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HOMOGENEOUS SASAKI AND VAISMAN MANIFOLDS OF UNIMODULAR LIE GROUPS

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Abstract. A Vaisman manifold is a special kind of locally conformally Kähler manifold, which is closely related to a Sasaki manifold. In this paper, we show a basic structure theorem of simply connected homogeneous Sasaki and Vaisman manifolds of unimodular Lie groups, up to holomorphic isometry. For the case of unimodular Lie groups, we obtain a complete classification of simply connected Sasaki and Vaisman unimodular Lie groups, up to modification.

Introduction

We recall that a *locally conformally Kähler manifold*, or shortly an *l.c.K. manifold*, is a Hermitian manifold (M, g, J), where g is a Hermitian metric with complex structure J whose associated fundamental 2-form Ω satisfies the condition

$$d\Omega = \Omega \wedge \theta \tag{$*$}$$

for a closed 1-form θ, called the *Lee form*. We may also define it as a *locally conformally symplectic manifold* with compatible complex structure (M, Ω, J), where Ω is a nondegenerate 2-form which satisfies $(*)$ for a closed 1-form θ and J is an integrable complex structure compatible with Ω. M is of *Vaisman type* if the Lee form θ is parallel with respect to g or, equivalently, if the *Lee field* ξ, the dual vector field of θ by the metric g, is Killing.

A homogeneous l.c.K. manifold (M, g, J) is a homogeneous Hermitian manifold whose associated fundamental form Ω satisfies the above condition $(*)$; in particular, the Lee form θ is also invariant. We can express M, if necessary, as G/H in an effective form, where G is a connected Lie group of automorphisms of (M, g, J) and H is a closed subgroup of G which does not contain any normal subgroup of G.

Recall that a connected Lie group G is *unimodular* if it admits a bi-invariant Haar measure or, equivalently, if its adjoint representation $\mathrm{ad}(X)$ in the Lie algebra $\mathfrak{g}$ has trace zero for any $X \in \mathfrak{g}$. Any compact, semisimple, nilpotent, reductive Lie groups, and Lie groups with a uniform lattice are unimodular. Note that we have obtained in the paper [8] a complete classification of four-dimensional unimodular Lie groups with and without lattices.

A homogeneous l.c.K. manifold of a compact Lie group is nothing but a compact homogeneous l.c.K. manifold; we have already shown in [9] a holomorphic structure theorem asserting that it is a holomorphic fiber bundle over a flag manifold with fiber a one-dimensional complex torus, and a metric structure theorem asserting that all of them are of Vaisman type. Note that we have an extended version of the above metric theorem for homogeneous l.c.K. manifolds in [1], showing a sufficient condition for being of Vaisman type; that is, the normalizer of the isotropy subgroup H in G is compact, while showing

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an example of a non-Vaisman l.c.K. structure on a reductive Lie group. For the four-dimensional case, we have shown in [8] that a Hopf manifold of homogeneous type is the only compact homogeneous l.c.K. manifold.

We recall that a *contact metric structure* $\{\phi, \eta, \widetilde{J}, g\}$ on M^{2n+1} is a contact structure $\phi, \phi \wedge (d\phi)^n \neq 0$ with the *Reeb field* $\eta, i(\eta)\phi = 1,\ i(\eta)\, d\phi = 0$, a $(1, 1)$-tensor $\widetilde{J}, \widetilde{J}^2 = -I + \phi \otimes \eta$, and a Riemannian metric g, satisfying $g(X, Y) = \phi(X)\phi(Y) + d\phi(X, \widetilde{J}Y)$. A *Sasaki structure* on M^{2n+1} is a contact metric structure $\{\phi, \eta, \widetilde{J}, g\}$ satisfying $\mathcal{L}_\eta g = 0$ (that is η is a Killing field) and the integrability of $J = \widetilde{J}|\mathcal{D}$, where $\mathcal{D} = \ker \phi$ is a CR structure.

An automorphism of a Sasaki manifold M is a diffeomorphism Ψ which satisfies

$$\Psi^*\phi = \phi, \qquad J\Psi_* = \Psi_* J.$$

Note that the automorphism group of a Sasaki manifold is a closed Lie subgroup of the isometry group of M. M is a *homogeneous Sasaki manifold* if a connected Lie group G of automorphisms acts transitively on M, that is, $M = G/H$ with isotropy subgroup H of G.

A *Sasaki (Vaisman) Lie group* G is a homogeneous Sasaki (Vaisman) manifold with trivial isotropy subgroup. We can define and study Sasaki (Vaisman) structures on the Lie algebra $\mathfrak{g}$ of G, which correspond uniquely to Sasaki (Vaisman) structures on G. For l.c.K. structure on $\mathfrak{g}$, we need to only consider the structure (g, J) or (Ω, J) on $\mathfrak{g}$ satisfying $(*)$, where g is a Riemannian metric and Ω is a nondegenerate 2-form on $\mathfrak{g}$ compatible with J. Since the Lee form θ is closed, the Vaisman condition is just

$$g([\xi, X], Y) + g(X, [\xi, Y]) = 0;$$

that is, ξ is Killing.

A homogeneous Hermitian or Sasaki manifold may have different coset expressions G/H and G'/H'. As a key strategy of proving a structure theorem of homogeneous l.c.K. or Sasaki manifolds G/H of a unimodular Lie group G, up to holomorphic isometry, we apply a *modification* of G/H into G'/H' (see Section 1 for definition), which preserves holomorphic isometry and unimodularity. For Hermitian or Sasaki Lie groups, we see that modification is an equivalence relation, which preserves Hermitian or Sasaki structures respectively and unimodularity.

As main results of the paper, we classify unimodular Sasaki and Vaisman Lie groups, up to modification (Theorem 2.1). More generally, we show a structure theorem for simply connected homogeneous Sasaki and Vaisman manifolds of unimodular Lie groups, up to holomorphic isometry (Theorem 4.1).

THEOREM 2.1. *A simply connected Sasaki unimodular Lie group is isomorphic to N, $\mathrm{SU}(2)$ or $\widetilde{\mathrm{SL}}(2, \mathbf{R})$, up to modification. Accordingly, a simply connected Vaisman unimodular Lie group is isomorphic to one of the following, up to modification:*

$$\mathbf{R} \times N, \qquad \mathbf{R} \times \mathrm{SU}(2), \qquad \mathbf{R} \times \widetilde{\mathrm{SL}}(2, \mathbf{R}),$$

where N is a real Heisenberg Lie group and $\widetilde{\mathrm{SL}}(2, \mathbf{R})$ is the universal covering Lie group of $\mathrm{SL}(2, \mathbf{R})$.

