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THEORETICAL PROPERTIES OF THE WEIGHTED GENERALIZED GAMMA AND RELATED DISTRIBUTIONS

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A new class of weighted generalized gamma distribution (WGGD) and related distributions are presented. Theoretical properties of the generalized gamma model, WGGD including the hazard function, reverse hazard function, moments, coefficient of variation, coefficient of skewness, coefficient of kurtosis, and entropy measures are derived. The results presented here generalizes the generalized gamma distribution and includes several distributions as special cases. The special cases include generalized gamma, weighted gamma, weighted exponential, weighted Weibull, weighted Rayleigh distributions, and their underlying or parent distributions.

1. INTRODUCTION

Well-defined sampling structures often do not exist in natural populations such as human, wildlife, insect, plant, or fish populations. Therefore, recorded observations on individuals in these populations are biased and will not have the original distribution unless every observation is given an equal chance of being recorded. Weighted distribution theory gives a unified approach to modeling these biased data (Patil and Rao [11]). The concept of weighted distributions has been employed in a wide variety applications in reliability and survival analysis, analysis of family data, meta-analysis, ecology, and forestry. Rao [14,15] identified the various sampling situations that can be modeled by what he called weighted distributions, extending the idea of the methods of ascertainment upon estimation of frequencies by Fisher [3]. Patil and Rao [10,11] investigated the applications of weighted distributions. Statistical applications of weighted distributions, especially to the analysis of data relating to human population and ecology can be found in Patil [12].

To introduce the concept of a weighted distribution, suppose X is a non-negative random variable (rv) with its natural probability density function (pdf) $f(x;\theta)$, where the natural parameter is $\theta \in \Omega$ (Ω is the parameter space). Suppose a realization x of X under $f(x;\theta)$ enters the investigator's record with probability proportional to $w(x;\beta)$, so that the recording (weight) function $w(x;\beta)$ is a non-negative function with the parameter β representing

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the recording (sighting) mechanism. Clearly, the recorded x is not an observation on X, but on the rv X_w, having a pdf

$$f_w(x;\theta,\beta) = \frac{w(x,\beta)f(x;\theta)}{\omega}, \tag{1}$$

where ω is the normalizing factor obtained to make the total probability equal to unity by choosing $0 < \omega = E[w(X,\beta)] < \infty$. The rv X_w is called the weighted version of X, and its distribution is related to that of X and is called the weighted distribution with weight function w.

The main objective of this paper is to construct and explore the properties of weighted generalized gamma distribution (WGGD). This paper is organized as follows. Section 2 contains some basic definitions, utility notions and useful functions. The pdf, cumulative distribution function (cdf), hazard function and reverse hazard function of the WGGD is derived in Section 3. In Section 4, moments and related measures are derived. Measures of uncertainty are presented in Section 5, followed by concluding remarks.

2. BASIC UTILITY NOTIONS

In this section, some basic utility notions and results are presented. Suppose the distribution of a continuous rv X has the parameter set $\theta^* = \{\theta_1, \theta_2, \ldots, \theta_n\}$. Let the pdf of X be given by $f(x;\theta^*)$. The cdf of X, is defined to be

$$F(x;\theta^*) = \int_{-\infty}^{x} f(t;\theta^*)\,dt. \tag{2}$$

The hazard rate function of X can be interpreted as the instantaneous failure rate or the conditional probability density of failure at time x, given that the unit has survived until time x. See Finkelstein [2] for additional details. The hazard function $h(x;\theta^*)$ is defined to be

$$h(x;\theta^*) = \lim_{\Delta x \to 0} \frac{P\left(x \le X \le x + \Delta x\right)}{\Delta x[1 - F(x;\theta^*)]} = \frac{-\bar{F}'(x;\theta^*)}{\bar{F}(x;\theta^*)} = \frac{f(x;\theta^*)}{1 - F(x;\theta^*)}, \tag{3}$$

where $\bar{F}(x;\theta^*) = 1 - F(x;\theta^*)$ is the survival or reliability function. The reverse Hazard function can be interpreted as an approximate probability of a failure in $[x, x + dx]$, given that the failure had occurred in $[0, x]$ (see Finkelstein [2] for additional details). The reverse hazard function $\tau(x;\theta^*)$ is defined to be

$$\tau(x;\theta^*) = \frac{f(x;\theta^*)}{F(x;\theta^*)}. \tag{4}$$

Some useful functions that are employed in subsequent sections are given below. The gamma and digamma functions are given by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt, \quad \text{and} \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{5}$$

where $\Gamma'(x) = \int_0^{\infty} t^{x-1}(\log t) e^{-t}\,dt$ is the first derivative of the gamma function. The second derivative of the gamma function is $\Gamma''(x) = \int_0^{\infty} t^{x-1}(\log t)^2 e^{-t}\,dt$. The lower and upper incomplete gamma functions are

$$\gamma(s,x) = \int_0^{x} t^{s-1} e^{-t}\,dt \quad \text{and} \quad \Gamma(s,x) = \int_x^{\infty} t^{s-1} e^{-t}\,dt, \tag{6}$$

respectively.

