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AN INVARIANT OF LEGENDRIAN AND TRANSVERSE LINKS FROM OPEN BOOK DECOMPOSITIONS OF CONTACT 3-MANIFOLDS

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Abstract. We introduce a generalization of the Lisca–Ozsváth–Stipsicz–Szabó Legendrian invariant $\mathfrak{L}$ to links in every rational homology sphere, using the collapsed version of link Floer homology. We represent a Legendrian link L in a contact 3-manifold (M, ξ) with a diagram D, given by an open book decomposition of (M, ξ) adapted to L, and we construct a chain complex $cCFL^-(D)$ with a special cycle in it denoted by $\mathfrak{L}(D)$. Then, given two diagrams D_1 and D_2 which represent Legendrian isotopic links, we prove that there is a map between the corresponding chain complexes that induces an isomorphism in homology and sends $\mathfrak{L}(D_1)$ into $\mathfrak{L}(D_2)$. Moreover, a connected sum formula is also proved and we use it to give some applications about non-loose Legendrian links; that are links such that the restriction of ξ on their complement is tight.

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1. Introduction. The Legendrian invariant $\mathfrak{L}$ was first introduced in [14]. In this paper, we generalize $\mathfrak{L}$ to Legendrian links in rational homology contact 3-spheres. For sake of simplicity, we consider only null-homologous links, which in this settings are links whose all components represent trivial classes in homology.

A *contact structure* ξ on a differentiable 3-manifold M is a smooth 2-plane field that is given as the kernel of a 1-form α such that $\alpha \wedge d\alpha > 0$. A *Legendrian n-component link* $L \hookrightarrow (M, \xi)$ is a link such that the tangent space T_pL is contained in ξ_p for every point p. A link $T \hookrightarrow (M, \xi)$ is *transverse* if $T_pT \oplus \xi_p = T_pM$ for every $p \in T$ and $\alpha > 0$ on T. A *Legendrian isotopy* between L and L' is a smooth isotopy F_t of M, where $t \in [0, 1]$, such that $F_0(L) = L$, $F_1(L) = L'$ and $F_t(L)$ is Legendrian for every t.

A 3-manifold M can be represented with an open book decomposition (B, π). Suppose that L is a Legendrian n-component *oriented* link in M, equipped with a contact structure ξ. When some compatibility conditions with ξ and L, which are explained in Section 2, are satisfied (B, π), together with an appropriate set A of embedded arcs in the page S_1, is said to be adapted to the Legendrian link L. In particular, we prove the following theorem.

THEOREM 1.1. *Every Legendrian oriented link L in a contact 3-manifold (M, ξ) admits an adapted open book decomposition (B, π, A) which is compatible with the triple (L, M, ξ). Moreover, the contact framing of L coincides with the framing induced on L by the page S_1.*

Suppose from now on that M is a rational homology sphere. A *multi-pointed Heegaard diagram* is defined in [19] as two collections of $g + n - 1$ curves and two sets of n basepoints in a genus g surface Σ which describe an oriented link in M. As shown in Section 3, one can associate, up to isotopy, a diagram of this kind to every adapted open book decomposition, compatible with the triple (L, M, ξ). This diagram is called a Legendrian Heegaard diagram and it is denoted with $D_{(B,\pi,A)}$. The surface Σ in $D_{(B,\pi,A)}$ is obtained by gluing the pages S_1 and $\overline{S_{-1}}$ together, moreover all the basepoints are contained in S_1.

In Heegaard Floer theory, the diagram $D_{(B,\pi,A)}$ gives a bigraded chain complex $\left(cCFL^-(D_{(B,\pi,A)}), \partial^-\right)$, generated by some discrete subsets of points in Σ [21] and whose homology is an $\mathbb{F}[U]$-module, where $\mathbb{F}$ is the field with two elements, called *collapsed link Floer homology*. The isomorphism type of such a group is a smooth link invariant and it is denoted by $cHFL^-\left(\overline{M}, L\right)$, see [16, 19]. Furthermore, there is only one cycle in $cCFL^-\left(D_{(B,\pi,A)}\right)$, see [11, 14], that lies on the page S_1: this cycle is denoted by $\mathfrak{L}\left(D_{(B,\pi,A)}\right)$.

THEOREM 1.2. *Let us consider a Legendrian Heegaard diagram D, given by an open book compatible with a triple (L, M, ξ), where M is a rational homology 3-sphere and L is a null-homologous Legendrian n-component oriented link.*

Let us take the cycle $\mathfrak{L}(D) \in cCFL^-(D, \mathfrak{t}_{\mathfrak{L}(D)})$; if D_1 and D_2 are diagrams as before for Legendrian isotopic links then we can find a map

$$cCFL^-(D_1, \mathfrak{t}_{\mathfrak{L}(D_1)}) \longrightarrow cCFL^-(D_2, \mathfrak{t}_{\mathfrak{L}(D_2)}) \quad \textit{such that} \quad \mathfrak{L}(D_1) \longrightarrow \mathfrak{L}(D_2)$$

and inducing a bigraded isomorphism in homology. Furthermore, the Spin^c structure $\mathfrak{t}_{\mathfrak{L}(D)}$ coincides with $\mathfrak{t}_\xi$.

As in [14], we introduce an equivalence relation on the family of bigraded $\mathbb{F}[U]$-modules with a distinguished element in them. Namely, we say that $(M, m) \sim (N, n)$ if and only if there is a bigraded isomorphism $M \to N$ such that $m \mapsto n$. Hence, Theorem 1.2 tells us that the equivalence class of the pair

$$\left(cHFL^-\left(\overline{M}, L, \mathfrak{t}_\xi\right),\ [\mathfrak{L}(D)]\right)$$

with respect to $\sim$ is a Legendrian invariant of (L, M, ξ); and we denote it with $\mathfrak{L}(L, M, \xi)$.

In [1], Baldwin, Vela-Vick, and Vértesi, using a different construction, introduced another invariant of Legendrian links in contact 3-manifolds which generalizes $\mathfrak{L}$ in the case of knots in the standard 3-sphere. The same argument in [1] implies that this invariant coincides with our $\mathfrak{L}$ for every Legendrian link in (S^3, ξ_{st}).

CONJECTURE 1.3. *The invariant $\mathfrak{L}$ concides with the invariant from* [1] *for every null-homologous link in a rational homology* 3*-sphere.*

REMARK 1.4. It is important to observe that we cannot define $\mathfrak{L}$ just as a fixed element in $cHFL^-$ without a naturality property, similar to the one given by Juhász, Thurston, and Zemke in [13]. In fact, even for Legendrian isotopic links, the homology groups have different presentations according to the choice of the corresponding Legendrian Heegaard diagrams: this fact justifies the statement of Theorem 1.2 and the definition of $\mathfrak{L}$. Nonetheless, in the rest of the paper, for the sake of simplicity, we will sometimes reduce the formalism and write $\mathfrak{L}(L, M, \xi)$ for the homology class $[\mathfrak{L}(D)]$ in $cHFL^-\left(\overline{M}, L, \mathfrak{t}_\xi\right)$.

Clearly, it is very difficult to claim that two Legendrian links have different invariant $\mathfrak{L}$ in general; nonetheless, we can still prove that $\mathfrak{L}$ has some properties which do not involve naturality, for example, its bigrading and U-torsion order in $cHFL^-$ and the behavior under connected sums. Such properties are all discussed in detail in the paper.

In addition, we note that in [**12**], Juhász, Miller, and Zemke showed that the transverse link invariant introduced in [**1**] has indeed a naturality property, but we recall that such an invariant is known to coincide with $\mathfrak{L}$ only in (S^3, ξ_{st}).

The smooth link type is clearly a Legendrian invariant. Together with the Thurston–Bennequin and the rotation numbers are called classical invariant of a Legendrian link. The *Thurston–Bennequin* and *rotation numbers* of a null-homologous link L in a rational homology contact 3-sphere (M, ξ) are defined as follows. The first one, denoted with $\text{tb}(L)$, is the linking number of the contact framing ξ_L of L respect to ξ and a Seifert surface for L. While, the second one, denoted with $\text{rot}(L)$, is defined as the numerical obstruction to extending a non-zero vector field, everywhere tangent to the knot, to a Seifert surface of K (see [**6**]).

If in (M, ξ) there is an embedded disk E, with Legendrian boundary, such that $\text{tb}(\partial E) = 0$ then the structure ξ is *overtwisted*; otherwise it is called *tight*. Furthermore, in overtwisted structures, there are two types of Legendrian and transverse links: *non-loose* if their complement is tight and *loose* if it is overtwisted. We have the following proposition.

PROPOSITION 1.5. *Let L be a loose Legendrian link in an overtwisted contact 3-manifold (M, ξ). Then we have that $\mathfrak{L}(L, M, \xi) = [0]$.*

We use the invariant $\mathfrak{L}$ to prove the existence of non-loose Legendrian n-component links, with loose components, in many overtwisted contact 3-manifolds.

THEOREM 1.6. *Suppose that (M, ξ) is an overtwisted 3-manifold such that there exists a contact structure ζ with $\mathfrak{t}_\xi = \mathfrak{t}_\zeta$ and $\widehat{c}(M, \zeta) \neq [0]$. Then there is a non-split Legendrian n-component link L, for every $n \geqslant 1$, such that $\mathfrak{L}(L, M, \xi)$ is non-zero and all of its sublinks are loose. In particular, L is non-loose and stays non-loose after a sequence of negative stabilizations.*

Furthermore, the transverse link T_L, obtained as transverse push-off of L, is again non-split and $\mathfrak{T}(T_L, M, \xi) := \mathfrak{L}(L, M, \xi)$ is non-zero, which means that T_L is also non-loose.

Furthermore, we give examples of *non-loose, non-simple Legendrian and transverse link types*. Here non-simple means that there exists a pair of Legendrian (transverse) links that have the same classical invariants and their components are Legendrian (transverse) isotopic knots, but they are not Legendrian (transverse) isotopic (as links). Moreover, with non-loose, we mean that such a pair consists of two non-loose Legendrian (transverse) links.

THEOREM 1.7. *Suppose that (M, ξ) is an overtwisted 3-manifold as in Theorem 1.6. Then in (M, ξ) there is a pair of non-loose, non-split Legendrian (transverse) n-component links, with the same classical invariants and Legendrian (transversely) isotopic components, but that are not Legendrian (transversely) isotopic.*

This paper is organized as follows. In Section 2, we define open book decompositions and we describe the compatibility condition with Legendrian links. Moreover, we show that such adapted open books always exist. In Section 3, we give some remarks on link Floer homology. In Section 4, we define $\mathfrak{L}$ and we prove its invariance under Legendrian isotopy. In Section 5, we show some properties of the invariant $\mathfrak{L}$, including the relations with the contact invariant $\widehat{c}$ and the classical invariants of Legendrian links. In Section 6, we generalize the transverse invariant $\mathfrak{T}$ to links and we describe some of its properties. Finally, in Section 7, we give some applications of our invariants.

2. Open book decompositions adapted to a Legendrian link.

2.1. Definition. Let us start with a contact 3-manifold (M, ξ). We say that an *open book decomposition* for M is a pair (B, π) where

- the binding B is a smooth link in M;
- the map $\pi : M \setminus B \to S^1$ is a locally trivial fibration such that $\overline{\pi^{-1}(\theta)} = S_\theta$ is a compact surface with $\partial S_\theta = B$ for every $\theta \in S^1$. The surfaces S_θ are called *pages* of the open book.

Moreover, the pair (B, π) supports ξ if, up to isotopy, we can find a 1-form α for ξ such that

- the 2-form $d\alpha$ is a positive area form on each page S_θ;
- we have $\alpha > 0$ on the binding B.

Let us take the page $S_1 = \overline{\pi^{-1}(1)}$, which is an oriented, connected, compact surface of genus g and with l boundary components. Assume from now on that links are always oriented; suppose that an n-component Legendrian link L in (M, ξ) sits inside S_1 and its components represent n independent elements in $H_1(S_1; \mathbb{F})$. We say that a collection of disjoint, simple arcs $A = \{a_1, \ldots, a_{2g+l+n-2}\}$ is a *system of generators for* S_1 *adapted to* L if the following conditions hold:

1. the subset $A^1 \sqcup A^2 \subset A$, where $A^1 = \{a_1, \ldots, a_n\}$ and $A^2 = \{a_{n+1}, \ldots, a_{2g+l-1}\}$, is a basis of $H_1(S_1, \partial S_1; \mathbb{F}) \cong \mathbb{F}^{2g+l-1}$;
2. we have that $L \pitchfork a_i = \{1 \text{ pt}\}$ for every $a_i \in A^1$ and $L \cap a_i = \emptyset$ for $i > n$. The arcs in the subset A^1 are called *distinguished arcs*;
3. the subset $A^3 = \{a_{2g+l}, \ldots, a_{2g+l+n-2}\}$ is such that the surface S given by the closure of $S_1 \setminus A$ has n disks as connected components, each one containing exactly one component of L.
   The arcs in $A^2 \sqcup A^3$ that appear twice on the same component of the boundary of S are called *dead arcs*; the others *living arcs*;
4. the elements in A^3 are all living arcs; moreover, they separates a unique pair of components of the disk $S_1 \setminus (A^1 \sqcup A^2)$.

With this definition in place, we say that the triple (B, π, A) is an open book decomposition *adapted* to the Legendrian link L if

- the pair (B, π) is compatible with (M, ξ);
- the link L is contained in the page S_1;
- the n components of L are independent in S_1;
- the set A is a system of generators for S_1 adapted to L.

In this case, we also say that the adapted open book decomposition (B, π, A) is compatible with the triple (L, M, ξ). It is important to observe that, since the components of L are required to be independent in homology, we only consider open book decompositions with pages not diffeomorphic to a disk.

2.2. Existence. We need to prove that open book decompositions adapted to Legendrian links always exist. In order to do this, we recall the definition of contact cell

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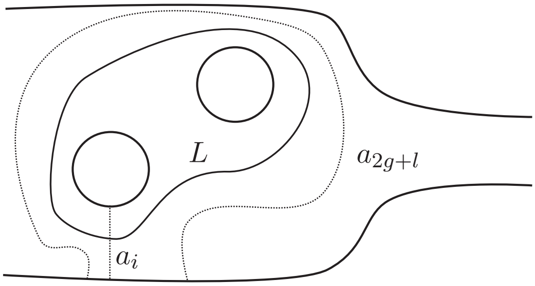

Figure 1. We add a new living arc which is parallel to L_i except near the distinguished arc.

decomposition (of a contact 3-manifold) and ribbon of a Legendrian graph. A *contact cell decomposition* of (M, ξ) is a finite CW-decomposition of M such that

1. the 1-skeleton is a connected Legendrian graph;
2. each 2-cell E satisfies $\text{tb}(\partial E) = -1$;
3. the contact structure ξ is tight when restricted to each 3-cell.