THEOREM 4.1. *A simply connected homogeneous Vaisman manifold M of a unimodular Lie group is holomorphically isometric to $M' = \mathbf{R} \times M_1$ with canonical Vaisman structure, where M_1 is a simply connected homogeneous Sasaki manifold of unimodular Lie group,* *which is a quantization of a simply connected homogeneous Kähler manifold* M_2 *of a reductive Lie group. As a Hermitian manifold* M *is a holomorphic principal bundle over a simply connected homogeneous Kähler manifold* M_2 *with fiber* $\mathbf{C}^1$ *or* $\mathbf{C}^*$.

In the above statement, we mean by a *quantization* of a homogeneous Kähler manifold M_2, a principal bundle M_1 over M_2 with fiber $\mathbf{R}$ or S^1 satisfying $d\psi = \omega$ for a contact form ψ on M_1 and the Kähler form ω on M_2.

A basic idea of the proofs is to show first that, up to modifications, a simply connected homogeneous Vaisman manifold of unimodular Lie group can be assumed to have the form $M = G/H$, where G is a simply connected unimodular Lie group of the form $G = \mathbf{R} \times G_1$ and H is a connected compact subgroup of G_1 with $\dim Z(G_1) = 1$, where $Z(G_1)$ denotes the center of G_1; combining with our previous results in [9], [1] yields the following structure theorem.

Let $\mathfrak{g}, \mathfrak{h}$ *be the Lie algebras of* G, H, *respectively. Then the pair* $\{\mathfrak{g}, \mathfrak{h}\}$ *is of the following form*:

$$\mathfrak{g} = \mathbf{R} \times \mathfrak{g}_1,$$

where $\mathfrak{g}_1 = \ker\theta \supset \mathfrak{h}$, *and* $\mathfrak{g}_1$ *can be expressed as a central extension of* $\mathfrak{g}_2$ *by* $\mathbf{R}$:

$$0 \to \mathbf{R} \to \mathfrak{g}_1 \to \mathfrak{g}_2 \to 0.$$

The Lee field ξ *and the Reeb field* $\eta = J\xi$ *generate* $Z(\mathfrak{g})$, *where* $Z(\mathfrak{g})$ *denotes the center of* $\mathfrak{g}$; *the l.c.K. form* Ω *can be written as*

$$\Omega = -\theta \wedge \psi + d\psi,$$

where ψ *is the Reeb form defining a contact form on the homogeneous Sasaki manifold* $M_1 = G_1/H$. *Let* $\mathfrak{k} = \pi(\mathfrak{h})$ *for the projection* $\pi : \mathfrak{g}_1 \to \mathfrak{g}_2$. *Then the pair* $\{\mathfrak{g}_2, \mathfrak{k}\}$ *defines a homogeneous Kähler manifold* $M_2 = G_2/K$ *with the Kähler form* $\omega = d\psi|\mathfrak{g}_2$, *where* G_1 *and* K *are the Lie groups corresponding to* $\mathfrak{g}_1$ *and* $\mathfrak{k}$, *respectively.*

We further observe, applying some basic results from the field of homogeneous Kähler manifolds, that the homogeneous Kähler manifold M_2 associated with $\{\mathfrak{g}_2, \mathfrak{k}\}$ is of reductive type. Hence, we can reduce the classification problem of homogeneous Vaisman manifolds of unimodular type to that of homogeneous Sasaki manifolds of the same type, which are quantizations of homogeneous Kähler manifolds of a reductive Lie group. We already know that a simply connected homogeneous Kähler manifold of a reductive Lie group is a Kählerian product of $\mathbf{C}^k$ and a homogeneous Kähler manifold of semisimple Lie group (which has the structure of a holomorphic fiber bundle over a symmetric domain with fiber a flag manifold).

Conversely, starting from a simply connected homogeneous Kähler manifold M_2 of a reductive Lie group, we may construct its quantization which will be a simply connected homogeneous Sasaki manifold M_1 and then take a product with $\mathbf{R}$, making it a simply connected homogeneous Vaisman manifold M of unimodular type. Here the quantization must be the one induced from a central extension of a Kähler algebra $(\mathfrak{g}_2, \mathfrak{k})$ of reductive Lie algebra as stated above. We assert, in general, that a simply connected homogeneous Kähler manifold M_2 of a reductive Lie group is $\mathbf{R}$-quantizable to a simply connected homogeneous Sasaki manifold M_1 if and only if M_2 is a product of $\mathbf{C}^k$ and a symmetric domain, which is exactly the case when M_2 contains no flag manifolds; M_2 is S^1-quantizable in all other cases.

The paper is organized as follows. In Section 1, we review some basic terminologies and results in the field of homogeneous manifolds; in particular, we discuss *modification*, which was a key strategy in proving a structure theorem of homogeneous Kähler manifolds (Fundamental Conjecture of Gindikin and Vinberg) [5], in a slightly more general setting. As an important observation, we see that modification in the category of unimodular Lie groups (Lie algebras) is an equivalence relation. In Section 2, we discuss Sasaki and Vaisman Lie algebras (Lie groups) and prove Theorem 2.1. In Section 3, we provide some examples of Vaisman and non-Vaisman l.c.K. Lie groups. In Section 4, we discuss homogeneous Sasaki and Vaisman manifolds and prove Theorem 4.1; we also prove, applying some results in the field of homogeneous Kähler manifolds, a more detailed structure theorem of homogeneous Sasaki manifolds of unimodular Lie groups (Theorem 4.2).

§1. Preliminaries

Let $M = G/H$ be a homogeneous space of a connected Lie group G with closed subgroup H. Then the tangent bundle of M is given as a G-bundle $G \times_H \mathfrak{g}/\mathfrak{h}$ over $M = G/H$ with fiber $\mathfrak{g}/\mathfrak{h}$, where the action of H on the fiber is given by $\mathrm{Ad}(x)$ $(x \in H)$. A vector field on M is a section of this bundle; a p-form on M is a section of the G-bundle $G \times_H \wedge^p(\mathfrak{g}/\mathfrak{h})^*$, where the action of H on the fiber is given by $\mathrm{Ad}(x)^*$ $(x \in H)$. A vector field (respectively p-form), which is invariant by the left action of G, is canonically identified with an element of $(\mathfrak{g}/\mathfrak{h})^H$ (respectively $(\wedge^p(\mathfrak{g}/\mathfrak{h})^*)^H$), which is the set of elements of $\mathfrak{g}/\mathfrak{h}$ (respectively $\wedge^p(\mathfrak{g}/\mathfrak{h})^*$) invariant by the adjoint action of H. An invariant complex structure J on M is likewise considered as an element J of $\mathrm{Aut}(\mathfrak{g}/\mathfrak{h})$ such that $J^2 = -1$ and $\mathrm{Ad}(x)J = J\,\mathrm{Ad}(x)$ $(x \in H)$. Note that we may also consider an invariant p-form as an element of $\wedge^p\mathfrak{g}^*$ vanishing on $\mathfrak{h}$ and invariant by the action $\mathrm{Ad}(x)^*$ $(x \in H)$ and an invariant complex structure as an element J of $\mathrm{End}(\mathfrak{g})$ such that $J^2 = -1 \pmod{\mathfrak{h}}$ and $J\mathfrak{h} \subset \mathfrak{h}$.