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2.1. Generalized Gamma Distribution (GGD)

The GGD was introduced by Stacy [16]. It is considered to be a useful life distribution model and is suitable for modeling data with different types of hazard rate function: increasing, decreasing and unimodal. It is a flexible family of distributions in terms of the varieties of shapes and hazard functions. The GGD has been used in several research areas such as engineering, hydrology and survival analysis. See Ali, Woo, and Nadarajah [1], Lehmann [8], and Nadarajah and Gupta [9] for additional details. Many distributions commonly used in survival analysis, such as Weibull, log normal are special cases of the generalized gamma distribution. The pdf of the generalized gamma distribution is given by

$$g(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma(k)}(\lambda x)^{k\beta-1}e^{-(\lambda x)^{\beta}},\quad \text{for}\quad x>0\quad\text{and}\quad \lambda,\beta,k>0. \tag{7}$$

Note that λ is a scale parameter, and k and β are shape parameters. When $k=\beta=1$, the GGD results in the exponential distribution. When $\beta=1$, it results in the gamma distribution. When $k=1$, Weibull distribution is obtained. Also, when $\beta=2$ and $k=1$, Rayleigh distribution is obtained. See Khodabin and Ahamadabadi [7] for additional details. The cdf of the GGD is given by

$$G(x;\lambda,\beta,k)=\frac{\gamma\left(k,(\lambda x)^{\beta}\right)}{\Gamma(k)},\quad \text{for}\quad x>0,\quad\text{and}\quad \lambda,\beta,k>0. \tag{8}$$

Basic properties of generalized gamma distribution are also derived by Huang and Hwang [6], and Khodabin and Ahamadabadi [7]. From these results we have

$$E(X^{c}e^{tX^{\beta}})=\frac{\lambda^{k\beta}\Gamma(k+\frac{c}{\beta})}{(\lambda^{\beta}-t)^{k+\frac{c}{\beta}}\Gamma(k)},\quad \lambda^{\beta}>t. \tag{9}$$

The mth non-central moment of the GGD is given by

$$E(X^{m})=\frac{\Gamma\left(k+\frac{m}{\beta}\right)}{\lambda^{m}\Gamma(k)},\qquad \text{for } m=1,2,\ldots. \tag{10}$$

The variance of GGD is given by

$$\sigma^{2}=\frac{\Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k)-\Gamma^{2}\left(k+\frac{1}{\beta}\right)}{\lambda^{2}\Gamma^{2}(k)}. \tag{11}$$

The coefficient of variation (CV) is given by

$$CV=\frac{\left[\Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k)-\Gamma^{2}\left(k+\frac{1}{\beta}\right)\right]^{\frac{1}{2}}}{\Gamma\left(k+\frac{1}{\beta}\right)}. \tag{12}$$

3. WEIGHTED GENERALIZED GAMMA DISTRIBUTIONS

In this section, we present results on the WGGD with the weight function $w(x)=x^{k}e^{tx^{\beta}}F^{i}(x)\overline{F}^{j}(x)$. When $i=j=0$, we have $w_2(x;c,\beta,t)=x^{c}e^{tx^{\beta}}$. In particular, we use the weight function $w_2(x;c,\beta,t)$ to construct the WGGD. This weight is useful in its on

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TABLE 1. Generalizations of GGD and Submodels

λ	β	k	c	t	Distribution	pdf
–	–	–	–	–	WGGD, $w_2(x) = x^c e^{tx^\beta}$	$g_{w_2}(x;\lambda,\beta,k,c,t)$
–	–	–	–	0	WGGD, $w(x) = x^c$	$g_w(x;\lambda,\beta,k,c)$
–	–	–	0	–	WGGD, $w_1(x) = e^{tx^\beta}$	$g_{w_1}(x;\lambda,\beta,k,t)$
–	–	–	0	0	GGD	$g(x;\lambda,\beta,k)$
–	1	–	0	0	Gamma	$GAM((1/\lambda),k)$
–	1	1	0	0	Exponential	$EXP(1/\lambda)$
(1/2)	1	–	0	0	Chi-Square	$\chi^2(2k)$
–	–	1	0	0	Weibull	$WEI((1/\lambda),\beta)$
–	2	1	0	0	Rayleigh	Rayleigh$(1/\lambda)$

right, since it included weights that leads to the length-biased or size biased, and moment generating functions versions of the WGGDs. Then we derive the cdf, survival or reliability function, and some other useful distributional properties.