Moreover, if we have a Legendrian link $L \hookrightarrow (M, \xi)$, then we also suppose that

4. the 1-skeleton contains L.

Denote the 1-skeleton of a contact cell decomposition of (M, ξ) with G. Then G is a Legendrian graph and its *ribbon* is a compact surface S_G satisfying:

- S_G retracts onto G;
- $T_pS_G = \xi_p$ for every $p \in G$;
- $T_pS_G \neq \xi_p$ for every $p \in S \setminus G$.

We say that an adapted open book decomposition (B, π, A), compatible with the triple (L, M, ξ), comes from a contact cell decomposition if $S = \overline{\pi^{-1}(1)}$ is a ribbon of the 1-skeleton of (M, ξ).

Proof of Theorem 1.1. Corollary 4.23 in [7] assures us that we can always find an open book decomposition (B, π) which comes from a contact cell decomposition of (M, ξ) and the link L is contained in S_1, where the page S_1 is precisely a ribbon of the 1-skeleton of (M, ξ). The proof of this corollary also gives that the two framings of L agree.

The components of L are independent because it is easy to see from the construction that there is a collection of disjoint, properly embedded arcs $\{a_1, \ldots, a_n\}$ in S_1 such that

$$L_i \pitchfork a_i = \{1 \text{ pt}\} \quad \text{and} \quad L_i \cap \left(\bigcup_{j \neq i} a_j\right) = \emptyset$$

for every i. To conclude, we only need to show that there exists a system of generators $A = \{a_1, \ldots, a_{2g+l+n-2}\}$ for S_1 which is adapted to L.

The arcs $a_1, \ldots, a_n$ are taken as before. If we complete L to a basis of $H_1(S; \mathbb{F})$, then Alexander duality gives a basis $\{a_1, \ldots, a_{2g+l-1}\}$ with the same property. We define $a_{2g+l}, \ldots, a_{2g+l+n-2}$ in the following way: each new living arc is parallel to L_i and extended by following the distinguished arc until the boundary of S_1 as in Figure 1. Clearly, it disconnects the surface, because the first $2g + l - 1$ arcs are already a basis of $H_1(S, \partial S; \mathbb{F})$. If one of the components contains no distinguished arcs, like in Figure 2, then we choose the other endpoint of a_i to extend the arc. □

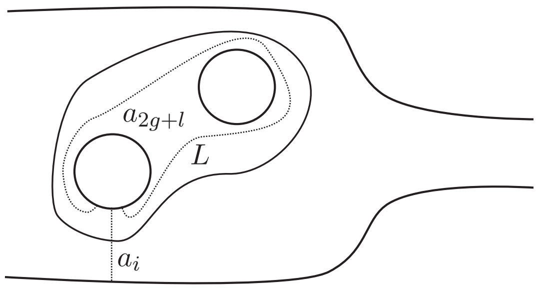

Figure 2. The picture appears similar to Figure 1, but this time the new arc follows the distinguished arc in the opposite direction.

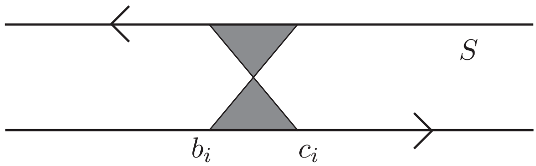

Figure 3. Two arcs in strip position.

In the following paper, we use adapted open book decompositions to present Legendrian links in contact 3-manifolds. Moreover, we study how to relate two different open book decompositions representing isotopic Legendrian links.

2.3. Abstract open books. An *abstract open book* is a quintuple $(S, \Phi, \mathcal{A}, z, w)$ defined as follows. We start with the pair (S, Φ). We have that $S = S_{g,l}$ is an oriented, connected, compact surface of genus g and with l boundary components, not diffeomorphic to a disk. While Φ is the isotopy class of a diffeomorphism of S into itself which is the identity on ∂S. The class Φ is called *monodromy*.

The pair (S, Φ) determines a contact 3-manifold up to contactomorphism. The construction is described in [7] Definition 2.3, Lemma 2.4, and Theorem 3.13.

The set $\mathcal{A}$ consists of two collections of properly embedded arcs, $B = \{b_1, \ldots, b_{2g+l+n-2}\}$ and $C = \{c_1, \ldots, c_{2g+l+n-2}\}$ in S with $n \geqslant 1$, such that all the arcs in B are disjoint, all the arcs in C are disjoint and each pair b_i, c_i appears as in Figure 3. We suppose that each strip, the gray area between b_i and c_i, is disjoint from the others. We also want B and C to represent two system of generators for the relative homology group $H_1(S, \partial S; \mathbb{F})$. In this way, if we name the strips $\mathcal{A}_i$, we have that $S \setminus \bigcup b_i$, $S \setminus \bigcup c_i$ and $S \setminus \bigcup \mathcal{A}_i$ have exactly n connected components.

Finally, z and w are two sets of basepoints: $w = \{w_1, \ldots, w_n\}$ and $z = \{z_1, \ldots, z_n\}$. We require these sets to have the following properties:

- there is a z_i in each component of $S \setminus \bigcup \mathcal{A}_i$, with the condition that every component contains exactly one element of z;
- each w_i is inside one of the strips $\mathcal{A}_i$, between the arcs b_i and c_i, with the property that every strip contains at most one element of w. See Figure 4;
- we choose z and w in a way that each component of $S \setminus B$ and $S \setminus C$ contains exactly one element of z and one element of w.

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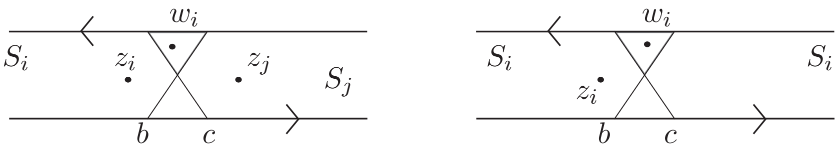

Figure 4. On the left S_i and S_j are different components of $S \setminus \bigcup \mathcal{A}_i$.

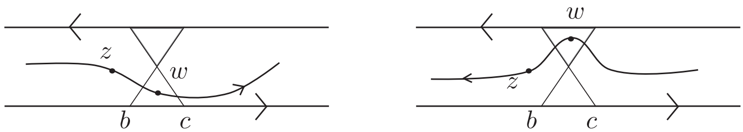

Figure 5. The link is oriented accordingly to the basepoints.

We can draw an n-component link inside S using the following procedure: we go from the z's to the w's by crossing B and from w to z by crossing C, as shown in Figure 5. Moreover, we observe that the components of the link are independent in S.

Using the Legendrian realization theorem (Theorem 2.7) in [**14**] and the procedure we described, we can prove that every abstract open book determines a Legendrian link in a contact 3-manifold up to contactomorphism. We are now interested in proving that an adapted open book decomposition (B, π, A) always determines an abstract open book.

PROPOSITION 2.1. *We can associate to an adapted open book decomposition* (B, π, A)*, compatible with the triple* (L, M, ξ)*, an abstract open book* $(S, \Phi, \mathcal{A}, z, w)$ *up to isotopy.*

Proof. The surface S is obviously the page $\overline{\pi^{-1}(1)}$. Now consider the subsets of unit complex numbers $I_\pm \subset S^1 \subset \mathbb{C}$ with non-negative and non-positive imaginary part. Since they are contractible, we have that $\pi|_{\pi^{-1}(I_\pm)}$ is a trivial bundle. This gives two diffeomorphisms between the pages S_1 and S_{-1}. The monodromy Φ is precisely the isotopy class of the composition of these diffeomorphisms.

At this point, we want to define the strips $\mathcal{A}$. Hence, we need the collections of arcs B and C: starting from the system of generators A, which is adapted to $L \subset S$, we take them to be both isotopic to A, in "strip position" like in Figure 3 and such that L does not cross the intersections of the arcs in B with the ones in C. We only have an ambiguity on, following the orientation of L, which is the first arc intersected by L. To solve this problem, we have to follow the rule that we fixed in Figure 5.

Now we need to fix the basepoints. We put the z's on L; exactly one on each component of $L \setminus (L \cap \mathcal{A})$. The points in z on different components of L stay in different domains because of Condition 3 in the definition of adapted system of generators. Then $S \setminus \mathcal{A}$ has n connected components, since the components of L are independent, and each of these contains exactly one element of z. Since the z's are outside of the strips, then we have that each component of $S \setminus B$ and $S \setminus C$ also contains only one z_i.

The w's are still put on L, but inside the strips containing the n distinguished arcs. The points $w_1, \ldots, w_n$ correspond to $\mathcal{A}_1, \ldots, \mathcal{A}_n$. □

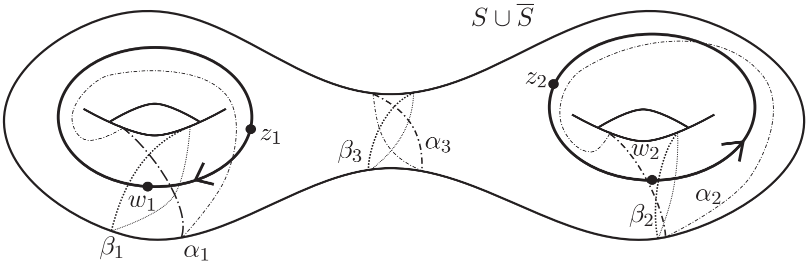

Figure 6. A diagram for the standard Legendrian 2-unlink in (S^3, ξ_{st}).

3. Heegaard Floer homology.

3.1. Legendrian Heegaard diagrams. Heegaard Floer homology has been introduced by Ozsváth and Szabó in [**18**]; later it was generalized to knots and links in [**19**] and independently by Rasmussen, in his PhD thesis [**23**]. In its original formulation, links and 3-manifolds were presented with Heegaard diagrams, but in this work we only use a specific type of these diagrams, obtained from adapted open book decompositions, that we call Legendrian Heegaard diagrams.

From now on, a 3-manifold M is always supposed to be a rational homology sphere. Given an adapted open book decomposition (B, π, A), compatible with the triple (L, M, ξ), a *Legendrian Heegaard diagram* consists of a quintuple $(\Sigma, \alpha, \beta, w, z)$ where Σ is a closed, oriented surface, α and β are two collections of curves in Σ, and w and z are two sets of n basepoints in Σ.

The surface Σ is $S_1 \cup \overline{S_{-1}}$, where $S_{\pm 1} = \overline{\pi^{-1}(\pm 1)}$; since π is a locally trivial fibration the pages S_1 and S_{-1} are diffeomorphic, but we reverse the orientation of the second one when we glue them together.

We have that $A = \{a_1, \ldots, a_{2g+l+n-2}\}$ and we choose $B = \{b_1, \ldots, b_{2g+l+n-2}\}$ in a way that A and B are like in Figure 3. We recall that g is the genus of S_1, l is the number of boundary components of S_1, and n is the number of components of L. Then, we define $\alpha_i = b_i \cup (h \circ \Phi)(b_i)$ and $\beta_i = a_i \cup h(a_i)$ for every $i = 1, \ldots, 2g + l + n - 2$, where $h : S_1 \to \overline{S_{-1}}$ is the identity and Φ is the monodromy, which is fixed by the open book as seen in Proposition 2.1. Finally, the z's and the w's are the set of basepoints that we introduced in Subsection 2.3.

In the settings of [**18**] and [**19**], the Legendrian Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ is a diagram for the (smooth) link L in the 3-manifold $\overline{M}$ ([**21**] Section 3), that is, M considered with the opposite orientation. We remark that here α and β are swapped respect to the papers of Ozsváth and Szabó.

We observe that, given (B, π, A), the only freedom in the choice of the Legendrian Heegaard diagram is in the arcs $(h \circ \Phi)(b_1), \ldots, (h \circ \Phi)(b_{2g+l+n-2})$ inside S_{-1}, which depend on the isotopy class of $\Phi(b_1), \ldots \Phi(b_{2g+l+n-2})$.

3.2. Basics of Heegaard Floer theory. Let us consider a Legendrian Heegaard diagram $D = (\Sigma, \alpha, \beta, w, z)$, coming from an adapted open book decomposition compatible with (L, M, ξ). Consider the $(2g + l + n - 2)$-dimensional tori

$$T_\alpha = \alpha_1 \times \ldots \times \alpha_{2g+l+n-2} \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times \ldots \times \beta_{2g+l+n-2}$$

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in the symmetric power $\mathrm{Sym}^{2g+l+n-2}(\Sigma)$ of Σ and define $\widehat{CF}(D)$ and $cCFL^-(D)$, respectively, as the $\mathbb{F}$-vector space and the free $\mathbb{F}[U]$-module generated by the elements of the transverse intersection $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Fix an appropriate symplectic and compatible almost-complex structure (ω, J) on $\mathrm{Sym}^{2g+l+n-2}(\Sigma)$. For every relative homology class $\phi \in \pi_2(x, y)$, we define $\mathfrak{M}(\phi)$ as the moduli space of J-holomorphic maps from the unit disk $D \subset \mathbb{C}$ to $\left(\mathrm{Sym}^{2g+l+n-2}(\Sigma), J\right)$ with the appropriate boundary conditions, see [18]. The formal dimension of $\mathfrak{M}(\phi)$, denoted by $\mu(\phi)$, is the Maslov index; moreover, we call $\widehat{\mathfrak{M}}(\phi)$ the quotient $\mathfrak{M}(\phi)/\mathbb{R}$ given by translation.

Since we are working on 3-manifolds, we use the definition of Spin^c structure given by Turaev in [24]: an isotopy class, away from a point, of nowhere vanishing vector fields on the manifold. As described in [21] Section 3.3, we have two well-defined maps

$$\mathfrak{s}_w, s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c\left(\overline{M}\right) \cong H^2(M; \mathbb{Z})$$

which send an intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ into the Spin^c structures $\mathfrak{s}_w(x)$ and $\mathfrak{s}_z(x)$ on $\overline{M}$. These maps are obtained by associating, to every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, two global 2-plane fields $\pi_w(x)$ and $\pi_z(x)$ on $\overline{M}$, whose restrictions are the corresponding Spin^c structures.

