CONSTANCY OF THE HILBERT–SAMUEL FUNCTION

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Abstract. We prove a criterion for the constancy of the Hilbert–Samuel function for locally Noetherian schemes such that the local rings are excellent at every point. More precisely, we show that the Hilbert–Samuel function is locally constant on such a scheme if and only if the scheme is normally flat along its reduction and the reduction itself is regular. Regularity of the underlying reduced scheme is a significant new property.

§1. Introduction

The Hilbert–Samuel function and the multiplicity function are fundamental locally defined invariants on Noetherian schemes. They have been playing an important role in desingularization for many years [3], [5], [6], [11], [15], [17], [18], and others. Bennett studied upper semicontinuity of the Hilbert–Samuel function on schemes and linked it with Hironaka's invariant ν^*. He also proved that it is nonincreasing under permissible blowing ups. The latter are blowing ups at regular subschemes along which the singular scheme is normally flat. For the definition of the Hilbert–Samuel function of $\mathcal{X}$ and the notion of normal flatness, we refer to Definitions 2.1 and 2.4(1), respectively.

For a reduced scheme, the Hilbert–Samuel function is locally constant if and only if it is regular: this translates the question of resolution of singularities into a problem of lowering the Hilbert–Samuel function. We show here that this result can be extended to *nonreduced* schemes, as follows.

THEOREM 1.1. *Let $\mathcal{X}$ be a locally Noetherian scheme such that $\mathcal{O}_{\mathcal{X},x}$ is excellent for every $x \in \mathcal{X}$. The Hilbert–Samuel function is locally constant on $\mathcal{X}$ if and only if $\mathcal{X}_{red}$ is everywhere regular and $\mathcal{X}$ is normally flat on $\mathcal{X}_{red}$.*

In the case where $\mathcal{X}$ is not reduced of characteristic 0 or of dimension ≤ 2, using Hironaka's results, there exists a projective morphism composition of permissible blowing ups (cf. Definition 2.4(2))

$$\mathcal{X}' \longrightarrow \mathcal{X}$$

such that the Hilbert–Samuel function is locally constant on $\mathcal{X}'$. As a consequence of our theorem, there exists a projective morphism $\mathcal{X}' \longrightarrow \mathcal{X}$ such that $\mathcal{X}'_{red}$ is everywhere regular and $\mathcal{X}'$ is normally flat along $\mathcal{X}'_{red}$ (see [6, Cor. 6.19 and Rem. 6.20] and [2, Claim 2.11]). At any rate, this sheds a new light on [13, Th. I*, p. 138] reformulated in [5, Th. 11.14].

Let us note that in [17, Prop. 6.14], Villamayor gets a similar and finer result for the multiplicity function instead of the Hilbert–Samuel function, in the case where $\mathcal{X}$ is an equidimensional scheme of finite type over a perfect field k: the strata defined by the multiplicity on $\mathcal{X}_{red}$ and $\mathcal{X}$ coincide. This is achieved from a local algebraic and differential

description of the maximal multiplicity locus of a variety (see also [1]). Nonetheless, the hypotheses of Theorem 1.1 are much more general.

Apart from classical results on Hilbert–Samuel functions, this article makes essential use of Hironaka's characteristic polyhedron [15] which is an essential ingredient in the proof of Theorem 1.1. These notions and relevant properties are recalled in the next section.

A substantial difficulty here consists in dealing with nonperfect residue fields of positive characteristic. For instance, Theorem 1.1 is not obvious even for cones in affine space over such a field. To overcome this, we recall part of Giraud's theory of presentations [12] in §3 and derive as a byproduct a general result on the Hilbert–Samuel stratum for affine cones (see Proposition 3.4). The notions of directrix and ridge of a cone play an important role, and we recall them beforehand in §2 (Definition 2.8). Finally, the proof of Theorem 1.1 is given in the last section, where the main argument involves characteristic polyhedra.

CONVENTIONS 1.2. We follow the convention that zero is a natural number, $\mathbb{N} = \mathbb{Z}_{\geq 0}$. Further, we denote by $\mathbb{N}^{\mathbb{N}}$ the set of functions $\mathbb{N} \to \mathbb{N}$. We equip $\mathbb{N}^{\mathbb{N}}$ with the product ordering: for $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, we write $\alpha \leq \beta$ if and only if $\alpha(n) \leq \beta(n)$ for every $n \in \mathbb{N}$.

We use multi-index notation. For example, we write $X^A = X_1^{A_1} \cdots X_n^{A_n}$ for $X = (X_1, \ldots, X_n)$ and $A = (A_1, \ldots, A_n) \in \mathbb{N}^n$.

All schemes are assumed to be finite-dimensional.

§2. Invariants of singularities

Fix a locally Noetherian scheme $\mathcal{X}$ and a point $x \in \mathcal{X}$ (not necessarily closed). We begin by recalling the definition of local invariants for the singularity of $\mathcal{X}$ at x. For a reference providing more details on the different notions, we refer to [6, Chaps. 2 and 3].

