Robotica (1999) volume 17, pp. 283–292. Printed in the United Kingdom © 1999 Cambridge University Press

# Influence of external forces on the behaviour of redundant manipulators

Leon Žlajpah

*Department of Automatics, Biocybernetics and Robotics, Jožef Stefan Institute, Jamova 39, 1000 Ljubljana (Slovenia)*
*E-mail: leon.zlajpah@ijs.si*

(Received in Final Form: October 30, 1998)

**SUMMARY**
The paper considers the influence of external forces on the behaviour of a redundant manipulator. It is assumed that the forces can act anywhere on the body of the manipulator. First, the equivalent generalized forces in the task space and the null space are defined and several special manipulator configurations regarding the equivalent forces and torques are identified. Next, two measures for the quantification of the influence of external forces on the task space are proposed. These measures are then used in the control algorithm to minimize the influence of external forces on the task space position accuracy. The control is based on the redundancy resolution at the acceleration level and the gradient projection technique. Improvement of the position accuracy is illustrated using the simulation of a four link planar manipulator.

KEYWORDS: Redundant manipulators; External forces; Control algorithm; Position accuracy.

## 1. INTRODUCTION

To apply a manipulator to a task which involves contact with the environment requires a control of the resulting forces. For that purpose different control approaches have been proposed like hybrid position/force control[1] or impedance control[2] which have also been applied to redundant manipulators.[3–6] Usually, the contact is supposed to occur between the end-effector or the handling object and the environment. Therefore, these forces act only in the task space. Common to all these approaches is that contact forces are control variables and that the manipulator must have "enough" degrees of freedom to control the desired position and force variables. The situation becomes complicated if the force can act anywhere on the body of the manipulator especially when the manipulator has redundant degrees of freedom. Thus if an external force is acting on the body of the manipulator "below" the end-effector then the effects of this force are noticeable in the task space and in the null space. Only some authors consider such forces.[5]

As the external forces can act anywhere it is questionable if they can be measured. Hence, it has to be assumed that they are not measurable and they have to be considered as disturbances. Our goal is to design a control algorithm which minimizes the influence of the external forces on the behaviour in the task space without measuring them but the locations of the application points are assumed to be known. As even this assumption may be problematic in some practical applications we give some guidelines how to use the proposed control when the forces and application points are unknown.

The paper consists of two parts: In the first part we analyze the characteristics of these forces. First the influence of external forces in the task space and in the null space is analyzed. We identify some special configurations regarding the equivalent forces and torques.

In the second part we propose a control algorithm which decreases the task space position errors due to external forces. Hence, two measures are defined to quantify the influence of the external forces on the behaviour of the manipulator. They are used in a control algorithm to minimize the task space position accuracy. In the end, a simulation examples illustrate the effectiveness of the proposed control.

## 2. MANIPULATOR KINEMATICS

The robotic systems under study are $n$ degrees of freedom serial manipulators. We consider the redundant systems which have more degrees of freedom than needed to accomplish the task, i.e. the dimension of the joint space $n$ exceeds the dimension of the task space $m$. Let the configuration of the manipulator be represented by the vector $\boldsymbol{q}$ of $n$ joint positions, and the end-effector position (and orientation) by $m$-dimensional vector $\boldsymbol{x}$ of task positions (and orientations). The joint and task positions are related by the following expression

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q}) \tag{1}$$

where $\boldsymbol{f}$ is $m$-dimensional vector function representing the manipulator forward kinematics. Differentiating Eq. (1) we obtain the relation between velocities

$$\dot{\boldsymbol{x}} = \mathbf{J}\dot{\boldsymbol{q}} \tag{2}$$

where $\mathbf{J} = \dfrac{\partial \mathbf{f}}{\partial \boldsymbol{q}}$ is the $m \times n$ manipulator Jacobian matrix. In the case of redundant manipulators there can exist also an internal motion which does not contribute to the motion of the end-effector. Hence, the general solution of Eq. (2) can be given as follows

$$\dot{\boldsymbol{q}} = \mathbf{J}^{\#}\dot{\boldsymbol{x}} + \mathbf{N}\dot{\boldsymbol{q}} \tag{3}$$

https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press

where $\mathbf{J}^{\#}$ is the generalized inverse of $\mathbf{J}$ and $\mathbf{N}$ is $n \times n$ matrix representing the projection of $\dot{\boldsymbol{q}}$ into the null space of $\mathbf{J}$, $\mathbf{N} = (\mathbf{I} - \mathbf{J}^{\#}\mathbf{J})$. Note that the decomposition of the system depends on the particular selection of $\mathbf{J}^{\#}$ and that there is an infinite number of generalized inverses $\mathbf{J}^{\#}$. In the following we will assume that the workspace of the manipulator excludes the singular configurations. Hence $\mathbf{J}(\mathbf{q})$ will always have a full rank, $\operatorname{rank}(\mathbf{J}) = m$.

Differentiating Eq. (2), we obtain the relation between joint space and task space accelerations

$$\ddot{\boldsymbol{x}} = \mathbf{J}\ddot{\boldsymbol{q}} + \dot{\mathbf{J}}\dot{\boldsymbol{q}} \tag{4}$$

Considering also the accelerations in the null space of $\mathbf{J}$ the general solution of Eq. (4) is typically given in the form

$$\ddot{\boldsymbol{q}} = \mathbf{J}^{\#}(\ddot{\boldsymbol{x}} - \dot{\mathbf{J}}\dot{\boldsymbol{q}}) + \mathbf{N}\ddot{\boldsymbol{q}} \tag{5}$$

Eq. (3) and (5) form a basis of the inverse kinematics of a redundant manipulator.

## 3. EXTERNAL FORCES

For the redundant manipulators the static relationship between the $m$-dimensional generalized force in task space $\boldsymbol{F} = [\mathbf{f}^T \,\vdots\, \boldsymbol{m}^T]^T$, where $\boldsymbol{f}$ represents the linear forces and $\boldsymbol{m}$ the moments, and the corresponding $n$-dimensional generalized joint space force $\boldsymbol{\tau}$ is expressed as

$$\boldsymbol{\tau} = \mathbf{J}^T\boldsymbol{F} + \mathbf{N}^T\boldsymbol{\tau}_n \tag{6}$$

where $\mathbf{N}^T$ is $n \times n$ matrix representing the projection into the null space of $\mathbf{J}^{\#T}$ and $\boldsymbol{\tau}_n$ is an arbitrary $n$-dimensional vector of joint torques. In the following we denote the generalized external forces as *external forces* and the generalized joint forces as *torques*. Note that there is an infinite number of joint torques within the null space of $\mathbf{J}^{\#T}$ that could be applied to the system without affecting the forces in the task space.

Usually, only generalized forces acting at the end-effector of the manipulated object are considered. If external forces, acting at any point on the manipulator body, are of interest, we cannot use Eq. (6) directly. Suppose that $\boldsymbol{F}_0$ is acting somewhere on the link $i$ (see Figure 1). Then, the static relation between the external force $\boldsymbol{F}_0$ and joint torques $\boldsymbol{\tau}$ is

$$\boldsymbol{\tau}_{\mathrm{F}} = \mathbf{J}_{\mathrm{F}}^T\boldsymbol{F}_0 \tag{7}$$

where the Jacobian matrix $\mathbf{J}_{\mathrm{F}}$ has the form

$$\mathbf{J}_{\mathrm{F}} = [\mathbf{J}_{\mathrm{A}} \,\vdots\, \mathbf{0}_{m \times (n-i)}] \tag{8}$$

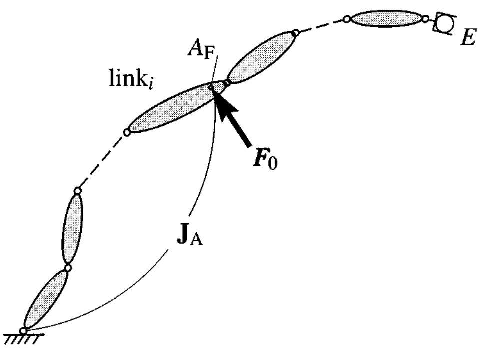


Fig. 1. An external force acting on the body of the manipulator.

and $\mathbf{J}_{\mathrm{A}}$ is a $m \times i$ Jacobian matrix associated with the manipulator between the base and the point $A_{\mathrm{F}}$ in which the force $\boldsymbol{F}_0$ is acting. Hence, Eq. (7) can be rewritten into the form

$$\boldsymbol{\tau}_{\mathrm{F}} = \begin{bmatrix} \mathbf{J}_{\mathrm{A}}^T\boldsymbol{F}_0 \\ \mathbf{0}_{(n-i)} \end{bmatrix} \tag{9}$$

It is apparent that this force does not influence directly the behaviour of the manipulator beyond the link $i$. Consequently, the static components $\tau_k$, $k > i$, are zero.

