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Closeness to singularities of robotic manipulators measured by characteristic angles

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SUMMARY
Singularities have a great influence on kinematics and dynamics of both serial and parallel robots. In order to prevent a robot from entering singular configurations, it needs to measure the "distance" between the robot current configuration and the singular configuration. This paper presents a novel approach based on characteristic angles to measure closeness to singularities. For the problem of inconsistent dimensions in the scalar product of screws, the physical meanings of twists and wrenches are reinterpreted. For the problem of the metric invariant to origin selection, the origin of the screw frame is required to coincide with the origin of the robotic tool frame. The major merit of the proposed metric lies in the identical result of measuring similar mechanisms with different sizes. Moreover, the measurement is insensitive to screw magnitude, since the metric expression is dimensionless. Furthermore, the geometrical meaning of the determinant of a screw matrix is clarified.

KEYWORDS: Parallel manipulators; Singularity; Metrics; Screw theory.

1. Introduction

Singularities have an important influence on the kinematics and dynamics of both serial and parallel robots, which have attracted many researchers.[1–8] There are various categories of singularities according to different perspectives,[9,10] and a significant classification is to divide the singularities as kinematic singularities and static singularities. The kinematic singularities emerge in the case that joint twists in a serial kinematic chain become linearly dependent, which may occur in serial robots or in a limb of parallel robots. The static singularities arise when constraint wrenches between kinematic chains become linearly dependent, which only occur in parallel robots. Once a serial or parallel robot moves into a kinematic singular configuration, its joint velocities cannot be solved for a given velocity of its end-effector or moving platform; in other words, the required joint velocities are infinite. Once a parallel robot goes into a static singular configuration, its limbs cannot provide necessary constraints to its moving platform; in other words, the required actuating forces or torques are infinite. Obviously parallel robots in singularities are uncontrollable, and even the robot structure may be destroyed.

Robots should be protected from reaching singular configurations, and the singularity margin that indicates the "distance" between the robot current configuration and the singular configuration should be inspected. There are some well-known metrics to detect singularities, such as the determinant of Jacobian, the smallest singular value, and the condition number.[11,12] These metrics, however, are unsuitable to measure closeness to singularities, since they lack some physical or geometrical meaning,[13] and they are not invariant to origin selection or scaling.[14] Hence, many researchers have studied metrics of closeness to singularities for robotic manipulators. Voglewede and Ebert-Uphoff[15] proposed three attributes to estimate different metrics, and adopted constrained optimization in both velocity and force domains to evaluate closeness to singularities that was represented as power, stiffness, kinetic energy, potential energy, or natural frequency. They declared that there was no invariant metric on the group of rigid body displacement, and any metric necessitated a choice to weight the translational and rotational portions together. Hubert and Merlet[16] detected closeness to

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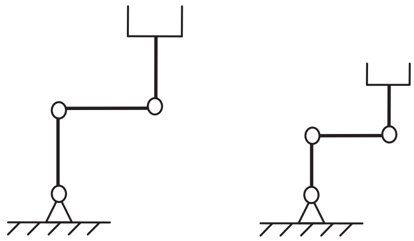

Fig. 1. Two similar robot mechanisms.

singularities based on a practical approach where certain thresholds of robot joint forces or torques were adopted to solve for the limit of the force workspace for a given orientation of the moving platform. Obviously the joint forces or torques not only depended on the robot configuration, but also on the action lines of external loads. Hartley and Kerr studied closeness to singularities of abstract screw systems. For a system containing six screws, they employed the reciprocal product of screws to find a metric invariant to origin selection.[17] For a system containing less than six screws, however, it was hard to prove the invariance of their proposed metric.[18] Liu *et al.*[19] adopted the reciprocal product between transmission wrenches and input or output twists to investigate closeness to singularities, and their approach had a physical meaning. For a limb with less than six-DOF (degrees of freedom), however, the transmission wrench could not be uniquely determined unless the orthogonal operation between the transmission wrench and the limb constraint wrench was employed; the orthogonality of screws was not invariant to origin selection.[14,20]

As declared by many researchers, it is hard to say there is a best metric of closeness to singularities, and this is still an open issue.[16] In the existing metrics, to the best of the author's knowledge, the results of measuring *similar mechanisms* are not the same. Here, the similar mechanisms are defined as the mechanisms with the same architecture, at the same posture, having the same length ratio of corresponding links, but with different sizes. As shown in Fig. 1, these two robots are at the same posture, with the same link ratio, and hence they are called similar mechanisms. The metric proposed in this paper can obtain the identical result for these similar mechanisms, which is a major merit comparing with other existing metrics. Moreover, the measurement is insensitive to screw magnitude, since the proposed metric expression is dimensionless. Furthermore, the geometrical meaning of the determinant[20,21] of a screw matrix is clarified.