DEFINITION 2.1 ([6, after Lem. 2.22 and Def. 2.28]).

1. Let $\mathcal{O}$ be a Noetherian local ring (which is not necessarily regular) with maximal ideal $\mathfrak{n}$ and residue field κ. The *Hilbert–Samuel function* $H_{\mathcal{O}}^{(0)} \colon \mathbb{N} \to \mathbb{N}$ *of* $\mathcal{O}$ is defined by

$$H_{\mathcal{O}}^{(0)}(n) := \dim_{\kappa}(\mathfrak{n}^n / \mathfrak{n}^{n+1}), \quad \text{for } n \in \mathbb{N}.$$

Furthermore, one defines $H_{\mathcal{O}}^{(t)} \colon \mathbb{N} \to \mathbb{N}$, for $t \in \mathbb{N}$ and $t > 0$, via the recursion

$$H_{\mathcal{O}}^{(t)}(n) := \sum_{i=0}^{n} H_{\mathcal{O}}^{(t-1)}(i), \quad \text{for } n \in \mathbb{N}.$$

2. Let $\mathcal{X}$ be a locally Noetherian catenary scheme and fix $N \geq \dim(\mathcal{X})$. The *modified Hilbert–Samuel function* $H_{\mathcal{X}} := H_{\mathcal{X}}^N \colon \mathcal{X} \to \mathbb{N}^{\mathbb{N}}$ is defined by

$$H_{\mathcal{X}}(x) := H_{\mathcal{O}_{\mathcal{X},x}}^{(\phi_{\mathcal{X}}(x))} \in \mathbb{N}^{\mathbb{N}},$$

where $\phi_{\mathcal{X}}(x) := N - \min\{\operatorname{codim}_{\mathcal{Y}}(x) \mid \mathcal{Y} \in \operatorname{Irr}(x)\}$ and $\operatorname{Irr}(x)$ denotes the set of irreducible components of $\mathcal{X}$ containing x. Often, we call $H_{\mathcal{X}}$ just the Hilbert–Samuel function of $\mathcal{X}$. Furthermore, we define, for $\nu \in \mathbb{N}^{\mathbb{N}}$,

$$\mathcal{X}(\geq \nu) := \{x \in \mathcal{X} \mid H_{\mathcal{X}}(x) \geq \nu\}$$
$$\mathcal{X}(\nu) := \{x \in \mathcal{X} \mid H_{\mathcal{X}}(x) = \nu\}$$

and call $\mathcal{X}(\nu)$ the *Hilbert–Samuel stratum of $\mathcal{X}$ for ν*.

Using the Cohen Structure Theorem for complete Noetherian local rings, we may assume, after some ring extension, and when necessary that at $x \in \mathcal{X}$, there is an embedding $(\mathcal{X}, x) \subset (\mathcal{Z}, x)$ in a regular $\mathcal{Z}$. Hence, we reduce the setting to a nonzero ideal $\mathcal{I} \subset R$ in a (complete) regular local ring R. More precisely, $R := \mathcal{O}_{\mathcal{Z}, x}$ is the local ring of $\mathcal{Z}$ at x and $\mathcal{I}$ is the ideal describing $\mathcal{X}$ locally at x. We denote by $\mathfrak{m} \subset R$ the maximal ideal of R and by $k := R/\mathfrak{m}$ the residue field of R.

The *graded ring of R at $\mathfrak{m}$* is defined as $\mathrm{gr}_{\mathfrak{m}}(R) := \bigoplus_{s \geq 0} \mathfrak{m}^s / \mathfrak{m}^{s+1}$. If we fix a regular system of parameters $(z) = (z_1, \ldots, z_n)$ for R, then there is an isomorphism $\mathrm{gr}_{\mathfrak{m}}(R) \cong k[X_1, \ldots, X_n]$, where $X_i := z_i \bmod \mathfrak{m}^2$ for $i \in \{1, \ldots, n\}$ and $n := \dim(R)$.

DEFINITION 2.2. Let $(R, \mathfrak{m}, k)$ be a regular local ring.

