ of S. In figure 1, $p(\phi)$ is the distance from $(0,0)$ to the tangent line normal to the direction ϕ. The area is $\frac{1}{2}\int(p^2 - p'^2)\,\mathrm{d}\phi$. The usual measure of perimeter (ℓ^2 norm) is $\int(p + p'')\,\mathrm{d}\phi = L$. Minimizing area with this constraint quickly yields a circular shape S: $p = \text{const.}$ from equation (3.2).

That computation of the optimal shape generalizes directly to other norms, and *the Euler–Lagrange equation (2.5) for the best $p(\phi)$ is linear.*

We met these new measures for the perimeter in identifying the *minimum cut* ∂S in a plane domain Ω [34],

$$\text{constrained isoperimetric problem:} \quad \min \frac{|\partial S|}{|S|} \quad \text{for } S \subset \Omega. \tag{1.1}$$

In the ℓ^2 norm for $\mathrm{d}s$, the minimum ratio is *Cheeger's constant* $h(\Omega)$. That number provides a lower bound $\lambda_1 \geqslant h^2/4$ for the eigenvalues of the Laplace–Dirichlet operator on Ω [9, 17]. The number h is also (by duality) the largest value λ such that $\operatorname{div} v = \lambda$ has a solution with $v_1^2 + v_2^2 \leqslant 1$ in Ω.

That *max flow–min cut theorem* [34] for flows in a plane domain Ω holds for any norm of the velocity vector $v(x,y)$. This application of duality is remarkable for the fact that the minimum cut ∂S may easily be determined from (1.1), while the flow vector $v(x,y)$ that fills the cut to capacity has only recently been approximated [26]. No analytic expression for v has been discovered [33, 35].

We end with these applications after finding the Euler–Lagrange equations for maximum area (with perimeter $= L$).

After the manuscript of this paper was completed, we learned of new applications [19–22] to landslide modelling (and blocking sets and the 1-Laplacian). Z. Milbers (2006, unpublished thesis, Köln Universität) has found the flow field $v(x,y)$ required in equation (6.1) with the maximal $\lambda = 2 + \sqrt{\pi}$.

2. A rotated dual ball is isoperimetrically optimal

To start, the only constraint $\int \|(\mathrm{d}x, \mathrm{d}y)\| = L$ is on the perimeter of S. The problem is to maximize the area. Classical arguments show that the optimal shape S is convex. We will compute its perimeter and area using the *support function* $p(\phi)$—the distance to the support line—rather than the position function $r(\theta)$.

The two equations in the caption of figure 1 express the fact that ∂S is the envelope of its tangent lines. Santaló [32] solves those equations for the point of tangency:

$$x = p\cos\phi - p'\sin\phi, \qquad y = p\sin\phi + p'\cos\phi. \tag{2.1}$$

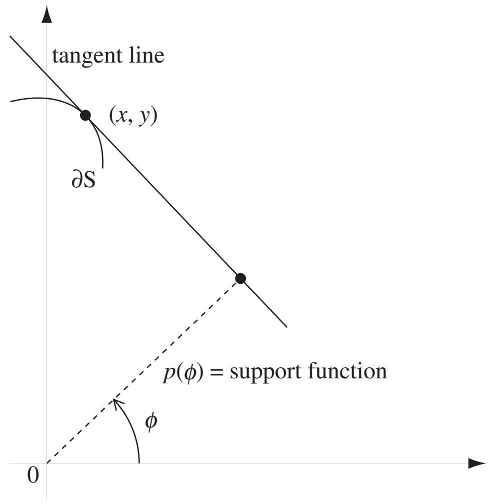

Figure 1. The line at distance p is tangent to ∂S at (x, y) [32, p. 2].
$x\cos\phi + y\sin\phi = p(\phi)$ and $-x\sin\phi + y\cos\phi = p'(\phi)$.

Differentiating these equations, we have

$$\mathrm{d}x = -(p + p'')\sin\phi\,\mathrm{d}\phi, \qquad \mathrm{d}y = (p + p'')\cos\phi\,\mathrm{d}\phi. \tag{2.2}$$

The convexity of S is equivalent to $p + p'' \geqslant 0$. Then the perimeter of S can be computed directly from $p(\phi)$ and the norm $N(\phi)$ of the unit tangent vector $(-\sin\phi, \cos\phi)$:

$$\text{perimeter of } S = \int_0^{2\pi} \|(\mathrm{d}x, \mathrm{d}y)\| = \int_0^{2\pi} (p + p'')N(\phi)\,\mathrm{d}\phi. \tag{2.3}$$

Since $N(\phi) = 1$ in the ℓ^2 norm and $p(\phi)$ is periodic, the usual measure of perimeter has the particularly nice form $\int p\,\mathrm{d}\phi$.