2. A Uniform Mathematical Representation of Two Singularities

Robot singularities may be classified as kinematic singularities and static singularities. The kinematic singularities are also known as serial singularities, which emerge in the case that joint twists in a serial kinematic chain become linearly dependent. The static singularities are also called parallel singularities, which arise when limb constraint wrenches applied to the moving platform become linearly dependent. These two kinds of singularities may be uniformly presented in the following equation.

$$\sum_{i=1}^{N} x_i \$_i = \$_E, \quad \text{where } N \leq 6. \tag{1}$$

As for kinematic singularities, $\$_i$, x_i, and $\$_E$ denote the joint unit twist, the joint velocity magnitude, and the end-effector twist, respectively; N denotes the number of joint twists in the serial chain. As for static singularities, $\$_i$, x_i, and $\$_E$ denote the unit constraint wrench, the wrench magnitude, and the external load wrench applied to the moving platform, respectively; N denotes the number of limb constraint wrenches. Since there are six components in $\$_i$ and $\$_E$, Eq. (1) is a set of six equations. Note that when $N < 6$, there are only N independent equations in Eq. (1) at most. From a mathematical viewpoint, a singularity means the unknown x_i for the given $\$_i$ and $\$_E$ in Eq. (1) cannot be solved; in other words, the singularity implies the number of independent equations in Eq. (1) is less than N.

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Equation (1) can be rewritten in a matrix form, where the N screws (twists or wrenches) constitute a $6\times N$ screw matrix as follows.

$$\boldsymbol{M} = \left[\,\$_1\ \$_2 \ldots \$_N\,\right].$$

From an algebraic viewpoint, a singularity indicates that

$$\text{Rank}\,(\mathbf{M}) < N.$$

When $N < 6$, $\mathbf{M}$ is not square and has no determinant. Since

$$\text{Rank}\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right) = \text{Rank}\,(\mathbf{M})\,.$$

Therefore, the singularity also implies

$$\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right) = 0.$$

If no singularity occurs, then $\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right) > 0$; if the robot is close to some singular configuration, the value of $\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)$ decreases toward zero. However, if $\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)$ is taken as the metric of closeness to singularities, there are several shortfalls: (1) $\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)$ is not invariant to the selection of origin; (2) $\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)$ has no explicit physical or geometrical meanings; (3) $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ refers to the scalar product of screws where the dimensions of linear and angular velocities, or forces and torques, are inconsistent. The remainder of this paper will investigate these three issues.

3. Weighted Scalar Product of Twists or Wrenches

For the problem of inconsistent dimensions in the scalar product of screws, as mentioned by Voglewede and Ebert-Uphoff,[15] any metric of closeness to singularities needs a choice to weight the linear and angular portions of screws together. Some researchers employed a weighted matrix $\mathbf{Q}$[13] to interpret $\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}$ as some physical quantity of the robotic end-effector, such as the kinetic energy. Obviously there are different weighted matrices for various robots, and it is difficult to provide a meaningful weighted matrix for an abstract screw system. To solve this problem, an alternative representation of rigid body kinematics and statics needs to be put forward.

It is well known that only linear velocity (no angular velocity) is needed to describe the velocity of a mass point. A rigid body has volume, so it requires both linear velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$, which may be expressed as a twist $\$_t = \left(\boldsymbol{\omega}\ \mathbf{v}\right)^{\mathrm{T}}$, where $\boldsymbol{\omega}$ and $\mathbf{v}$ are called the real part and the dual part of the twist respectively. In fact, the velocity of a rigid body can also be described by linear velocities at two points; in other words, the angular velocity can be numerically expressed as the linear velocity at a point l meters away from the twist axis. Thus a twist may be represented as

$$\$_{lt} = \left(l\cdot\boldsymbol{\omega}\ \mathbf{v}\right)^{\mathrm{T}}.$$

This novel expression of a twist still satisfies Eq. (1). Similarly, as for a wrench $\$_w = \left(\mathbf{f}\ \mathbf{t}\right)^{\mathrm{T}}$, where $\mathbf{f}$ and $\mathbf{t}$ are called the real part and the dual part of the wrench respectively, the force $\mathbf{f}$ can be numerically described by the torque at a point l meters away from the wrench axis. Thus a wrench may be represented as

$$\$_{lw} = \left(l\cdot\mathbf{f}\ \mathbf{t}\right)^{\mathrm{T}}.$$

In this way, the six components of the twist $\$_{lt}$ or of the wrench $\$_{lw}$ have the same dimension. Therefore the scalar product of two twists or two wrenches has the consistent dimension as the square of velocity or torque. Actually, this weighted scalar product of screws, $\$_{lt}^{\mathrm{T}}\$_{lt}$ or $\$_{lw}^{\mathrm{T}}\$_{lw}$, is equivalent