Because $\boldsymbol{F}_0$ is not acting in general at the end-effector we have to be aware of the fact that it can also affect the torques in the null space of $\mathbf{J}^{\#T}$ (which is not a case for forces acting at the end-effector). Therefore, $\boldsymbol{F}_0$ can be substituted with an equivalent force acting in the task space $\boldsymbol{F}_{\mathrm{eq}}$ and equivalent joint torques in the null space $\tau_{\mathrm{eq}}$ (see Figure 2).

**Proposition 1.** *The external force $\boldsymbol{F}_0$ can be substituted by a force acting in the task space (at the end-effector)*

$$\boldsymbol{F}_{\mathrm{eq}} = \mathbf{J}^{\#T}\mathbf{J}_{\mathrm{F}}^T\boldsymbol{F}_0 = \mathbf{J}^{\#T}\tau_{\mathrm{F}} \tag{10}$$

*and by joint torques acting in the null space of* $\mathbf{J}^{\#T}$

$$\tau_{\mathrm{eq}} = \mathbf{N}^T\mathbf{J}_{\mathrm{F}}^T\boldsymbol{F}_0 = \mathbf{N}^T\tau_{\mathrm{F}} \tag{11}$$

**Proof:** Combining Eqs. (10) and (11) results in

$$\mathbf{J}^T\boldsymbol{F}_{\mathrm{eq}} + \tau_{\mathrm{eq}} = \mathbf{J}_{\mathrm{F}}^T\boldsymbol{F}_0 = \tau_{\mathrm{F}} \tag{12}$$

■

This decomposition enables the analysis of the behaviour of the redundant manipulator separately for the task space and the null space, which has a practical significance in the control design.

### 3.1. *Special configurations*

The proposed equivalent forces and torques depend on the configuration of the manipulator. The configurations where the equivalent force $\boldsymbol{F}_{\mathrm{eq}}$ and/or equivalent torques $\tau_{\mathrm{eq}}$ equal zero or the equivalent force $\boldsymbol{F}_{\mathrm{eq}}$ equals the external force $\boldsymbol{F}_0$ are especially interesting.

**Lemma 2.** *For any generalized external force $\boldsymbol{F}_0 \neq 0$ applied to the redundant manipulator there will always exist*

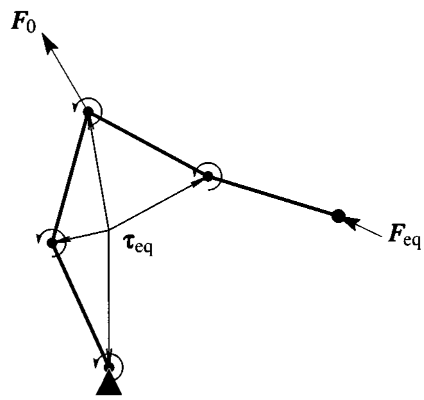


Fig. 2. Substitution of an external force $\boldsymbol{F}_0$ by an equivalent force $\boldsymbol{F}_{\mathrm{eq}}$ and equivalent joint torques $\boldsymbol{\tau}_{\mathrm{eq}}$.

https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press

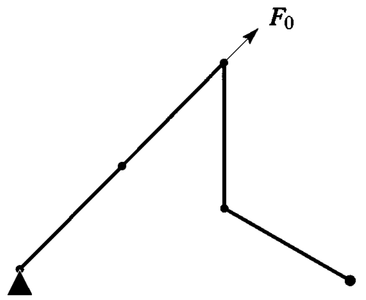


Fig. 3. Configuration of a 4R planar manipulator where $\mathbf{J}_F^T \boldsymbol{F}_0 = 0$.

*$\boldsymbol{F}_{eq} \neq 0$ or $\tau_{eq} \neq 0$ except when $\mathbf{J}_F$ is singular and $\boldsymbol{F}_0$ is in the null space of $\mathbf{J}_F^T$.*

**Proof:** If $\boldsymbol{F}_0$ is in the null space of $\mathbf{J}_F^T$ then $\mathbf{J}_F^T \boldsymbol{F}_0 = 0$. It is obvious that if the part of the manipulator between the base and the point $A_F$ is in singular configuration and $\mathbf{J}_F^T \boldsymbol{F}_0 = 0$ then the influence of $\boldsymbol{F}_0$ is transmitted through the construction to the base of the manipulator and $\boldsymbol{F}_0$ is compensated by the reaction forces in the base. Figure 3 shows the configuration of a manipulator where $\mathbf{J}_F^T \boldsymbol{F}_0 = 0$.
■

**Lemma 3.** *For any generalized external force $\boldsymbol{F}_0 = [\boldsymbol{f}_0^T \,\vdots\, \boldsymbol{m}_0^T]^T$ applied to the redundant manipulator the external force $\boldsymbol{F}_{eq}$ is equivalent to $\boldsymbol{F}_0$, $\boldsymbol{F}_{eq} = \boldsymbol{F}_0$, if:*

1. *$\mathbf{J}_B$ is singular and $\mathbf{J}_B^T \boldsymbol{F}_0 = 0$*
2. *$\boldsymbol{r}_B \times \boldsymbol{f}_0 = 0$*

*where $\mathbf{J}_B$ is the basic Jacobian matrix of the part of the manipulator between the point $A_F$ and the end-effector, and $\boldsymbol{r}_B$ a vector connecting the point $A_F$ and the end-effector, both expressed with the respect to the reference frame $\Re_0$.*

**Proof:** Suppose that the $n$ degree of freedom manipulator consists of a serial combination of two structures. The first structure, referred to as the part A, is connected to the base and ends at the point $A_F$. It has $n_A$ degrees of freedom and is described by the first $n_A$ generalized coordinates $\boldsymbol{q}_A$. The second structure, referred to as the part B, is connected to the end of the first structure. It has $n_B$ degrees of freedom, $n_B = n - n_A$, and is described by the rest of the generalised coordinates $\boldsymbol{q}_B$. Hence, the whole structure is described by the $n$ generalized coordinates $\boldsymbol{q} = [\boldsymbol{q}_A^T \,\vdots\, \boldsymbol{J}_B]^T$. The corresponding configuration is illustrated in Figure 4.

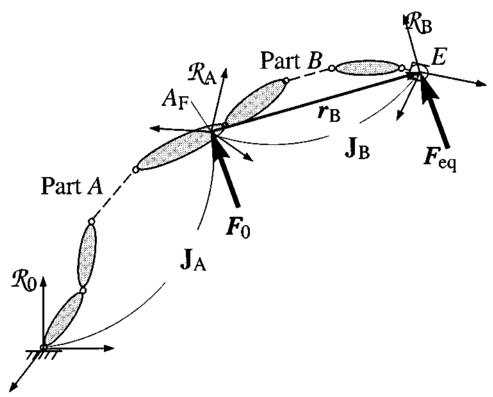


Fig. 4. Statics of a manipulator when the manipulator is virtually divided into two parts.