The pdf of the WGGD with weight function $w_2(x;c,\beta,t) = x^c e^{tx^\beta}$, for $\beta > 0$, $c > 0$ is given by

$$g_{w_2}(x;\lambda,\beta,k,c,t) = \frac{\beta(\lambda^\beta - t)^{k+\frac{c}{\beta}} x^{k\beta+c-1} e^{-(\lambda^\beta - t)x^\beta}}{\Gamma(k+\frac{c}{\beta})}, \tag{13}$$

for $\lambda^\beta > t$, $k\beta + c > 1$ and $x,\lambda,\beta,k,t,c > 0$.

The corresponding cdf is given by

$$G_{w_2}(x;\lambda,\beta,k,c,t) = \frac{\gamma((k+\frac{c}{\beta}),(\lambda^\beta - t)x^\beta)}{\Gamma(k+\frac{c}{\beta})}, \tag{14}$$

for $\lambda^\beta > t$, $k\beta + c > 1$ and $x,\lambda,\beta,k,t,c > 0$.

Also, note that the survival function is given by

$$\bar{G}_{w_2}(x;\lambda,\beta,k,c,t) = \frac{\Gamma((k+\frac{c}{\beta}),(\lambda^\beta - t)x^\beta)}{\Gamma(k+\frac{c}{\beta})}, \tag{15}$$

for $\lambda^\beta > t$, $k\beta + c > 1$ and $x,\lambda,\beta,k,t,c > 0$, where $\Gamma((k+\frac{c}{\beta}),(\lambda^\beta - t)x^\beta)$ is the upper incomplete gamma function given in Eq. (6).

Several distributions can be obtained from the WGGD with the weight function $w_2(x;c,\beta,t)$. These distributions are summarized in Table 1. The graphs of the pdfs for different values of the parameters λ,β, and k are also presented (Figures 1 and 2).

3.1. Properties of WGGD

In this section, we present the mode, the behavior of the hazard function and the reverse hazard function of the WGGD.

1. **Mode**

   Consider the pdf of the WGGD given by Eq. (13).

$$\ln[g_{w_2}(x;\lambda,\beta,k,c,t)] = \ln\frac{\beta(\lambda^\beta - t)^{k+\frac{c}{\beta}}}{\Gamma(k+\frac{c}{\beta})} + (k\beta + c - 1)\ln(x) - (\lambda^\beta - t)x^\beta. \tag{16}$$

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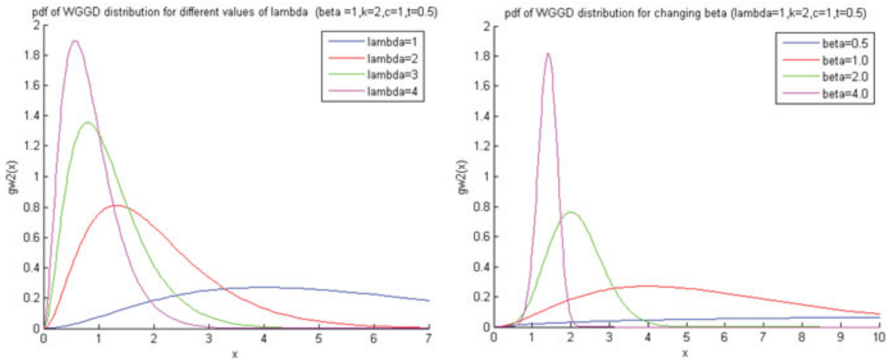

Figure 1. PDF of WGGD for Different Values of λ and β, respectively.