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to $\$_t^{\mathrm{T}}\mathbf{Q}\$_t$ or $\$_w^{\mathrm{T}}\mathbf{Q}\$_w$, where $\mathbf{Q}$ is the following weighted matrix:

$$\mathbf{Q} = \begin{bmatrix} l^2 & & & & & \\ & l^2 & & & 0 & \\ & & l^2 & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{bmatrix}_{6\times 6}.$$

The matrix $\mathbf{Q}$ is positive definite since $l > 0$. Therefore the following equation holds.

$$\mathrm{Rank}\left(\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}\right) = \mathrm{Rank}\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right) = \mathrm{Rank}\left(\mathbf{M}\right). \tag{2}$$

When $l = 1$, $\mathbf{M}^{\mathrm{T}}\mathbf{Q}\mathbf{M}$ turns to $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ which has no inconsistent problem now. This treatment is similar to the "characteristic length" proposed by Angeles.[22,23] However, the physical or geometrical meaning of $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ and $\det\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)$ is still unknown.

4. Origin Selection of the Screw Frame

Equation (1) indicates that the singularity is a property of the mapping between joint twists (or limb constraint wrenches) and the end-effector twist (or the external wrench at the moving platform). When the robot is close to a singular configuration, the mapping is hard to execute. Obviously the values of the end-effector velocity and the external torque may change, as the point of interest varies. Hence, the metric value of closeness to singularities is related to the location of the point of interest at the end-effector or moving platform.

The velocity or torque at the point of interest is not always equal to the dual part of the twist or wrench, since the elements in the dual part depend on the origin position of the screw frame. Hence, for conveniently measuring closeness to singularities, the origin of the screw frame is demanded to be translated to the point of interest at the end-effector or moving platform; in other words, the origin of the screw frame should coincide with the origin of the robotic tool frame.

When the origin of the screw frame has been selected, the norm of a screw, regarded as the weighted scalar product as interpreted in Section 3, can be proved invariant to the rotation of the screw frame. Suppose $\$_l = \left(l\cdot\mathbf{S}\ \mathbf{S}_0\right)^{\mathrm{T}}$, where $l\cdot\mathbf{S}$ and $\mathbf{S}_0$ denote the real part and dual part of the screw respectively, and its norm is

$$\|\$_l\| = \sqrt{\$_l^{\mathrm{T}}\$_l} = \sqrt{\left(l\cdot\mathbf{S}^{\mathrm{T}}\quad \mathbf{S}_0^{\mathrm{T}}\right)\left(l\cdot\mathbf{S}\quad \mathbf{S}_0\right)^{\mathrm{T}}} = \sqrt{l^2\cdot\mathbf{S}^{\mathrm{T}}\mathbf{S}+\mathbf{S}_0^{\mathrm{T}}\mathbf{S}_0} = \sqrt{l^2\,\|\mathbf{S}\|^2+\|\mathbf{S}_0\|^2}.$$

The transformation matrix for screws[24] is

$$\mathrm{H} = \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{TR} & \mathbf{R} \end{pmatrix}_{6\times 6},$$

where $\mathbf{R}$ and $\mathbf{T}$ denote the rotation matrix and the translation matrix respectively. For pure rotation, $\mathbf{T} = \mathbf{0}$. After the transformation of pure rotation, the weighted scalar product of screws becomes $\$'^{\mathrm{T}}_l\$'_l$, which can be expressed as

$$\$'^{\mathrm{T}}_l\$'_l = \left(\mathbf{H}\$_l\right)^{\mathrm{T}}\mathbf{H}\$_l = \$_l^{\mathrm{T}}\begin{pmatrix} \mathbf{R}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{pmatrix}\$_l = \$_l^{\mathrm{T}}\$_l.$$

Therefore $\|\$'_l\| = \|\$_l\|$.