The Euclidean area of S is also given by a line integral involving $p(\phi)$. Santaló notes that if S is decomposed into triangles of height $p(\phi)$ and base $\mathrm{d}s = (p+p'')\,\mathrm{d}\phi$ (vertex at the origin, Euclidean length along ∂S), then

$$\text{area of } S = \frac{1}{2}\int_0^{2\pi} p(p + p'')\,\mathrm{d}\phi = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\,\mathrm{d}\phi. \tag{2.4}$$

The last step was integration by parts of pp'', using the periodicity $pp'(2\pi) = pp'(0)$. A non-smooth support function (where ∂S contains a line segment) can be approximated by a smoother $p(\phi) \in C^2$, to justify (2.3) and (2.4).

Those formulae produce a beautifully simple restatement of the isoperimetric problem with Minkowski perimeter:

$$\text{maximize } \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\,\mathrm{d}\phi \text{ subject to } \int_0^{2\pi} (p + p'')N(\phi)\,\mathrm{d}\phi = L.$$

A Lagrange multiplier λ builds the perimeter constraint into the functional

$$F(p, \lambda) = \text{area} - \lambda(\text{perimeter} - L).$$

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The calculus of variations sets the first variation of F to zero. Equation (4.1) will recall this step, and here we go directly to the unusually neat form of the optimality condition:

$$\text{Euler–Lagrange:} \quad p(\phi) + p''(\phi) - \lambda[N(\phi) + N''(\phi)] = 0. \tag{2.5}$$

The equation is linear in $p(\phi)$, *not in the position variable* $r(\theta)$. A particular solution to (2.5) is $p(\phi) = \lambda N(\phi)$, and the complete solution includes arbitrary constants A and B:

$$p(\phi) = \lambda N(\phi) + A\cos\phi + B\sin\phi. \tag{2.6}$$

Centring S horizontally and vertically, so that $p(0) = p(\pi)$ and $p(\pi/2) = p(3\pi/2)$, achieves $A = 0$ and $B = 0$. Then the support function of the optimal shape is $\lambda N(\phi)$:

$$\text{Optimal shape } S: \quad p(\phi) = \lambda N(\phi) = \lambda\|(-\sin\phi, \cos\phi)\|. \tag{2.7}$$

This shape will be seen as a ball in the dual norm, scaled by λ and rotated by $\pi/2$.

3. The classical isoperimetric problem

The classical problem uses the ℓ^2 norm, with $N(\phi) = \|(-\sin\phi, \cos\phi)\| = 1$. We repeat the steps in detail for this special case, to emphasize how the isoperimetric problem is simplified by using the support function $p(\phi)$:

$$\text{maximize } \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\,\mathrm{d}\phi \text{ subject to } \int p\,\mathrm{d}\phi = L. \tag{3.1}$$

Suppose an optimal $p(\phi)$ is perturbed by $q(\phi)$. Then $\int q\,\mathrm{d}\phi = 0$ to maintain the perimeter constraint $\int (p+q)\,\mathrm{d}\phi = L$. The new area is half of

$$\int_0^{2\pi} [(p+q)^2 - (p'+q')^2]\,\mathrm{d}\phi = \int (p^2 - p'^2)\,\mathrm{d}\phi - 2\int (pq - p'q')\,\mathrm{d}\phi + \int (q^2 - q'^2)\,\mathrm{d}\phi. \tag{3.2}$$

To ensure that $p(\phi)$ is maximizing, we need the first-order condition

$$\int (pq - p'q')\,\mathrm{d}\phi = 0$$

whenever $\int q\,\mathrm{d}\phi = 0$. We also need $\int (q^2 - q'^2)\,\mathrm{d}\phi \leqslant 0$ for all q.

The first requirement, after integration by parts, produces the Euler–Lagrange equation $p + p'' = \text{const.}$:

$$\int (pq - p'q')\,\mathrm{d}\phi = \int (p + p'')q\,\mathrm{d}\phi = 0 \quad \text{when} \int q\,\mathrm{d}\phi = 0 \text{ needs } p + p'' = \lambda. \tag{3.3}$$

This is the argument that becomes automatic with the introduction of a Lagrange multiplier. *If* $(u, q) = 0$ *implies* $(v, q) = 0$, *then* $u = \lambda v$. Here $p + p'' = \lambda$ is (2.5) with $N = 1$. The solution $p = \lambda$ produces a circle. The circle can be translated as in (2.6) by including in p the homogeneous solution $A\cos\phi + B\sin\phi$.