Since our manifold M already comes with a contact structure ξ, we have that both M and $\overline{M}$ are equipped with a specific Spin^c structure, induced by ξ, that we denote with $\mathfrak{t}_\xi$. The elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ can be partitioned according to the Spin^c structures on $\overline{M}$, resulting in decompositions

$$\widehat{CF}(D) = \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(\overline{M})} \widehat{CF}(D, \mathfrak{t}) \qquad \text{and} \qquad cCFL^-(D) = \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(\overline{M})} cCFL^-(D, \mathfrak{t}),$$

where $\widehat{CF}(D, \mathfrak{t})$ is generated by the intersection points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $s_z(x) = \mathfrak{t}$, while $cCFL^-(D, \mathfrak{t})$ by the ones such that $s_w(x) = \mathfrak{t}$. Note that the Spin^c structures in the two splittings may be different.

For every $\phi \in \pi_2(x, y)$, we call

$$n_{z_i}(\phi) = \#\left|\phi \cap \{z_i\} \times \mathrm{Sym}^{2g+l+n-3}(\Sigma)\right| \qquad \text{and} \qquad n_{w_i}(\phi) = \#\left|\phi \cap \{w_i\} \times \mathrm{Sym}^{2g+l+n-3}(\Sigma)\right|$$

where here we mean algebraic intersection. Moreover, we have

$$n_z(\phi) = \sum_{i=1}^{n} n_{z_i}(\phi) \qquad \text{and} \qquad n_w(\phi) = \sum_{i=1}^{n} n_{w_i}(\phi).$$

We define the differential $\widehat{\partial}_z : \widehat{CF}(D, \mathfrak{t}) \to \widehat{CF}(D, \mathfrak{t})$ as follows:

$$\widehat{\partial}_z x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}}} \sum_{\substack{\phi \in \pi_2(x,y), \\ \mu(\phi)=1, n_z(\phi)=0}} \#\left|\widehat{\mathfrak{M}}(\phi)\right| \cdot y$$

for every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}}$. We note that, since we are interested in ϕ's that are image of some J-holomorphic disks, we have that $n_{z_i}(\phi), n_{w_i}(\phi) \geqslant 0$ (Lemma 3.2 in [18]). This means that $n_z(\phi) = n_w(\phi) = 0$ if and only if $n_{z_i}(\phi) = n_{w_i}(\phi) = 0$ for every $i = 1, \ldots, n$.

The map $\widehat{\partial}_z$ is well defined if the diagram D is admissible, which means that every $\phi \in \pi_2(x, x)$ with $n_w(\phi) = 0$, representing a non-trivial homology class, has both positive and negative local multiplicities. Usually, we have a distinction between weak and strong admissibility, but if M is a rational homology 3-sphere then the weakly and strongly admissible conditions coincide; see Remark 4.11 in [18] and Definition 3.5 and Subsection 4.1 in [21]. Given a diagram, we can always achieve admissibility with isotopies.

PROPOSITION 3.1. *Suppose $D=(\Sigma,\alpha,\beta,w,z)$ is a Legendrian Heegaard diagram given by an adapted open book decomposition compatible with (L,M,ξ), where M is a rational homology 3-sphere. Then D is always admissible up to isotope the arcs in $\overline{B}$.*

Proof. It follows from Proposition 3.6 in [21] and Theorem 2.1 in [22]. □

The fact that $\widehat{\partial}_z\circ\widehat{\partial}_z=0$ is proved in [18]. This gives that the pair $\left(\widehat{CF}(D,\mathfrak{t}),\widehat{\partial}_z\right)$ is a chain complex. In the definition of $\left(\widehat{CF}(D,\mathfrak{t}),\widehat{\partial}_z\right)$, we never use the basepoints in w, in fact the complex does not depend on the link L, but only on the number of its components. Moreover, in [18], it is proved that the homology $\widehat{HF}(D,\mathfrak{t})$ of $\left(\widehat{CF}(D,\mathfrak{t}),\widehat{\partial}_z\right)$ is an invariant of the Spin^c 3-manifold $\left(\overline{M},\mathfrak{t}\right)$ if the number of basepoints in D is fixed. When $n=1$, the homology group is usually denoted with $\widehat{HF}\left(\overline{M},\mathfrak{t}\right)$.

In addition, since M is a rational homology sphere, $\widehat{CF}(D,\mathfrak{t})$ comes with an additional $\mathbb{Q}$-grading, called *Maslov grading* [17], given by $M_z(x)=d_3(\pi_z(x))$, where d_3 is the Hopf invariant of a nowhere zero vector field. The differential $\widehat{\partial}_z$ drops the Maslov grading by one and then we have that

$$\widehat{HF}(D,\mathfrak{t})=\bigoplus_{d\in\mathbb{Q}}\widehat{HF}_d(D,\mathfrak{t}),$$

where each $\widehat{HF}_d(D,\mathfrak{t})$ is a finite dimensional $\mathbb{F}$-vector space.

3.3. Link Floer homology. We consider the free $\mathbb{F}[U]$-module $cCFL^-(D,\mathfrak{t})$ and we define a new differential

$$\partial^-:cCFL^-(D,\mathfrak{t})\to cCFL^-(D,\mathfrak{t})$$

in the following way:

$$\partial^-x=\sum_{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta|_\mathfrak{t}}\sum_{\substack{\phi\in\pi_2(x,y),\\ \mu(\phi)=1,n_z(\phi)=0}}\#\left|\widehat{\mathfrak{M}}(\phi)\right|\cdot U^{n_w(\phi)}y$$

and

$$\partial^-(Ux)=U\cdot\partial^-x$$

for every $x\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta|_\mathfrak{t}$. If D is admissible then ∂^- is also well defined. Moreover, the fact that $\partial^-\circ\partial^-=0$ is proved in Lemma 4.3 in [21]. Hence, the pair $\left(cCFL^-(D,\mathfrak{t}),\partial^-\right)$ is a chain complex. From [21], we know that the homology of $\left(cCFL^-(D,\mathfrak{t}),\partial^-\right)$, that is denoted with $cHFL^-\left(\overline{M},L,\mathfrak{t}\right)$, is a smooth isotopy invariant of L in $\overline{M}$.

As before, we can define the Maslov grading as $M(x)=M_w(x)=d_3(\pi_w(x))$ for every intersection point and we extend it by taking $M(Ux)=M(x)-2$. Note that this definition of Maslov grading is different from the one used in the previous subsection; in fact, now the set w appears in place of z. In order to avoid confusion, we denote the Maslov grading of x with $M(x)$ only in the case of links; otherwise, we specify which set of basepoints is used in the definition. Again the Maslov grading gives an $\mathbb{F}$-splitting

$$cHFL^-\left(\overline{M},L,\mathfrak{t}\right)=\bigoplus_{d\in\mathbb{Q}}cHFL^-_d\left(\overline{M},L,\mathfrak{t}\right),$$

where $cHFL^-\left(\overline{M}, L, \mathfrak{t}\right)$ is a finitely generated $\mathbb{F}[U]$-module and each $cHFL^-_d\left(\overline{M}, L, \mathfrak{t}\right)$ is a finite dimensional $\mathbb{F}$-vector space.

In the case of null-homologous links, we can also define a $\frac{\mathbb{Z}}{2}$-grading on $cCFL^-(D, \mathfrak{t})$ called *Alexander grading* and denoted with A. Let us call $\overline{M}_L$ the 3-manifold with boundary $\overline{M} \setminus \nu(\mathring{L})$. Since L has n components, we have that $\partial\overline{M}_L$ consists of n disjoint tori. On this kind of 3-manifold, we define a relative Spin^c structure as in [21]: the isotopy class, away from a point, of a nowhere vanishing vector field such that the restriction on each boundary torus coincides with the canonical one (see [24]). We denote the set of the relative Spin^c structures on $\overline{M}_L$ by Spin$^c\left(\overline{M}, L\right)$; then we have an identification of Spin$^c\left(\overline{M}, L\right)$ with the relative cohomology group $H^2\left(M_L, \partial\nu(L); \mathbb{Z}\right)$. Moreover, from [21], we have the following map

$$\mathfrak{s}_{w,z} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \text{Spin}^c\left(\overline{M}, L\right).$$

Clearly, the relative Spin^c structure $\mathfrak{s}_{w,z}(x)$ extends to the actual Spin^c structure $\mathfrak{s}_w(x)$. Poincaré duality gives that

$$H^2\left(M_L, \partial\nu(L); \mathbb{Z}\right) \cong H_1\left(M_L; \mathbb{Z}\right) \cong H_1\left(L; \mathbb{Z}\right) \oplus H_1\left(M; \mathbb{Z}\right) \cong \mathbb{Z}^n \oplus H^2\left(M; \mathbb{Z}\right),$$

where we recall that $H^2(M; \mathbb{Z})$ is a finite group. A basis of the $\mathbb{Z}^n$ summand is given by the cohomology classes $\{\text{PD}[\mu_i]\}_{i=1,\ldots,n}$, where μ_i is a meridian of the i-th component of L, oriented coherently. Then we have that

$$\mathfrak{s}_{w,z}(x) = \sum_{i=1}^{n} 2s_i \cdot \text{PD}[\mu_i] + \mathfrak{s}_w(x),$$

where each s_i is an integer. Since L admits a Seifert surface F, we define the Alexander absolute grading as follows:

$$A(x) = \sum_{i=1}^{n} s_i = \frac{\mathfrak{s}_{w,z}(x)[F]}{2},$$

extended on the whole $cCFL^-(D, \mathfrak{t})$ by saying that $A(Ux) = A(x) - 1$.

We have that ∂^- preserves the Alexander grading and then there is another $\mathbb{F}$-splitting

$$cHFL^-\left(\overline{M}, L, \mathfrak{t}\right) = \bigoplus_{d,s\in\mathbb{Q}} cHFL^-_{d,s}\left(\overline{M}, L, \mathfrak{t}\right),$$

where each $cHFL^-_{d,s}\left(\overline{M}, L, \mathfrak{t}\right)$ is a finite dimensional $\mathbb{F}$-vector space.

4. The invariant.

4.1. Special intersection points in Legendrian Heegaard diagrams. In this subsection, we define a cycle in the link Floer chain complex that we previously introduced. The corresponding homology class will be our Legendrian invariant. Let us consider the only intersection point of $D = (\Sigma, \alpha, \beta, w, z)$ which lies in the page S_1. We recall that, in general, the intersection points live in the space $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, but they can be represented inside Σ. We denote this element with $\mathfrak{L}(D)$.

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PROPOSITION 4.1. *The intersection point* $\mathfrak{L}(D)$ *is such that* $\partial^-\mathfrak{L}(D)=0$ *and then* $\mathfrak{L}(D)$ *is a cycle in* $cCFL^-\left(D,\mathfrak{t}_{\mathfrak{L}(D)}\right)$, *where* $\mathfrak{t}_{\mathfrak{L}(D)}$ *is the* Spin^c *structure that it induces on* $\overline{M}$.

Proof. Every $\phi\in\pi_2(\mathfrak{L}(D),y)$ such that $\mu(\phi)=1$, where $y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, has the property that $n_z(\phi)>0$. The claim follows easily from the definition of the differential. □

In Subsection 4.4, we show that more can be said on the Spin^c structure $\mathfrak{t}_{\mathfrak{L}(D)}$. Now we spend a few words about the Ozsváth–Szabó contact invariant $c(\xi)$, introduced in [20]. Given a Legendrian Heegaard diagram D, let us call $c(D)$ the only intersection point which lies on the page S_1 as before, but now considered as an element in $\left(\widehat{CF}(D,\mathfrak{t}_{c(D)}),\widehat{\partial}_z\right)$. The proof of Proposition 4.1 says that $c(D)$ is also a cycle. Moreover, we have the following theorem.

THEOREM 4.2 (Ozsváth and Szabó). *Let us consider a Legendrian Heegaard diagram* D *with a single basepoint* z, *given by an open book compatible with a pair* (M,ξ), *where* M *is a rational homology 3-sphere. Let us take the cycle* $c(D)\in\widehat{CF}(D,\mathfrak{t}_{c(D)})$.

Then the equivalence class of $(\widehat{HF}\left(\overline{M},\mathfrak{t}_\xi\right),[c(D)])$ *is a contact invariant of* (M,ξ) *and we denote it with* $\widehat{c}(M,\xi)$. *Furthermore, we have the following properties:*

- *the* Spin^c *structure* $\mathfrak{t}_{c(D)}$ *coincides with* $\mathfrak{t}_\xi$;
- *the Maslov grading of* $c(D)$ *is given by* $M_z(c(D))=-d_3(M,\xi)$.

The proof of this theorem comes from [20], where Ozsváth and Szabó first define the invariant $\widehat{c}(M,\xi)$, and [11], where Honda, Kazez, and Matić give the reformulation using open book decompositions that we use in this paper.

We want to prove a result similar to Theorem 4.2, but for Legendrian links. In other words, we show that the isomorphism class of the element $[\mathfrak{L}(D)]$ inside the homology group $cHFL^-\left(\overline{M},L,\mathfrak{t}_{\mathfrak{L}(D)}\right)$ can be considered, as we are going to explain, a Legendrian invariant of the triple (L,M,ξ), where here it is helpful to remember the observations in Remark 1.4. Let us be more specific: we consider a Legendrian link $L\hookrightarrow(M,\xi)$ in a rational homology contact 3-sphere; we associate two open book decompositions compatible with (L,M,ξ), say (B_1,π_1,A_1) and (B_2,π_2,A_2), and these determine (up to isotopy) two Legendrian Heegaard diagrams, that we call $D_1=(\Sigma_1,\alpha_1,\beta_1,w_1,z_1)$ and $D_2=(\Sigma_2,\alpha_2,\beta_2,w_2,z_2)$, respectively. Then we want to find a chain map

$$\Psi_{(D_1,D_2)}:cCFL^-(D_1,\mathfrak{t})\longrightarrow cCFL^-(D_2,\mathfrak{t}),\tag{1}$$

that induces an isomorphism in homology and preserves the bigrading, and it is such that $\Psi_{(D_1,D_2)}(\mathfrak{L}(D_1))=\mathfrak{L}(D_2)$, where $\mathfrak{t}=\mathfrak{t}_{\mathfrak{L}(D_1)}=\mathfrak{t}_{\mathfrak{L}(D_2)}\in\text{Spin}^c\left(\overline{M}\right)$.

Our strategy is to study how Legendrian isotopic triples are related to one another, which means to define a finite set of moves between two adapted open book decompositions. Then find the maps $\Psi_{(D_1,D_2)}$ in these particular cases.

4.2. Open books adapted to Legendrian isotopic links. We want to show that, given two Legendrian isotopic links $L_1,L_2\hookrightarrow(M,\xi)$, two open book decompositions (B_i,π_i,A_i), compatible with the triples (L_i,M,ξ), are related by a finite sequence of moves.

4.2.1. Global isotopies

The first lemma follows easily.