1. Let $f \in R \setminus \{0\}$ be a nonzero element in R. The *initial form* $\mathrm{in}_{\mathfrak{m}}(f) \in \mathrm{gr}_{\mathfrak{m}}(R)$ *of f (with respect to $\mathfrak{m}$)* is defined as the class of f in $\mathfrak{m}^d / \mathfrak{m}^{d+1}$, where $d := \mathrm{ord}_{\mathfrak{m}}(f) := \sup\{a \in \mathbb{N} \mid f \in \mathfrak{m}^a\}$ is the *order of f at $\mathfrak{m}$*. Moreover, we set $\mathrm{in}_{\mathfrak{m}}(0) := 0$.
2. Let $\mathcal{I} \subset R$ be a nonzero ideal. The *ideal of initial forms* $\mathrm{in}_{\mathfrak{m}}(\mathcal{I})$ of $\mathcal{I}$ *(with respect to $\mathfrak{m}$)* is defined as the ideal in $\mathrm{gr}_{\mathfrak{m}}(R)$ generated by the initial forms of all $f \in \mathcal{I}$,

$$\mathrm{in}_{\mathfrak{m}}(\mathcal{I}) := (\mathrm{in}_{\mathfrak{m}}(f) \mid f \in \mathcal{I}) \subseteq \mathrm{gr}_{\mathfrak{m}}(R).$$

In our setting, $x \in \mathcal{X} \subset \mathcal{Z}$ with $\mathcal{O}_{\mathcal{X}, x} \cong R/\mathcal{I}$, the initial ideal defines the *tangent cone $C_x(\mathcal{X})$ of $\mathcal{X}$ at x*,

$$C_x(\mathcal{X}) := \mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}}(R) / \mathrm{in}_{\mathfrak{m}}(\mathcal{I})) \subseteq \mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}}(R)) \cong \mathbb{A}_k^n,$$

which is a first approximation of the singularity of $\mathcal{X}$ at x.

DEFINITION 2.3. Let $\mathcal{X}$ be a locally Noetherian scheme, which is embedded in a regular $\mathcal{Z}$, and let $x \in \mathcal{X}$ be any point. Let $\mathcal{I} \subset R$ be the ideal defining X locally at x, where $(R := \mathcal{O}_{\mathcal{Z}, x}, \mathfrak{m}, k)$ is the respective regular local ring.

1. A system of elements $(f) = (f_1, \ldots, f_m) \in R^m$ is called a *standard basis of $\mathcal{I}$ at $\mathfrak{m}$* if:
   (a) $\mathrm{in}_{\mathfrak{m}}(\mathcal{I}) = (\mathrm{in}_{\mathfrak{m}}(f_1), \ldots, \mathrm{in}_{\mathfrak{m}}(f_m))$,
   (b) $\mathrm{in}_{\mathfrak{m}}(f_i) \notin (\mathrm{in}_{\mathfrak{m}}(f_1), \ldots, \mathrm{in}_{\mathfrak{m}}(f_{i-1}))$, for every $i \in \{2, \ldots, m\}$, and
   (c) if we set $\nu_i := \mathrm{ord}_{\mathfrak{m}}(f_i)$ for $i \geq 1$, we have $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$.
2. The *ν^*-invariant of $\mathcal{X} \subset \mathcal{Z}$ at x* is defined as

$$\nu_x^*(\mathcal{X}, \mathcal{Z}) := \nu_{\mathfrak{m}}^*(\mathcal{I}, R) := (\nu_1, \ldots, \nu_m, \nu_{m+1}, \ldots,),$$

where $(\nu_1, \ldots, \nu_m)$ is determined by a standard basis of $\mathcal{I}$ at $\mathfrak{m}$ and $\nu_j = \infty$ for all $j \geq m+1$.

Even though the definition of the ν^*-invariant seems to depend on the choice of a standard basis, this is not the case. For details, we refer to [6, Chap. 2] or [14]. Following Hironaka, we denote

$$\nu_x^*(\mathcal{X}, \mathcal{Z}) =: (\nu_x^1(\mathcal{I}, R), \ldots, \nu_x^m(\mathcal{I}, R), \nu_x^{m+1}(\mathcal{I}, R), \ldots,). \tag{2.1}$$

Hironaka's ν^*-invariant is an invariant measuring the complexity of the singularity of $\mathcal{X}$ at x, which is closely related to the Hilbert–Samuel function in the embedded case. In order to make the latter more concrete, we introduce the following notation.

DEFINITION 2.4 ([6, Def. 3.1]). Let $\mathcal{X}$ be a locally Noetherian scheme, and let $D \subset \mathcal{X}$ be a reduced closed subscheme. Let $I_D \subset \mathcal{O}_\mathcal{X}$ be the ideal sheaf of D in $\mathcal{X}$.

1. The scheme $\mathcal{X}$ is *normally flat along D at $x \in D$* if the stalk $\mathrm{gr}_{I_D}(\mathcal{O})_x$ of $\mathrm{gr}_{I_D}(\mathcal{O}) := \bigoplus_{t \geq 0} I_D^t / I_D^{t+1}$ is a flat $\mathcal{O}_{D,x}$-module. Furthermore, $\mathcal{X}$ is *normally flat along D* if $\mathcal{X}$ is normally flat along D at every point of D.
2. We say that D is *permissible for $\mathcal{X}$ at $x \in D$* if the following three conditions hold:
   (a) D is regular at x,
   (b) $\mathcal{X}$ is normally flat along D at x, and
   (c) D contains no irreducible component of $\mathcal{X}$ containing x.
   Moreover, D is *permissible for $\mathcal{X}$*, if D is permissible for $\mathcal{X}$ at every point of D.