The second requirement, $\int q^2 \leqslant \int q'^2$, is exactly Wirtinger's inequality for any periodic $q(\phi) = \sum a_n \mathrm{e}^{\mathrm{i}n\phi}$, when we know that $a_0 = \int q\,\mathrm{d}\phi/2\pi = 0$:

$$\text{Wirtinger:} \quad \int_0^{2\pi} (q^2 - q'^2)\,\mathrm{d}\phi = 2\pi\Big[\sum |a_n|^2 - \sum n^2|a_n|^2\Big] \leqslant 0. \tag{3.4}$$

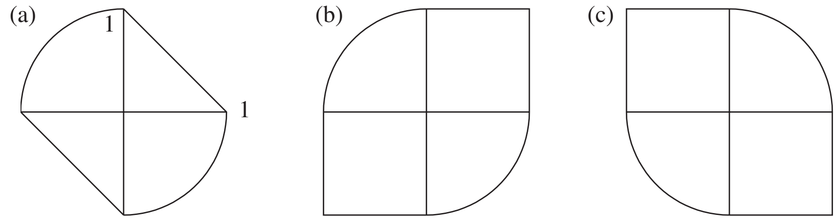

Figure 2. Dual unit balls have $|xu + yv| \leqslant 1$. Rotated ball has least perimeter. (a) $\|(u,v)\| = u + v$ or $\sqrt{u^2+v^2}$; (b) $\|(x,y)\|_{\mathrm{D}} = 1$, N-perimeter 8; (c) $\|(x,y^{\perp})\|_{\mathrm{D}} = 1$, N-perimeter $4+\pi$.

This is an equality when the only non-zero Fourier coefficients have $n^2 = 1$. That allows the arbitrary $q(\phi) = A\cos\phi + B\sin\phi$ which we met in (2.6). If any non-zero coefficient of $q(\phi)$ has $n^2 > 1$, then (3.4) is a strict inequality. In that case the perturbation $q(\phi)$ of the optimal $p(\phi) = \lambda$ will decrease the area in (3.2).

The set S with $p(\phi) = \text{const.}$ is a circle, the isoperimetric shape that was conjectured by the Greeks. Satisfactory proofs were found much later! For guides to the fascinating history of this problem see [4, 29, 31]. A short analytic proof was given by Lax [24].

Returning to more general norms, with the factor $N(\phi) = \|(-\sin\phi, \cos\phi)\|$ in the perimeter integral, the same steps (3.2), (3.3) lead to the Euler–Lagrange equation. Now $\int(p+p'')q = 0$ whenever $\int(N+N'')q = 0$, so that $p + p''$ is proportional to $N + N''$. *The solution is* $p = \lambda N$.

We note a small twist to the usual proof that S must be convex. If not convex, there will be a line not crossed by S but tangent at two points. Normally we can move a region (between the line and ∂S) across that line. The mirror image leads to increased area with the same perimeter. With $N(\phi)$ included in the perimeter, we have to *reverse the direction* of that piece of ∂S, for the move to maintain a fixed perimeter.

Dacorogna and Pfister [10, 11] have extended Wirtinger's inequality (3.4) to this case of more general norms. Their analysis uses a parametric form $(x(s), y(s))$ for the boundary ∂S, and gives the proof in full detail.

The area-maximizing shape S has $p(\phi)$ given by $\lambda N(\phi)$. By definition, the unit ball $\|(x,y)\|_{\mathrm{D}} \leqslant 1$ in the dual norm has the support function $\|(\cos\phi, \sin\phi)\|$. Therefore S has the same shape as that ball, after it is rescaled by λ and rotated by $\pi/2$. Figure 2 shows a particular unit ball and its dual.