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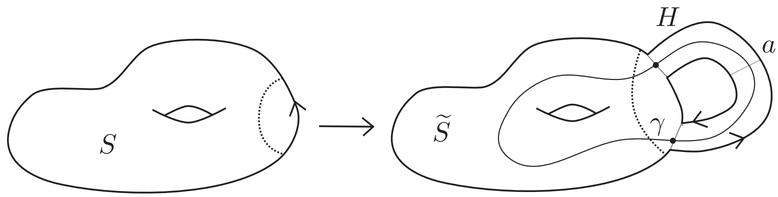

Figure 7. The arc $\gamma \cap \mathring{S}$ is a generic arc in the interior of S.

LEMMA 4.3. *Let us consider an adapted open book decomposition* (B_1, π_1, A_1)*, compatible with the triple* (L_1, M, ξ)*, and suppose that there is a contact isotopy of* (M, ξ)*, sending* L_1 *into* L_2.

*Then the time-*1 *map of the isotopy is a diffeomorphism* $F : M \to M$ *such that* $\big(F(B_1), \pi_1 \circ F^{-1}, F(A_1)\big)$ *is an adapted open book decomposition, compatible with* (L_2, M, ξ).

This lemma says that, up to global contact isotopies, we can consider (B_i, π_i, A_i) to be both compatible with a triple (L, M, ξ), where the link L is Legendrian isotopic to L_i for $i = 1, 2$. In other words, we can just study the relation between two open book decompositions compatible with a single triple (L, M, ξ).

4.2.2. *Positive stabilizations*

Let us start with a pair (S, Φ). A positive stabilization of (S, Φ) is the pair $\big(\widetilde{S}, \widetilde{\Phi}\big)$ obtained in the following way:

- the surface $\widetilde{S}$ is given by adding a 1-handle H to S;
- the monodromy $\widetilde{\Phi}$ is isotopic to $\Phi' \circ D_\gamma$. The map Φ' concides with Φ on S and it is the identity on H. While D_γ is the right-handed Dehn twist along a curve γ which intersects $S \cap H$ transversely precisely in the attaching sphere of H. See Figure 7.

We say that (B', π', A') is a positive stabilization of (B, π, A) if

- the pair (S', Φ'), obtained from (B', π'), is a positive stabilization of (S, Φ), the one coming from (B, π);
- the system of generators A' is isotopic to $A \cup \{a\}$, where a is the cocore of H as in Figure 7.

We recall the following theorem, proved by Giroux [**10**]. More details can be found in [7] Sections 3 and 4.

THEOREM 4.4. *Two open book decompositions* (B_i, π_i, A_i) *are compatible with contact isotopic triples* (L_i, M, ξ_i) *if and only if they admit isotopic positive stabilizations.*

In other words, we may need to stabilize both open books many times and eventually we obtain other two open books $\big(B, \pi, \widetilde{A}_i\big)$, both compatible with $L \hookrightarrow (M, \xi)$, which is contact isotopic to (L_i, M, ξ_i) for $i = 1, 2$.

4.2.3. *Admissible arc slides*

Take an adapted system of generators A for an n-component link L, lying inside a surface S. We define admissible arc slide, a move that change $A = \{a_1, ..., a_i, ..., a_j, a_{2g+l+k-2}\}$

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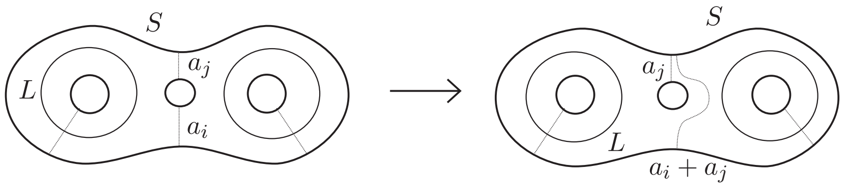

Figure 8. The arc $a_i + a_j$ is obtained by sliding a_j over a_i.

into $A' = \{\dots, a_i + a_j, \dots, a_j, \dots\}$, where a_j is not a distinguished arc and one of the endpoints of a_i and a_j are adjacent, like in Figure 8. We can prove the following proposition.

PROPOSITION 4.5. *Let us consider two open book decompositions (B, π, A_i), compatible with the Legendrian link $L \hookrightarrow (M, \xi)$. Then, after a finite number of admissible arc slides and isotopies on A_i, the open books coincide.*

We need three preliminary lemmas. We recall that, according to the definition in Subsection 2.1, we have $A = A^1 \sqcup A^2 \sqcup A^3$, where A^1 is the set of distinguished arcs, the set A^3 contains only living arcs and the arcs in A^2 can be either dead or alive.

LEMMA 4.6. *An admissible arc slide, from A to A', can be inverted. In the sense that we can perform another admissible arc slide, now from A' to A'', such that A'' is isotopic to A.*

Proof. If the arc slide changes the arc a_i into $a_i + a_j$ then it is easy to see that we can just slide $a_i + a_j$ over an arc a'_j, isotopic to a_j; in a way that $a_i + a_j + a'_j$ is isotopic to a_i. □

LEMMA 4.7. *Suppose that $a_i \in A^2 \subset A$ is a living arc in an adapted system of generators. Then we can permute the arcs in A in a way that $a_i \in A^3$ and A is still an adapted system of generators.*

Proof. From Condition 4 in the definition of adapted system of generators, we have that $S \setminus A$ is the disjoint union of n disks $D_1, \dots, D_n$ and each arc in A^3 connects exactly one pair of them. This means that if we build a graph $\mathcal{G}$ by taking the D_i's as vertices and the arcs in A^3, which are all alive by definition, as edges then $\mathcal{G}$ is a tree.

Since a_i is a living arc, we obtain that a_i appears on the boundary of two distinct disks D_1 and D_2. Consider the minimal path in $\mathcal{G}$ from D_1 to D_2 and denote with D_3 the first disk that we meet after D_1. We call a_j the arc in A^3 which connects D_1 to D_3; we claim that swapping a_i and a_j in A yields again an adapted system of generators B.

This follows because $S \setminus (B^1 \sqcup B^2)$ is a disk, which implies $B^1 \sqcup B^2$ is a basis of $H_1(S, \partial S; \mathbb{F})$, and B disconnects S into the same collection of disks as A does. □

LEMMA 4.8. *Suppose that we perform an admissible arc slide that changes a_i into $a_i + a_j$. Then the set A', obtained from A by sliding a_i over a_j, is still an adapted system of generators, possibly after rearranging the arcs. Furthermore, we have the following facts:*

(a) the arc a_i is distinguished if and only if $a_i + a_j$ is a distinguished arc;
(b) the arc a_i is alive if and only if $a_i + a_j$ is a living arc;
(c) the arc a_i is dead if and only if $a_i + a_j$ is a dead arc.

Proof. (a) The arc $a_i + a_j$ represents the sum of the relative homology classes of a_i and a_j. At this point, since a_j cannot be distinguished from the definition of admissible arc slide, we just need basic linear algebra. The converse follows in the same way, applying Lemma 4.6.

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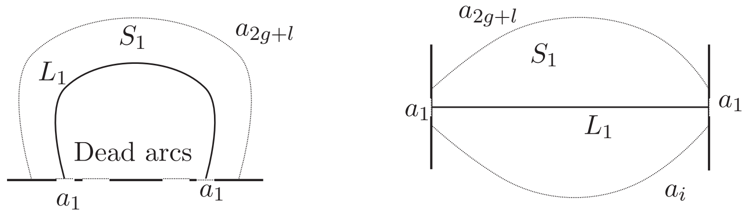

Figure 9. Case 1 is on the left. The figure actually shows a portion of $S \setminus A_1$.

(b) We use Lemma 4.7 to assume that $a_i \in A^3$. Now, we observe that $a_i + a_j$ disconnects $S \setminus (A^1 \sqcup A^2)$ because $A^1 \sqcup A^2$ is a basis of $H_1(S, \partial S; \mathbb{F})$; moreover, when considering $S \setminus A'$, the arc $a_i + a_j$ also appears on the boundary of two distinct disks. This argument proves both that A' is still an adapted system of generators and $a_i + a_j$ is alive. The other implication follows easily from Lemma 4.6.

(c) It follows from (b) and (c), but after an additional observation, it is not obvious that if a_i is dead and $a_j \in A^3$ then $(A')^1 \sqcup (A')^2$ is still a basis of $H_1(S, \partial S; \mathbb{F})$. To see this, we assume that it is not the case; then using linear algebra, we obtain that swapping $a_i + a_j$ and a_j in A' yields an adapted system of generators. Hence, from Lemma 4.6, we can slide $a_i + a_j$ over a_j and get (a permutation of) A back, but now a_i is alive because of Point (b). This is a contradiction. □

As an example we explain in detail what happens in Figure 8. Denote by d_1 and d_2 the two distinguished arcs, then we have $A = \{d_1, d_2, a_i, a_j\}$ and $A' = \{d_1, d_2, a_j, a_i + a_j\}$. In A the arcs a_i and a_j are both alive, while a_j dies in A' (Lemma 4.8 does not say anything about that); accordingly to Lemma 4.8, we needed to rearrange the ordering of the arcs to guarantee that A' remains an adapted system of generators. Notice however that being dead or alive does not depend on the ordering of the arcs in A.

The strategy of the proof of Proposition 4.5 is, say $S_1, \ldots, S_n$ and $S'_1, \ldots, S'_n$ are the components of S minus the living arcs of A_1 and A_2, respectively, we modify all the living arcs, in both A_1 and A_2, with admissible arc slides; in order for S_i to concide with S'_i for every $i = 1, \ldots, n$. Moreover, we also want that each living arc in A_1 becomes isotopic to a living arc in A_2.

At the end, each of the components of S minus the living arcs will contain a unique component of L, a unique distinguished arc, and S_i will have the same number of dead arcs with respect to S'_i; in particular, this means that A_1 and A_2 will also have the same number of living arcs. We then conclude applying n times Proposition 3.2 in [**14**], which is the knot case of this proposition.

Proof of Proposition 4.5. We want to prove that there is an adapted system of generators A for L in $S = \overline{\pi^{-1}(1)}$ such that A is obtained, from A_1 and A_2, by a sequence of admissible arc slides (and isotopies).

We start from the component S_1. We can suppose that ∂S_1 contains the living arc $a_{2g+l} \subset A_1^3 \subset A_1$ and the distinguished arc $a_1 \subset A_1^1 \subset A_1$, with almost adjacent endpoints. We can have two cases: in the first one, there are no other living arcs in the portion of ∂S_1 on the opposite side of a_{2g+l}. In the second case, they appear; possibly more than one, but we can suppose that there is exactly one of them. See Figure 9.

Case 1. When we consider S'_1, which contains the same component of L that is in S_1, after some arc slides we have that it appears as in Figure 9 (left). This is because

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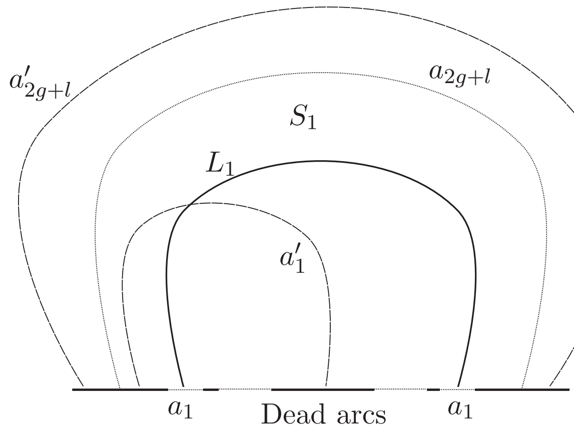

Figure 10. There are no other living arcs, except for a_{2g+l} in S_1.

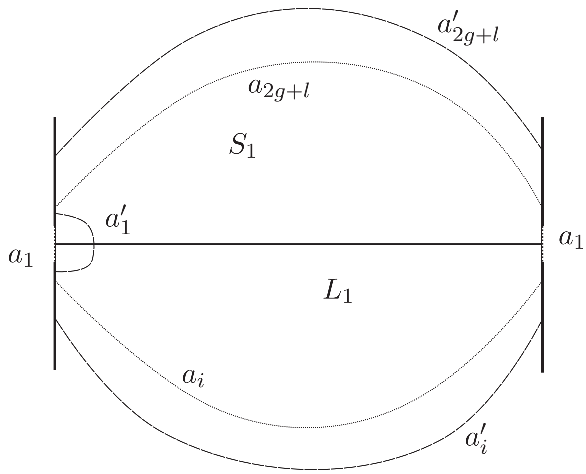

Figure 11. There are exactly two living arcs, namely a_{2g+l} and a_i in S_1.

in the same figure we see that S_1 is split into two pieces by L_1 and the innermost one is not connected in any way to other components of S; in fact, there are no living arcs on that side. This means that the same holds for S'_1 too. At this point, it is easy to see that S_1 can be modified to be like in Figure 10; more explicitly, the living arcs are parallel and the distinguished arcs lie in the region where L_1 is.

Case 2. As before, we have that also S'_1 appears like in the right part of Figure 9 (always after some arc slides). The reason is the same of previous case. Hence, now we can modify S_1 to be like in Figure 11; just in the same way as we did in Case 1.

We have obtained that the living arcs are fixed on S_1 and S'_1 and then the surfaces now have isotopic boundaries. Hence, we can move to another component S_2, whose boundary contains a living arc that has not yet been fixed, and we repeat the same process described before. We may need to slide some living arcs over the ones in A^3 (or $(A')^3$) that we have already fixed in the previous step, but this is not a problem. We just iterate this procedure until all the S_i coincides with S'_i and this completes the proof. □

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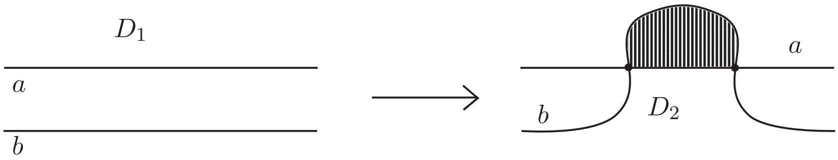

Figure 12. By moving the curve b through a Hamiltonian isotopy, we can introduce a pair of canceling intersection points.

The results of this subsection prove the following theorem.

THEOREM 4.9. *If $L_1, L_2 \hookrightarrow (M, \xi)$ are Legendrian isotopic links, then the open book decompositions (B_i, π_i, A_i), which are compatible with the corresponding triples, are related by a finite sequence of global contact isotopies, positive stabilizations, and admissible arc slides.*

Though they are easy to deal with, we do not have to forget that, when we define the corresponding Legendrian Heegaard diagrams, we need to consider the choices of the monodromy Φ and the families of arcs a, b and basepoints z, w inside their isotopy classes.