Let $\mathbf{J}_A$ and $\mathbf{J}_B$ be the basic Jacobian matrices associated with two parts of the manipulator and $\boldsymbol{r}_B$ a vector connecting the origins of frames $\Re_A$ (attached to the point $A_F$) and $\Re_B$ (attached to the end-effector), all expressed with the respect to the reference frame $\Re_0$. The Jacobian matrix of the whole manipulator can be expressed as[7]

$$\mathbf{J} = [\mathbf{V}\mathbf{J}_A \,\vdots\, \mathbf{J}_B] \tag{13}$$

where

$$\mathbf{V} = \begin{bmatrix} \mathbf{I} & -\hat{\boldsymbol{r}}_B \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{14}$$

and $-\hat{\boldsymbol{r}}_B$ is the cross-product operator associated with the vector $\boldsymbol{r}_B$.

Thus, the joint torques due to the external force $\boldsymbol{F}_{eq} = [\boldsymbol{f}_{eq}^T \,\vdots\, \boldsymbol{m}_{eq}^T]^T$ acting at the end-effector are

$$\begin{aligned} \tau_{F_{eq}} &= \mathbf{J}^T \boldsymbol{F}_{eq} \\ &= [\mathbf{V}\mathbf{J}_A \,\vdots\, \mathbf{J}_B]^T \boldsymbol{F}_{eq} \\ &= \begin{bmatrix} \left( \mathbf{J}_A^T \boldsymbol{F}_{eq} + \begin{bmatrix} \mathbf{0} \\ -\boldsymbol{r}_B \times \boldsymbol{f}_{eq} \end{bmatrix} \right) \\ \hline \mathbf{J}_B^T \boldsymbol{F}_{eq} \end{bmatrix} \end{aligned} \tag{15}$$

For $\boldsymbol{F}_{eq} = \boldsymbol{F}_0$ to be satisfied it is necessary that $\boldsymbol{F}_0$ applied at the end-effector results in the same joint torques $\tau_F$ as when $\boldsymbol{F}_0$ is applied at the point $A_F$. Comparing the terms in Eqs. (9) and (15) it can be seen that this is true only if $\mathbf{J}_B^T \boldsymbol{F}_0 = 0$ and $\boldsymbol{r}_B \times \boldsymbol{f}_0 = 0$. Figure 5 shows the configuration of a manipulator where $\boldsymbol{F}_{eq} = \boldsymbol{F}_0$.

## 4. OPTIMAL CONFIGURATIONS

The derivation of optimal configurations relies on the selection of the criterion. Before defining the criterion for the optimality we have to analyze the nature of external forces in detail. First of all, we assume that the occurrence of these forces is not connected to the completion of the task. Next, we consider only the external forces which are not acting at the end-effector. Thus, external forces acting on the end-effector are usually necessary to accomplish the task and are controlled by the task controller. So, the external forces are considered in the following as a disturbance to the task controller. Under such assumptions it is reasonable to define the optimal configuration as:

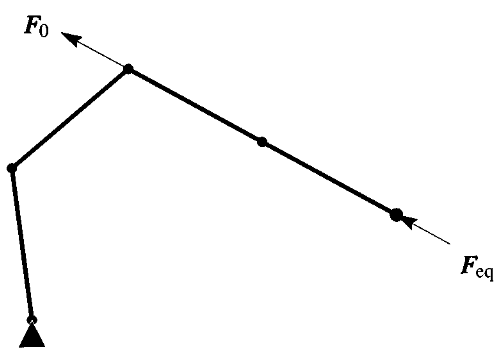


Fig. 5. Configuration of a 4R planar manipulator where $\boldsymbol{F}_{eq} = \boldsymbol{F}_0$.

https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press

**Definition 4** *The* **optimal** *configuration of a manipulator from the viewpoint of the influence of the external force* $\boldsymbol{F}_0$ *is the configuration where influence of* $\boldsymbol{F}_0$ *on the behaviour in the task space is minimal.*

This definition is a general one. Hence, depending on the particular task, the optimality has to be specified more precisely.

*4.1 Static force sensitivity measure*

Considering only the statics, the optimal configuration could be defined as the configuration where the equivalent force at the end-effector $\boldsymbol{F}_{\mathrm{eq}}$ has its minimal value. For that purpose, the following criterion can be used

$$\psi_{\mathrm{F}} = \|\mathbf{J}^{\#T}(\boldsymbol{q})\mathbf{J}_{\mathrm{F}}^{T}(\boldsymbol{q}, A_{\mathrm{F}})\tilde{\boldsymbol{F}}_0\| \tag{16}$$

where the normalized vector $\tilde{\boldsymbol{F}}_0$, $\tilde{\boldsymbol{F}}_0 = \boldsymbol{F}_0/\|\boldsymbol{F}_0\|$, represents the direction of $\boldsymbol{F}_0$. Actually, $\psi_{\mathrm{F}}$ represents the amplitude of the equivalent force caused by the normalized external force

$$\|\boldsymbol{F}_{\mathrm{eq}}\| = \psi_{\mathrm{F}}\|\boldsymbol{F}_0\| \tag{17}$$

A specific external force is considered in the above criterion. To determine the optimal configuration of the manipulator where the influence of the external force is minimal regardless of the direction of the force, the following measure is proposed.

**Proposition 5.** *A measure* $\psi$ *which quantifies the relationship between any external force acting on the manipulator at a certain point* $A_{\mathrm{F}}$ *and the maximal equivalent task space force, denoted as* **static force sensitivity** *measure, is defined as*

$$\psi(\boldsymbol{q}, A_{\mathrm{F}}) = \|\mathbf{J}^{\#T}(\boldsymbol{q})\mathbf{J}_{\mathrm{F}}^{T}(\boldsymbol{q}, A_{\mathrm{F}})\| \tag{18}$$

*where the norm* $\|\cdot\|$ *denotes the maximal singular value of the matrix.*

The following two lemmas consider the relation between $\psi_{\mathrm{F}}$ and $\psi$.

**Lemma 6.** *Measure* $\psi(\boldsymbol{q}, A_{\mathrm{F}})$ *is the supremum of* $\psi_{\mathrm{F}}(\boldsymbol{q}, A_{\mathrm{F}}, \tilde{\boldsymbol{F}}_0)$

$$\psi_{\mathrm{F}}(\boldsymbol{q}, A_{\mathrm{F}}, \tilde{\boldsymbol{F}}_0) \leq \psi(\boldsymbol{q}, A_{\mathrm{F}}) \tag{19}$$

*i.e. it represents the upper bound for* $\boldsymbol{F}_{\mathrm{eq}}$*, for the normalized external force.*

**Proof:** Using the relation $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|$ in Eq. (16) and knowing that $\|\tilde{\boldsymbol{F}}_0\| = 1$ yields

$$\psi_{\mathrm{F}} = \|\mathbf{J}^{\#T}(\boldsymbol{q})\mathbf{J}_{\mathrm{F}}^{T}(\boldsymbol{q}, A_{\mathrm{F}})\tilde{\boldsymbol{F}}_0\| \leq \|\mathbf{J}^{\#T}(\boldsymbol{q})\mathbf{J}_{\mathrm{F}}^{T}(\boldsymbol{q}, A_{\mathrm{F}})\|\|\tilde{\boldsymbol{F}}_0\| = \psi \tag{20}$$

Next combining Eqs. (17) and (19) and rewriting yields

$$\boldsymbol{F}_{\mathrm{eq}} = \psi_{\mathrm{F}}\|\boldsymbol{F}_0\| \leq \psi(\boldsymbol{q}, A_{\mathrm{F}})\|\boldsymbol{F}_0\| \tag{21}$$

and the Lemma is proved.