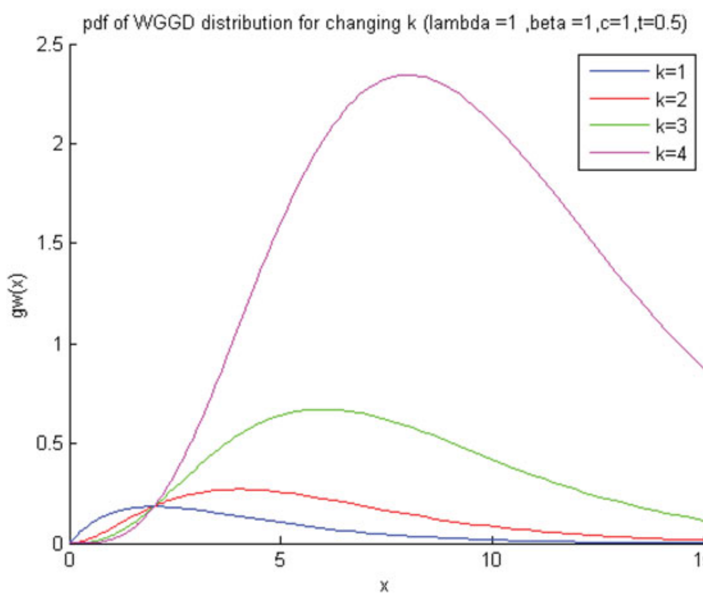

Figure 2. PDF of WGGD for different values of k.

Differentiating Eq. (16) with respect to x, we get

$$\frac{\partial}{\partial x}\ln[g_{w_2}(x;\lambda,\beta,k,c,t)] = \frac{k\beta + c - 1 - (\lambda^\beta - t)x^\beta}{x}.$$

We can find the mode by equating $(\partial/\partial x)\ln g_{w_2}(x;\lambda,\beta,k,c,t)$ to zero. Hence the mode of the WGGD is

$$x_0 = \left[\frac{k\beta + c - 1}{\beta(\lambda^\beta - t)}\right]^{\frac{1}{\beta}}, \tag{17}$$

for $k\beta + c > 1$ and $\lambda^\beta > t$. Note that

$$\frac{\partial^2}{\partial x^2}\ln[g_{w_2}(x;\lambda,\beta,k,c,t)] = \frac{-(k\beta + c - 1) - (\beta - 1)(\lambda^\beta - t)x^\beta}{x^2} < 0, \tag{18}$$

for $\forall x > 0$, $k\beta + c > 1$ and $\lambda^\beta > t$.

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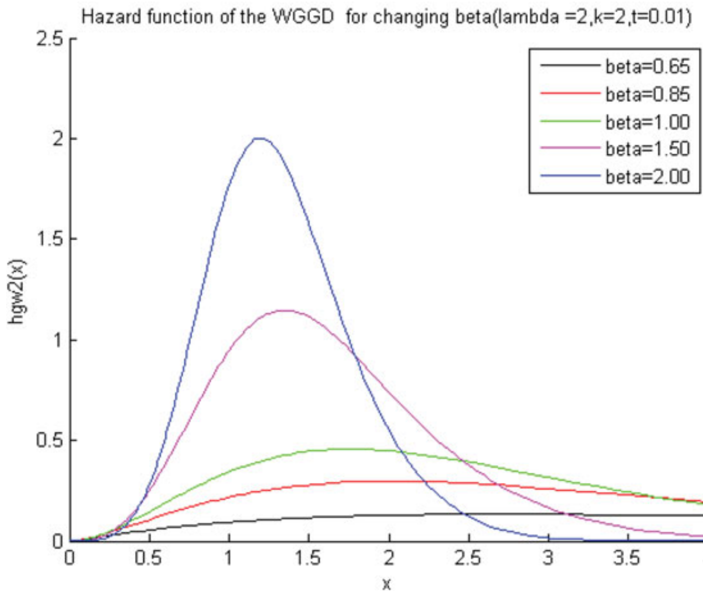

FIGURE 3. Graph of the hazard function for different values of β.

2. **Hazard Function**
The hazard function for WGGD is given by

$$h_{g_{w_2}}(x;\lambda,\beta,k,c,t) = \frac{\beta(\lambda^\beta - t)^{k+\frac{c}{\beta}} x^{(k\beta+c-1)} e^{-(\lambda^\beta - t)x^\beta}}{\Gamma(k+\frac{c}{\beta}) - \gamma\left((k+\frac{c}{\beta}),(\lambda^\beta - t)x^\beta\right)}, \quad \textbf{(19)}$$

for $x > 0$, $\lambda^\beta > t$, $k\beta + c > 1$ and $\Gamma(k + (c/\beta)) > \gamma\left((k + (c/\beta)), (\lambda^\beta - t)x^\beta\right)$.
The following theorem describes the behavior of the hazard function. The theorem is proved using Glaser's result [4].

THEOREM 3.1: 1. *If $\beta > 1$, then the hazard function is monotonically increasing.*
2. *If $0 < \beta < 1$, then the hazard function is upside down bathtub shape(UBT).*

PROOF:

(a) If $\beta > 1$, then $\eta'_{g_{w_2}}(x;\lambda,\beta,k,c,t) > 0$ for all $x > 0$. Consequently, the hazard function is monotonically increasing.