With $p(\phi) = \lambda N(\phi)$, the ratio of perimeter to area is $2/\lambda$:

$$\frac{\int(p+p'')N\,\mathrm{d}\phi}{\int(p+p'')p\,\mathrm{d}\phi/2} = \frac{\lambda\int(N+N'')N\,\mathrm{d}\phi}{\lambda^2\int(N+N'')N\,\mathrm{d}\phi/2} = \frac{2}{\lambda}. \tag{3.5}$$

The Lagrange multiplier λ is determined by the value L of the perimeter:

$$L = \int_0^{2\pi}(p+p'')N\,\mathrm{d}\phi = \lambda\int_0^{2\pi}(N+N'')N\,\mathrm{d}\phi = \lambda\int_0^{2\pi}(N^2 - N'^2)\,\mathrm{d}\phi.$$

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4. The isoperimetrix

When those paragraphs were written, using the calculus of variations to identify the optimal shape S, I did not know whether this problem (in an arbitrary norm) was new. It is definitely not new! Sixty years ago Busemann [7] asked exactly the same question, and gave S the name *isoperimetrix*. This construction became a key result in convex geometry [37], following Minkowski's fundamental idea that every norm on $\mathbb{R}^n$ defines a length and area and volume.

One family of inequalities is at the centre of that subject. The *Brunn–Minkowski inequality* states that the nth root of the volume is a concave function on the set of convex bodies. When Busemann reduced the isoperimetric problem to that inequality, his proof was complete. He did not need an Euler–Lagrange equation for the boundary curve ∂S, and we have not found this equation for $p(\phi)$ in the literature.

Geometry leads to Minkowski's concept of a mixed volume $V(K_1, K_2)$, which is the coefficient of ab in the volume of a convex combination $aK_1 + bK_2$. The two-dimensional inequality that Brunn proved becomes $V^2(K_1, K_2) \geqslant V(K_1)V(K_2)$. In fact, Gromov has pointed to a mention of the isoperimetrix in Brunn's 1887 inaugural dissertation. The inequalities that Minkowski uncovered made it possible for Busemann to extend his isoperimetrix to $\mathbb{R}^n$ [8].

To an outsider, the Euler–Lagrange equation is more familiar. Integration by parts quickly produces the first variation of $F = \int f(p, p', p'')\,\mathrm{d}\phi$:

$$\text{Euler–Lagrange:}\quad \frac{\partial f}{\partial p} - \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{\partial f}{\partial p'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{\partial f}{\partial p''}\right) = 0. \tag{4.1}$$

Applied to $f = p(p + p'')/2 - \lambda(p + p'')N$, this gives (2.5) and the solution $p(\phi) = \lambda N(\phi)$. Euclidean geometry has $N(\phi) = 1$, and then $p(\phi) = \lambda$ produces a circle. We apologise for asking any reader to choose between geometry and analysis.

One beautiful sidelight on the isoperimetrix appears in [40]. Wallen noticed that Kepler's law of *equal areas swept out in equal times* holds for a planet that travels along the optimal ∂S with constant Minkowski speed. In our notation, this is the statement that $p(p + p'')/2$ is in fixed proportion to $(p + p'')N$ because $p = \lambda N$.

We also owe to Wallen [39] a new appreciation of Wirtinger's inequality in (3.4). This can yield the Brunn–Minkowski inequality [5, p. 114], Bonnesen's refinement [13, 40] and the classical isoperimetric inequality [10, 11]. Every argument for a minimum or maximum needs convexity or concavity at some point. Wirtinger's inequality provides that here, by confirming that the last term in (3.2) decreases the area.

When the unit balls are polyhedra (as for ℓ^1 and ℓ^∞) the vertex directions in one ball are normal to the flat sides of the other ball. The isoperimetrix in this polyhedral case is the Wulff shape [41] in the theory of crystals. The energy is given by the (normed) surface area of the faces, and minimized.

The isoperimetrix has a long and important history in geometric measure theory. The existence of a minimizing S and its uniqueness and form have been established for increasingly general domains and integrands [1, 6, 14, 28, 36]. Morgan [27] outlines a history of this effort (see also [26, 30]) that is fascinatingly independent of convex geometry. It was Gromov who connected both approaches to the work of Levy [25]

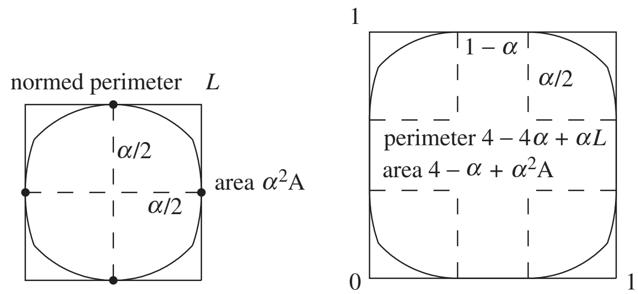

Figure 3. Four pieces of α (isoperimetrix) yield the constrained isoperimetrix.

by a new proof of the Brunn–Minkowski inequality [3, 18]. That proof rests on the divergence theorem, which was applied to the symmetrization map by Knothe [23].