■

**Lemma 7.** *Let* $\boldsymbol{q}_a$ be the optimal configuration when $\psi_{\mathrm{F}}$ *is used*

$$\boldsymbol{q}_a = \arg\min_{q}(\psi_{\mathrm{F}}(\boldsymbol{q}, A_{\mathrm{F}}, \tilde{\boldsymbol{F}}_0)) \tag{22}$$

*and* $\boldsymbol{q}_b$ *be the optimal configuration for* $\psi$

$$\boldsymbol{q}_b = \arg\min_{q}(\psi(\boldsymbol{q}, A_{\mathrm{F}})) \tag{23}$$

*then* $\boldsymbol{q}_a$ *and* $\boldsymbol{q}_b$ *are not necessarily the same configurations but the following relation is always true*

$$\psi_{\mathrm{F}}(\boldsymbol{q}_a, A_{\mathrm{F}}, \tilde{\boldsymbol{F}}_0) \leq \psi(\boldsymbol{q}_b, A_{\mathrm{F}}) \tag{24}$$

**Proof:** If $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ are the same configuration, $\boldsymbol{q}_a \equiv \boldsymbol{q}_b$, then by the Lemma 6 the relation (24) is true. Next, suppose that there exists such an optimal configuration $\boldsymbol{q}_a$, $\boldsymbol{q}_a \neq \boldsymbol{q}_b$, for which the following relations would hold

$$\psi_{\mathrm{F}}(\boldsymbol{q}_a, A_{\mathrm{F}}, \tilde{\boldsymbol{F}}_0) > \psi(\boldsymbol{q}_b, A_{\mathrm{F}}) \tag{25}$$

Applying Lemma 6 to Eq. (25) yields

$$\psi_{\mathrm{F}}(\boldsymbol{q}_a, A_{\mathrm{F}}, \tilde{\boldsymbol{F}}_0) > \psi_{\mathrm{F}}(\boldsymbol{q}_b, A_{\mathrm{F}}, \tilde{\boldsymbol{F}}_0)$$

what indicates that $\boldsymbol{q}_b$ is "better" configuration than $\boldsymbol{q}_a$. Therefore, $\boldsymbol{q}_a$ cannot be an optimal configuration and the relation (24) must be always true.

■

To illustrate these relations an example using a 4 degree-of-freedom planar manipulator is given. The external force is supposed to act at the end of the second link. In Figure 6 two contours are shown. One represents the value of $\psi_{\mathrm{F}}$ versus the direction of the external force and the other is the circle with the radius $\psi$. In the case A the optimal configuration is $\boldsymbol{q}_a$ (for $\boldsymbol{F}_0 = [1, 0]^T$) and in the case B the optimal configuration is $\boldsymbol{q}_b$. It can be seen the value of $\psi$ is better in the case B, $\psi(\boldsymbol{q}_a) > \psi(\boldsymbol{q}_b)$, but for the selected external force $\boldsymbol{F}_0 = [1, 0]^T$, the value of $\psi_{\mathrm{F}}(\tilde{\boldsymbol{F}}_0)$ is better in the case A (almost zero), $\psi_{\mathrm{F}}(\boldsymbol{q}_a, \tilde{\boldsymbol{F}}_0) < \psi_{\mathrm{F}}(\boldsymbol{q}_b, \tilde{\boldsymbol{F}}_0)$.

*4.2. Selection of the generalized inverse*

Analyzing the right side of Eq. (10) and (11) we can see that the equivalent forces in the task space and the null space torques depend on the selection of the generalized inverse. Although there exist many generalized inverses $\mathbf{J}^{\#}$ only some of them are suitable for the robotic systems.

Most authors use the Moore-Penrose pseudoinverse[8] which is defined for $n > m$ as

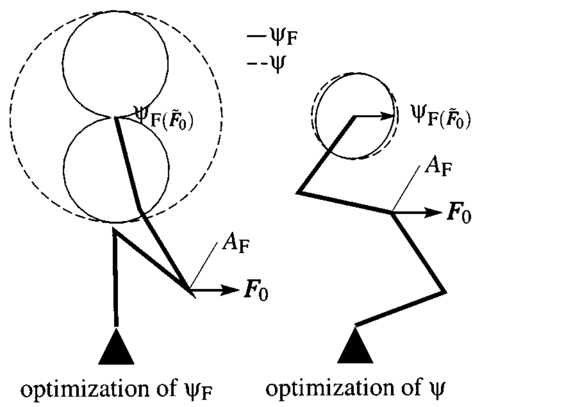


Fig. 6. Optimal configurations of a 4R planar manipulator considering the static force sensitivity measure $\psi$ for $\mathbf{J}^{\#} = \mathbf{J}^{+}$.

https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press

$$\mathbf{J}^{+}=\mathbf{J}^{T}(\mathbf{J}\mathbf{J}^{T})^{-1} \tag{26}$$

or its “weighted” counterpart[9] defined as

$$\mathbf{J}_w^{+}=\mathbf{W}^{-1}\mathbf{J}^{T}(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^{T})^{-1} \tag{27}$$

where $\mathbf{W}$ is $n\times n$ weighting matrix. A special form of $\mathbf{J}_w^{+}$ is when $\mathbf{W}=\mathbf{H}$. Khatib[7] has proved that

$$\bar{\mathbf{J}}=\mathbf{H}^{-1}\mathbf{J}^{T}(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}^{T})^{-1} \tag{28}$$

is the only pseudoinverse which is dynamically consistent, i.e. the task space acceleration $\ddot{\boldsymbol{x}}$ is not affected by any arbitrary torques $\boldsymbol{\tau}_n$ applied through the associated null space, $\bar{\mathbf{N}}^{T}\boldsymbol{\tau}_n$, $\bar{\mathbf{N}}^{T}=(\mathbf{I}-\mathbf{J}^{T}\bar{\mathbf{J}}^{T})$. Additionally, the dynamically consistent generalized inverse $\bar{\mathbf{J}}$ is the only generalized inverse which assures that an external force does not produce a null space acceleration.[10]

As well as the equivalent forces, the measures $\psi_{\mathrm{F}}$ and $\psi$ also depend on the selection of pseudoinverses. Optimal configurations of the system have been compared for $\mathbf{J}^{+}$ and $\bar{\mathbf{J}}$ used in the measure $\psi$. As the task space is defined by the end-effector position in the plane (two-dimensional space), the systems has 2 redundant degrees-of-freedom and $q_1$ and $q_2$ have been selected as independent variables. Figure 7 shows the value of $\psi$ for a 4R planar manipulator for the task space position $\boldsymbol{x}=[0;\ 2.2]^{T}$ verus $q_1$ and $q_2$ for both generalized inverses. The markers $\times$ show where $\psi$ has a minimum, $\psi=\psi_{\min}$, and the contour lines show the region where $\psi\le 1.2\psi_{\min}$. The shaded region represents the redundant space $\mathfrak{A}$ defined as a set $\{(q_1,q_2),\ \exists \underline{\boldsymbol{q}}=\boldsymbol{f}^{-1}(\boldsymbol{x})\}$. It can be seen that the values of $\psi$ are lower $\mathbf{J}^{\#}=\bar{\mathbf{J}}$ compared to the case when $\mathbf{J}^{\#}=\mathbf{J}^{+}$. In Figure 8 optimal configurations using $\psi_{\mathrm{F}}$ and $\psi$ for $\mathbf{J}^{\#}=\bar{\mathbf{J}}$ are shown. The external force is the same as in the Figure 6. As before, the value of $\psi$ is better in the case B, and the value of $\psi_{\mathrm{F}}(\tilde{\boldsymbol{F}}_0)$ is better in the case A. Note that the optimal configurations in Figure 8 are different from that on the Figure 6, and that all values of $\psi$ and $\psi_{\mathrm{F}}$ are better (lower) when $\bar{\mathbf{J}}$ is used.