(b) Suppose $\beta < 1$ and let $x^* = ((k\beta + c - 1))/(\beta(\lambda^\beta - t)(1-\beta))$. Then $\eta'_{g_{w_2}}(x;\lambda,\beta,k,c,t) = 0$ if $x = x^*$. Also, $\eta'_{g_{w_2}}(x;\lambda,\beta,k,c,t) > 0$ if $x < x^*$, and $\eta'_{g_{w_2}}(x;\lambda,\beta,k,c,t) < 0$ if $x > x^*$.

■

The graphs of the hazard function given by Eq. (19) are displayed in the Figure 3 for different values of the parameter β. When $\beta \leq 1$, the hazard function is UBT shape. When $\beta > 1$, the hazard function is monotonically increasing. This attractive flexibility makes the WGGD hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

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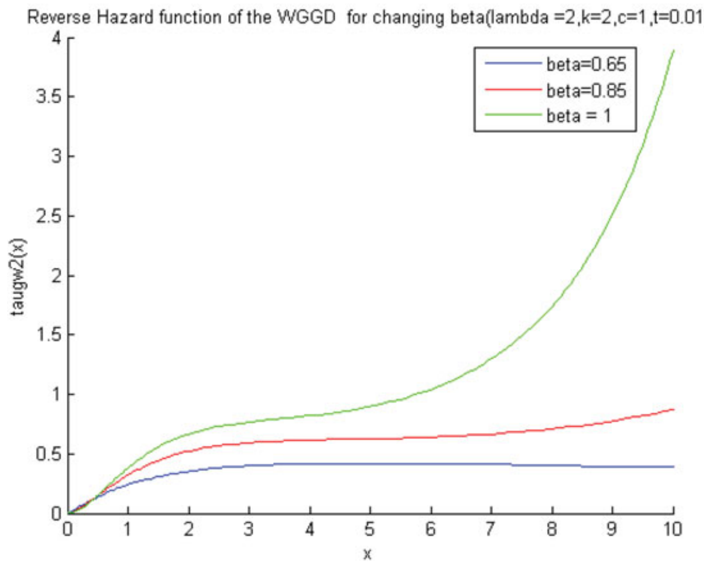

FIGURE 4. Graph of the reverse hazard function for different values of β.

3. **Reverse Hazard Function** The reverse hazard function for the WGGD when $w_2(x;c,\beta,t) = x^c e^{tx^\beta}$ is given by

$$\tau_{g_{w_2}}(x;\lambda,\beta,k,c,t) = \frac{\beta(\lambda^\beta - t)^{k+\frac{c}{\beta}} x^{k\beta+c-1} e^{-(\lambda^\beta - t)x^\beta}}{\gamma\left((k+\frac{c}{\beta}), (\lambda^\beta - t)x^\beta\right)}, \tag{20}$$

for $\lambda^\beta > t, k\beta + c > 1$ and $x > 0, \lambda > 0, \beta > 0$.

The graphs of the reverse hazard function given by Eq. (20) are displayed in Figure 4. The shape of the reverse hazard function close to the origin is similar to the pdf. It is monotonically increasing for $\beta < 1$. If $\beta \geq 1$, the reverse hazard function drastically increases as x increases.

4. MOMENTS

The rth non-central moment of WGGD with weight function $w_2(x;c,\beta,t)$ is given by

$$E_{G_{w_2}}(X^r) = \frac{\Gamma(k+\frac{r+c}{\beta})}{(\lambda^\beta - t)^{\frac{r}{\beta}} \Gamma(k+\frac{c}{\beta})}, \quad \text{for } \lambda^\beta > t. \tag{21}$$

Using Eq. (21), the mean, variance, CV, coefficient of skewness (CS), and coefficient of kurtosis (CK) are readily obtained. The mean and variance are given by

$$\mu_{G_{w_2}} = \frac{\Gamma(k+\frac{1+c}{\beta})}{(\lambda^\beta - t)^{\frac{1}{\beta}} \Gamma(k+\frac{c}{\beta})}, \quad \text{for } \lambda^\beta > t, \tag{22}$$

and

$$\sigma^2_{G_{w_2}} = \frac{\Gamma(k+\frac{c+2}{\beta})\Gamma(k+\frac{c}{\beta}) - \Gamma^2(k+\frac{c+1}{\beta})}{(\lambda^\beta - t)^{\frac{2}{\beta}} \Gamma^2(k+\frac{c}{\beta})}, \quad \text{for } \lambda^\beta > t, \tag{23}$$