We next define and discuss *modification* in the category of homogeneous Hermitian manifolds and Lie groups. Let $\mathfrak{g}$ be a Lie algebra with Hermitian structure (g, J) and $\mathrm{Der}(\mathfrak{g})$ the derivation algebra of $\mathfrak{g}$, which is a Lie subalgebra of $\mathrm{End}(\mathfrak{g})$. Let $\mathfrak{k}$ be a subalgebra of $\mathrm{Der}(\mathfrak{g})$ consisting of skew-symmetric derivations σ compatible with J:

$$g(\sigma(X), Y) + g(X, \sigma(Y)) = 0, \qquad J\sigma = \sigma J \tag{1.1}$$

for any $X, Y \in \mathfrak{g}$. We define the Lie algebra $\hat{\mathfrak{g}}$ by setting

$$\hat{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{k},$$

on which the new Lie brackets are defined by

$$[(X, \sigma), (Y, \sigma')] = ([X, Y] + \sigma(Y) - \sigma'(X), [\sigma, \sigma']).$$

We extend the metric g and the complex structure J to $\hat{\mathfrak{g}}$, setting $\hat{g}(\hat{\mathfrak{g}}, \mathfrak{k}) = 0$ and $\hat{J}(\mathfrak{k}) = 0$. We have a *modification* $\bar{\mathfrak{g}}$ of $\mathfrak{g}$:

$$\bar{\mathfrak{g}} = \hat{\mathfrak{g}}/\mathfrak{k},$$

which is isomorphic to $\mathfrak{g}$ as Hermitian vector space. Let G (resp. $\hat{G}$) be the simply connected Lie group with Lie algebra $\mathfrak{g}$ (resp. $\hat{\mathfrak{g}}$) and K a compact subgroup of $\hat{G}$ with Lie algebra $\mathfrak{k}$.

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Then, G is isomorphic to $\hat{G}/K$ as a homogeneous Hermitian manifold. It should be noted that the unimodularity is preserved by the modification since it is a skew-symmetric operation (1.1).

Any subgroup G' of $\hat{G}$ canonically acts on G; the action is simply transitive if and only if the Lie algebra $\mathfrak{g}'$ of G' is of the form

$$\mathfrak{g}' = \{(X, \phi(X)) \in \hat{\mathfrak{g}} \mid X \in \mathfrak{g}\},$$

where ϕ is a linear map from $\mathfrak{g}$ to $\mathfrak{k}$. The Lie bracket on $\mathfrak{g}'$ is defined by

$$[(X, \phi(X)), (Y, \phi(Y))]' = ([X, Y] + \phi(X)(Y) - \phi(Y)(X), [\phi(X), \phi(Y)]).$$

In case $\mathfrak{k}$ is abelian, the projection of $\hat{\mathfrak{g}}$ onto $\bar{\mathfrak{g}} = \hat{\mathfrak{g}}/\mathfrak{k}$ maps $\mathfrak{g}'$ isomorphically (as a vector space) onto $\bar{\mathfrak{g}}$, defining a Lie algebra structure on $\bar{\mathfrak{g}}$,

$$[X, Y]^{-} = [X, Y] + \phi(X)(Y) - \phi(Y)(X) \tag{1.2}$$

for $X, Y \in \bar{\mathfrak{g}}$. The Lie group $\bar{G}$ with Lie algebra $\bar{\mathfrak{g}}$ is called a *modification* of the Lie group G. Note that $\bar{G}$ preserves the original Hermitian structure on G.

We consider the set $\mathfrak{L}$ of linear maps $\phi : \mathfrak{g} \to \mathfrak{k}$ satisfying the condition $\phi([\mathfrak{g}, \mathfrak{g}]) = 0$, $\phi(\sigma(X)) = 0$ for any $X \in \mathfrak{g}$ and $\sigma \in \mathfrak{k}$. Since $\mathfrak{k}$ is abelian, $\mathfrak{L}$ may also be considered as the set of Lie algebra homomorphisms $\phi : \mathfrak{g} \to \mathfrak{k}$ satisfying the condition

$$\phi(\sigma(X)) = 0 \tag{1.3}$$

for any $X \in \mathfrak{g}$ and $\sigma \in \mathfrak{k}$. In particular, we have $\phi_1(\phi_2(X)Y) = 0$ for any X, $Y \in \mathfrak{g}$ and $\phi_1, \phi_2 \in \mathfrak{L}$. It is easy to see that $\mathfrak{L}$ is a linear vector space, and any element $\phi(X)$ ($X \in \bar{\mathfrak{g}}$) defines a skew-symmetric derivation compatible with J (condition (1.1)) with respect to the new Lie bracket (1.2). In particular, the modification of $\bar{\mathfrak{g}}$ by $-\phi$ defines the original Lie algebra $\mathfrak{g}$; the composite of two modifications ϕ_1, ϕ_2 is given by $\phi_1 + \phi_2$. We also see that the modification is an equivalence relation in the set of Hermitian Lie algebras (groups).

EXAMPLE 1.1. Let $\mathfrak{g}'$ be a Lie algebra with a basis $\{X, Y, Z, W\}$ for which the bracket multiplication is defined by

$$[X, Y] = -Z, \qquad [W, X] = -Y, \qquad [W, Y] = X,$$

and other brackets vanish. A complex structure J on $\mathfrak{g}'$ is defined by

$$JX = -Y, \qquad JY = X, \qquad JZ = -W, \qquad JW = Z. \tag{1.4}$$

A Hermitian metric g is defined such that X, Y, Z, W is an orthogonal basis. Let $\mathfrak{n}$ be the Heisenberg Lie algebra with a basis $\{X, Y, Z\}$ for which the bracket multiplication is defined by

$$[X, Y] = -Z,$$

and other brackets vanish. We see that $\bar{\mathfrak{g}}$ is a modification of $\mathfrak{g} = \mathfrak{n} \times \mathbf{R}$. A linear map $\phi : \mathfrak{g} \to \mathrm{Der}\,(\mathfrak{g})$ is defined as

$$\phi(X) = \phi(Y) = \phi(Z) = 0, \qquad \phi(W) = \mathrm{ad}_W,$$

where ad_W is defined by

$$\mathrm{ad}_W(X) = -Y, \qquad \mathrm{ad}_W(Y) = X, \qquad \mathrm{ad}_W(Z) = 0, \qquad \mathrm{ad}_W(W) = 0.$$

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It is clear that ad_W is skew-symmetric with respect to g and compatible with J. Hence, setting $\mathfrak{k} = \langle \mathrm{ad}_W \rangle$, we get a modification $\bar{\mathfrak{g}}$ of $\mathfrak{g}$:

$$\bar{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{k}/\mathfrak{k}.$$

Since ϕ clearly satisfies the condition (1.3), $\bar{\mathfrak{g}}$ can be identified with $\mathfrak{g}'$ through the map $\psi : \mathfrak{g}' \to \mathfrak{g} \rtimes \mathfrak{k} \to \bar{\mathfrak{g}}$ defined by $\psi(X) = \mathrm{pr}(X, \phi(X))$.