REMARK 2.5. It follows from [6, Th. 3.3] that if D is regular and y is the generic point of the irreducible component of D containing x, then $\mathcal{X}$ is normally flat along D at x if and only if $H_\mathcal{X}(x) = H_\mathcal{X}(y)$.

The following result, which is included in [6, Th. 3.10], shows the close connection between the Hilbert–Samuel function and Hironaka's ν^*-invariant from the perspective of resolution of singularities.

THEOREM 2.6 (cf. [6, Th. 3.10]). *Let $\mathcal{X}$ be an excellent scheme, or a scheme which is embeddable in a regular scheme $\mathcal{Z}$. Let $D \subset \mathcal{X}$ be a permissible closed subscheme, and let $\pi_\mathcal{X}: \mathcal{X}' := \mathrm{Bl}_D(\mathcal{X}) \to \mathcal{X}$ (resp. $\pi_\mathcal{Z}: \mathcal{Z}' := \mathrm{Bl}_D(\mathcal{Z}) \to \mathcal{Z}$) be the blowing up with center D. Take any points $x \in D$ and $x' \in \pi_\mathcal{X}^{-1}(x)$. Then:*

1. $H_{\mathcal{X}'}(x') \leq H_\mathcal{X}(x)$ *(with respect to the product ordering on $\mathbb{N}^\mathbb{N}$).*
2. $\nu_{x'}^*(\mathcal{X}', \mathcal{Z}') \leq \nu_x^*(\mathcal{X}, \mathcal{Z})$ *(with respect to the lexicographical ordering).*
3. $H_{\mathcal{X}'}(x') = H_\mathcal{X}(x)$ *if and only if $\nu_{x'}^*(\mathcal{X}', \mathcal{Z}') = \nu_x^*(\mathcal{X}, \mathcal{Z})$.*

DEFINITION 2.7 ([7, Def. 3.13(1)]). Let the hypothesis be as in Theorem 2.6. A point $x' \in \pi_\mathcal{X}^{-1}(x)$ is *near to x* if $H_{\mathcal{X}'}(x') = H_\mathcal{X}(x)$.

Using the notation of Theorem 2.6, if $\mathcal{X}$ is embedded in a regular $\mathcal{Z}$, then $x' \in \pi_\mathcal{X}^{-1}(x)$ is near to x if and only if $\nu_{x'}^*(\mathcal{X}', \mathcal{Z}') = \nu_x^*(\mathcal{X}, \mathcal{Z})$.

If $x' \in \pi_\mathcal{X}^{-1}(x)$ is not near to x, the Hilbert–Samuel function (resp. the ν^*-invariant) detects a strict improvement of the singularity. Hence, for proving resolution of singularities, it is necessary to find additional invariants (resp. tools) able to detect an improvement at x' if the center is chosen suitably. The directix and the ridge of $\mathcal{X}$ at x are objects, which reveal information on the singularities of the tangent cone $C_x(\mathcal{X})$. They play a crucial role for the task of controlling the locus of near points.

Recall that a polynomial $f(X) = f(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ is called *additive* if $f(X + Y) = f(X) + f(Y)$, where $(Y) = (Y_1, \ldots, Y_n)$ is a system of indeterminates and we abbreviate $(X + Y) := (X_1 + Y_1, \ldots, X_n + Y_n)$.

DEFINITION 2.8. Let $\mathcal{I} \subset S := k[X_1, \ldots, X_n]$ be an ideal, which is generated by homogeneous elements.

1. Let $\mathcal{T}(\mathcal{I}) \subset \bigoplus_{i=1}^n k X_i$ be the smallest k-vector subspace such that

$$\big(\mathcal{I} \cap k[\mathcal{T}(\mathcal{I})]\big) \cdot S = \mathcal{I},$$

where $k[\mathcal{T}(\mathcal{I})] = \mathrm{Sym}_k(\mathcal{T}(\mathcal{I})) \subseteq S$. The *directix of the cone* $\mathrm{Spec}(S/\mathcal{I})$ is the closed subscheme $\mathrm{Dir}(S/\mathcal{I}) \subseteq \mathrm{Spec}(S/\mathcal{I})$ defined by the surjection $S/\mathcal{I} \to S/\mathcal{T}(\mathcal{I})S$.

2. The *ridge of the cone* $\mathrm{Spec}(S/\mathcal{I})$ is the maximal additive subgroup of $\mathrm{Spec}(S) \cong \mathbb{A}^n_K$ (considered as an additive group scheme), which leaves the cone $\mathrm{Spec}(S/\mathcal{I})$ stable under translation.