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respectively. The CV is given by

$$CV = \frac{\sigma_{G_{w_2}}}{\mu_{G_{w_2}}} = \frac{\left(\Gamma(k+\frac{c+2}{\beta})\Gamma(k+\frac{c}{\beta}) - \Gamma^2(k+\frac{c+1}{\beta})\right)^{\frac{1}{2}}}{\Gamma(k+\frac{c+1}{\beta})}. \quad \textbf{(24)}$$

Let $\delta_j = \Gamma(k + ((c+j)/\beta))$, for $j \geq 0$. Then the CV can be written as $\mathrm{CV} = ((\delta_2\delta_0 - \delta_1^2)^{\frac{1}{2}})/(\delta_1)$. The CS is given by

$$\begin{aligned} \mathrm{CS} &= \frac{E(X-\mu_{G_{w_2}})^3}{(E(X-\mu_{G_{w_2}})^2)^{3/2}} \\ &= \frac{\Gamma^2(k+\frac{c}{\beta})\Gamma(k+\frac{c+3}{\beta}) - 3\Gamma(k+\frac{c}{\beta})\Gamma(k+\frac{c+1}{\beta})\Gamma(k+\frac{c+2}{\beta}) + 2\Gamma^3(k+\frac{c+1}{\beta})}{\left(\Gamma(k+\frac{c+2}{\beta})\Gamma(k+\frac{c}{\beta}) - \Gamma^2(k+\frac{c+1}{\beta})\right)^{\frac{3}{2}}} \\ &= \frac{\delta_0^2\delta_3 - 3\delta_0\delta_1\delta_2 + 2\delta_1^3}{[\delta_2\delta_0 - \delta_1^2]^{\frac{3}{2}}}. \quad \textbf{(25)} \end{aligned}$$

Similarly, the CK is given by

$$\mathrm{CK} = \frac{E(Y-\mu_{G_{w_2}})^4}{(E(X-\mu_{G_{w_2}})^2)^2} = \frac{\delta_0^3\delta_4 - 4\delta_0^2\delta_1\delta_3 + 6\delta_0\delta_1^2\delta_2 - 3\delta_1^4}{[\delta_2\delta_0 - \delta_1^2]^2}. \quad \textbf{(26)}$$

In particular, if we set $k = 1$, $c = 0$ and $t = 0$, we obtain these measures for the Weibull distribution. That is,

$$\mu_{\mathrm{W}} = \frac{\Gamma(1+\frac{1}{\beta})}{\lambda}, \quad \sigma^2_{\mathrm{W}} = \frac{\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})}{\lambda^2},$$

$$\mathrm{CV}_{\mathrm{W}} = \frac{\left(\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})\right)^{\frac{1}{2}}}{\Gamma(1+\frac{1}{\beta})},$$

$$\mathrm{CS}_{\mathrm{W}} = \frac{\Gamma(1+\frac{3}{\beta}) - 3\Gamma(1+\frac{1}{\beta})\Gamma(1+\frac{2}{\beta}) + 2\Gamma^3(1+\frac{1}{\beta})}{\left(\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})\right)^{\frac{3}{2}}},$$

and

$$\mathrm{CK}_{\mathrm{W}} = \frac{\Gamma(1+\frac{4}{\beta}) - 4\Gamma(1+\frac{1}{\beta})\Gamma(1+\frac{3}{\beta}) + 6\Gamma^2(1+\frac{1}{\beta})\Gamma(1+\frac{2}{\beta}) - 3\Gamma^4(1+\frac{1}{\beta})}{\left(\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})\right)^2}.$$

5. SOME MEASURES OF UNCERTAINTY

The concept of entropy plays a vital role in information theory. The entropy of an rv is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. In this section, we present Shannon entropy, Renyi entropy, generalized entropy, and beta entropy for the WGGD.

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5.1. Shannon Entropy

Shannon entropy $H[g_{w_2}(x;\lambda,\beta,k,c,t)]$ is defined to be

$$H\left[g_{w_2}(x;\lambda,\beta,k,c,t)\right] = E_{G_{w_2}}\left[-\log(g_{w_2}(X;\lambda,\beta,k,c,t))\right]. \qquad \textbf{(27)}$$

Using the WGGD g_{w_2}, we note that

$$\begin{aligned} E_{G_{w_2}}(\log X) &= \int_0^\infty (\log x)\frac{\beta(\lambda^\beta - t)^{k+\frac{c}{\beta}} x^{k\beta+c-1} e^{-(\lambda^\beta - t)x^\beta}}{\Gamma(k+\frac{c}{\beta})}\,dx \\ &= \frac{\Psi(k+\frac{c}{\beta}) - \log(\lambda^\beta - t)}{\beta}, \quad \text{for } \lambda^\beta > t. \end{aligned}$$