4.3. Invariance.

4.3.1. Definition of the diagrams and global isotopies

If two open book decompositions are related by a global isotopy, then it easy to see that the induced abstract open book coincide, up to conjugation of the monodromy and isotopy of the curves and the basepoints in the diagrams.

Hence, let us consider an abstract open book (S, Φ, A, z, w) and recall that in S we have another set of arcs B, as explained in Subsections 2.3 and 3.1. The first check is easy: in fact, if we perturb the basepoints inside the corresponding components of $S \setminus A \cup B$, then even the complex $(cCFL^-(D, \mathfrak{t}), \partial^-)$ does not change, where D is a Legendrian Heegaard diagram obtained from (S, Φ, A, z, w). The same is true for an isotopy of S.

Now for what it concerns S we are done, but when we define the chain complex we also consider the closed surface Σ, obtained by gluing together S and $\overline{S}$. We still have some choices on $\overline{S}$, in fact by Proposition 3.1 we may need to modify the arcs $(h \circ \Phi)(b_i)$ (see Subsection 3.1) in their isotopy classes to achieve admissibility for $D = (\Sigma, \alpha, \beta, w, z)$. Then the proof rests on the following proposition.

PROPOSITION 4.10. *Suppose that two curves in a Heegaard diagram are related by the move shown in Figure 12, then we can find a map $\Psi_{(D_1,D_2)}$ as in equation (1).*

Proof. The map $\Psi_{(D_1,D_2)}$ is constructed using a Hamiltonian diffeomorphism of the surface, as described in [**18**] Subsection 7.3. Since the new disks appear in $\overline{S}$, we have that $\mathfrak{L}(D_1)$ is sent to $\mathfrak{L}(D_2)$ which lie both in S. □

4.3.2. Admissible arc slides

An arc slide $\{..., a_i, ..., a_j, ...\} \to \{..., a_i + a_j, ..., a_j, ...\}$ in S corresponds to a handleslide $\{..., \alpha_i, ..., \alpha_j, ...\} \to \{..., \alpha_i', ..., \alpha_j, ...\}$ in Σ, where $\alpha_i' = a_i + a_j \cup h(a_i + a_j) \subset \Sigma$, again see Subsection 3.1. Thus, a chain map $\Psi_{(D_1,D_2)}$, which induces an isomorphism in homology, is obtained by counting holomorphic triangles, as explained in [**21**] Subsection 6.3 and Section 7. The admissibility of the arc slide is required only to avoid crossing

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a basepoint in w. Remember that for every arc slide, we actually have two handleslides; in fact, we need to slide both the α and the β curves. The fact that $\Psi_{(D_1,D_2)}(\mathfrak{L}(D_1)) = \mathfrak{L}(D_2)$ follows from Lemma 3.5 in [11], where the arc slides invariance is proved in the open books setting.

4.3.3. Positive stabilizations

At this point, in order to complete the proof of the invariance of $[\mathfrak{L}(D)]$, it would be enough to define $\Psi_{(D,D^+)}$, such that $\Psi_{(D,D^+)}(\mathfrak{L}(D)) = \mathfrak{L}(D^+)$, in the case where D and D^+ are obtained from an open book and one of its positive stabilizations. Nevertheless, this is not what we prove. In fact, we define *L-elementary positive stabilizations* as the ones such that the curve γ, which is the curve in the page S' that we used to perform the stabilization (see Figure 7), intersects the link L (that sits in S and then also in S') in at most one point transversely. Then what we actually prove is the existence of $\Psi_{(D,D^+)}$ for an L-elementary stabilization. To do this, we only need the following theorem, which is a modification of Giroux's Theorem 4.4 and whose proof is explained in Section 4 in [7].

THEOREM 4.11. *If (B_i, π_i) are two open book decompositions, compatible with the triple (L, M, ξ), then they admit isotopic L-elementary stabilizations. Namely, there is another compatible open book (B, π, A) which is isotopic to both (B_1, π_1) and (B_2, π_2), after an appropriate sequence of L-elementary stabilizations.*

Since we have already proved invariance under admissible arc slides, we can complete the open books (B, π) and (B^+, π^+) with every possible adapted system of generators and then eventually define the map $\Psi_{(D,D^+)}$.

PROPOSITION 4.12. *Let us consider the page $S = S_{g,l} = \overline{\pi^{-1}(1)} \supset L$ of (B, π). Then we can always find A, an adapted system of generators for L in S, with the property that A is disjoint from the arc $\gamma' = \gamma \cap S$, where γ is the curve that we used to perform the L-elementary stabilization.*

Proof. We have to study four cases:

(a) the arc γ' intersects L (transversely in one point);
(b) the intersection $\gamma' \cap L$ is empty and γ' does not disconnect S;
(c) the intersection $\gamma' \cap L$ is empty, the arc γ' disconnects S, and L is not contained in one of the two resulting connected components of S;
(d) the intersection $\gamma' \cap L$ is empty, the arc γ' disconnects S, and the link L lies in one of the resulting connected components of S.

Let us start with Case (a). We observe that γ' does not disconnect S; in fact if this was the case, then a component of L would be split inside the two resulting components of S, thus we would have at least another intersection point between L and γ', which is forbidden. We define $A = \{a_1, ..., a_{2g+l+n-2}\}$ as follows. Take a_1 as a push-off of γ'; clearly, a_1 is a distinguished arc, because it intersects L_1, a component of L, in one point. Now call a' the arc given by taking a push-off of L_1 and extend it through a_1, on one side of L_1; this is the same procedure described in the proof of Theorem 1.1. If a' disconnects S into S_1, S_2 and there are components of L that lie in both S_i, then we take a' as a living arc; thus, we extend $\{a_1, a'\}$ to an adapted system of generators A using Theorem 1.1. On the other hand, if L is contained in S_2 and S_1 is empty, then we consider $\{a_1, a''\}$, where a'' is another push-off of

L_1, this time extended through a_1 on the other side of L_1. We still have problems when a' does not disconnect S. We can fix this by taking $\{a_1, a', a''\}$, which together disconnect S into two connected components, one of them containing only L_1 and the other one $L \setminus L_1$. Again, we extend the set $\{a_1, a', a''\}$ to A applying Theorem 1.1.

The other three cases are easier. In Case (b), we just denote with a_{n+1} the push-off of γ' and we complete it to A.

In Case (c), we denote with a_{2g+l} our push-off of γ' and we take it as a living arc. Then, we can complete it to A.

Finally, Case (d) is as follows. Since in this case the push-off is trivial in homology and it bounds a surface disjoint from L, we can actually ignore it and easily find a set A which never intersects γ'. □

Now we have (S, Φ, A, z, w) the abstract open book obtained from (B, π, A). Denote with (S^+, Φ^+, A^+, z, w) the one coming from (B^+, π^+, A^+), where $S^+ = \overline{(\pi^+)^{-1}(1)}$ is S with a 1-handle attached; $\Phi^+ = \Phi' \circ D_\gamma$ where Φ' coincides with Φ on S, extended with the identity on the new 1-handle; and $A^+ = A \cup \{a\}$ with a being the cocore of the new 1-handle (see Figure 7). Then, we call D and D^+ the corresponding Legendrian Heegaard diagrams.

We define $\Psi_{(D,D^+)}$ in the following way. For every $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta|_{\mathfrak{t}_{\mathfrak{L}(D)}}$ one has $\Psi_{(D,D^+)}(x) = x'$, where $x' = x \cup \{a \cap b\}$ with b being the arc in strip position with a, as in Figure 3. It results that $\Psi_{(D,D^+)}$ is a chain map because the curve $\alpha = b \cup (h \circ \Phi)(b)$ only intersects the curve $\beta = a \cup h(a)$, and moreover one has $\alpha \cap \beta = \{1 \text{ pt}\}$, since we choose A^+ in a way that every arc in it is disjoint from γ'. We have that $\Psi_{(D,D^+)}$ induces an isomorphism in homology, because it is an isomorphism also on the level of chain complexes, and sends $\mathfrak{L}(D)$ into $\mathfrak{L}(D^+)$.

4.4. Proof of the main theorem.

We prove our main result which defines the Legendrian invariant $\mathfrak{L}$.

Proof of Theorem 1.2. We proved that $\mathfrak{L}(D)$ is a cycle in Proposition 4.1. The invariance follows from the results obtained in this section: in fact, we proved that if D_1 and D_2 are Legendrian Heegaard diagrams, representing Legendrian isotopic links, then we have a chain map $\Psi_{(D_1,D_2)}$ that preserves the bigrading and the Spin^c structure and sends $\mathfrak{L}(D_1)$ to $\mathfrak{L}(D_2)$.

Now to see which is the corresponding Spin^c structure, we recall that

$$\mathfrak{s}_w(\mathfrak{L}(D)) - \mathfrak{s}_z(\mathfrak{L}(D)) = \mathrm{PD}[L]$$

from Lemma 2.19 in [18]. Then, we have

$$\mathfrak{t}_{\mathfrak{L}(D)} = \mathfrak{s}_w(\mathfrak{L}(D)) = \mathfrak{s}_z(\mathfrak{L}(D)) + \mathrm{PD}[L] = \mathfrak{t}_{c(D)} = \mathfrak{t}_\xi .$$

□

We note that the invariant can be a U-torsion class in the group $cHFL^-\left(\overline{M}, L, \mathfrak{t}_\xi\right)$, which means that there is a $k \geqslant 0$ such that $U^k \cdot \mathfrak{L}(L, M, \xi) = [0]$. Moreover, the cycle $\mathfrak{L}(D)$ possesses a bigrading (d, s); such bigrading is induced on the invariant $\mathfrak{L}(L, M, \xi)$, because all the maps Ψ defined in this section preserve both the Maslov and the Alexander grading.

5. Properties of $\mathfrak{L}$ and connected sums.

5.1. Multiplication by U+1 in the link Floer homology group.

Take a Legendrian Heegaard diagram D, obtained from an adapted open book decomposition compatible with

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the triple (L, M, ξ), where M is a rational homology 3-sphere and L is a null-homologous Legendrian n-component link. We have the following surjective $\mathbb{F}$-linear map:

$$F : cCFL^{-}(D, \mathfrak{t}_\xi) \xrightarrow{U=1} \widehat{CF}(D, \mathfrak{t}_\xi),$$

which is given by setting U equals to 1.

The map F clearly commutes with the differentials and it is such that $F(\mathfrak{L}(D)) = c(D)$. It is well defined and surjective, because every intersection point x is such that $F(x) = x$ and if $s_w(x) = \mathfrak{t}_\xi$ then $s_z(x) = \mathfrak{t}_\xi$, since L is null-homologous. Moreover, it respects the gradings in the following sense.

LEMMA 5.1. *The map F sends an element with bigrading (d, s) into an element with Maslov grading $d - 2s$.*

Proof. We have that

$$M_z(F(x)) = d_3\left(\overline{M}, \pi_z(x)\right) = d_3\left(\overline{M}, \pi_w(x)\right) - \mathfrak{s}_{w,z}(x)[S] = M(x) - 2A(x),$$

where S is a Seifert surface for L. □

The map F induces F_* in homology:

$$F_* : cHFL^{-}\left(\overline{M}, L, \mathfrak{t}_\xi\right) \xrightarrow{U=1} \widehat{HF}\left(\overline{M}, \mathfrak{t}_\xi\right) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)}.$$

Then, we can prove the following two lemmas.

LEMMA 5.2. *The map F defined above is such that $F_*(\mathfrak{L}(L, M, \xi)) = \widehat{c}(M, \xi) \otimes (\boldsymbol{e}_{-1})^{\otimes(n-1)}$, where $\boldsymbol{e}_{-1}$ is the generator of $\mathbb{F}_{(-1)}$.*

Proof. Denote with D_1 a Legendrian Heegaard diagram for the standard Legendrian unknot in (M, ξ). If we perform n consecutive stabilizations on D_1, then we easily obtain a diagram D_n for the standard Legendrian n-component unlink.

Since D and D_n are Legendrian Heegaard diagrams for the same contact manifold (M, ξ) and they have the same number of basepoints, from [**18**], we know that

$$\widehat{HF}(D, \mathfrak{t}_\xi) \cong \widehat{HF}\left(\overline{M}, \mathfrak{t}_\xi\right) \otimes \left(\mathbb{F}_{(-1)} \oplus \mathbb{F}_{(0)}\right)^{\otimes(n-1)}. \tag{2}$$

Moreover, a little variation of the maps Ψ that we define in Section 4 tells us that the isomorphism in equation (2) also identifies the class $[c(D)]$ with $\widehat{c}(M, \xi) \otimes (\mathbf{e}_{-1})^{\otimes(n-1)}$. □

Before proving the second lemma, we recall that, since the link Floer homology group is an $\mathbb{F}[U]$-module and $\mathbb{F}[U]$ is a principal ideal domain, we have $cHFL^{-}\left(\overline{M}, L, \mathfrak{t}_\xi\right) \cong \mathbb{F}[U]^r \oplus T$, where r is an integer and T is the torsion $\mathbb{F}[U]$-module.

LEMMA 5.3. *The following two statements hold*

1. *$F(x) = 0$ and x is homogeneous with respect to the Alexander grading if and only if $x = 0$;*
2. *$F_*[x] = [0]$ and $[x_i]$ is homogeneous with respect to the Alexander grading, for every $i = 1, \ldots, r$, if and only if $[x]$ is torsion. Here the $[x_i]$'s are a decomposition of $[x]$ in the torsion-free quotient of $cHFL^{-}\left(\overline{M}, L, \mathfrak{t}_\xi\right)$.*

Proof. 1. The if implication is trivial. For the only if, suppose that $F(x)=0$, this gives that

$$x=(1+U)\lambda_1(U)y_1+\ldots+(1+U)\lambda_t(U)y_t,$$

where $\lambda_i(U)\in\mathbb{F}[U]$ for every $i=1,\ldots,t$ and $y_1,\ldots,y_t$ are all the intersection points that induce the Spin^c structure $\mathfrak{t}_\xi$.
Since each y_i is homogeneous and the monomial U drops the Alexander grading by 1, we have that $\lambda_i(U)=0$ for every $i=1,\ldots,t$ and then $x=0$.

2. Again the if implication is trivial. Now we have that

$$[x]=\sum_{i=1}^{r}[x_i]+[x]_T,$$

where $[x]_T$ is the projection of $[x]$ on the torsion submodule T. Since $F_*[x]=[0]$, we have

$$[x]=(1+U)[z]=(1+U)\lambda'_1(U)[z_1]+\ldots+(1+U)\lambda'_r(U)[z_r]+[x]_T,$$

where one has $[x_i]=(1+U)\lambda'_i(U)[z_i]$ for every $i=1,\ldots,r$ and the $[z_i]'s$ are a homogeneous basis of the torsion-free quotient.
The same argument of 1 implies that the polynomials $\lambda'_i(U)$ are all 0 and then $[x]=[x]_T$. □

Now we use Lemma 5.3 to show that there is a correspondence between the torsion-free quotient of the link Floer homology group and the multi-pointed hat Heegaard Floer homology.