Furthermore, $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ can be proved invariant to the transformation of pure rotation. Suppose $\mathbf{M}'$ is the screw matrix after the transformation of pure rotation.

$$\mathbf{M}' = \left[\,\mathbf{H}\$_1\ \mathbf{H}\$_2\ \ldots\ \mathbf{H}\$_N\,\right]$$

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Then,

$$\mathbf{M}'^{\mathrm{T}}\mathbf{M}' = \begin{bmatrix} \mathbf{\$}_1^{\mathrm{T}}\mathbf{H}^{\mathrm{T}} \\ \mathbf{\$}_2^{\mathrm{T}}\mathbf{H}^{\mathrm{T}} \\ \vdots \\ \mathbf{\$}_N^{\mathrm{T}}\mathbf{H}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{H}\$_1 \ \mathbf{H}\$_2 \ \ldots \ \mathbf{H}\$_N \end{bmatrix} = \mathbf{M}^{\mathrm{T}}\mathbf{M}.$$

Note that the origin of the screw frame is artificially selected, and it may be set at other points instead of the origin of the robotic tool frame. Nevertheless, it is reasonable to choose the point of interest at the end-effector or moving platform as the origin of the screw frame, since this point is vital in kinematics and dynamics analysis.

5. Characteristic Angles between Screw Submanifolds

Screws of an N order system ($N \leq 6$) span an N dimensional manifold, if the N screws are independent with each other; excluding the ith screw $\mathbf{\$}_i$, the remaining screws span an N-1 dimensional submanifold. And a screw $\mathbf{\$}_{ni}$ normal to this N-1 dimensional submanifold can be found in the entire N dimensional manifold. When the angle between $\mathbf{\$}_i$ and $\mathbf{\$}_{ni}$ equals 90°, it indicates that $\mathbf{\$}_i$ is linearly dependent on the N-1 dimensional submanifold, and a singularity occurs. When the angle between $\mathbf{\$}_i$ and $\mathbf{\$}_{ni}$ equals 0°, it implies that $\mathbf{\$}_i$ is perpendicular to the N-1 dimensional submanifold and the robot configuration is far away from singularities. Hence, this angle may be adapted to measure closeness to singularities, which is named the characteristic angle between screw submanifolds. Note that the calculation of the normal screw $\mathbf{\$}_{ni}$ and the characteristic angle refers to the weighted scalar product which is not invariant to origin selection, and therefore the origin of the screw frame should be artificially translated to the origin of the robotic tool frame; in this way, the characteristic angle is definite.

The normal screw $\mathbf{\$}_{ni}$ belongs to the N dimensional manifold, and can be expressed as

$$\mathbf{\$}_{ni} = \sum_{j=1}^{N} y_j \mathbf{\$}_j, \tag{3}$$

where y_j is the undetermined coefficient. According to the definition of the normal screw, the following equation holds.

$$\$_{ni}^{\mathrm{T}}\$_k = 0, \quad \text{where } k = 1, 2, \ldots, N \text{ and } k \neq i. \tag{4}$$

By substituting Eq. (3) into Eq. (4) yields

$$\sum_{j=1}^{N} y_j \mathbf{\$}_j^{\mathrm{T}}\mathbf{\$}_k = 0, \quad \text{where } k = 1, 2, \ldots, N \text{ and } k \neq i. \tag{5}$$

From Eq. (3) the following equations hold:

$$\|\mathbf{\$}_{ni}\|^2 = \sum_{j=1}^{N}\sum_{p=1}^{N} y_j y_p \mathbf{\$}_j^{\mathrm{T}}\mathbf{\$}_p, \tag{6}$$

$$\mathbf{\$}_{ni}^{\mathrm{T}}\mathbf{\$}_i = \sum_{j=1}^{N} y_j \mathbf{\$}_j^{\mathrm{T}}\mathbf{\$}_i. \tag{7}$$

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By substituting Eq. (5) into Eq. (6) yields

$$\left\|\$_{ni}\right\|^2 = \sum_{j=1}^{N} y_i y_j \$_j^{\mathrm{T}}\$_i. \tag{8}$$

The characteristic angle between $\$_{ni}$ and $\$_i$ is expressed as

$$\theta_i = \arccos \frac{\$_{ni}^{\mathrm{T}}\$_i}{\left\|\$_{ni}^{\mathrm{T}}\right\| \left\|\$_i\right\|}. \tag{9}$$