### 4.3. *Dynamic force sensitivity measure*

In this section the influence of the external force on the task space motion is analyzed. Therefore, the dynamics of the manipulator has to be taken into the consideration. Assuming the manipulator consists of rigid bodies the joint space equations of motion can be written in a form

$$\boldsymbol{\tau}=\mathbf{H}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})-\boldsymbol{\tau}_{\mathrm{F}} \tag{29}$$

where $\boldsymbol{\tau}$ is $n$-dimensional vector of control torques, $\mathbf{H}$ is $n\times n$ inertia matrix, $\boldsymbol{h}$ is $n$-dimensional vector of Coriolis and centrifugal forces, $\boldsymbol{g}$ is $n$-dimensional vector of gravity forces, and vector $\boldsymbol{\tau}_{\mathrm{F}}$ represents the torques due to the external force $\boldsymbol{F}_0$ acting on the manipulator. Using Eqs. (29), (4), (6) and (7), and ignoring the coupling and gravity terms, leads to the following equation

$$\ddot{\boldsymbol{x}}=\mathbf{J}\mathbf{H}^{-1}\mathbf{J}_{\mathrm{F}}^{T}\boldsymbol{F}_0 \tag{30}$$

which describes the relation between the task space acceleration and the external force. Note that if the force is acting at the end-effector then the term $\mathbf{J}\mathbf{H}^{-1}\mathbf{J}_{\mathrm{F}}^{T}$ equals the inverse of the task space inertia matrix. Hence, we denoted the matrix $\mathbf{H}_R=(\mathbf{J}\mathbf{H}^{-1}\mathbf{J}_{\mathrm{F}}^{T})^{-1}$ as the *reduced inertia* matrix. Using $\mathbf{H}_R$ a new measure has been defined to quantify the influence of the external force on the task space accelerations.

**Proposition 8.** *A measure* $\mathfrak{S}$, *denoted as* **dynamic force sensitivity** *measure, is defined*

$$\mathfrak{S}(\boldsymbol{q},A_{\mathrm{F}})=\|\mathbf{J}(\boldsymbol{q})\mathbf{H}^{-1}(\boldsymbol{q})\mathbf{J}_{\mathrm{F}}^{T}(\boldsymbol{q},A_{\mathrm{F}})\|=\|\mathbf{H}_R\| \tag{31}$$

*where the norm* $\|\cdot\|$ *denotes the maximal singular value of the matrix.*

**Lemma 9.** *Measure* $\mathfrak{S}(\boldsymbol{q},A_{\mathrm{F}})$ *represents the upper bound for the task space acceleration* $\ddot{\mathbf{x}}$ *due to the normalized external force* $\boldsymbol{F}_0/\|\boldsymbol{F}_0\|$.

**Proof:** Using the relation $\|\mathbf{AB}\|\le\|\mathbf{A}\|\,\|\mathbf{B}\|$ in Eq. (30) yields

$$\|\ddot{\mathbf{x}}\|=\|\mathbf{J}\mathbf{H}^{-1}\mathbf{J}_{\mathrm{F}}^{T}\frac{\boldsymbol{F}_0}{\|\boldsymbol{F}_0\|}\|<\|\mathbf{J}\mathbf{H}^{-1}\mathbf{J}_{\mathrm{F}}^{T}\|=\mathfrak{S} \tag{32}$$

and the Lemma is proved.

■

Note that in Eqs. (30) and (31) there is no generalized inverse of $\mathbf{J}$ present. This points out an important fact that

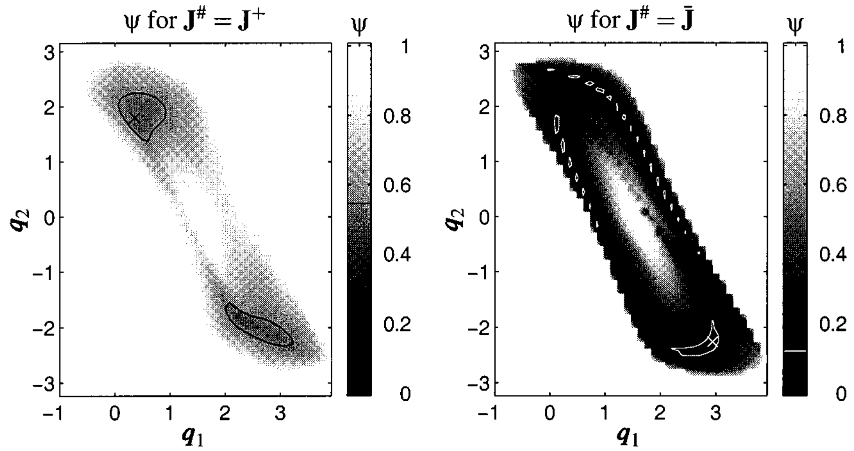


Fig. 7. The static force sensitivity measure $\psi$ for a 4R planar manipulator (the end-effector position $\boldsymbol{x}=[0,\ 2.2]^{T}$; the external force $\boldsymbol{F}_0$ is acting at the end of link 2).

https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press

the task space motion due to the external force is not directly dependent on the selection of generalized inverse of **J**. This dependency is introduced by the control algorithm, i.e. by the part controlling the null space motion. Namely, the term $\mathbf{J}\mathbf{H}^{-1}\mathbf{J}_F^T$ depends on the configuration of the manipulator which depends on the particular null space motion. So, the task space motion is indirectly dependent on the selection of $\mathbf{J}^{\#}$.

In order to compare the measures $\psi$ and $\mathfrak{S}$, we give the values of $\mathfrak{S}$ and the optimal configuration using $\mathfrak{S}$ for the same situation as before (see Figure 9). The marker $\times$ shows the minimum $\mathfrak{S}_{\min}$ and the contour lines the region where $\mathfrak{S} \leq 1.2\mathfrak{S}_{\min}$. Although the values of the measures cannot be compared directly, one can observe that distributions of measures $\psi$ and $\mathfrak{S}$ are similar, i.e. they have their minima and maxima at similar configurations.

### 4.4. *Multiple external forces*

Until now only one external force has been assumed. When more then one forces act on the manipulator at different points the question arises what the overall optimal configuration is. In general, the optimal configurations for each force are distinct. However, the range of sensitivity of the task space motion on the external force is very different with regard to the application point $A_F$. For example, if $A_F$ is at the end-effector, then the force is acting actually in the task space and the change of the sensitivity is minimal. To get better insight in this dependency Figure 10 shows the values of $\mathfrak{S}$ for $A_F$ being at the end of links. One can observe that the sensitivity increases when $A_F$ is moving toward the end-effector. Next, the range of $\mathfrak{S}$ is significantly smaller (the improvement in the task space motion characteristics is not so extensive when the configuration of the manipulator changes) when $A_F$ is near the base of the manipulator or near the end-effector. Hence, in the calculation of the overall optimal configuration it is essential to include those forces which are acting in the middle of the structure. Therefore, satisfactory results can be obtained also in situations when the locations of application points $A_F$ are not known. Nmely, the optimal configuration can be determined as the location of $A_F$ would be somewhere in the middle of the manipulator structure. In order to find out which locations of applications points should be considered in the optimization process, such an analysis can be made for any particular manipulator mechanical structure. The areas where the external forces could act on the body of the manipulator may also be bounded by the task.

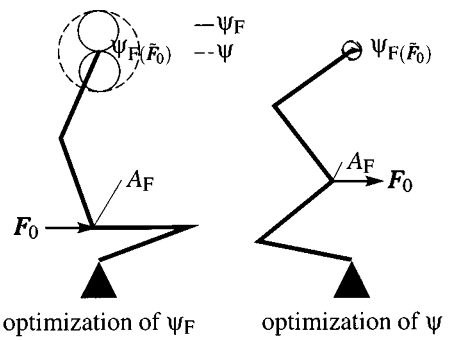


Fig. 8. Optimal configurations of a 4R planar manipulator considering the static force sensitivity measure $\psi$ for $\mathbf{J}^{\#} = \bar{\mathbf{J}}$.

## 5. CONTROL

Most tasks performed by a redundant manipulator can be divided into several subtasks with different priority. In the following it is assumed that the subtask with the highest priority, referred to as the main task, is associated with the positioning of the end-effector in the task space. The control problem we want to solve can be defined as how to move the manipulator to the optimal configuration, where the influence of external forces is minimal, by using the redundancy of the mechanism.