Note that $\mathfrak{g}$ is a nilpotent Lie algebra and $\mathfrak{g}'$ is a unimodular non-nilpotent solvable Lie algebra. The corresponding Lie groups G and G' with the integrable complex structure (1.4) admit uniform lattices, defining primary and secondary Kodaira surfaces, respectively. Both of them are Vaisman Lie groups with a l.c.K. form Ω defined by

$$\Omega = x \wedge y + z \wedge w$$

with the Lee form w, where x, y, z, w are the Maurer–Cartan forms corresponding to X, Y, Z, W, respectively.

We can define a modification of a pair $(\mathfrak{g}, \mathfrak{h})$ of a Hermitian Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ under the additional condition,

$$\sigma(\mathfrak{h}) \subset \mathfrak{h}, \qquad J\sigma = \sigma J \ (\mathrm{mod}\ \mathfrak{h}) \tag{1.5}$$

for any $\sigma \in \mathfrak{k}$. We get a modification $(\mathfrak{g}', \mathfrak{h}')$ of $(\mathfrak{g}, \mathfrak{h})$ as

$$\mathfrak{g}' = \mathfrak{g} \rtimes \mathfrak{k}, \qquad \mathfrak{h}' = \mathfrak{h} \rtimes \mathfrak{k}.$$

Let G (resp. G') be the simply connected Lie group with Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}'$) and H (resp. H') be its closed subgroup with Lie algebra $\mathfrak{h}$ (resp. $\mathfrak{h}'$). G'/H' is isomorphic to G/H as a Hermitian manifold.

For modification in the category of homogeneous Sasaki manifolds G/H or the corresponding Lie algebras $(\mathfrak{g}, \mathfrak{h})$ with Sasaki structure $\{\phi, \eta, \widetilde{J}, g\}$, we consider a subalgebra $\mathfrak{k}$ of $\mathrm{Der}(\mathfrak{g})$ consisting of skew-symmetric derivations σ compatible with $\widetilde{J}$:

$$g(\sigma(X), Y) + g(X, \sigma(Y)) = 0, \qquad \widetilde{J}\sigma = \sigma\widetilde{J} \tag{1.1$'$}$$

for any $X, Y \in \mathfrak{g}$. Then we can define the modification of Sasaki algebras $(\mathfrak{g}, \mathfrak{h})$ in the same way as for the case of Hermitian algebras.

The following lemma is a key in the whole arguments for the proofs of our main results.

LEMMA 1.1. *Let $M = G/H$ be a simply connected homogeneous Vaisman manifold, where H is a connected subgroup of a simply connected Lie group G. Then, we can modify, if necessary, $\mathfrak{g}/\mathfrak{h}$ into $\mathfrak{g}'/\mathfrak{h}'$ with $\dim Z(\mathfrak{g}') = 2$ and $\dim \mathfrak{h}' \leqslant \dim \mathfrak{h} + 1$. Hence, G/H is isomorphic to G'/H' as a homogeneous Vaisman manifold, where (G', H') is the corresponding Lie groups of $(\mathfrak{g}', \mathfrak{h}')$. Similarly, for a simply connected homogeneous Sasaki manifold G/H, we can modify, if necessary, $\mathfrak{g}/\mathfrak{h}$ into $\mathfrak{g}'/\mathfrak{h}'$ with $\dim Z(\mathfrak{g}') = 1$.*

Proof. In fact, the set of invariant vector fields can be identified with $(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$; since the Lee field ξ and Reeb field $\eta = J\xi$ are invariant, they belong to this set. Since ξ and η are Killing and compatible with the complex structure J, they define ad_ξ and ad_η in $\mathrm{Der}(\mathfrak{g})$, which commute with each other and are compatible with J. They are also $\mathrm{ad}\, h$-invariant for $h \in \mathfrak{h}$.

Let $\mathfrak{k} = \langle \mathrm{ad}_\xi \rangle$, and $\hat{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{k}$, $\hat{\mathfrak{h}} = \mathfrak{h} \times \mathfrak{k}$. We have $\mathfrak{g}/\mathfrak{h} = \hat{\mathfrak{g}}/\hat{\mathfrak{h}}$, where $\hat{\mathfrak{g}}$ has a central element $\zeta = \xi - \mathrm{ad}_\xi$ in $\hat{\mathfrak{g}}$ which is identified with $\xi \,(\mathrm{mod}\ \hat{\mathfrak{h}})$. Since the Lee form θ is closed and $\theta(\xi) = 1$, we have $\xi \notin [\mathfrak{g}, \mathfrak{g}]$. Hence, we have a modification of $\mathfrak{g}$ into $\hat{\mathfrak{g}}/\mathfrak{k} = \mathfrak{g}'$ and $\hat{\mathfrak{h}}/\mathfrak{k} = \mathfrak{h}' = \mathfrak{h}$ through the map $X \to (X, \phi(X))$. Therefore, we have

$$\mathfrak{g}/\mathfrak{h} = \mathfrak{g}'/\mathfrak{h}' = \mathfrak{g}'/\mathfrak{h}$$

with $\xi \in Z(\mathfrak{g}')$. In particular, we have $\mathfrak{g}' = \mathbf{R} \times \mathfrak{g}_1$ with $\mathfrak{g}_1 = \ker \theta \supset \mathfrak{h}$, where $\mathbf{R}$ is generated by ξ. Similarly, we can modify $\mathfrak{g}'/\mathfrak{h}'$ into $\mathfrak{g}''/\mathfrak{h}''$ with $\xi, \eta \in Z(\mathfrak{g}'')$. Note that in case ξ or η is already in $Z(\mathfrak{g})$, ad_ξ or ad_η is trivial; thus $\mathfrak{g}' = \mathfrak{g} \times \mathfrak{k}$, $\mathfrak{h}' = \mathfrak{h} \times \mathfrak{k}$ without any modification on $\mathfrak{g}$ and $\mathfrak{h}$. Since for a homogeneous Vaisman manifold G''/H'', the dimension of the center is not greater than 2 [9], [1], the Lee field and the Reeb field generate $Z(\mathfrak{g}'')$. Since $\mathfrak{h}' = \mathfrak{h}$, we have $\dim \mathfrak{h}'' \leqslant \dim \mathfrak{h} + 1$. ∎

We review some basic and historical results on a classification of homogeneous Kähler manifolds (due to Dorfmeister, Nakajima, Vinberg, Gindikin, Piatetski-Shapiro, Matsushima, Borel, Hano, and Shima; see [3], [5], [7], [15], and references therein).