THEOREM 5.4. *If L is a Legendrian n-component link in* (M,ξ)*, then there exists an isomorphism of* $\mathbb{F}[U]$*-modules*

$$\frac{cHFL^-\left(\overline{M},L,\mathfrak{t}_\xi\right)}{T}\longrightarrow \left(\widehat{HF}\left(\overline{M},\mathfrak{t}_\xi\right)\otimes_{\mathbb{F}}\mathbb{F}[U]\right)\otimes_{\mathbb{F}[U]}\left(\mathbb{F}[U]_{(-1)}\oplus\mathbb{F}[U]_{(0)}\right)^{\bigotimes(n-1)},$$

which sends a homology class of bigrading (d,s) *into one of Maslov grading* $d-2s$.

Proof. We just have to show that F_* sends $\{[z_1],\ldots,[z_r]\}$, a homogeneous $\mathbb{F}[U]$-basis of the torsion-free quotient of $cHFL^-\left(\overline{M},L,\mathfrak{t}_\xi\right)$, into an $\mathbb{F}$-basis of $\mathcal{X}=\widehat{HF}\left(\overline{M},\mathfrak{t}_\xi\right)\otimes\left(\mathbb{F}_{(-1)}\oplus\mathbb{F}_{(0)}\right)^{\bigotimes(n-1)}$. Statement 1 in Lemma 5.3 tells us that F_* is surjective. In fact, if $[y]\in\mathcal{X}$ then one has $0=\widehat{\partial}_z y=F(\partial^-x)$, where $F(x)=y$. We apply Lemma 5.3 to $\partial^-(x)$, since we can suppose that both x and y are homogeneous, and we obtain that $\partial^-(x)=0$; then $[x]$ is indeed a homology class. At this point, it is easy to see that the set $\{F_*[z_1],\ldots,F_*[z_r]\}$ is a system of generators of $\mathcal{X}$.

In order to prove that $F_*[z_1],\ldots,F_*[z_r]$ are also linearly independent in $\mathcal{X}$, we suppose that there is a subset $\left\{F_*[z_{i_1}],\ldots,F_*[z_{i_k}]\right\}$ such that

$$F_*[z_{i_1}]+\ldots+F_*[z_{i_k}]=F_*[z_{i_1}+\ldots+z_{i_k}]=[0].$$

Then we apply Statement 2 of Lemma 5.3 to $[z_{i_1}+\ldots+z_{i_k}]$ and we obtain that it is a torsion class. This is a contradiction, because $[z_{i_1}],\ldots,[z_{i_k}]$ are part of an $\mathbb{F}[U]$-basis of a torsion-free $\mathbb{F}[U]$-module. □

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Lemmas 5.1 and 5.2 and Theorem 5.4 immediately give the following corollary.

COROLLARY 5.5. *The Legendrian invariant $\mathfrak{L}(L, M, \xi)$ is non-torsion if and only if the contact invariant $\widehat{c}(M, \xi)$ is non-zero. Furthermore, if D is a Legendrian Heegaard diagram compatible with (L, M, ξ), then the Maslov and Alexander grading of the element $\mathfrak{L}(D)$ are related by the following equality:*

$$M(\mathfrak{L}(D)) = -d_3(M, \xi) + 2A(\mathfrak{L}(D)) + 1 - n,$$

where n is the number of component of L.

In particular, this corollary says that the Legendrian link invariant $\mathfrak{L}$ is always a torsion class if ξ is overtwisted and always non-torsion if (M, ξ) is strongly symplectically fillable. In fact, from [**20**], we know that $\widehat{c}(M, \xi)$ is zero in the first case and non-zero in the second one.

5.2. Relation with classical Legendrian invariants. We proved that the isomorphism class $\mathfrak{L}(L, M, \xi)$ is a Legendrian invariant. This is true also for the Alexander (and Maslov) grading of the element $\mathfrak{L}(D) \in cCFL^-(D, \mathfrak{t}_\xi)$, that we denote with $A(\mathfrak{L}(D))$. From Corollary 5.5, we know that $\widehat{c}(M, \xi) \neq [0]$ implies that $\mathfrak{L}(L, M, \xi)$ is non-torsion, hence it determines $A(\mathfrak{L}(D))$. On the other hand, if $\widehat{c}(M, \xi)$ is zero, then a priori the gradings of the element $\mathfrak{L}(D)$ could give more information. Starting from these observations, we want to express the value of $A(\mathfrak{L}(D))$ in terms of the other known invariants of the Legendrian link L. Note that Corollary 5.5 also tells us that the Maslov grading of $\mathfrak{L}(D)$ is determined, once we know $A(\mathfrak{L}(D))$.

First we recall that, from the definitions of Thurston–Bennequin and rotation number, we have

$$\text{tb}(L) = \sum_{i=1}^{n} \text{tb}_i(L), \qquad \text{where } \text{tb}_i(L) = \text{tb}(L_i) + \text{lk}(L_i, L \setminus L_i)$$

and

$$\text{rot}(L) = \sum_{i=1}^{n} \text{rot}_i(L), \qquad \text{where } \text{rot}_i(L) = \text{rot}(L_i).$$

THEOREM 5.6. *Consider $L \hookrightarrow (M, \xi)$ a null-homologous Legendrian n-component link in a rational homology contact 3-sphere and D a Legendrian Heegaard diagram, that comes from an open book compatible with (L, M, ξ). Then, we have that*

$$(\mathfrak{L}(D)) = \frac{\text{tb}(L) - \text{rot}(L) + n}{2} \quad \textit{and} \quad M(\mathfrak{L}(D)) = -d_3(M, \xi) + \text{tb}(L) - \text{rot}(L) + 1.$$

Proof. If L is a knot, then the claim has been proved by Ozsváth and Stipsicz in [**15**] (Theorem 1.6). At this point, in order to obtain the claim for links, we need to relate $A(\mathfrak{L}(D)$ with the Alexander grading of the Legendrian invariants of the components L_i of L.

A Legendrian Heegaard diagram D_i for the knot L_i is easily gotten from D by removing some curves and basepoints. We denote the intersection point representing the Legendrian

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invariant of L_i with $\mathfrak{L}(D_i)$. Then, we have

$$\begin{aligned} A(\mathfrak{L}(D)) &= \sum_{i=1}^{n}\left(A\left(\mathfrak{L}(D_i)\right)+\frac{1}{2}\mathrm{lk}(L_i, L\setminus L_i)\right) \\ &= \sum_{i=1}^{n}\left(\frac{\mathrm{tb}(L_i)-\mathrm{rot}(L_i)+1}{2}+\frac{\mathrm{lk}(L_i, L\setminus L_i)}{2}\right)=\sum_{i=1}^{n}\frac{\mathrm{tb}_i(L)-\mathrm{rot}_i(L)+1}{2} \\ &= \frac{\mathrm{tb}(L)-\mathrm{rot}(L)+n}{2}. \end{aligned}$$ □

5.3. The link Floer homology group of a Legendrian connected sum. Take two null-homologous Legendrian oriented links L_1, L_2, respectively, in the connected contact manifolds (M_1, ξ_1) and (M_2, ξ_2). We can define a Legendrian connected sum of the two links [9] and we denote it with $L_1\#L_2 \hookrightarrow (M_1\#M_2, \xi_1\#\xi_2)$.

While the contact 3-manifold $(M_1\#M_2, \xi_1\#\xi_2)$ is uniquely defined, the Legendrian link $L_1\#L_2$ depends on the choice of the components used to perform the connected sum. Moreover, we have the following properties [7, 9]:

- $d_3(M_1\#M_2, \xi_1\#\xi_2) = d_3(M_1, \xi_1) + d_3(M_2, \xi_2)$;
- $\mathrm{tb}(L_1\#L_2) = \mathrm{tb}(L_1) + \mathrm{tb}(L_2) + 1$;
- $\mathrm{rot}(L_1\#L_2) = \mathrm{rot}(L_1) + \mathrm{rot}(L_2)$.

Let us consider two adapted open book decompositions (B_i, π_i, A_i), compatible with the triples (L_i, M_i, ξ_i). We can define a third open book (B, π, A), for the manifold $M_1\#M_2$, with the property that $\pi^{-1}(1)$ is a Murasugi sum of the pages $\pi_1^{-1}(1)$ and $\pi_2^{-1}(1)$; see [7] for the definition. The resulting open book is compatible with the triple $(L_1\#L_2, M_1\#M_2, \xi_1\#\xi_2)$, where the Murasugi sum is done along the components involved in the connected sum.

Denote with D_1, D_2, and D the Legendrian Heegaard diagrams obtained from the open book decompositions that we introduced before. Then, we have the following theorem.

THEOREM 5.7. *For every Spin^c structure on M_1 and M_2, there is a chain map*

$$cCFL^-(D, \mathfrak{t}_1\#\mathfrak{t}_2) \longrightarrow cCFL^-(D_1, \mathfrak{t}_1) \otimes_{\mathbb{F}[U]} cCFL^-(D_2, \mathfrak{t}_2)$$

that preserves the bigrading and the element $\mathfrak{L}(D)$ is sent into $\mathfrak{L}(D_1)\otimes\mathfrak{L}(D_2)$.

Furthermore, this map induces an isomorphism in homology, which means that

$$\begin{aligned} &cHFL^-\left(\overline{M}_1\#\overline{M}_2, L_1\#L_2, \mathfrak{t}_{\xi_1\#\xi_2}\right) \cong \\ &\qquad cHFL^-\left(\overline{M}_1, L_1, \mathfrak{t}_{\xi_1}\right) \otimes_{\mathbb{F}[U]} cHFL^-\left(\overline{M}_2, L_2, \mathfrak{t}_{\xi_2}\right) \end{aligned}$$

as bigraded $\mathbb{F}[U]$-modules and

$$\mathfrak{L}(L_1\#L_2, M_1\#M_2, \xi_1\#\xi_2) = \mathfrak{L}(L_1, M_1, \xi_1) \otimes \mathfrak{L}(L_2, M_2, \xi_2).$$

Proof. It follows from [14] Section 7, [19] Section 7, and [21] Section 11. □

We note that the link Floer homology group of the connected sum does not depend on the choice of the components. In particular, this means that we can compute the link Floer homology group and the Legendrian invariant of a disjoint union $L_1 \sqcup L_2$. See [2] for the definition.

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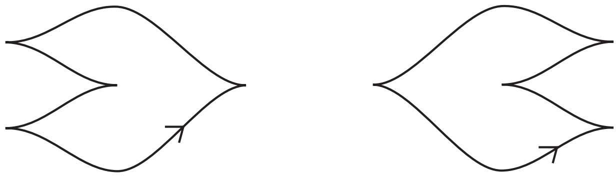

Figure 13. Front projection of $\mathcal{O}^+$ (left) and $\mathcal{O}^-$ (right).

PROPOSITION 5.8. *If we denote with* $\bigcirc_2$ *a smooth* 2*-component unlink and with* $\mathcal{O}_2$ *the Legendrian* 2*-component unlink in* (S^3, ξ_{st}) *such that* $\text{tb}(\mathcal{O}_2) = -2$*, then we have*

$$cHFL^-\left(\overline{M}, L_1 \sqcup L_2, \mathfrak{t}_\xi\right) \cong cHFL^-\left(\overline{M}_1, L_1, \mathfrak{t}_{\xi_1}\right) \otimes_{\mathbb{F}[U]} cHFL^-\left(\overline{M}_2, L_2, \mathfrak{t}_{\xi_2}\right) \otimes_{\mathbb{F}[U]} cHFL^-(\bigcirc_2)$$

and $\mathfrak{L}(L_1 \sqcup L_2, M, \xi) = \mathfrak{L}(L_1, M_1, \xi_1) \otimes \mathfrak{L}(L_2, M_2, \xi_2) \otimes \mathfrak{L}(\mathcal{O}_2)$.

Proof. We just apply Theorem 5.7 twice, each time on one of the two components of $\mathcal{O}_2$. □

The homology group $cHFL^-(\bigcirc_2)$ is isomorphic, as bigraded $\mathbb{F}[U]$-module, to $\mathbb{F}[U]_{(-1,0)} \oplus \mathbb{F}[U]_{(0,0)}$ (a proof can be found in [**16**]). Furthermore, Theorem 5.6 tells us that

$$\mathfrak{L}(\mathcal{O}_2) = \mathbf{e}_{-1,0},$$

that is the generator of $\mathbb{F}[U]$ with bigrading $(-1, 0)$.

This means that, if $\widehat{c}(M, \xi)$ is non-zero, we have that

$$M(\mathfrak{L}(L_1 \sqcup L_2, M, \xi)) = M(\mathfrak{L}(L_1, M_1, \xi_1)) + M(\mathfrak{L}(L_2, M_2, \xi_2)) - 1$$

and

$$A(\mathfrak{L}(L_1 \sqcup L_2, M, \xi)) = A(\mathfrak{L}(L_1, M_1, \xi_1)) + A(\mathfrak{L}(L_2, M_2, \xi_2)).$$

We also observe that:

- $\text{tb}(L_1 \sqcup L_2) = \text{tb}(L_1) + \text{tb}(L_2)$;
- $\text{rot}(L_1 \sqcup L_2) = \text{rot}(L_1) + \text{rot}(L_2)$.

5.4. Stabilizations of a Legendrian link. We know that Legendrian links in the tight S^3 can be represented with front projections, see [**6**] for more details. Then, we define positive (negative) stabilization of a Legendrian link L in (S^3, ξ_{st}), with front projection P, the Legendrian link $L^\pm$ represented by the front projection $P^\pm$, which is obtained by adding two consecutive downward (upward) cusps to P. Stabilizations are well defined up to Legendrian isotopy, in the sense that they do not depend on the choice of the point of P where we add the new cusps.

At this point, it is easy for us to define stabilizations in every contact manifold. In fact, we say that $L^\pm$ is the positive (negative) stabilization of L, a Legendrian link in (M, ξ), if $L^\pm = L\#\mathcal{O}^\pm$, where $\mathcal{O}^\pm$ is the positive (negative) stabilization of the standard Legendrian unknot $\mathcal{O}$ in (S^3, ξ_{st}). The Legendrian knots $\mathcal{O}^\pm$ are shown as front projections in Figure 13.

The link type remains unchanged after stabilizations. The behavior of the other classical invariants is given by the following proposition.