Now, we can derive Shannon Entropy. Note that

$$\begin{aligned} H[g_{w_2}(x;\lambda,\beta,k,c,t)] &= E_{G_{w_2}}(-\log(g_w(X;\lambda,\beta,k,c,t))) \\ &= -\log\left[\frac{\beta(\lambda^\beta - t)^(k+\frac{c}{\beta})}{\Gamma(k+\frac{c}{\beta})}\right] - (k\beta+c-1)E_{G_{w_2}}(\log X) + (\lambda^\beta - t)E_{G_{w_2}}(X^\beta), \end{aligned} \qquad \textbf{(28)}$$

and substituting for $E_{G_{w_2}}(\log X)$ and $E_{G_{w_2}}(X^\beta)$, we obtain

$$\begin{aligned} H[g_{w_2}(x;\lambda,\beta,k,c,t)] &= -\log\left[\frac{\beta(\lambda^\beta - t)(k+\frac{c}{\beta})}{\Gamma(k+\frac{c}{\beta})}\right] - (k\beta+c-1)\frac{\Psi(k+\frac{c}{\beta}) - \log(\lambda^\beta - t)}{\beta} \\ &\quad + (\lambda^\beta - t)\frac{\Gamma(k+\frac{\beta+c}{\beta})}{(\lambda^\beta - t)\Gamma(k+\frac{c}{\beta})}. \end{aligned} \qquad \textbf{(29)}$$

Values of Shannon entropy for WGGD with the weight functions $w(x;c) = x^c$ and $w_1(x;\beta,t) = e^{tx^\beta}$ for different values of the parameters λ, β, k and c are listed in Table 2.

5.2. Generalized Entropy

Generalized entropy is often used in econometrics. See Golan [5] for additional details. It is indexed by a single parameter α. The generalized entropy is defined to be $I_\alpha = (v_\alpha \mu^{-\alpha} - 1)/(\alpha(\alpha-1))$, where $\alpha \neq 0, 1$ and $v_\alpha = \int_0^\infty x^\alpha g_{w_2}(x;\lambda,\beta,k,c,t)\,dx = E_{G_{w_2}}(X^\alpha)$. Substituting for v_α and μ, we obtain

$$\begin{aligned} I_\alpha &= \frac{\dfrac{\Gamma(k+\frac{\alpha+c}{\beta})}{(\lambda^\beta - t)^{\frac{\alpha}{\beta}}\Gamma(k+\frac{c}{\beta})}\left[\dfrac{\Gamma(k+\frac{1+c}{\beta})}{(\lambda^\beta - t)^{\frac{1}{\beta}}\Gamma(k+\frac{c}{\beta})}\right]^{-\alpha} - 1}{\alpha(\alpha-1)} \\ &= \frac{\Gamma(k+\frac{\alpha+c}{\beta})\Gamma^{\alpha-1}(k+\frac{c}{\beta})\Gamma^{-\alpha}\left(k+\frac{1+c}{\beta}\right) - 1}{\alpha(\alpha-1)}, \quad \text{for } \alpha \neq 0, 1. \end{aligned} \qquad \textbf{(30)}$$

5.3. Renyi Entropy

Renyi entropy is an extension of Shannon entropy (Renyi [13]). Renyi entropy is defined to be $H_\alpha = (\log(\int_0^\infty g_{w_2}^\alpha(x;\lambda,\beta,k,c,t)\,dx))/(1-\alpha$, where $\alpha \neq 1)$. Renyi entropy tends to

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Table 2. Table of Shannon entropy for WGGD using $w(x;c)$ and $w_1(x;\beta,t)$.

λ	β	k	c	$H[g_w(x;\lambda,\beta,k,t)]$	$H[g_{w_1}(x;\lambda,\beta,k,t)]$
1	1	1	1	1.5772	1.6931
			3	2.0234	1.6931
			5	2.2569	1.6931
			8	2.4795	1.6931
			10	2.5869	1.6931
1	1	0.5	1	1.361	1.1303
		1		1.5772	1.6931
		2.5		1.9431	1.3834
		5		2.2569	0.0741
		8		2.4795	−1.7432
1	0.5	1	1	2.0772	2.4443
	1			1.5772	1.6931
	2.5			2.2581	0.603
	5			3.0131	−0.38
	8			3.5329	−1.1978
1	1	1	1	1.5772	1.6931
2.5				0.6609	0.3069
5				−0.0322	−0.5041
8				−0.5022	−1.0149