Let $M = G/H$ be a homogeneous Kähler manifold, where H is a closed subgroup of a simply connected Lie group G. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H, respectively. Then, we can consider a Kähler structure on G/H as a pair (J, ω) of a complex structure $J \in \mathrm{End}(\mathfrak{g})$ and a skew-symmetric bilinear form ω on $\mathfrak{g}$, satisfying the following conditions:

(i) $J\,\mathfrak{h} \subset \mathfrak{h}$, $J^2 = -I \,(\mathrm{mod}\ \mathfrak{h})$;
(ii) $\mathrm{ad}_X J = J\,\mathrm{ad}_X \,(\mathrm{mod}\ \mathfrak{h})$ for $X \in \mathfrak{h}$;
(iii) $[JX, JY] = [X, Y] + J\,[JX, Y] + J\,[X, JY] \,(\mathrm{mod}\ \mathfrak{h})$;
(iv) $\omega(\mathfrak{h}, \mathfrak{g}) = 0$, $\omega(JX, JY) = \omega(X, Y)$;
(v) $\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0$;
(vi) $\omega(JX, X) \neq 0$ for $X \notin \mathfrak{h}$.

A *Kähler algebra* $(\mathfrak{g}, \mathfrak{h}, J, \omega)$ is a Lie algebra $\mathfrak{g}$ with subalgebra $\mathfrak{h}$, $J \in \mathrm{End}(\mathfrak{g})$ and a skew-symmetric bilinear form ω on $\mathfrak{g}$, satisfying the above conditions. A *Kähler algebra* $(\mathfrak{g}, \mathfrak{h}, J, \omega)$ is *effective* if $\mathfrak{h}$ includes no nontrivial ideals of $\mathfrak{g}$. A *J-algebra* is a Kähler algebra $(\mathfrak{g}, \mathfrak{h}, J, \omega)$ with a linear form ρ such that $d\rho = \omega$. Note that the condition $d\rho = \omega$ is often referred to as *nondegenerate*; for a Kähler algebra of effective form, it is actually equivalent to nondegeneracy of the Ricci curvature form $\mathfrak{r}$ of the Kähler structure (due to Nakajima [12]).

Structure theorem of homogeneous Kähler manifolds. A homogeneous Kähler manifold is a holomorphic fiber bundle over a homogeneous bounded domain with fiber a Kählerian product of a locally flat Kähler manifold and a flag manifold. In particular, due to the Grauert–Oka principle [6], it is biholomorphic to the product of these complex manifolds.

A key idea of the proof [5] for the theorem is to show, applying modifications if necessary, that there exists an abelian ideal $\mathfrak{a}$ and a J-algebra $\mathfrak{f}$ containing $\mathfrak{h}$ such that

$$\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{f} \tag{1.6}$$

which is a semidirect sum, and $\mathfrak{g}$ is *quasinormal*, that is, $\mathrm{ad}(X)$ has only real eigenvalues for any element $X \in \mathrm{rad}(\mathfrak{g})$, where $\mathrm{rad}(\mathfrak{g})$ is the radical of $\mathfrak{g}$. There also exists a compact J-subalgebra $\mathfrak{q}$ of $\mathfrak{f}$ satisfying $\mathfrak{f} \supset \mathfrak{q} \supset \mathfrak{h}$ for which we can express M as a fiber bundle:

$$P/H \to M = G/H \to G/P, \tag{1.7}$$

where $P = AQ$ and A, Q are the Lie groups associated with $\mathfrak{a}$, $\mathfrak{q}$, respectively; $P/H = A/\Gamma \times Q/H_0$ with $H = H_0\Gamma$ for the connected component H_0 of H and a discrete subgroup Γ of A. The base space G/P defines a homogeneous bounded domain, A/Γ a locally flat complex manifold, Q/H_0 a flag manifold, and the fibration is holomorphic.

§2. Sasaki and Vaisman unimodular Lie algebras

A Lie group G is a homogeneous space with its own transitive action on the left. It is a homogeneous l.c.K. manifold if it admits a left-invariant Hermitian structure (g, J) satisfying

$$d\Omega = \Omega \wedge \theta$$

for its associated fundamental form Ω and a closed 1-form θ (Lee form). Note that θ must also be left-invariant. It is clear that G admits a left-invariant l.c.K. structure if and only if its Lie algebra $\mathfrak{g}$ admits an l.c.K. form Ω. We call $\mathfrak{g}$ with an l.c.K. form Ω an l.c.K. Lie algebra.

We already know a classification of l.c.K. reductive Lie algebras ([8], [1]) and l.c.K. nilpotent Lie algebras ([13], [8]), determining, at the same time, which l.c.K. structures are of Vaisman type. In this section, we determine all Vaisman unimodular Lie algebras, up to modifications.

THEOREM 2.1. *A Sasaki unimodular Lie algebra is, up to modification, isomorphic to one of the three types:* $\mathfrak{n}$, $\mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbf{R})$. *Accordingly, Vaisman unimodular Lie algebra is, up to modification, isomorphic to one of the following:*

$$\mathbf{R} \times \mathfrak{n}, \qquad \mathbf{R} \times \mathfrak{su}(2), \qquad \mathbf{R} \times \mathfrak{sl}(2, \mathbf{R}),$$

where $\mathfrak{n}$ *is a Heisenberg Lie algebra. In terms of Lie groups, a simply connected Sasaki unimodular Lie group is, up to modification, isomorphic to one of the three types:* N, $\mathrm{SU}(2)$, $\widetilde{\mathrm{SL}}(2, \mathbf{R})$. *Accordingly, a simply connected Vaisman unimodular Lie group is, up to modification, isomorphic to one of the following:*

$$\mathbf{R} \times N, \qquad \mathbf{R} \times \mathrm{SU}(2), \qquad \mathbf{R} \times \widetilde{\mathrm{SL}}(2, \mathbf{R}).$$

Proof. Let $\mathfrak{g}$ be a Vaisman unimodular Lie algebra of dimension $2k + 2$ with an l.c.K. form Ω and Lee form θ. Applying modification, if necessary, we can assume that

$$\mathfrak{g} = \mathbf{R} \times \mathfrak{g}_0, \tag{2.1}$$

where $\mathfrak{g}_0 = \ker\theta$, and $\mathbf{R}$ is generated by the Lee field ξ. $\mathfrak{g}_0$ is a Sasaki Lie algebra with Reeb field η. Let ψ be the contact form and $\mathfrak{k} = \langle\eta\rangle$, then $(\mathfrak{g}_0, \mathfrak{k}, J|\mathfrak{g}_0, d\psi)$ defines a Kähler algebra. The Ricci curvature form $\mathfrak{r}$ of the Kähler structure is given by

$$\mathfrak{r}(X, Y) = -\kappa([X, Y]),$$

where κ is the *Koszul form* defined by

$$\kappa(X) = \mathrm{Tr}_{\mathfrak{g}_0/\mathfrak{k}}(\mathrm{ad}\, JX - J\mathrm{ad}\, X),$$

which is well defined on $\mathfrak{g}_0/\mathfrak{k}$ [11]. Now, in case $\dim Z(\mathfrak{g}_0) = 1$, $Z(\mathfrak{g}_0) = \mathfrak{k}$, and $\mathfrak{g}_0/\mathfrak{k}$ is a unimodular Kähler Lie algebra. Then due to Hano [7], $\mathfrak{g}_0/\mathfrak{k}$ is meta-abelian and locally flat