Hybrid position/force control[1,3,4] or impedance control[2] are usually used when a manipulator performs a task which involves the contact with the environment. Common to these control methods is that they control positions and forces and that the forces are measurable. If the forces act at the end-effector than it is easy to measure the forces by using a force/torque sensor mounted at the end of the effector. The situation is more complex if the force can act

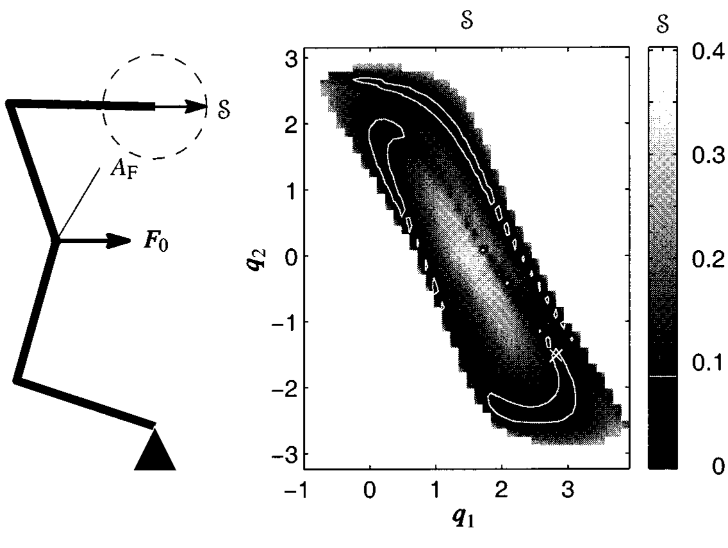


Fig. 9. The optimal configuration of a 4R planar manipulator considering the dynamic force sensitivity measure $\mathfrak{S}$.

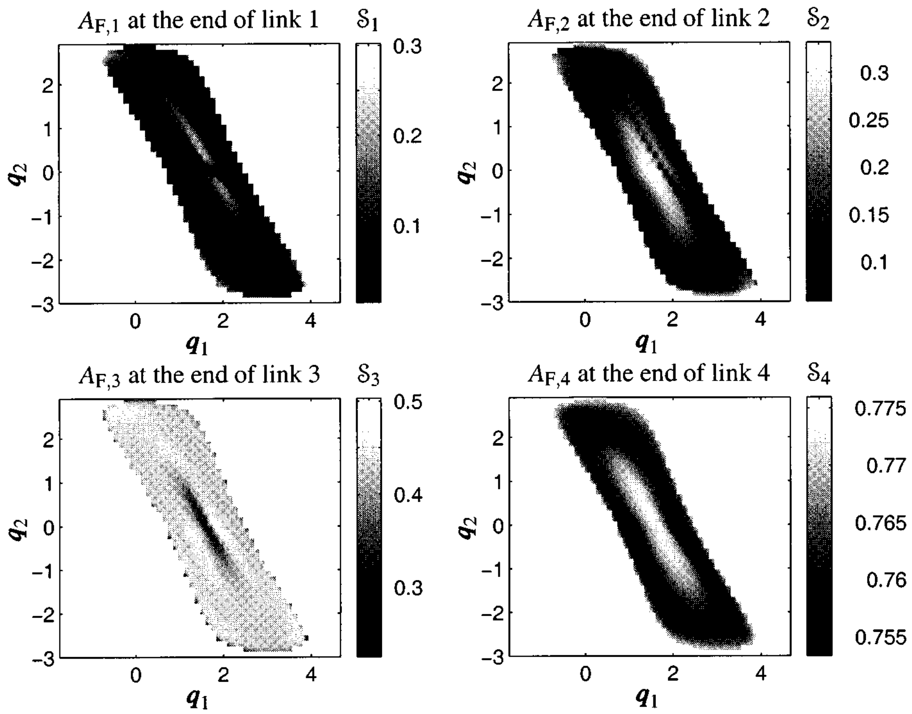


Fig. 10. The dynamic force sensitivity measure $\mathfrak{S}$ for 4R planar manipulator versus the location of $A_F$.

anywhere on the body of the manipulator. First of all, it is questionable if it is practically possible to measure forces which can act anywhere and secondly, many forces can be present at the same time and a sensor is needed for each of them. Hence, we suppose that external forces are not measurable. Consequently, they are considered as disturbances to the controller. To minimize the influence of external forces we propose to move the manipulator into a "better" configuration by using the redundant DOF and the earlier defined measures.

In the following, redundancy resolution at the acceleration level is used. A general formulation of the control law is given in the form

$$\boldsymbol{\tau} = \mathbf{H}(\mathbf{J}^{\#}(\ddot{\boldsymbol{x}}_d + \mathbf{K}_v\dot{\boldsymbol{e}} + \mathbf{K}_p\boldsymbol{e} - \dot{\mathbf{J}}\dot{\boldsymbol{q}}) + \mathbf{N}(\ddot{\boldsymbol{\varphi}} + \mathbf{K}_n(\dot{\boldsymbol{\varphi}} - \dot{\boldsymbol{q}}))) + \boldsymbol{h} + \boldsymbol{g} \tag{33}$$

where $\boldsymbol{e}$, $\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{x}$, is the tracking error, $\ddot{\boldsymbol{x}}_d$ is the desired task space acceleration, $\mathbf{K}_v$ and $\mathbf{K}_p$ are $n \times n$ constant gain matrices, and $\dot{\boldsymbol{\varphi}}$ is the desired $n$-dimensional null space velocity vector. Note that in Eq. (33) no term is present to compensate the external forces. As we focus our attention only to the influence of the external force on the task space error, a simple null space controller is used and no analysis of null space dynamics is made.

Combining Eqs. (29) and (33) yields

$$\begin{aligned}\mathbf{H}\ddot{\boldsymbol{q}} + \boldsymbol{h} + \boldsymbol{g} - \boldsymbol{\tau}_{\text{ext}} = \mathbf{H}(\mathbf{J}^{\#}(\ddot{\boldsymbol{x}}_d + \mathbf{K}_v\dot{\boldsymbol{e}} + \mathbf{K}_p\boldsymbol{e} - \dot{\mathbf{J}}\dot{\boldsymbol{q}})\\ + \mathbf{N}(\ddot{\boldsymbol{\varphi}} + \mathbf{K}_n(\dot{\boldsymbol{\varphi}} - \dot{\boldsymbol{q}}))) + \boldsymbol{h} + \boldsymbol{g}\end{aligned} \tag{34}$$

Using Eq. (5) in Eq. (34) and rearranging it yields

$$\mathbf{J}^{\#}(\ddot{\boldsymbol{x}}_d + \mathbf{K}_v\dot{\boldsymbol{e}} + \mathbf{K}_p\boldsymbol{e} - \ddot{\boldsymbol{x}}) + \mathbf{N}(-\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{\varphi}} + \mathbf{K}_n(\dot{\boldsymbol{\varphi}} - \dot{\boldsymbol{q}})) = -\mathbf{H}^{-1}\boldsymbol{\tau}_{\text{ext}} \tag{35}$$

Premultiplying Eq. (35) with $\mathbf{J}$ yields

$$\ddot{\boldsymbol{e}} + \mathbf{K}_v\dot{\boldsymbol{e}} + \mathbf{K}_p\boldsymbol{e} = -\mathbf{J}\mathbf{H}^{-1}\mathbf{J}_F^T\boldsymbol{F}_0 \tag{36}$$

since $\mathbf{J}\mathbf{J}^{\#} = \mathbf{I}$.