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PROPOSITION 5.9. *For every null-homologous Legendrian link L in a contact 3-manifold (M, ξ), one has*

$$\mathrm{tb}\left(L^{\pm}\right) = \mathrm{tb}(L) - 1 \quad \textit{and} \quad \mathrm{rot}\left(L^{\pm}\right) = \mathrm{rot}(L) \pm 1.$$

Furthermore, we have that

$$\mathfrak{L}(L^{+}, M, \xi) = U \cdot \mathfrak{L}(L, M, \xi) \quad \textit{and} \quad \mathfrak{L}(L^{-}, M, \xi) = \mathfrak{L}(L, M, \xi)$$

in $cHFL^{-}\left(\overline{M}, L, \mathfrak{t}_{\xi}\right)$.

Proof. The first claim is a standard fact (see [6]). The second one follows from Theorem 5.7, which says that we just need to determine $\mathfrak{L}(\mathcal{O}^{\pm})$, the fact that $cHFK^{-}(\bigcirc) \cong \mathbb{F}[U]_{(0,0)}$, which says that $\mathfrak{L}(\mathcal{O}^{\pm})$ is fixed by the classical invariants of $\mathcal{O}^{\pm}$, and the first part of this proposition, which tells us that $\mathrm{tb}(\mathcal{O}^{\pm}) = -2$ and $\mathrm{rot}(\mathcal{O}^{\pm}) = \pm 1$. □

5.5. Loose Legendrian links. Since $\widehat{c}(M, \xi)$ is always zero for overtwisted contact manifold, we have that the Legendrian link invariant $\mathfrak{L}$ is always torsion in this case. But Proposition 1.5 says more in the case of loose Legendrian links.

Proof of Proposition 1.5. The complement of L in M contains an overtwisted disk E. Since E is contractible, we can find a ball U such that $E \subset U \subset M \setminus L$. The restriction of ξ to U is obviously overtwisted; moreover, we can choose E such that ∂U has trivial dividing set. Thus, we have that $(M, \xi) = (M, \xi_1) \# (S^3, \xi')$, where ξ_1 concides with ξ near L and ξ' is an overtwisted structure on S^3.

We now use the fact that the standard Legendrian unknot $\mathcal{O}$ is well defined, up to Legendrian isotopy, and Theorem 5.7 to say that

$$\mathfrak{L}(L, M, \xi) = \mathfrak{L}(L, M, \xi_1) \otimes \mathfrak{L}(\mathcal{O}, S^3, \xi').$$

But since $cHFK^{-}(\bigcirc) \cong \mathbb{F}[U]_{(0,0)}$, and then there is no torsion, we know that the Legendrian invariant of an unknot is zero in overtwisted 3-spheres. Then, $\mathfrak{L}(L, M, \xi)$ is also zero. □

This proposition says something about stabilizations. In fact, in principle, a stabilization of a non-loose Legendrian link $L \hookrightarrow (M, \xi)$ can be loose, but if $\mathfrak{L}(L, M, \xi)$ is non-zero then all its negative stabilizations are also non-loose.

6. The transverse case. We recall that there is a way to associate a Legendrian oriented link to a transverse link and vice versa. Given a Legendrian link L, we denote with T_L the transverse push-off of L, which is transverse. The construction is found in Section 2.9 in [6]; here, we recall that transverse links have a canonical orientation induced by the contact form. Any two transverse push-offs are transversely isotopic and then T_L is uniquely defined, up to transverse isotopy.

In the same way, if T is a transverse link, then we can define a Legendrian approximation L_T of T. The procedure is also described in Section 2.9 of [6]. The Legendrian link L_T is not well defined up to Legendrian isotopy, but only up to negative stabilizations. Then from [8], we have the following theorem.

THEOREM 6.1 (Epstein). *Two transverse links in a contact manifold are transversely isotopic if and only if they admit Legendrian approximations which differ by negative stabilizations.*

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The only classical invariant of a null-homologous transverse link T, other than the smooth link type is the *self-linking number* $\mathrm{sl}(T)$. In every (M, ξ) rational homology contact 3-sphere, we define the number $\mathrm{sl}(T)$ as follows:

$$\mathrm{sl}(T) = \mathrm{tb}(L_T) - \mathrm{rot}(L_T).$$

Clearly, from Theorem 6.1 and since negative stabilizations drop both the Thurston–Bennequin and rotation number by 1, we have that the self-linking number is a well-defined transverse invariant. Moreover, from the definition of self-linking number, we have the following properties:

- $\mathrm{sl}(T_1 \# T_2) = \mathrm{sl}(T_1) + \mathrm{sl}(T_2) + 1$;
- $\mathrm{sl}(T_1 \sqcup T_2) = \mathrm{sl}(T_1) + \mathrm{sl}(T_2)$.

We can now define a transverse invariant from link Floer homology by taking

$$\mathfrak{T}(T, M, \xi) = \mathfrak{L}(L_T, M, \xi),$$

where L_T is a Legendrian approximation of T. The invariant $\mathfrak{T}$ has the same basic properties of $\mathfrak{L}$ that we recall in the following theorem.

THEOREM 6.2. *The isomorphism class $\mathfrak{T}(T, M, \xi)$ in $cHFL^-\left(\overline{M}, T, \mathfrak{t}_\xi\right)$ is a transverse link invariant. If n is the number of components of T, we have that the bigrading of $\mathfrak{T}$ is*

$$(\mathfrak{T}(T, M, \xi)) = \frac{\mathrm{sl}(T) + n}{2} \quad \textit{and} \quad M(\mathfrak{T}(T, M, \xi)) = -d_3(M, \xi) + \mathrm{sl}(T) + 1.$$

Furthermore, one has

$$\mathfrak{T}(T_1 \# T_2, M_1 \# M_2, \xi_1 \# \xi_2) = \mathfrak{T}(T_1, M_1, \xi_1) \otimes \mathfrak{T}(T_2, M_2, \xi_2).$$

Proof. Proposition 5.9 and Theorem 6.1 tell us that $\mathfrak{T}(T, M, \xi)$ is an invariant. The other properties follow from Theorem 5.6, the definition of self-linking number and the fact that the operations of Legendrian approximation and connected sum commute. □

In the case of knots, the invariant $\mathfrak{T}$ has been introduced first in [**14**].

7. Applications.

7.1. A different version of the Legendrian invariant. Let us consider a Legendrian Heegaard diagram D, given by an open book compatible with a triple (L, M, ξ), where M is a rational homology 3-sphere and L is a null-homologous Legendrian n-component oriented link with link type $\mathcal{L}$. We recall that, when M admits a diffeomorphism that reverses the orientation, we can identify $(\overline{M}, \mathcal{L})$ with $(M, \mathcal{L}^*)$. We denote by $\mathcal{L}^*$ the *mirror image* of the oriented link type $\mathcal{L}$.

We can define another chain complex by taking the $\mathbb{F}$-vector space

$$\widehat{CFL}(D, \mathfrak{t}) = \frac{cCFL^-(D, \mathfrak{t})}{U = 0}$$

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for every Spin^c structure $\mathfrak{t}$ on M. The corresponding differential is $\widehat{\partial}=\partial^-|_{U=0}$. We obtain the hat link Floer homology group

$$\widehat{HFL}(D,\mathfrak{t})=\bigoplus_{d,s\in\mathbb{Q}}\widehat{HFL}_{d,s}(D,\mathfrak{t})\,;$$

given by

$$\widehat{HFL}_{d,s}(D,\mathfrak{t})=\frac{\operatorname{Ker}\widehat{\partial}_{d,s}}{\operatorname{Im}\widehat{\partial}_{d+1,s}}.$$

The group $\widehat{HFL}(D,\mathfrak{t})$ is a finite dimensional, bigraded $\mathbb{F}$-vector space and its isomorphism type is invariant under smooth isotopy of the link L [19]. Hence, we can denote $\widehat{HFL}(D,\mathfrak{t})$ with $\widehat{HFL}\left(\overline{M},L,\mathfrak{t}\right)$.

The intersection point $\mathfrak{L}(D)$ is a cycle also in $\widehat{CFL}(D,\mathfrak{t})$ and it determines the Spin^c structure $\mathfrak{t}_\xi$. Then we have the following theorem.

THEOREM 7.1. *The equivalence class of* $(\widehat{HFL}\left(\overline{M},L,\mathfrak{t}_\xi\right),[\mathfrak{L}(D)])$ *is a Legendrian invariant of* (L,M,ξ) *and we denote it with* $\widehat{\mathfrak{L}}(L,M,\xi)$. *Furthermore, if* $\widehat{\mathfrak{L}}(L,M,\xi)$ *is non-zero then* $\mathfrak{L}(L,M,\xi)$ *is also non-zero.*

For a null-homologous transverse link $T\hookrightarrow(M,\xi)$, *we have that* $\widehat{\mathfrak{T}}(T,M,\xi)=\widehat{\mathfrak{L}}(T_L,M,\xi)$, *where* T_L *is a Legendrian push-off of* T, *is a transverse invariant of* T *and it has the same non-vanishing property of* $\widehat{\mathfrak{L}}$.

The proof of this theorem is the same as the one of Theorems 1.2 and 6.2, except for the non-vanishing property, which follows from the fact that $\widehat{CFL}(D,\mathfrak{t})$ is a quotient of $cCFL^-(D,\mathfrak{t})$.

The invariant $\widehat{\mathfrak{L}}(L,M,\xi)$ can be refined using a naturality property of the link Floer homology group of a connected sum. Suppose that L is a Legendrian oriented link, with link type $\mathcal{L}$, in a contact 3-sphere (S^3,ξ) such that $\widehat{\mathfrak{L}}(L,S^3,\xi)\neq[0]$. Let S be a convex, splitting sphere with connected dividing set, which intersects L transversely in exactly two points. Such a splitting sphere expresses L as a connected sum of two links L_1 and L_2.

Since $L=L_1\#_S L_2$, then its hat Heegaard Floer homoloy group admits the splitting

$$\widehat{HFL}(\mathcal{L}^*)\cong\widehat{HFL}\left(\mathcal{L}_1^*\right)\otimes_{\mathbb{F}}\widehat{HFL}\left(\mathcal{L}_2^*\right),$$

where the mirror images appear because S^3 has a diffeomorphism that reverses the orientation.

The Alexander grading of $\widehat{\mathfrak{L}}(L,S^3,\xi)$ is well defined, because we suppose that the invariant is non-zero. Moreover, we have that

$$A\left(\widehat{\mathfrak{L}}(L,S^3,\xi)\right)=A\left(\widehat{\mathfrak{L}}(L_1,S^3,\xi_1)\right)+A\left(\widehat{\mathfrak{L}}(L_2,S^3,\xi_2)\right).$$

The pair (s_1,s_2), where

$$s_i=A\left(\widehat{\mathfrak{L}}(L_i,S^3,\xi_i)\right)$$

for $i=1,2$, is called *Alexander pair* of $\widehat{\mathfrak{L}}(L,S^3,\xi)$ with respect to S and we denote it with $A_S(L,S^3,\xi)$. We have that the Alexander pair is an invariant of L in the sense of the following theorem.

THEOREM 7.2. *Suppose that* L *is a Legendrian link in* (S^3,ξ) *such that* $\widehat{\mathfrak{L}}(L,S^3,\xi)$ *is non-zero. We also assume that there are two convex, splitting spheres* S_1 *and* S_2, *which* *decompose L as Legendrian connected sums, such that we can find a smooth isotopy of M that fix L and sends S_1 into S_2.*

Then the two Alexander pairs of $\widehat{\mathfrak{L}}(L, S^3, \xi)$, with respect to S_1 and S_2, coincide, which means that $A_{S_1}(L, S^3, \xi) = A_{S_2}(L, S^3, \xi)$.

Proof. The proof is a link version of the one of Theorems 8.4 and 9.1 in [14]. □

There is a version of Theorem 7.2 for transverse links.

COROLLARY 7.3. *Suppose that T is a transverse link in (S^3, ξ). Assume also that one of its Legendrian approximations L_T respects the hypothesis of Theorem 7.2. Then one has*

$$A_{S_1}(T, S^3, \xi) = A_{S_2}(T, S^3, \xi),$$

where the Alexander pair is now defined as $A_S(T, S^3, \xi) = A_S(L_T, S^3, \xi)$.

Proof. It is a consequence of the fact that Legendrian approximations of the same transverse link differ by negative stabilizations. Therefore, the Alexander gradings are the same because negative stabilizations do not change the invariant. □

The Alexander pair can be useful in distinguishing Legendrian and transverse links that are not isotopic.

PROPOSITION 7.4. *Suppose that L_1 and L_2 are smoothly isotopic Legendrian (transverse) links in (S^3, ξ) which appear as follows. Say $L_1 \approx L_2$ is a 2-component Legendrian (transverse) link, obtained from three Legendrian (transverse) knots K, H, and J with prime knot types, defined as follows: take the connected sum of $K \#_S H$ with a (standard) positive Hopf link $\mathcal{H}_+$ and J, in the way that $K \#_S H$ is summed on the first component of $\mathcal{H}_+$ and J on the second one.*

We have that if $A_S(L_1, S^3, \xi) \neq A_S(L_2, S^3, \xi)$, then L_1 is not Legendrian (transverse) isotopic to L_2.

Proof. It follows from Theorem 7.2 and Corollary 7.3 and the fact that, if there is a Legendrian (transverse) isotopy F between L_1 and L_2, the isotopy F is such that $F(S) = S'$ and we can smoothly isotope S' onto S. □

7.2. Non-loose Legendrian links with loose sublinks. It is easy to prove that we can always find loose Legendrian links in every overtwisted contact 3-manifold; in fact, Legendrian links inside a Darboux ball need to be loose. On the other hand, it is not harder to show that the same holds for non-loose Legendrian links. In fact, it is a known [7] that every overtwisted contact 3-manifold is obtained from some -1-surgeries and exactly one $+1$-surgery on Legendrian knots in (S^3, ξ_{st}); then we just take the Legendrian link given by n parallel contact push-offs of J^+, the knot where we perform the $+1$-surgery. Moreover, it is easy to check that this link also has non-loose components.

A more interesting result is to show that, under some hypothesis on (M, ξ), we can also find non-split Legendrian n-components links such that $\mathfrak{L}(L, M, \xi) \neq [0]$, which means that they are non-loose from Proposition 1.5, and all of their sublinks are now loose. We start by constructing Legendrian knots with non-trivial invariant in all the overtwisted structures on S^3.