If $x \in \mathcal{X}$ is a point of a locally Noetherian scheme $\mathcal{X}$, with an embedding $(\mathcal{X}, x) \subset (\mathcal{Z}, x)$ for some regular $\mathcal{Z}$, then the directrix $\mathrm{Dir}_x(\mathcal{X})$ (resp. ridge $\mathrm{Rid}_x(\mathcal{X})$) of $\mathcal{X}$ at x is defined as directrix (resp. ridge) of the tangent cone $C_x(\mathcal{X})$ of $\mathcal{X}$ at x, embedded in the Zariski tangent space $T_x(\mathcal{Z})$.

In fact, one can define the directrix and the ridge of a locally Noetherian scheme $\mathcal{X}$ at x without the assumption of an embedding in a regular $(\mathcal{Z}, x)$. Both definitions of directrix and ridge coincide via the embedding $T_x(\mathcal{X}) \subset T_x(\mathcal{Z})$. For details, we refer to [6, Chap. 2], [11, Chap. I, §5)], or [4].

The last notion which we need to recall is Hironaka's characteristic polyhedron [15] (see also [6, Chap. 18]). It captures refined information on the singularity of $\mathcal{X}$ at x, which is not detected by the tangent cone, the directrix, or the ridge.

DEFINITION 2.9.   Let $(R, \mathfrak{m}, k)$ be a regular local ring, and let $\mathcal{I} \subset R$ be a nonzero ideal. We fix a system $(u) = (u_1, \ldots, u_e)$ in R, which can be extended to a regular system of parameters for R and such that $\mathrm{gr}_{\mathfrak{m}}(R)/\mathcal{T}(\mathrm{in}_{\mathfrak{m}}(\mathcal{I})) \cong k[U_1, \ldots, U_e]$ (using the notation of Definition 2.8), where $U_i := u_i \bmod \mathfrak{m}^2$ is the image of u_i in the graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$.

1. Let $(y) = (y_1, \ldots, y_r)$ be a system of elements in R such that (u, y) is a regular system of parameters for R. Let $(f) = (f_1, \ldots, f_m)$ be a standard basis for $\mathcal{I}$. For every $i \in \{1, \ldots, m\}$, consider expansions $f_i := \sum_{(A,B) \in \mathbb{Z}^{e+r}_{\geq 0}} C_{A,B,i} u^A y^B$ with coefficients $C_{A,B,i} \in R^\times \cup \{0\}$. The *projected polyhedron* $\Delta(f; u; y)$ *of* (f) *with respect to* (u, y) is defined as the smallest convex subset of $\mathbb{R}^e$ which contains all points of the set

$$\left\{ \frac{A}{\nu_i - |B|} + \mathbb{R}^e_{\geq 0} \,\middle|\, C_{A,B,i} \neq 0 \,\wedge\, |B| < \nu_i \right\},$$

   where $\nu_i := \mathrm{ord}_{\mathfrak{m}}(f_i)$, for all $i \in \{1, \ldots, m\}$.
   For a vertex $v \in \Delta(f; u; y)$ and $i \in \{1, \ldots, m\}$, the *initial form of* f_i *at* v is defined as

$$\mathrm{in}_v(f_i) := \sum_{B:|B|=\nu_i} \overline{C_{0,B,i}} Y^B + \sum_{(A,B): \frac{A}{\nu_i - |B|} = v} \overline{C_{A,B,i}} U^A Y^B \in k[U, Y],$$

   where $\overline{C_{A,B,i}} := C_{A,B,i} \bmod \mathfrak{m}$ and $Y_j := y_j \bmod \mathfrak{m}^2$ for $j \in \{1, \ldots, r\}$.
2. For fixed (u), the *characteristic polyhedron* $\Delta(\mathcal{I}; u)$ *of* $\mathcal{I}$ *at* $\mathfrak{m}$ is defined as the intersection of all projected polyhedra $\Delta(f; u; y)$, where one varies the choice of $(f; y)$ fulfilling the properties of the first part of the definition,

$$\Delta(\mathcal{I}; u) := \bigcap_{(f;y)} \Delta(f; u; y).$$

Given a vertex v of $\Delta(g; u; z)$, we say that $(g; z)$ is *prepared at* v if $(g; u; z)$ is normalized at v [6, Defs. 8.12 and 8.11] and v is not solvable [6, Def. 8.13]. We say that $(g; z)$ is a *suitable choice* if $(g; u; z)$ is prepared at every vertex v of $\Delta(g; u; z)$ [6, Defs. 8.11–8.13]. The first condition (normalized) is an appropriate choice for the generators of $\mathcal{I}$, while the second means that for every vertex $v \in \Delta(g; u; z)$ it is impossible to find a change in (z) such that corresponding projected polyhedron is contained in $\Delta(g; u; z)$, v not being a vertex.