The last thing to do is define $\boldsymbol{\varphi}$. Suppose that $p$ is a function representing the desired performance criterion. A widely implemented technique to optimize $p$ is to select $\dot{\boldsymbol{\varphi}}$ as[11]

$$\dot{\boldsymbol{\varphi}} = \mathbf{K}\nabla p \tag{37}$$

where $\nabla p$ is the gradient of $p$ and $\mathbf{K}$ is an $n \times n$ matrix. The aim of $\mathbf{K}$ is to assure that the form

$$(\nabla p)^T\mathbf{N}\mathbf{K}\nabla p \tag{38}$$

is positive semidefinite so that the optimization converges.[12]

Let $\boldsymbol{F}_0$ be not known. One will recognize from Eq. (36) that by minimizing the $\mathfrak{S}$ we minimize also the upper bound of the steady state task position error. Hence, it is reasonable to define $\dot{\boldsymbol{\varphi}}$ as

$$\dot{\boldsymbol{\varphi}} = k_g\mathbf{K}\nabla\mathfrak{S}(A_F) \tag{39}$$

If no more than one external force is supposed to act on the manipulator, $\dot{\boldsymbol{\varphi}}$ should be defined actually as

$$\dot{\boldsymbol{\varphi}} = \mathbf{H}^{-1}\sum_i k_i\nabla\mathfrak{S}(A_{F,i}) \tag{40}$$

https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press

where the summation indicates all possible points $A_{F,i}$ where the force can act, and $k_i$ are weighting factors for particular points. As the calculation $\dot{\boldsymbol{\varphi}}$ becomes very complex if many points have to be considered, it is necessary to reduce the number of terms in Eq. (40). It could be shown that the optimal configuration is similar for points acting on the same link. Hence, it may be enough to consider only one point per link. Unfortunately, even with these simplifications the control is very complicated and time consuming.

## 6. SIMULATION EXAMPLE

To illustrate the behaviour of a manipulator a simulation example is given. The simulation has been done in MATLAB/SIMULINK using the Planar Manipulators Toolbox.[13] The Planar Manipulators Toolbox is based on a kinematic and dynamic model of a planar manipulator with revolute joints and permits simulation of manipulators with many DOF.

The example deals with the minimization of the influence of the external force on the task space position accuracy when the force is independent of $\boldsymbol{q}$. In the simulation a four-link planar manipulator with revolute joints is used. The manipulator is supposed to hold a position in the task space, $\boldsymbol{x}=[0,\ 2.2]^T m$. The initial configuration of the manipulator is $\boldsymbol{q}_0 \approx [2.2,\ -1.2,\ 1.77,\ -2.58]^T$.

The controller is based on the algorithm (33). The controller gains are $\mathbf{K}_p = 1000\mathbf{I}s^{-2}$, $\mathbf{K}_v = 80\mathbf{I}s^{-1}$, and $\mathbf{K}_n = 50\mathbf{I}s^{-1}$ ($\mathbf{I}$ is the identity matrix). The null space velocity is calculated using the gradient of the measure $\mathfrak{S}$ (Eq. (39)). In order to observe the process of optimization, the desired null space motion is started when $t=t_b$, $t_b=3.5s$. Therefore, the gain $k_g$ is selected as

$$k_g = \begin{cases} 0, & t < t_b \\ -20, & t \geq t_b \end{cases} \tag{41}$$

In the example, the influence of one external force on the manipulator is considered. The external force is acting at the end of the second link and equals

$$\boldsymbol{F}_0 = \begin{cases} 0, & t < t_a \\ [50,\ -86.6]^T \mathrm{N}, & t \geq t_a \end{cases} \tag{42}$$

where $t_a$ is the starting time of the force, $t_a = 0.2s$. The simulation results are given in Figures 11 and 12. Figure 11 presents the position error in the task space, dynamic force sensitivity measure and the configurations of the manipulators at the time $t_a$ and $t_b$, and the final configuration ($t=t_c$). One can see that the external force pushes the manipulator into the configuration where one part of the manipulator is in the singular configuration. i.e. at the edge of the space $\mathfrak{A}$. In this example this is the upper part of the manipulator. Hence, the influence of the external force is high and also the error is big (see situation at $t=t_b$). After the manipulator is moved into the optimal configuration, the influence of the external force decreases and consequently the position error is lower. The optimization progression can be viewed also in Figure 12 which is showing the values of $\mathfrak{S}$ versus $q_1$ and $q_2$ and the path $[q_1(t), q_2(t), \mathfrak{S}(t)]$. After $t=t_b$ the manipulator is moved fast from the edge of $\mathfrak{A}$ and then it is moving along the valley until the final configuration is reched. (The values of $\mathfrak{S}$ for the whole space $\mathfrak{A}$ can be seen in Figure 9). Note that the proposed control converges to the local optimum.

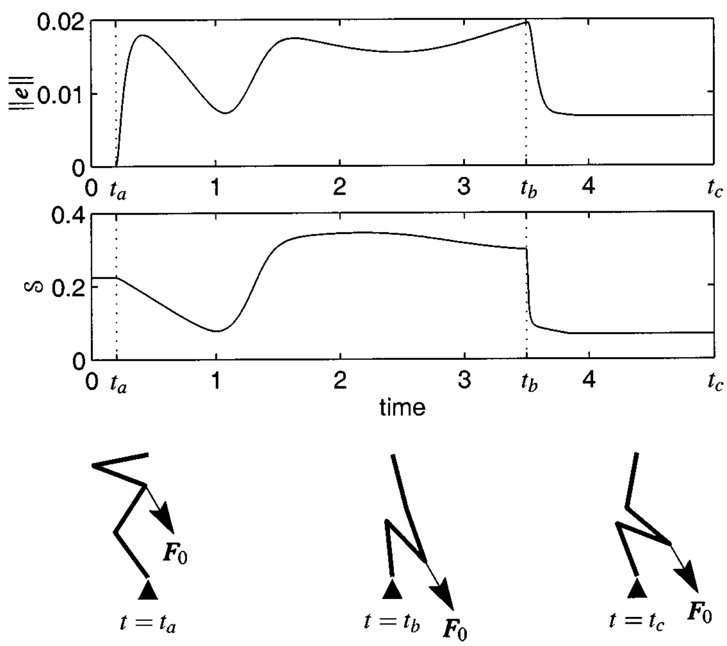


Fig. 11. The position error norm $\|\boldsymbol{e}\|$, the dynamic force sensitivity measure $\mathfrak{S}$ and manipulator configurations when $\boldsymbol{F}_0$ acts as the end of link 2.

https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press

Our studies have shown that the situation may be different in the case when the force acts in an another direction. Sometimes, when the manipulator is pushed into the configuration where the lower part is in the singular configuration and the influence of $\boldsymbol{F}_0$ is zero, moving the manipulator into another configuration increases the influence of the force and the position error is even bigger as without the optimization. Let the desired motion of the end-effector be, for example, to move along a line. The desired trajectory has a trapezoidal velocity profile ($\ddot{x} = 2\mathrm{ms}^{-2}$ and $\dot{x}_{\max} = 0.5\mathrm{ms}^{-1}$). The external force is the same as in the previous example. We have made two simulation runs. In the first run the desired null space velocity has been zero. In the second run the measure $\mathfrak{S}$ has been optimized. All other controller parameters have been equal as before. Figure 13 presents the position error in the task space and the configurations of the manipulators for both simulation runs.

In the first case, when no optimization is included in the control, the external force pushes the manipulator into a configuration where the lower part of the manipulator is almost in the singular configuration and consequently, the influence of $\boldsymbol{F}_0$ is minimal. However, the second case shows us that the maximal position error is smaller when $\mathfrak{S}$ is

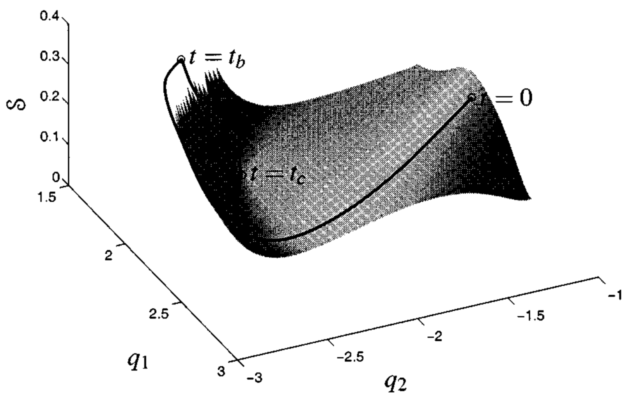


Fig. 12. The measure $\mathfrak{S}$ and the resulting path in the space $\mathfrak{A}$.

optimized during the motion. Therefore, we can conclude that it is in general reasonable to optimize one of the proposed measures when an unknown external force is acting somewhere on the body of the manipulator.