Shannon entropy as $\alpha \to 1$. Note that

$$\int_0^\infty g_{w_2}^{\alpha}(x;\lambda,\beta,k,c,t)\,dx = \frac{(\lambda^\beta - t)^{\frac{\alpha-1}{\beta}}\beta^{\alpha-1}\Gamma(k\alpha + \frac{c\alpha-\alpha+1}{\beta})}{\alpha^{k\alpha+\frac{c\alpha-\alpha+1}{\beta}}\Gamma^\alpha(k+\frac{c}{\beta})}. \tag{31}$$

Consequently, Renyi entropy is given by

$$H_\alpha(g_{w_2}(x;\lambda,\beta,k,c,t)) = \frac{\log\left(\frac{(\lambda^\beta-t)^{\frac{\alpha-1}{\beta}}\beta^{\alpha-1}\Gamma(k\alpha+\frac{c\alpha-\alpha+1}{\beta})}{\alpha^{k\alpha+\frac{c\alpha-\alpha+1}{\beta}}\Gamma^\alpha(k+\frac{c}{\beta})}\right)}{1-\alpha}, \quad \text{for } \alpha \neq 1. \tag{32}$$

5.4. β-Entropy

β-entropy is a one parameter generalization of the Shannon entropy. Applications of the β-entropy can be found in many physical systems (Yaghoobi, Borzadaran, and Yari [17]). β-entropy is defined by

$$H_{\tilde{\beta}}(g) = \frac{1}{\tilde{\beta}-1}\left[1 - \int_0^\infty g^{\tilde{\beta}}(x)\,dx\right], \quad \text{for } \tilde{\beta} \neq 1. \tag{33}$$

β-entropy for WGGD using the weight function $w_2(x;c,\beta,t) = x^c e^{tx^\beta}$ is given by

$$H_{\tilde{\beta}}(g_{w_2}(x;\lambda,\beta,k,c,t)) = \frac{1}{\tilde{\beta}-1}\left[1 - \frac{(\lambda^\beta - t)^{\frac{\tilde{\beta}-1}{\beta}}\beta^{\tilde{\beta}-1}\Gamma(k\tilde{\beta} + \frac{c\tilde{\beta}-\tilde{\beta}+1}{\beta})}{(1-\tilde{\beta})\tilde{\beta}^{k\tilde{\beta}+\frac{c\tilde{\beta}-\tilde{\beta}+1}{\beta}}\Gamma^{\tilde{\beta}}(k+\frac{c}{\beta})}\right], \tag{34}$$

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TABLE 3. Table of β-Entropy for WGGD using $w(x;c)$ and $w_1(x;\beta,t)$.

λ	β	k	c	$\tilde{\beta}$	$H_{\tilde{\beta}}(g_w(x;\lambda,\beta,k,c))$	$H_{\tilde{\beta}}(g_{w_1}(x;\lambda,\beta,k,t))$
1	1	1	1	0.5	3.01326	0.828427125
				1.5	1.03520	0.585786438
				2	0.75	0.5
				5	0.24808	0.141741071
				8	0.14281	0.585786438
1	1	1	1	1.5	1.03520	0.585786438
			3		1.23412	0.585786438
			5		1.31974	0.585786438
			8		1.39201	0.585786438
			10		1.42400	0.585786438
1	1	1	1	2	0.75000	0.363380228
		1			0.83023	0
		5			0.87695	−34
		8			0.90181	−171.5
1	1	1	1	5	0.24808	0.156187237
	2.5				0.11686	−0.253314137
	5				−1.55921	−3.054083332
	8				−10.24869	−5.994509322
1	1	1	1	8	0.14281	0
2.5					0.11666	−0.5
5					−3.20990	−3.5
8					−89.85714	−6.5

for $\tilde{\beta} \neq 1$. Values of β-entropy for different values of parameters λ, β, k, c, and α, using the weight functions $w(x;c)$ and $w_1(x;\beta,t)$ are listed in Table 3. Note that β-entropy of WGGD for $w(x;c)$ is increasing as the values of the parameters c and k increase. But it decrease as the values of λ, β increase.

6. CONCLUDING REMARKS

Some theoretical properties of the generalized gamma distribution and WGGDs were presented. The pdf, cdf, moments, cv, cs, and ck, failure rate function or hazard function and the reverse hazard function are presented. The behavior of the hazard function for WGGD was also established. Entropy measures including Shannon entropy, Renyi entropy, generalized entropy, and β-entropy for WGGD were also derived.

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