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and, thus, up to modification, isomorphic to $\mathbf{C}^n$ as Kähler algebra. Therefore, we get $\mathfrak{g}_0 = \mathfrak{n}$, up to modification. In case $\dim Z(\mathfrak{g}_0) = 0$, we see that the Ricci form κ is nondegenerate. In fact, since we have $i(\eta)\phi = 1$, $i(\eta)\, d\phi = 0$, and $\mathrm{ad}\,(\eta)$ is not trivial, $\mathfrak{k}$ is not an ideal of $\mathfrak{g}_0$. Therefore, the Kähler algebra $(\mathfrak{g}_0, \mathfrak{k}, d\psi)$ is in effective form. Since the Kähler algebra $(\mathfrak{g}_0, \mathfrak{k}, d\psi)$ is nondegenerate (that is, it defines a J-algebra), the Ricci form $\mathfrak{r}$ is nondegenerate [12]; it follows (due to Hano [7]) that $\mathfrak{g}_0$ must be semisimple. Then it is well known [2] that $\mathfrak{g}_0$ must be either $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbf{R})$. □

REMARK 2.1. A Vaisman unimodular solvable Lie algebra $\mathfrak{g}$ is, up to modification, isomorphic to $\mathbf{R} \times \mathfrak{n}$ (see Example 3.1 for a non-nilpotent case). Since modification ϕ is a skew-symmetric operation, its eigenvalues are all pure-imaginary; in particular, a Vaisman unimodular completely solvable Lie algebra is isomorphic to $\mathbf{R} \times \mathfrak{n}$ [14].

REMARK 2.2. We have determined all homogeneous l.c.K. structures on $\mathbf{R} \times \mathfrak{n}$ and $\mathbf{R} \times \mathfrak{su}(2)$, which are all of Vaisman type [8]. We have also determined all homogeneous l.c.K. structures on $\mathbf{R} \times \mathfrak{sl}(2, \mathbf{R})$, some of them are of non-Vaisman type, as we will see in the next section.

§3. l.c.K. unimodular Lie groups of non-Vaisman type

In this section, we show examples of l.c.K. reductive Lie algebras of non-Vaisman type (which we already discussed in our previous papers [9], [1]), illustrating how Vaisman and non-Vaisman structures can be defined on $\mathbf{R} \times \mathfrak{sl}(2, \mathbf{R})$.

EXAMPLE 3.1. There exists a homogeneous l.c.K. structure on $\mathfrak{g} = \mathbf{R} \times \mathfrak{sl}(2, \mathbf{R})$ which is not of Vaisman type. Take a basis $\{X, Y, Z\}$ for $\mathfrak{sl}(2, \mathbf{R})$ with bracket multiplication defined by

$$[X, Y] = -Z, \qquad [Z, X] = Y, \qquad [Z, Y] = -X \tag{3.1}$$

and T as a generator of the center $\mathbf{R}$ of $\mathfrak{g}$, where we set

$$X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let t, x, y, z be the Maurer–Cartan forms corresponding to T, X, Y, Z, respectively; then we have

$$dt = 0, \qquad dx = z \wedge y, \qquad dy = x \wedge z, \qquad dz = x \wedge y \tag{3.2}$$

and an l.c.K. structure $\Omega = z \wedge t + x \wedge y$ compatible with an integrable homogeneous complex structure J on $\mathfrak{g}$ defined by

$$JY = X, \qquad JX = -Y, \qquad JT = Z, \qquad JZ = -T.$$

We can generalize Ω to an l.c.K. structure of the form

$$\Omega_\psi = \psi \wedge t + d\psi \tag{3.3}$$

compatible with the above complex structure J on $\mathfrak{g}$, where $\psi = ax + by + cz$ with a, b, $c \in \mathbf{R}$.

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We see that the symmetric bilinear form $g_\psi(U,V)=\Omega_\psi(JU,V)$ is represented, with respect to the basis $\{T,X,Y,Z\}$, by the matrix

$$A=\begin{pmatrix} c & -b & a & 0\\ -b & c & 0 & a\\ a & 0 & c & b\\ 0 & a & b & c\end{pmatrix},$$

which has the characteristic polynomial $\Phi_A(u)=\{(u-c)^2-(a^2+b^2)\}^2$ and has only positive eigenvalues if and only if $c>0,\ c^2>a^2+b^2$. The Lee form is $\theta=t$ and the Lee field is

$$\xi=\frac{1}{D}(cT+bX-aY),$$

with $D=c^2-a^2-b^2$. We have also

$$g_\psi(\xi,\xi)=\frac{c}{D}.$$

We can see that $g_\psi([\xi,U],V)+g_\psi(U,[\xi,V])\not\equiv 0$ unless $a=b=0$. In fact, for $U=V=Z$,

$$g_\psi([\xi,Z],Z)+g_\psi(Z,[\xi,Z])=2g_\psi([\xi,Z],Z)=-\frac{2}{D}(a^2+b^2)=0$$

if and only if $a=b=0$. Conversely, for the case $a=b=0$, it is easy to check that $g_\psi([\xi,U],V)+g_\psi(U,[\xi,V])\equiv 0$. Therefore, we have shown the following.

For J and Ω_ψ defined above, g_ψ defines a (positive definite) l.c.K. metric if and only if $c>0, c^2>a^2+b^2$. It is of Vaisman type if and only if $c>0, a=b=0$. And it is of non-Vaisman type if and only if $c>0, c^2>a^2+b^2>0$.

We see that $\mathfrak{g}$ can be modified into $\mathfrak{g}'/\langle S\rangle$, where $\mathfrak{g}'=\mathbf{R}\times\mathfrak{gl}(2,\mathbf{R})$ for which the basis consists of X,Y,Z, and

$$W=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

and we set

$$S=\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

Since we have $W=Z+S\in\mathfrak{gl}(2,\mathbf{R})$, ad_S defines a skew-symmetric action on $\mathfrak{g}$ and $Z=W \pmod S$. Hence, we get $\mathfrak{g}=\mathfrak{g}'/\langle S\rangle$ as an l.c.K. algebra with the original l.c.K. form Ω, which is of Vaisman type. Note that $\dim_{\mathbf{R}} Z(\mathfrak{g}')=2$. We see that for $\mathfrak{g}$ with the l.c.K. form Ω_ψ of non-Vaisman type, ad_S is not compatible with the metric g_ψ. In fact, for $U=bX-aY$,

$$g_\psi([S,U],Z)+g_\psi(U,[S,Z])=g_\psi([Z,U],Z)=a^2+b^2=0$$

if and only if $a=b=0$. Hence, we cannot modify $\mathfrak{g}$ with the l.c.K. form Ω_ψ of non-Vaisman type into $\mathfrak{g}=\mathfrak{g}'/\langle S\rangle$ with a compatible Vaisman structure.