By substituting Eqs. (7) and (8) into Eq. (9) yields

$$\theta_i = \arccos \frac{\sqrt{\sum_{j=1}^{N} y_j \$_j^{\mathrm{T}}\$_i}}{\sqrt{y_i}\left\|\$_i\right\|}. \tag{10}$$

Equation (5) can be rewritten in the following matrix form.

$$\begin{bmatrix} \$_1^{\mathrm{T}}\$_1 & \$_2^{\mathrm{T}}\$_1 & \cdots & \$_{i-1}^{\mathrm{T}}\$_1 & \$_i^{\mathrm{T}}\$_1 & \$_{i+1}^{\mathrm{T}}\$_1 & \cdots & \$_N^{\mathrm{T}}\$_1 \\ \$_1^{\mathrm{T}}\$_2 & \$_2^{\mathrm{T}}\$_2 & \cdots & \$_{i-1}^{\mathrm{T}}\$_2 & \$_i^{\mathrm{T}}\$_2 & \$_{i+1}^{\mathrm{T}}\$_2 & \cdots & \$_N^{\mathrm{T}}\$_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \$_1^{\mathrm{T}}\$_{i-1} & \$_2^{\mathrm{T}}\$_{i-1} & \cdots & \$_{i-1}^{\mathrm{T}}\$_{i-1} & \$_i^{\mathrm{T}}\$_{i-1} & \$_{i+1}^{\mathrm{T}}\$_{i-1} & \cdots & \$_N^{\mathrm{T}}\$_{i-1} \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \$_1^{\mathrm{T}}\$_{i+1} & \$_2^{\mathrm{T}}\$_{i+1} & \cdots & \$_{i-1}^{\mathrm{T}}\$_{i+1} & \$_i^{\mathrm{T}}\$_{i+1} & \$_{i+1}^{\mathrm{T}}\$_{i+1} & \cdots & \$_N^{\mathrm{T}}\$_{i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \$_1^{\mathrm{T}}\$_N & \$_2^{\mathrm{T}}\$_N & \cdots & \$_{i-1}^{\mathrm{T}}\$_N & \$_i^{\mathrm{T}}\$_N & \$_{i+1}^{\mathrm{T}}\$_N & \cdots & \$_N^{\mathrm{T}}\$_N \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{i-1} \\ y_i \\ y_{i+1} \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{11}$$

According to Cramer's rule, the variable y_i can be used to represent other y_k as

$$y_k = \frac{D_k}{D}, \quad \text{where} \quad k = 1, 2, \ldots, N \quad \text{and} \quad k \neq i, \tag{12}$$

where D denotes the determinant of the $N \times N$ matrix in Eq. (11), and D_k denotes the determinant of the $N \times N$ matrix whose kth column is replaced with the right side vector of Eq. (11). Suppose $\mathbf{M}_N$ denotes an N order screw matrix consisting of the N screws, and $\mathbf{M}_{(N-1)i}$ denotes an $N-1$ order screw matrix consisting of the $N-1$ screws without the ith screw. Thus, D can be expressed as

$$D = \det\left(\mathbf{M}_{(N-1)i}^{\mathrm{T}}\mathbf{M}_{(N-1)i}\right). \tag{13}$$

It can be proved that

$$\sum_{j=1}^{N} D_j \$_j^{\mathrm{T}}\$_i = y_i \det\left(\mathbf{M}_N^{\mathrm{T}}\mathbf{M}_N\right). \tag{14}$$

By substituting Eqs. (12)–(14) into Eq. (10) yields

$$\theta_i = \arccos \frac{\sqrt{\det\left(\mathbf{M}_N^{\mathrm{T}}\mathbf{M}_N\right)}}{\left\|\$_i\right\| \sqrt{\det\left(\mathbf{M}_{(N-1)i}^{\mathrm{T}}\mathbf{M}_{(N-1)i}\right)}}. \tag{15}$$

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Equation (15) indicates that, if a singularity occurs in the N order screw system, then $\theta_i = 90°$, since $\det\left(\mathbf{M}_N^{\mathrm{T}}\mathbf{M}_N\right) = 0$. Note that there are N characteristic angles for an N order system, and the largest angle can be chosen as the metric of closeness to singularities. The geometrical meaning of the determinant of a screw matrix appears in Eq. (15) where the characteristic angle can merely be expressed by two determinants.

Note that the characteristic angle expressed in Eq. (15) is dimensionless, which implies the proposed metric is independent of screw magnitude. In other words, the result of the measurement is invariant no matter whether the screws used in Eq. (15) are unit screws.