## 7. CONCLUSION

In the paper the influence of external forces on the behaviour of the manipulator was considered. The manipulator under study was redundant with $n$ degrees of freedom. The external forces acting anywhere on the body of the manipulator were analyzed. For these forces the equivalent generalized forces in the task space and in the null space were defined. Some special configurations of the manipulator where the equivalent force in the task space and/or equivalent torques in the null space equal zero or the equivalent force in the task space equals the external force were identified.

Next, we define what the optimal configuration regarding the influence of external forces is. To quantify the influence of external forces on the behaviour of the manipulator in the task space two measures are proposed. The static force sensitivity measure $\psi$ is related to the maximal magnitude of the equivalent task space force considering the application point of the external force. The dynamic force sensitivity measure $\mathfrak{S}$ quantifies the relation between the external force and the task space acceleration. The measure $\psi$ depends on the selection of the generalized inverse, but the measure $\mathfrak{S}$ does not. Analyzing the measure $\psi$ we can see that the values of $\psi$ when $\bar{\mathbf{J}}$ is selected are lower compared to the values of $\psi$ when $\mathbf{J}^+$ is selected. This means that the equivalent forces in the task space are reduced if $\bar{\mathbf{J}}$ is used instead of $\mathbf{J}^+$. Consequently, the equivalent null space torques are increased.

One of the goals in the control design is to minimize the influence of external forces. As it is questionable if the external forces can be measured they were considered as disturbances and were not included in the control algorithm. Hence, in order to reach the optimal configurations the proposed measures can be used in the control. We have

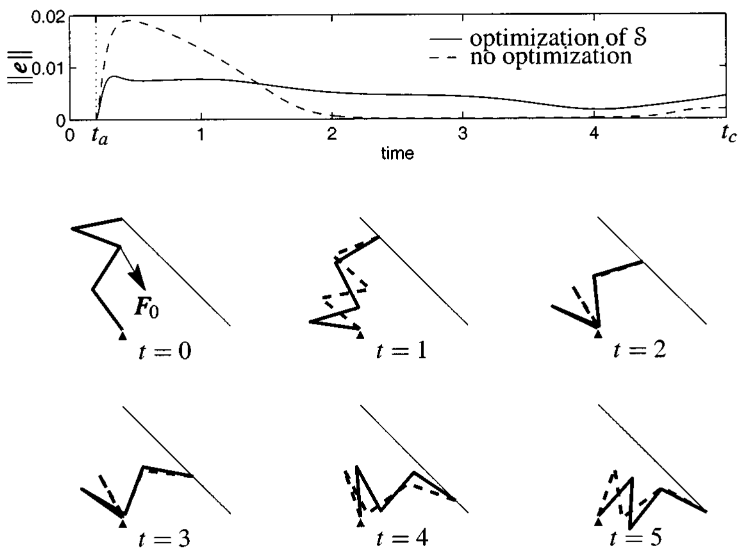


Fig. 13. Tracking a line: the position error norm $\|\boldsymbol{e}\|$ and manipulator configurations when $\boldsymbol{F}_0$ acts at the end of link 2.

https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press

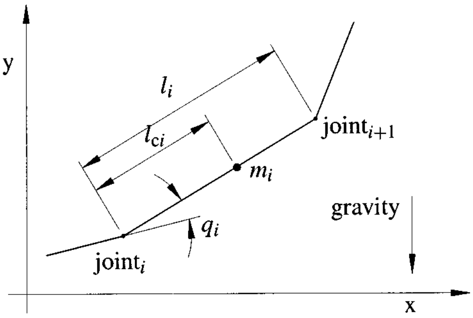


Fig. 14. The structure of $n$-DOF planar manipulator.

applied the redundancy resolution at the acceleration level and gradient projection technique. The simulation examples using $\mathfrak{S}$ show that it is possible to decrease the influence of external forces on the task space position error by using the proposed control. The degree of improvement depends on the number of redundant DOF, on the kinematical structure of the manipulator and on the location of the application point of the external force. Another important conclusion is that if the task controller has the form (33) then the task space position error due to external forces does not depend on the selection of generalized inverse in the control algorithm.

## APPENDIX A

In all the examples kinematic or dynamic models of $n$ degree-of-freedom planar manipulators with revolute joints are used. The manipulator is supposed to move in the vertical plane $x$-$y$ as shown in Figure 14. The task coordinates $\boldsymbol{x}$ are the positions in $x$-$y$ plane, $\boldsymbol{x} = [x, y]^T$. All the links are equal and are modelled as rods with the following parameters:

| | |
|---|---|
| link lengths | $l = 1$m |
| link masses | $m = 1$kg |
| link centre of mass | $l_c = 0.5$m |
| link inertias | $I = m\,l^2/12$ |
| joint viscose friction coef. | $B_v = 5$Nms |

There is additional load mass at the end of the last link, $m_l = 1$kg. The manipulator models are described in detail in.[13,14]

## References


1. M.H. Raibert and J.J. Craig, "Hybrid position/force control of manipulators" *Trans. of ASME J. of Dynamic Systems, Measurement, and Control* **102**, 126–133 (1981).
2. N. Hogan, "Impedance control: An approach to manipulation: Part 1: Thoery, part 2: Implementationm, part 3: Applications" *Trans. of ASME J. of Dynamic Systems, Measurement, and Control* **107**, 1–24 (1985).
3. O. Khatib, "A unified approach for motion and force control of robot manipulators: the operational space formulation" *IEEE Trans. on Robotics and Automation* **3**(1). 43–53 (1987).
4. Tsuneo Yoshikawa, "Dynamic hybrid position/force control of robot manipulators description of hand constraints and calculation of joint driving" *IEEE Trans. on Robotics and Automation* **3**(5), 386–392 (1987).
5. C. Woernle, "Nonlinear Control of Constrained Redundant Manipulators" **In:** (J. Angeles et al., editors) *Computational Kinematics* (Kluwer Academic Publishers, 1993) pp. 119–128.
6. J. Park, W. Chung and Y. Youm, "Design of Compliant Motion Controllers for Kinematically Redundant Manipulators" *Proc. IEEE Conf. Robotics and Automation* (1996) pp. 3538–3544.
7. O. Khatib, "The Impact of Redundancy on the Dynamic Performance of Robots" *Laboratory Robotics and Automation* **8**(1), 37–48 (1996).
8. C.A. Klein and C.H. Huang, Review of pseudoinverse control for use with kinematically redundant manipulators. *IEEE Trans. on Systems, Man, Cybernetics* SMC-**13**(3), 245–250 (1983).
9. John M. Hollerbach and Ki C. Suh, "Redundancy resolution of manipulators through torque optimization" *IEEE Trans. on Robotics and Automation*, **RA-3**(4), 308–316 (1987).
10. R. Featherstone and O. Khatib, "Load Independence of the Dynamically Consistent Inverse of the Jacobian Matrix" *Int. J. Robotic Research* **16**(2), 168–170 (1997).
11. D.N. Nenchev, "Redundancy resolution through local optimization: A review" *J. Robotic Systems* **6**(6), 769–798 (1989).
12. Tsuneo Yoshikawa, *Foundations of Robotics: Analysis and Control* (MIT Press, Cambridge, Mass., 1990).
13. L. Žlajpah, "Planar Manipulators Toolbox: User's Guide." *Technical Report DP 7791* (Jožef Stefan Institute, 1997; URL: http://www2.ijs.sii/~L./planman.html).
14. L. Žlajpah, "Simulation of $n$-r planar manipulators" *Simulation Practice and Theory* **6**(3), 305–321 (1998).


https://doi.org/10.1017/S0263574799001423 Published online by Cambridge University Press