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§4. Homogeneous Sasaki and Vaisman manifolds of unimodular Lie group

In this section, we prove Theorems 4.1 and 4.2 as our main results.

For any Sasaki manifold N, its *Kähler cone* $C(N)$ is defined as $C(N)=\mathbf{R}_+\times N$ with the Kähler form $\omega = r\,dr\wedge\psi + (r^2/2)\,d\psi$, where a compatible complex structure $\widehat{J}$ is defined by $\widehat{J}\eta = (1/r)\partial_r$ and $\widehat{J}|_{\mathcal{D}} = J$. Note that a contact metric manifold N^{2n+1} with $\{\psi, \eta, \widetilde{J}\}$ is Sasaki if and only if the Kähler cone $C(N)$ with $(\omega, \widetilde{J})$ is Kählerian.

For any Sasaki manifold N with contact form ψ, we can define an l.c.K. form $\Omega = (2/r^2)\omega = (2/r)\,dr\wedge\psi + d\psi$; or taking $t=-2\log r$, $\Omega = -dt\wedge\psi + d\psi$ on $M=\mathbf{R}\times N$ or $S^1\times N$ respectively, which is of Vaisman type. We can define a family of complex structures J compatible with Ω by

$$J\partial_t = b\partial_t + (1+b^2)\eta, \qquad J\eta = -\partial_t - b\eta, \tag{4.1}$$

where $b\in\mathbf{R}$ and the Lee field is $J\eta$. We call M a *canonical Vaisman manifold* associated with a Sasaki manifold N.

We have shown in Lemma 1.1 that, up to modifications, a simply connected homogeneous Vaisman manifold of unimodular Lie group can be assumed to have the form $M=G/H$, where G is a simply connected unimodular Lie group of the form $G=\mathbf{R}\times G_1$ and H is a connected compact subgroup of G_1 with $\dim Z(G_1)=1$; combining with our previous results in [9], [1] yields the following.

Proposition 4.1. *Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H, respectively. Then the pair $\{\mathfrak{g}, \mathfrak{h}\}$ is of the following form:*

$$\mathfrak{g} = \mathbf{R}\times\mathfrak{g}_1,$$

where $\mathfrak{g}_1 = \ker\theta \supset \mathfrak{h}$, and $\mathfrak{g}_1$ can be expressed as a central extension of a Lie algebra $\mathfrak{g}_2$ by $\mathbf{R}$:

$$0\to\mathbf{R}\to\mathfrak{g}_1\to\mathfrak{g}_2\to 0.$$

The Lee field ξ and the Reeb field $\eta = J\xi$ generate $Z(\mathfrak{g})$; the l.c.K. form Ω can be written as

$$\Omega = -\theta\wedge\psi + d\psi,$$

where ψ is the Reeb form defining a contact form on the homogeneous Sasaki manifold G_1/H, G_1 being the simply connected unimodular Lie group corresponding to $\mathfrak{g}_1$. Let $\mathfrak{k} = \pi(\mathfrak{h})$ for the projection $\pi:\mathfrak{g}_1\to\mathfrak{g}_2$. Then the pair $\{\mathfrak{g}_2, \mathfrak{k}\}$ defines a homogeneous Kähler manifold G_2/K with the Kähler form $\omega = d\psi|\mathfrak{g}_2$, where G_2 and K are the Lie groups corresponding to $\mathfrak{g}_2$ and $\mathfrak{k}$, respectively.

Let $\mathfrak{g}_1$ be the Sasaki algebra with the Reeb field η and the Kähler form ω in Proposition 4.1. Then, the Lie bracket on $\mathfrak{g}_1$ is the extension of $\mathfrak{g}_2$ given by

$$[X,Y]_{\mathfrak{g}_1} = [X,Y]_{\mathfrak{g}_2} - \omega(X,Y)\eta, \qquad [\eta, Z]_{\mathfrak{g}_2} = 0 \tag{4.2}$$

for $X, Y, Z\in\mathfrak{g}_2$. Conversely, given a Kähler algebra $\{\mathfrak{g}_2, \mathfrak{k}\}$ with a Kähler form ω, we can define a Sasaki Lie algebra $\mathfrak{g}_1$, which is a central extension with a generator η of $\mathbf{R}$ by the above formula. Since η is Killing, $\mathfrak{g}_1$ is unimodular if and only if $\mathfrak{g}_2$ is unimodular. Hence, $M_1 = G_1/H$ is of unimodular type if and only if $M_2 = G_2/K$ is of the same type.

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We have then the following, which is one of our main results.

THEOREM 4.1. *A simply connected homogeneous Vaisman manifold M of unimodular Lie group is holomorphically isometric to $M' = \mathbf{R} \times M_1$ with a canonical Vaisman structure, where M_1 is a simply connected homogeneous Sasaki manifold of unimodular Lie group, which is a quantization of a simply connected homogeneous Kähler manifold M_2 of reductive Lie group. As a Hermitian manifold M is a holomorphic principal bundle over a simply connected homogeneous Kähler manifold M_2 with fiber $\mathbf{C}^1$ or $\mathbf{C}^*$.*

For the proof of Theorem 4.1, since we have already discussed and proved the first part of the theorem, we need to only show the last part that *a simply connected homogeneous Sasaki manifold M_1 of unimodular Lie group has the structure as stated in the theorem*, which is covered and more detailed in the following theorem.

THEOREM 4.2. *A simply connected homogeneous Sasaki manifold M_1 of unimodular Lie group is a quantization of a simply connected homogeneous Kähler manifold M_2 of reductive Lie group; that is, M_1 is a principal bundle over M_2 with fiber $\mathbf{R}$ or S^1 satisfying $d\psi = \omega$ for a contact form ψ on M_1 and the Kähler form ω on M_2.*

The simply connected homogeneous Kähler manifold M_2 of reductive Lie group is a Kählerian product of $\mathbf{C}^k$, a flag manifold Q/V with a compact semisimple Lie group Q and a parabolic subgroup V, and a homogeneous Kähler manifold P/U with a noncompact semisimple Lie group P and a closed subgroup U:

$$M_2 = \mathbf{C}^k \times Q/V \times P/U. \tag{4.3}$$

The homogeneous Kähler manifold P/U has a structure of a holomorphic fiber bundle over a symmetric domain P/L with fiber a flag manifold L/U for a maximal compact subgroup L of P.