6. Screws for Measuring Closeness to Singularities

As for a serial robot, each joint twist is unique and definite; as for a parallel robot, the limb twists are also definite for measuring closeness to kinematic singularities. When static singularities of a parallel robot are taken into account, the constraint wrenches after locking actuators are definite only for the limb with six-DOF; if the limb has five-DOF, these constraint wrenches are not uniquely definite. This problem also arises in the Jacobian analysis for lower mobility parallel robots,[25] and can be solved as follows[26]: first the original constraint wrench of a limb before locking actuators is gained, and then a new constraint wrench after locking an actuator in the limb is obtained, and finally this new wrench is required to be orthogonal to the original constraint wrench. In this way, constraint wrenches of a limb with five-DOF are uniquely determined. Note that the orthogonality of screws is not invariant to origin selection,[14,20] and the origin of the screw frame should be artificially set at the origin of the robotic tool frame.

For an overconstrained parallel robot, there are common constraint wrenches between limbs, and they are linearly dependent. Therefore it is meaningless to solve the characteristic angles among these common constraint wrenches, and the base screws can be chosen arbitrarily in the submanifold of the common wrenches. It only needs to calculate the angles between other wrenches and the normals to the submanifolds containing these common wrenches.

7. Closeness to Singularities of Similar Mechanisms

The weighted factor l proposed in Section 3 can be regarded as a property of the mechanism at a certain posture, which may be defined as the mean or maximum value of distances from the origin of the screw frame to axes of all the revolute joints. Under this definition, this factor varies as the mechanism moves to different configurations. Alternatively, the weighted factor l may be defined as a constant, for example, the length of some link in the mechanism, or the height of the mechanism at its initial configuration. Suppose there are two similar mechanisms as shown in Fig. 1, and the ratio of similitude is α, which means the length ratio of the corresponding links in two mechanisms is α, and the ratio of distances from the origins of two screw frames to the corresponding revolute joints in two mechanisms is α, and the ratio of two weighted factors is also α.

Suppose $\$_{Ai}$ and $\$_{Bi}$ $(i = 1, 2, \ldots, N)$ denote the corresponding screws of two similar mechanisms A and B respectively, and for each mechanism there are p $(0 \leq p \leq N)$ screws whose real parts are $\mathbf{0}$. The remaining $N - p$ screws with non-zero real parts are expressed as $\$_{Ai} = (l_A \cdot \mathbf{S}_{Ai}\ \mathbf{S}_{0Ai})^{\mathrm{T}}$ and $\$_{Bi} = (l_B \cdot \mathbf{S}_{Bi}\ \mathbf{S}_{0Bi})^{\mathrm{T}}$ for the two similar mechanisms respectively, where l_A and l_B are the weighted factors, and $l_A/l_B = \alpha$; $\mathbf{S}_{Ai}$ and $\boldsymbol{S}_{Bi}$ denote the real parts without weighted factors, and it is required that $\mathbf{S}_{Ai} = \mathbf{S}_{Bi}$ and $||\mathbf{S}_{Ai}|| = ||\mathbf{S}_{Bi}|| = 1$ for simplification of deduction; $\mathbf{S}_{0Ai}$ and $\mathbf{S}_{0Bi}$ denote the dual parts. Since $\mathbf{S}_{0Ai} = \mathbf{r}_{Ai} \times \mathbf{S}_{Ai}$ and $\mathbf{S}_{0Bi} = \mathbf{r}_{Bi} \times \mathbf{S}_{Bi}$, where $\mathbf{r}_{Ai}$ and $\mathbf{r}_{Bi}$ denote distances from the origins of two screw frames to the screw axes, $\mathbf{r}_{Ai} = \alpha\mathbf{r}_{Bi}$, therefore $\mathbf{S}_{0Ai} = \alpha\mathbf{S}_{0Bi}$, and furthermore, $\$_{Ai} = \alpha\$_{Bi}$.