For example, if $\mathcal{I}$ is the ideal generated by $f_1 := y_1^2 - 2u_1^2 y + u_1^4 - u_2^5$, then $\Delta(f_1; u; y_1)$ has two vertices, namely $v := (2, 0)$ and $w := (0, \frac{5}{2})$. It is easy to verify that v is a solvable vertex: if we introduce $z_1 := y_1 - u_1^2$, then $f_1 = z_1^2 - u_2^5$ and the unique vertex w is not solvable. In particular, it follows that $(f_1; z_1)$ is a suitable choice.

By [15, Th. 4.8], we have the following equality for a suitable choice $(g; z)$

$$\Delta(g; u; z) = \Delta(\mathcal{I}; u),$$

and in particular this proves that $\Delta(\mathcal{I}; u)$ is actually a polyhedron. In [15], Hironaka proved that at least in the $\mathfrak{m}$-adic completion $\widehat{R}$ of R, there exists a suitable choice $(\widehat{g}, \widehat{u})$ for $(\mathcal{I}\widehat{R}; u)$ such that $\Delta(\widehat{g}; u; \widehat{z}) = \Delta(\mathcal{I}; u)$. In general, it is not clear, whether there exists a suitable choice without passing to the completion. It is shown in [8], [10] that Hironaka's result holds without passing to the completion if we assume R to be excellent and if we additionally require mild technical conditions, which are fulfilled in many cases, for example, if the residue field of R is perfect, or if R is Henselian, or in polynomial situations.

While the definition of the characteristic polyhedron depends on an embedding, it is still a useful source for invariants of the singularity of $\mathcal{X}$ at x. For example, the number $\delta(\Delta(\mathcal{I}; u))$, defined just below, is actually an invariant of $\mathrm{Spec}(R/\mathcal{I})$. This topic has been investigated in great details in [7].

DEFINITION 2.10. Let $\Delta \subseteq \mathbb{R}_{\geq 0}^e$ be a nonempty, closed, convex subset such that $\Delta + \mathbb{R}_{\geq 0}^e = \Delta$, where $+$ denotes the Minkowski sum. Set

$$\delta(\Delta) := \min\{v_1 + \cdots + v_e \mid v = (v_1, \ldots, v_e) \in \Delta\}.$$

We define the *first face of* Δ as the face of Δ consisting of all points $v = (v_1, \ldots, v_e) \in \Delta$ with $v_1 + \cdots + v_e = \delta(\Delta)$.

§3. Giraud's presentations and their application to cones

To prove Theorem 1.1, we need Giraud's Theory of presentations [12]. In particular, we require Theorem 3.1 below, which is a refinement of [12, Prop. 4.3]. A *presentation* $\mathcal{P}$ consists of the following data:

$$(S; R; z_1, \ldots, z_e; \mathcal{I}; f_1, \ldots, f_m; n_1, \ldots, n_m; E; P_1, \ldots, P_e; q_1, \ldots, q_e; s_1, \ldots, s_e),$$

where:

(a) $S \subseteq R$ are regular local rings with respective maximal ideals N and M, such that the residue field extension is trivial, say $k := R/M = S/N$, and such that the natural morphism of graded rings $\mathrm{gr}_N(S) \to \mathrm{gr}_M(R)$ is flat.

Further, $(z_1, \ldots, z_e)$ are differential local coordinates of R/S (in the sense of [12, définition 2.2(iii)]).

(b) $\mathcal{I} \subset R$ is an ideal such that $\mathrm{gr}_M(R/\mathcal{I})$ is flat over $\mathrm{gr}_N(S)$.

(c) $f_1, \ldots, f_m$ are elements in $\mathcal{I}$ such that their images in $R_0 := R/NR$ form a standard basis[1] for the ideal $\mathcal{I}_0 := \mathcal{I}R_0$ and $n_1, \ldots, n_m$ are positive integers such that $\nu_{M_0}^*(\mathcal{I}_0, R_0) = (n_1, \ldots, n_m, \infty, \ldots)$, for $M_0 := MR_0$.

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[1] To ask that it is a standard basis is slightly more restrictive than Giraud's original definition, but it is important for our setting.

Moreover, $E \subseteq \mathbb{Z}_{\geq 0}^e$ is a certain subset fulfilling $E + \mathbb{Z}_{\geq 0}^e = E$ (the so-called exponents of the ideal $\mathcal{I}_0$ with respect to (z)) such that for all $i \in \{1, \ldots, m\}$ and $A = (A_1, \ldots, A_e) \in E$ with $|A| = A_1 + \cdots + A_e < n_i$, we have $D_A^{(z)} f_i \equiv 0$. Here, $D_A^{(z)} f_i$ denotes the Hasse–Schmidt derivative of $f_i = f_i(z)$ with respect to z^A, meaning the coefficient of T^A in the Taylor expansion of $f_i(z + T) = f_i(z_1 + T_1, \ldots, z_e + T_e)$ with respect to $T = (T_1, \ldots, T_e)$ (when it does exist, e.g., in the completion of R).