Furthermore, M_1 is $\mathbf{R}$-quantization of M_2 if and only if M_2 is a product of $\mathbf{C}^k$ and a symmetric domain P/L with $L = U$, and S^1-quantization of M_2 in all other cases.

Note that a homogeneous Sasaki manifold, and more generally a homogeneous contact manifold, is necessarily regular (see [2], [9]). Note also that Theorem 4.2 may be considered independently as a result on classification of homogeneous Sasaki manifolds of unimodular Lie groups, which extends a known result on compact homogeneous Sasaki manifolds (see [4]).

First, we prove the following key result on homogeneous Kähler manifolds of unimodular Lie group, which could be of independent interest.

PROPOSITION 4.2. *A simply connected homogeneous Kähler manifold $M = G/K$ of unimodular Lie group G is of reductive type; that is, the Kähler algebra $\{\mathfrak{g}, \mathfrak{k}\}$ of M has, up to modification, a decomposition*

$$\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{l},$$

where $\mathfrak{a}$ is an abelian Kähler ideal of dimension k and $\mathfrak{l}$ is a semisimple Kähler subalgebra which contains $\mathfrak{k}$. As a Kähler manifold, M is a product of $\mathbf{C}^k$ and a homogeneous Kähler manifold $N = L/K$ of a semisimple Lie group L:

$$M = \mathbf{C}^k \times N.$$

Furthermore, N can be decomposed into a Kählerian product of flag manifolds and noncompact homogeneous Kähler manifolds each of which is a holomorphic fiber bundle over a symmetric domain with fiber a flag manifold.

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Proof. Let $M = G/K$ be a simply connected homogeneous Kähler manifold, where G is a unimodular Lie group and K its closed subgroup. We have a decomposition,

$$\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{f},$$

where $\mathfrak{a}$ is a maximal abelian J-ideal of $\mathfrak{g}$ isomorphic to $\mathbf{C}^k$ and $\mathfrak{f}$ is a J-subalgebra which contains $\mathfrak{k}$. Moreover, due to [15], $\mathfrak{f}$ decomposes into a product of a solvable J-subalgebra $\mathfrak{s}$, a reductive J-subalgebra $\mathfrak{q}$,

$$\mathfrak{f} = \mathfrak{s} \times \mathfrak{q},$$

where $\mathfrak{q}$ contains $\mathfrak{k}$ and the center of $\mathfrak{q}$ is contained in $\mathfrak{k}$. We see, applying the De Rham decomposition of homogeneous Kähler manifolds (see [10]), that $\mathfrak{s}$ is actually the radical of $\mathfrak{f}$, which is a maximal solvable ideal of $\mathfrak{f}$. We see also that $\mathfrak{a} \rtimes \mathfrak{s}$ is the radical of $\mathfrak{g}$. Since $\mathfrak{g}$ is, by assumption, a unimodular Lie algebra, so is $\mathfrak{a} \rtimes \mathfrak{s}$. It follows, due to Hano [7], that $\mathfrak{s}$ must be trivial. Since the center of $\mathfrak{q}$ is contained in $\mathfrak{k}$, we may express $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{l},$$

where $\mathfrak{l}$ is the semisimple part of $\mathfrak{q}$ and $\mathfrak{k}$ is contained in $\mathfrak{l}$. Since M is, by assumption, simply connected, $\mathfrak{a}$ corresponds to $\mathbf{C}^k$ as a flat Kähler manifold, and thus the action of L (the Lie group corresponding to $\mathfrak{l}$) on $\mathbf{C}^k$ is holomorphically isometric. Thus, as a Kähler manifold M is isomorphic to $\mathbf{C}^k \times L/K$ (see [5]), where L/K is a product of homogeneous Kähler manifolds of compact semisimple Lie groups and homogeneous Kähler manifolds of noncompact semisimple Lie groups each of which is a holomorphic fiber bundle over a symmetric domain with fiber a flag manifold (see [3]). ∎

Next, we discuss a quantization of a homogeneous Kähler manifold $M_2 = G_2/K$ of reductive type. In case $M_2 = \mathbf{C}^k$, its quantization is the Heisenberg Lie group N, which is a central extension of $\mathbf{R}$ by $\mathbf{C}^k$. In case $M_2 = L/K$ is a flag manifold, being a simply connected Hodge manifold, where L is a compact semisimple Lie group, it is quantizable to a compact simply connected homogeneous Sasaki manifold with fiber S^1. In case L is a noncompact semisimple Lie group, M_2 is a holomorphic fiber bundle over a symmetric domain L/B with fiber a flag manifold B/K, where B is a maximal compact Lie subgroup of L containing K. Since the flag manifold B/K is a Kähler submanifold of $M_2 = G/K$ and S^1-quantizable, M_2 is also S^1-quantizable. In general cases, for two or more homogeneous Kähler manifolds each of which is quantizable, we can construct naturally a quantization of their products in the following way. For two Kähler algebras $\mathfrak{g}_2$ and $\mathfrak{g}_2'$ with their central extension $\mathfrak{g}_1$ and $\mathfrak{g}_1'$, respectively, we can define a new central $\mathbf{R}$-extension of $\mathfrak{g}_2 \times \mathfrak{g}_2'$ by taking $\mathbf{R} \times \mathbf{R}/\Delta = \mathbf{R}$ with $\Delta = \{(X, -X) \mid X \in \mathbf{R}\}$:

$$0 \to \mathbf{R} \to \mathfrak{g}_1 \times_\Delta \mathfrak{g}_1' \to \mathfrak{g}_2 \times \mathfrak{g}_2' \to 0, \tag{4.4}$$

where $\mathfrak{g}_1 \times_\Delta \mathfrak{g}_1' = (\mathfrak{g}_1 \times \mathfrak{g}_1')/\Delta$, the quotient Lie algebra by the canonical action of Δ on $\mathfrak{g}_1 \times \mathfrak{g}_1'$. Correspondingly, we obtain a quantization $G_1 \times_\Delta G_1'$ of $G_2 \times G_2'$; in general, the quantization $G_1/H \times_\Delta G_1'/H'$ of $G_2/K \times G_2'/K'$. Now, in case M_2 is a product of $\mathbf{C}^k$ and a symmetric domain (which is the case $B = K$), since M_2 is contractible, it is $\mathbf{R}$-quantizable (but not S^1-quantizable) to a simply connected homogeneous Sasaki manifold. In all other cases, as we have seen in the above, since L is a noncompact semisimple Lie group and K is not a maximal compact subgroup of L (that is, $B \supsetneq K$), M_2 is S^1-quantizable to a simply connected homogeneous Sasaki manifold.

This completes the proof of Theorem 4.2, and thus of Theorem 4.1 as well.

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