Let $\mathbf{M}_{NA}$ and $\mathbf{M}_{NB}$ be the N order screw matrices of the two similar mechanisms, and then

$$\det\left(\mathbf{M}_{NA}^{\mathrm{T}}\mathbf{M}_{NA}\right) = \alpha^{2(N-p)}\det\left(\mathbf{M}_{NB}^{\mathrm{T}}\mathbf{M}_{NB}\right).$$

In the calculation of the characteristic angle, if the real part of the ith screw is $\mathbf{0}$, then the remaining $N - 1$ order screw matrix still contains $N{-}p$ screws with non-zero real parts, and therefore

$$\det\left(\mathbf{M}_{A(N-1)i}^{\mathrm{T}}\mathbf{M}_{A(N-1)i}\right) = \alpha^{2(N-p)}\det\left(\mathbf{M}_{B(N-1)i}^{\mathrm{T}}\mathbf{M}_{B(N-1)i}\right),$$
$$\|\$_{Ai}\| = \|\$_{Bi}\| = 1. \tag{16}$$

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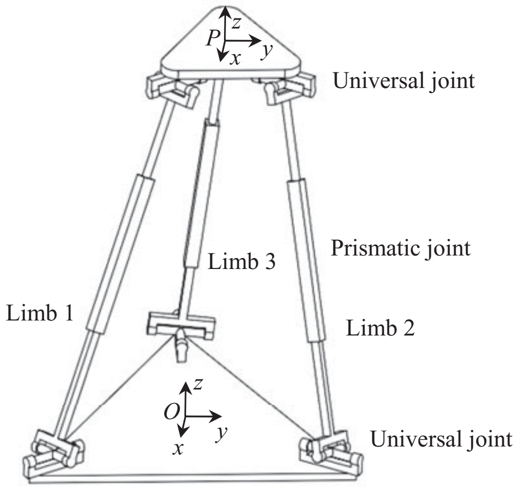

Fig. 2. The three-UPU parallel robot.

Note that Eq. (16) is only used to simplify the deduction, and actually the characteristic angle is independent of screw magnitude as interpreted in Section 5. According to Eq. (15), the closeness to singularities of mechanism A can be expressed as

$$\theta_{Ai} = \arccos \frac{\sqrt{\det\left(\mathbf{M}_{AN}^{\mathrm{T}}\mathbf{M}_{AN}\right)}}{\|\$_{Ai}\|\sqrt{\det\left(\mathbf{M}_{A(N-1)i}^{\mathrm{T}}\mathbf{M}_{A(N-1)i}\right)}} = \arccos \frac{\sqrt{\det\left(\mathbf{M}_{BN}^{\mathrm{T}}\mathbf{M}_{BN}\right)}}{\|\$_{Bi}\|\sqrt{\det\left(\mathbf{M}_{B(N-1)i}^{\mathrm{T}}\mathbf{M}_{B(N-1)i}\right)}}.$$

If the real part of the ith screw is not $\mathbf{0}$, then the remaining $N-1$ order screw matrix contains $N-p-1$ screws with non-zero real parts, and therefore

$$\det\left(\mathbf{M}_{A(N-1)i}^{\mathrm{T}}\mathbf{M}_{A(N-1)i}\right) = \alpha^{2(N-p-1)}\det\left(\mathbf{M}_{B(N-1)i}^{\mathrm{T}}\mathbf{M}_{B(N-1)i}\right),$$
$$\|\$_{Ai}\| = \alpha\|\$_{Bi}\|.$$

According to Eq. (15), the closeness to singularities of mechanism A can be expressed as

$$\theta_{Ai} = \arccos \frac{\sqrt{\det\left(\mathbf{M}_{AN}^{\mathrm{T}}\mathbf{M}_{AN}\right)}}{\|\$_{Ai}\|\sqrt{\det\left(\mathbf{M}_{A(N-1)i}^{\mathrm{T}}\mathbf{M}_{A(N-1)i}\right)}} = \arccos \frac{\sqrt{\det\left(\mathbf{M}_{BN}^{\mathrm{T}}\mathbf{M}_{BN}\right)}}{\|\$_{Bi}\|\sqrt{\det\left(\mathbf{M}_{B(N-1)i}^{\mathrm{T}}\mathbf{M}_{B(N-1)i}\right)}}.$$

Hence, no matter whether the real part of the ith screw is $\mathbf{0}$, the equation $\theta_{Ai} = \theta_{Bi}$ holds, where θ_{Bi} denotes the closeness to singularities of mechanism B. It can be seen that the proposed metric achieves the identical result for similar mechanisms with different sizes, which is an outstanding feature compared with other existing metrics.

8. An Example

A three-UPU parallel robot is shown in Fig. 2, whose moving platform is connected to the base with three limbs. Each limb consists of two universal joints and a prismatic joint equipped with an actuator. The three universal joints on the base form an equilateral triangle with the side length 0.5m, and the origin O of the base frame is at the center of this triangle. Similarly, the three universal joints on the moving platform form an equilateral triangle with the side length 0.2m, and the origin P of the tool frame is at the center of this triangle. Suppose the moving platform goes from point (0, 0, 1m) to location (0, 0.4m, 1m) keeping with the constant horizontal posture, and the closeness to singularities is measured during its motion.