(d) $P_1, \ldots, P_e \in S[X_{A,i} \mid A \in \mathbb{Z}_{\geq 0}^e, i \in \{1, \ldots, m\} : |A| < n_i]$ are polynomials with coefficients in S that are homogeneous of degree q_i if we assign to $X_{A,i}$ the degree $n_i - |A|$.

Finally, $s_1, \ldots, s_e \in R$ are elements in R such that $s_j = P_j(D_A^{(z)} f_i)$ for all $j \in \{1, \ldots, e\}$. Additionally, there is also a condition on the Hilbert–Samuel series of $R_0/(s)R_0$, which will be automatically fulfilled in our setting as the elements s_i will form a triangular system. Thus, we do not recall the details here.

For the precise definition of a presentation, we refer to [12, définition 3.1].

THEOREM 3.1. *Let $\mathcal{P}$ be a presentation as recalled just before. Suppose that S is excellent and contains a field. Let $x \in \mathcal{X}$ be the closed point corresponding to the maximal ideal $M \subset R$. Using the notation of the definition, we further set*

$$\mathcal{X} := \operatorname{Spec}(R/\mathcal{I}) \quad \mathcal{X}_i := \operatorname{Spec}(R/(f_i)), \quad and \quad \mathcal{Y}_j := \operatorname{Spec}(R/(s_j)),$$

for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, e\}$. The Hilbert–Samuel stratum of $\mathcal{X}$ at x fulfills the following equality:

$$\operatorname{HS}_{\mathcal{X},x} = \bigcap_{i=1}^m \operatorname{HS}_{\mathcal{X}_i,x} \subset \bigcap_{j=1}^e \operatorname{HS}_{\mathcal{Y}_j,x},$$

where we abbreviate $\operatorname{HS}_{\mathcal{X},x} := \mathcal{X}(H_{\mathcal{X}}(x))$ for the Hilbert–Samuel stratum (Definition 2.1) and analogously for $\operatorname{HS}_{\mathcal{X}_i,x}$ and $\operatorname{HS}_{\mathcal{Y}_j,x}$.

Proof. Let $\mathcal{H}$ be a reduced irreducible subscheme of $\mathcal{Z} := \operatorname{Spec}(R)$. In the case that $\mathcal{H}$ is regular, we have by [12, Prop. 4.3],

$$\begin{aligned} \mathcal{H} \subseteq \operatorname{HS}_{\mathcal{X},x} &\iff \mathcal{H} \subseteq \bigcap_{i=1}^m \operatorname{HS}_{\mathcal{X}_i,x} \\ \text{and } \mathcal{H} \subseteq \bigcap_{j=1}^e \operatorname{HS}_{\mathcal{Y}_j,x}, &\text{ if } \mathcal{H} \subseteq \operatorname{HS}_{\mathcal{X},x}. \end{aligned} \tag{3.1}$$

Suppose that $\mathcal{H}$ is singular at x. We choose a curve Γ such that $x \in \Gamma \subset \mathcal{H}$, $\Gamma \not\subset \operatorname{Sing}(\mathcal{H})$. We perform a sequence of blowing ups

$$\mathcal{Z} =: \mathcal{Z}_0 \xleftarrow{\pi_1} \mathcal{Z}_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} \mathcal{Z}_n \longleftarrow \cdots, \tag{3.2}$$

where $\pi_1: \mathcal{Z}_1 \to \mathcal{Z}_0$ is the blowing up with center x, and for $i \in \{1, \ldots, n-1\}$, $\pi_{i+1}: \mathcal{Z}_{i+1} \to \mathcal{Z}_i$ is the blowing up with center some closed point $x_i \in \mathcal{Z}_i$, exceptional for π_i and on the strict transform Γ_i of Γ. By [3, (2.2.3), p. 71], there exists $N \in \mathbb{N}$ such that, for $n \geq N$, $\nu_{x_n}^*(\mathcal{H}_n, \mathcal{Z}_n) = \nu_{x_N}^*(\mathcal{H}_N, \mathcal{Z}_N)$, where $\mathcal{H}_n$ (resp. $\mathcal{H}_N$) is the strict transform of $\mathcal{H}$ and $\mathcal{Z}_n$ (resp. $\mathcal{Z}_N$) of $\mathcal{Z} := \operatorname{Spec}(R)$. Of course, for N big enough, Γ_n is regular at x_n for $n \geq N$. Then, by [3, Prop. (3.1), p. 74], Γ_n is permissible for $\mathcal{H}_n$ at x_n: the Hilbert–Samuel function of $\mathcal{H}_n$ is constant along Γ_n in a neighborhood of x_n, so $\mathcal{H}_n$ is regular at x_n. Furthermore,

n is defined to be any natural number such that the strict transform $\mathcal{H}_n$ of $\mathcal{H}$ is regular at x_n.