Certainly this deeper analysis admits candidate shapes for which the first variation in (4.1) is not immediately defined. The proofs are far from elementary. The reader may wish to regard the Euler–Lagrange equation as decisive only when ∂S is piecewise smooth.

5. The constraint $S \subset \Omega$

Suppose we remove the constraint of a fixed perimeter, and require instead that S must be a subset of a given plane domain Ω. The ratio of perimeter to area will be minimized by a set that touches the boundary $\partial\Omega$. (Rescaling S by a factor $a > 1$ will decrease the ratio, so the optimal S cannot lie strictly inside Ω.) In fact, ∂S is tangent to $\partial\Omega$ at points where they meet.

This is the opposite of Queen Dido's semicircle, which was perpendicular to $\partial\Omega$ because the piece $\partial\Omega \cap \partial S$ was not counted in the perimeter of S. In both problems, S has the shape of an isoperimetrix in the interior of Ω. (In the language of optimization, the constraint $S \subset \Omega$ is not active inside Ω.) This section concentrates on calculating S and understanding it as a *minimum cut*, without proofs.

For an explicit calculation we take Ω to be a unit square. A typical isoperimetrix is drawn in figure 3, and its four pieces can fit (tangent to $\partial\Omega$) into the corners of the square. For simplicity we suppose that $(1, 0)$ and $(0, 1)$ lie on the dual unit ball, so the flat pieces $\partial S \cap \partial\Omega$ are measured by their length. Our problem is to determine the optimal scaling factor $\alpha \leqslant 1$ to minimize the ratio of perimeter to area:

$$\frac{\text{perimeter}}{\text{area}} = \frac{4 - 4\alpha + \alpha L}{1 - \alpha^2 + \alpha^2 A}. \tag{5.1}$$

The derivative of this ratio is zero when

$$(1 - \alpha^2 + \alpha^2 A)(L - 4) = (4 - 4\alpha + \alpha L)2\alpha(A - 1). \tag{5.2}$$

That quadratic equation can be written as

$$\alpha^2 CD - 8\alpha D + C = 0 \quad \text{with } C = 4 - L \text{ and } D = 1 - A. \tag{5.3}$$

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Comparing (5.1) and (5.2), the minimum ratio will be $C/2\alpha D$. It is simpler to solve (5.3) for the reciprocal $\beta = 1/\alpha \geqslant 1$:

$$\beta = \frac{4D + \sqrt{(4D)^2 - C^2 D}}{C}.$$

Then the optimum ratio is

$$\frac{\text{perimeter}}{\text{area}} = \frac{\beta C}{2D} = 2 + \sqrt{4 - \frac{C^2}{4D}}. \tag{5.4}$$

In the ℓ^2 norm, the isoperimetrix inside the unit square is a circle of perimeter π and area $\frac{1}{4}\pi$. Cheeger's constant with four quarter circles and boundary segments is then $2 + \sqrt{\pi}$:

$$\frac{C^2}{4D} = \frac{(4-\pi)^2}{4-\pi} = 4 - \pi \quad \text{and} \quad \frac{\text{perimeter}}{\text{area}} = 2 + \sqrt{\pi}. \tag{5.5}$$

6. Max flow–min cut theorem

This final section describes the application which led us to the constraint $S \subset \Omega$ and to the minimum problem for the ratio perimeter/area. The starting point is a maximum problem for flow through the domain Ω. We begin with the specific question that is answered by the number $2 + \sqrt{\pi}$ (the minimum ratio computed just above).

6.1. Maximum flow problem

Find the vector $v = (v_1(x, y), v_2(x, y))$ that has the largest possible divergence λ, constant in the unit square Ω, with the constraint $v_1^2 + v_2^2 \leqslant 1$ in Ω:

$$\text{maximize } \lambda \text{ such that } \operatorname{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \lambda \quad \text{with } \|v\| \leqslant 1 \text{ at all } x, y \in \Omega. \tag{6.1}$$

This can be seen as a model of physical phenomena that are linear up to a point—but that point $\|v\| = 1$ cannot be passed. If we twist a long plastic cylinder with cross-section Ω, then at some level of torsion the stresses v_1 and v_2 cannot be sustained. The equilibrium $\operatorname{div} v = \lambda$ becomes incompatible (for any distribution of stresses) with $\|v\| \leqslant 1$. The cylinder reaches a yield limit and collapses. The collapse will occur along a 'minimum cut', where perimeter/area $= 2 + \sqrt{\pi}$.