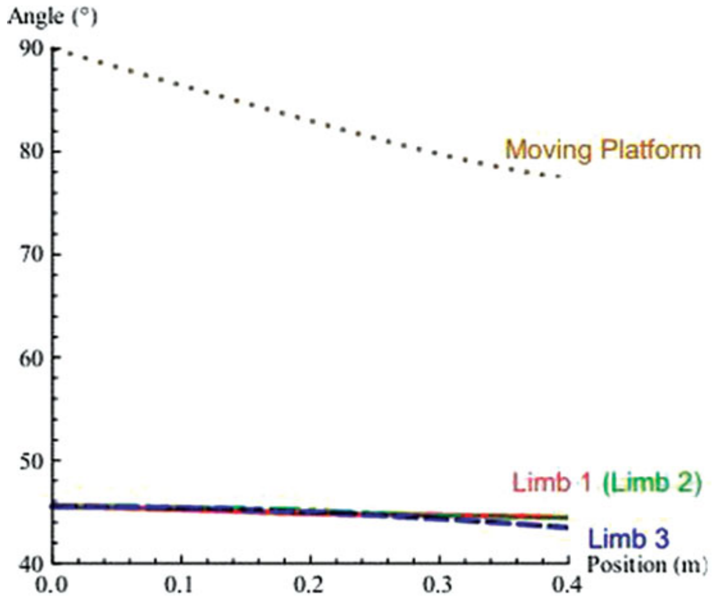

Fig. 3. Closeness to singularities of the three limbs and the moving platform.

Note that for the calculation of joint twists and limb wrenches, the origin of the screw frame should be set at point P. The transformation matrix for screws is

$$\mathbf{H} = \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{TR} & \mathbf{R} \end{pmatrix}_{6\times 6},$$

where $\mathbf{R}$ denotes the rotation matrix, and $\mathbf{T}$ denotes the translation matrix as follows.

$$\mathbf{T} = \begin{pmatrix} 0 & t_z & -t_y \\ -t_z & 0 & t_x \\ t_y & -t_x & 0 \end{pmatrix},$$

where t_x, t_y and t_z denote coordinates of point P described in the base frame along axes x, y and z respectively.

Here the weighted factor l is defined as the height of the mechanism at its initial configuration, i.e. $l = 1$m. According to Eq. (15), the curves of closeness to both kinematic singularities of the three limbs and static singularities of the moving platform are obtained as shown in Fig. 3, where the red solid curve, green dot-dash curve, and blue dashed curve stand for limbs one, two and three respectively, and the brown dotted curve is for the moving platform. Note that the red solid curve almost coincides with the green dot-dash curve. From Fig. 3 it can be found that at the initial location, values of closeness to kinematic singularities of the three limbs are the same (about 45.5°) since the three-UPU parallel robot has a symmetrical structure at the initial location; furthermore, the value of closeness to static singularities of the moving platform is 90° which indicates the robot is in a static singular configuration at the initial location. In fact, the singularity of the three-UPU parallel robot is related to the arrangement of the universal joints.[27]

Note that if a similar mechanism is half the size of the original mechanism, the height of the similar mechanism at its initial configuration is also reduced to one half, and therefore the weighted factor is $l = 0.5$m. According to Eq. (15), the closeness to singularities of the similar mechanism coincides with that of the original one.

9. Conclusions

Singularities have a great influence on kinematics and dynamics of both serial and parallel robots. In order to prevent the robot from entering singular configurations, it needs to measure the "distance"

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between the robot current configuration and the singular configuration. A novel metric based on characteristic angles is presented in this paper to measure closeness to singularities. In this approach, the characteristic angle of 90° indicates a singularity occurs, and the angle of 0° implies the robot is far away from singularities. For the problem of inconsistent dimensions in the scalar product of screws, the physical meanings of twists and wrenches are reinterpreted. For the problem of the metric invariant to origin selection, the origin of the screw frame is required to coincide with the origin of the robotic tool frame.

The major merit of the proposed metric lies in the identical result of measuring similar mechanisms with different sizes. Moreover, the result is invariant no matter whether the screws used in calculation are unit screws, since the metric expression is dimensionless. Furthermore, the geometrical meaning of the determinant of a screw matrix is clarified in the formula of the characteristic angle, which can merely be expressed by two determinants.

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