Consider the family of Legendrian knots $L(j)$, where $j \geqslant 1$, given by the surgery diagram in Figure 14. Using Kirby calculus, we easily see that $L(j)$ is a positive torus knot

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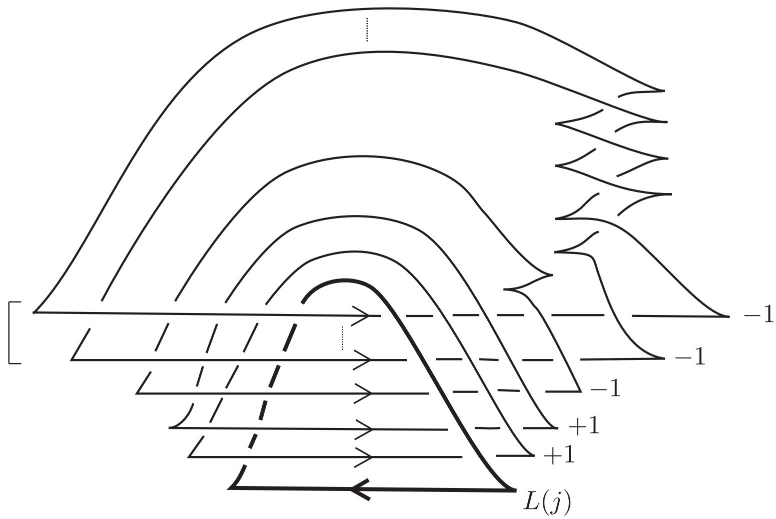

Figure 14. Contact surgery presentation for the Legendrian knot $L(j)$.

$T_{2,2j+1}$ in S^3. On the other hand, the Legendrian invariants of $L(j)$ and the contact structure where it lives are determined in [14] Chapter 6. Namely, the knots in Figure 14 are Legendrian knots in $\left(S^3, \xi_{1-2j}\right)$ and their invariants are

- $\mathrm{tb}(L(j)) = 6 + 4(j-1)$;
- $\mathrm{rot}(L(j)) = 7 + 6(j-1)$.

Furthermore, Proposition 6.2 in [14] tells us that $\widehat{\mathfrak{L}}(L(j), S^3, \xi_{1-2j}) \neq [0]$ and then $\mathfrak{L}(L(j), S^3, \xi_{1-2j})$ is a non-zero torsion class in $HFK^-(T_{2,-2j-1})$. Moreover, both have bigrading $(1, 1-j)$.

Now we want to consider another family of Legendrian knots: the knots $L_{k,l}$, with $k, l \geqslant 0$, shown in Figure 15. From [14] Chapter 6, we also know that $L_{k,l}$ is a negative torus knot $T_{2,-2k-2l-3}$ in $\left(S^3, \xi_{2l+2}\right)$ and its invariants are

- $\mathrm{tb}(L_{k,l}) = -6 - 4(k+l)$;
- $\mathrm{rot}(L_{k,l}) = -7 - 2k - 6l$.

In this case, from Theorems 6.8, 6.9, and 6.10 in [14], we have that the invariant $\widehat{\mathfrak{L}}$ of the Legendrian knots $L_{0,l}$, $L_{1,1}$, and $L_{1,2}$ is non-zero with bigrading $(-2k, 1-k+l)$ in the homology group $\widehat{HFK}(T_{2,2k+2l+3})$. Obviously, the fact that $\widehat{\mathfrak{L}}$ is non-zero again implies that the same is true for the invariant $\mathfrak{L}$.

At this point, we define the Legendrian knots K_i, for every $i \in \mathbb{Z}$, in the following way:

$$K_i = \begin{cases} L(j)\#L(1) & \text{if } i = -2j < 0 \\ L(j) & \text{if } i = 1-2j < 0 \\ L_{0,j-1} & \text{if } i = 2j > 0 \\ L_{0,j-1}\#L(1) & \text{if } i = 2j-1 > 0 \\ L_{0,0}\#L(1)^2 & \text{if } i = 0. \end{cases}$$

Then we have the following result, which is Theorem 7.2 in [14].

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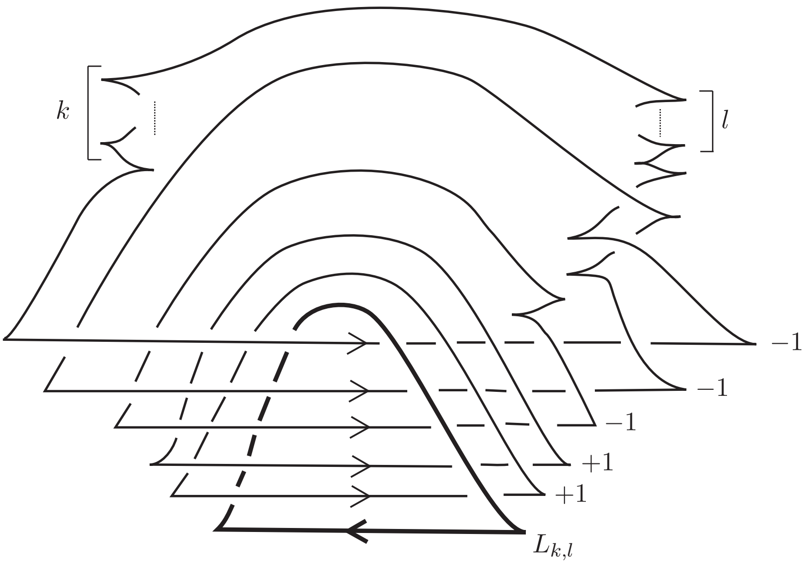

Figure 15. Contact surgery presentation for the Legendrian knot $L_{k,l}$.

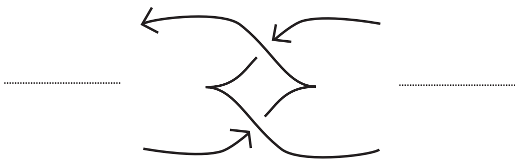

Figure 16. The two connected sums with the standard Legendrian positive Hopf link.

PROPOSITION 7.5 (Lisca, Ozsváth, Stipsicz, and Szabó). *The Legendrian knot* $K_i \hookrightarrow (S^3, \xi_i)$ *is such that* $\mathfrak{L}(K_i, S^3, \xi_i) \neq [0]$ *and then it is non-loose for every* $i \in \mathbb{Z}$.

Proof. It follows easily from the previous computation and the connected sum formula. □

We can now go back to links. Let us take an overtwisted 3-manifold (M, ξ) such that there exists another contact structure ζ on M with $\widehat{c}(M, \zeta) \neq [0]$ and $\mathfrak{t}_\xi = \mathfrak{t}_\zeta$; in particular ζ is tight. Consider $\mathcal{O}$ the standard Legendrian unknot in (M, ζ). We have that $\mathfrak{L}(\mathcal{O}, M, \zeta)$ coincides with $\mathbf{e}_{-d_3(M,\zeta),0} \neq [0]$ and the invariant is non-torsion; this is because $HFK^-\left(\overline{M}, \mathcal{O}, \mathfrak{t}_\zeta\right) \cong \mathbb{F}[U]_{(-d_3(M,\zeta),0)}$ and Theorem 5.6.

Now we connect sum two copies of $L(1)$ to the standard positive Hopf link H_+ in (S^3, ξ_{st}) in the way that one copy is summed on the first component of H_+ and the other one on the second component, see Figure 16. We repeat this procedure a total of $n-1$ times, where every time the connected sum is performed in the way that the resulting link has components with form a single chain.

Let us denote these Legendrian links with C_n. Since $L(1)$ lives in (S^3, ξ_{-1}), we have that C_n is an n-component Legendrian link in (S^3, ξ_{-n}). The invariant $\mathfrak{L}(C_n, S^3, \xi_{-n})$ is the tensor product of n times $\mathfrak{L}(L(1), S^3, \xi_{-1})$ with $n-1$ times $\mathfrak{L}(H_+)$ the invariant in the standard S^3. An easy computation shows that $\mathfrak{L}(H_+)$ is the only non-torsion element in the group

$$cHFL^-(H_-) \cong \mathbb{F}[U]_{(0,0)} \oplus \mathbb{F}[U]_{(1,1)} \oplus \left(\frac{\mathbb{F}[U]}{U \cdot \mathbb{F}[U]}\right)_{(0,0)}$$

with bigrading $(1, 1)$. This means that not only $\mathfrak{L}(H_+)$ is non-torsion, but it is also represented by the top generator of one of the $\mathbb{F}[U]$-towers of $cHFL^-(H_-)$.

Now we perform a connected sum between C_n and the Legendrian unknot $\mathcal{O}$ that introduced before. Therefore, we can now see C_n as a Legendrian link in (M, ξ'), where ξ' is an overtwisted structure such that $\mathfrak{t}_{\xi'} = \mathfrak{t}_\zeta = \mathfrak{t}_\xi$ and $d_3(M, \xi') = d_3(M, \zeta) - n$.

If we perform another connected sum with the Legendrian knot $K_{d_3(M,\xi)-d_3(M,\zeta)+n}$, then we obtain the Legendrian link

$$L = K_{d_3(M,\xi)-d_3(M,\zeta)+n} \# C_n,$$

which is a link in M equipped with a contact structure that has the same Hopf invariant as (M, ξ) and induces the same Spin^c structure of ξ. By the Eliashberg's classification of overtwisted structures [**5**], we conclude that L is a Legendrian n-component link in (M, ξ). We can now prove Theorem 1.6.

Proof of Theorem 1.6. We already saw that the link L exists if the hypothesis of the theorem holds. So first we check that $\mathfrak{L}(L, M, \xi)$ is non-zero. In fact, the invariant is represented by the tensor product of a non-zero torsion element with $\mathfrak{L}(C_n, S^3, \xi_{-n})$. We are working with $\mathbb{F}[U]$-modules and we recall that

$$\mathbb{F}[U] \otimes_{\mathbb{F}[U]} \left(\frac{\mathbb{F}[U]}{U \cdot \mathbb{F}[U]}\right) \cong \frac{\mathbb{F}[U]}{U \cdot \mathbb{F}[U]}$$

and, more precisely, in the $\mathbb{F}[U]$-factor only the generator survives.

Then, from what we said before, we have that $\mathfrak{L}(L, M, \xi)$ remains non-zero. In fact, the tensor products of the invariant $\mathfrak{L}$ of K_n with $\mathfrak{L}(L(1), S^3, \xi_{-1})$ is non-zero because the hat versions $\widehat{\mathcal{L}}$ are non-zero in this case ([**14**]). Note that this conclusion would be false if instead we took the negative Hopf link. Now the invariant $\mathfrak{L}$ does not lie in the top of an $\mathbb{F}[U]$-tower of the homology group and then it vanishes after the tensor product.

This immediately implies that $\mathfrak{T}(T, M, \xi)$ is also non-zero and then the theorem holds for transverse links. Moreover, it is easy to see that the sublinks of L are all loose; in fact, if L' is a sublink of L then there is at least one component of C_n which has been removed, say the i-th component. Since in an overtwisted 3-manifold, we can always find an overtwisted disk disjoint from a finite number of Darboux balls, this means that in the i-th (S^3, ξ_{-1})-summand we can find an overtwisted disk that happens to lie in the complement of L'.

It is only left to prove that L is non-split. From [**3**], we know that the connected sum of two tight contact manifolds is still tight. This implies that a non-loose Legendrian link is split if and only if its smooth link type is split. Hence, we just have to show that L is non-split as a smooth n-component link. But L is a connected sum of torus links in a 3-ball inside M and we know that L is non-split as a link in S^3. Furthermore, if L is split in M, then it would be split also in the 3-sphere and this is a contradiction. □

7.3. Non-simple link types. In the previous subsection, we saw that, under some hypothesis, in an overtwisted 3-manifold (M, ξ) we can find non-loose, non-split Legendrian n-components links L_n. Consider the links L'_n obtained as the connected sum

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of L_n with the standard Legendrian unknot in (S^3, ξ_0), where ξ_0 is the overtwisted S^3 with zero Hopf invariant.

Since (M, ξ) is already overtwisted, we have that $(M, \xi)\#(S^3, \xi_0)$ is contact isotopic to (M, ξ). This means that L'_n is also a non-split Legendrian link in (M, ξ), which is smoothly isotopic to L_n for every $n \geqslant 1$, but unlike L_n it is clearly loose. Each component of L'_n has the same classical invariants of a component of L_n. Moreover, if $n \geqslant 2$, then there is an overtwisted disk in their complement. From a result of Dymara in [**4**], the components of L'_n are Legendrian isotopic to the ones of L_n. Hence, we have the following corollary.

COROLLARY 7.6. *The link type of L_n and L'_n in M, which is denoted with $\mathcal{L}$, is both Legendrian and transverse non-simple.*

On the other hand, we can also find non-simple link types where the two Legendrian and transverse representatives are non-loose.

PROPOSITION 7.7. *Let us consider the links $L_1 = (L_{0,2}\#L_{1,2})\#H_+\#L(1)$ and $L_2 = (L_{1,1}\#L_{0,3})\#H_+\#L(1)$ in the contact manifold (S^3, ξ_{11}), where in L_i the knots on the left are summed on the first component of H_+ and $L(1)$ on the second one. Then, L_1 and L_2 are two non-loose, non-split Legendrian 2-component links, with the same classical invariants and Legendrian isotopic components, but that are not Legendrian isotopic.*

In the same way, the transverse push-offs of L_1 and L_2 are two non-loose, non-split transverse 2-component links, with the same classical invariants and transversely isotopic components, but that are not transversely isotopic.

Proof. We apply Theorem 7.2. The Legendrian invariant of L_1 and L_2 is computed in [**14**] and it is non-zero; moreover, the Alexander pairs of $\widehat{\mathfrak{L}}(L_1, S^3, \xi_{11})$ and $\widehat{\mathfrak{L}}(L_2, S^3, \xi_{11})$ are different. The fact that the components are Legendrian isotopic follows from Dymara's result [**4**]. The same argument proves the theorem in the transverse setting. □

Using the same construction, the refined version of $\widehat{\mathfrak{L}}$ and $\widehat{\mathfrak{T}}$ can be applied to find such examples for links with more than two components in every contact manifold as in Theorem 1.6.

Proof of Theorem 1.7. Let us take a standard Legendrian (transverse) positive Hopf link H_+ in (S^3, ξ_{st}). On the first component of H_+, we perform a connected sum with the knot $L_{0,2}\#L_{1,2}$ in one case and $L_{1,1}\#L_{0,3}$ in the other. While, on the second component of H_+, we sum a non-split Legendrian (transverse) n-component link in the overtwisted manifold (M, ξ'), where $d_3(M, \xi') = d_3(M, \xi) - 12$, with non-zero invariants; those links exist as we know from Theorem 1.6. We conclude by applying the same argument of the proof of Proposition 7.7. □

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