The key is the divergence theorem applied to subsets $S \subset \Omega$:

$$\iint_S \operatorname{div} v \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial S} v \cdot n \, \mathrm{d}s \quad \text{so that } \lambda \times \text{area } |S| \leqslant \text{perimeter } |\partial S|. \tag{6.2}$$

That inequality came from $|v \cdot n| \leqslant 1$. The *continuous max flow–min cut theorem* (established by duality theory in [34]) leads to the optimality condition: equality will hold on the extremal set S, where $|\partial S|/|S|$ is a minimum. At that moment, equation (6.2) ensures that $v = n$ along the cut. But we do not know the vector field $v(x, y)$ that achieves $\lambda_{\max} = 2 + \sqrt{\pi}$. This maximum flow has been approximately computed by discrete quadratic programming [26].

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For the *discrete problem* of maximum flow through a network, the minimum cut separates a subset of nodes including the 'source' from the complementary subset including the 'sink'. All edges across the cut are filled to capacity. A finite algorithm increases the flow up to this maximum, and at that point the minimal cut (the bottleneck in the flow) is identified. This source–sink framework corresponds to $\operatorname{div} v = 0$ in the interior of Ω, given in the discrete case by Kirchhoff's current law that flow into a node equals flow out. Our specific problem has source term λ inside Ω rather than on $\partial\Omega$, and the general case [34] has both interior sources $\lambda F(x, y)$ and boundary sources $\lambda f(x, y)$.

A more widely studied example is the Monge–Kantorovich continuous analogue [12,15] of the transportation problem on a discrete network. Again the applications are remarkable but the theory is not entirely constructive. We are asked to transport a mass from one density distribution to another, with minimum work. (For Monge, those were earth defences demanded by Napoleon.) The key rule is that the shortest paths of mass movement will never cross.

The maximum flow problem is finding unexpected uses in medical image segmentation [2] and in stereo reconstruction. We refer the reader to [35] for the essential role of the *co-area formula* in converting the dual to maximum flow into the constrained isoperimetric problem that directly identifies the minimum cut ∂S.

(i) Maximum flow. Maximize λ such that $\operatorname{div} v = \lambda$ and $\|v\| \leqslant 1$ in Ω.

(ii) Formal dual. Minimize $\iint_\Omega \|\operatorname{grad} u\| \,\mathrm{d}x\,\mathrm{d}y$ with $\iint u \,\mathrm{d}x\,\mathrm{d}y = 1$.

(iii) Minimum cut. Minimize (perimeter of S)/area of S) for $S \subset \Omega$.

The co-area formula expresses $\iint \|\operatorname{grad} u\| \,\mathrm{d}x\,\mathrm{d}y$ as an integral over the perimeters of level sets $S_t = \{u(x, y) \leqslant t\}$. This identifies the extreme points of the unit ball in the bounded variation norm $\|u\|_{\mathrm{BV}} = \iint \|\operatorname{grad} u\| \,\mathrm{d}x\,\mathrm{d}y$. Those extreme points are multiples of characteristic functions χ_S (equal to 1 on S and elsewhere zero, with $\|\chi\|_{\mathrm{BV}} =$ perimeter of S). When the search for a minimum in the formal dual is restricted to $u = \chi_S/(\text{area of } S)$, we have the constrained isoperimetric problem of minimizing $|\partial S|/|S|$ for $S \subset \Omega$.

Finally, other norms on $\mathbb{R}^2$ will appear as soon as we change the pointwise constraint on $v(x, y)$ in the original maximum problem:

> if the constraint becomes $\|v\| \leqslant 1$ for a different norm, then the dual of that norm appears in $\iint \|\operatorname{grad} u\|_{\mathrm{D}} \,\mathrm{d}x\,\mathrm{d}y$ and in the co-area formula and in the perimeter of S.

Any part of ∂S inside $\partial\Omega$ will have the form of the Busemann–Wulff isoperimetrix. Our contribution was to find the support function of that set as $p(\phi) = \lambda N(\phi)$, directly from the Euler–Lagrange equation $p + p'' = \lambda(N + N'')$.

Acknowledgments

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