Notice that $\mathcal{H}_n$ is an irreducible component of the Hilbert–Samuel stratum of the strict transform of $\mathcal{X}$ (resp. $\mathcal{X}_i$, resp. $\mathcal{Y}_j$) at x_n in $\mathcal{Z}_n$ if and only if $\mathcal{H}$ is an irreducible component of the Hilbert–Samuel stratum of $\mathcal{X}$ (resp. $\mathcal{X}_i$, resp. $\mathcal{Y}_j$) at x. Let $\mathcal{P}_n$ be the transform of $\mathcal{P}$, that is, the presentation defined by an induction of [12, Th. 5.2] (which requires the assumption that S contains a field). Since $\mathcal{H}_n$ is regular at x_n, we may apply [12, Prop. 4.3] locally at x_n. Therefore, (3.1) holds for $\mathcal{H}_n$ (with the respective transforms of $\mathcal{X}, \mathcal{X}_i$, and $\mathcal{Y}$) and thus the statements also hold for $\mathcal{H}$. This proves the assertion. □

Next, we apply Theorem 3.1 in the setting of cones. Let us fix the notation: let k be a field, and let $\mathcal{I} \subset k[X_1, \ldots, X_n]$ be an ideal generated by homogeneous polynomials. We set $C := V(\mathcal{I})$. Let $J, \mathcal{J}$ be the ideals of the directrix and of the ridge of $\mathcal{I}$, respectively (Definition 2.8). Then, by [12, §1.5, Lem. 1.6] or [11, Prop. 5.4, p. I-27], up to renumbering the variables, there exist $(F_1, \ldots, F_m)$ homogeneous generators of $\mathcal{I}$ of degree $n_i = \deg(F_i)$, $1 \le i \le m$, with

$$D_A^{(X)} F_i \equiv 0 \text{ for } A \in \exp(\mathcal{I}) =: E, \ i \in \{1, \ldots, m\} \text{ with } |A| < \deg F_i, \tag{3.3}$$

where $\exp(\mathcal{I}) := \{\exp(h) \mid h \in \mathcal{I} \setminus \{0\} \text{ homogeneous}\}$ and $\exp(h)$ denotes the dominant exponent of the polynomial h with respect to the lexicographical ordering on $\mathbb{N}^n$. Furthermore, $\mathcal{J}$ is generated by a triangular basis of additive homogeneous polynomials $(s_1, \ldots, s_e)$ with

$$s_i = X_i^{q_i} + \sum_{j>i} c_{i,j} X_j^{q_i}, \ c_{i,j} \in k, \ 1 \le i \le e, \text{ with } q_1 \le q_2 \le \cdots \le q_e. \tag{3.4}$$

By [12, §1.5, Lem. 1.7] or [11, Prop. 5.4, p. I-27], there are polynomials $P_1, \ldots, P_e$, $P_i \in k[X_{A,j} \mid 1 \le j \le m, \ |A| < n_i]$, $1 \le i \le m$, homogeneous of degree q_i when we give to $X_{A,j}$ the degree $n_i - |A|$, such that

$$s_j = P_j(D_A^{(Z)} F_i). \tag{3.5}$$

To draw the connection with the notion of presentation of [12], we denote

$$Z_i := X_i, \ 1 \le i \le e, \ R := k[Z_1, \ldots, Z_e, X_{e+1}, \ldots, X_n]_M, \ M := (X_1, \ldots, X_n),$$
$$S := k[X_{e+1}, \ldots, X_m]_N, N := (X_{e+1}, \ldots, X_m).$$

The reader will verify that:

PROPOSITION 3.2.
$$\begin{aligned} \mathcal{P} \ := \ & (S; R; Z_1, \ldots, Z_e; \mathcal{I}; F_1, \ldots, F_m; n_1, \ldots, n_m; E; \\ & P_1, \ldots, P_e; q_1, \ldots, q_e; s_1, \ldots, s_e) \end{aligned} \tag{3.6}$$

is a presentation as defined by Giraud.

By [12, lemme 1.6], this presentation has the following supplementary property:

$$F_i \in k[s_1, \ldots, s_e], \ 1 \le i \le m. \tag{3.7}$$

REMARK 3.3. By Theorem 3.1 and (3.7), the Hilbert Samuel strata of the origin of the cone C and of its ridge locally coincide.

The word *locally* may be skipped; indeed, these strata coincide in the full affine scheme $\operatorname{Spec} k[X_1, \ldots, X_n]$, as stated in the proposition below which makes more precise M.-J. Pomerol's theorem [16, Prop. 2.2].

PROPOSITION 3.4. *With the notations of the beginning of this section, the Hilbert–Samuel strata of the origin of the cone C and of its ridge coincide in $\operatorname{Spec} k[X_1, \ldots, X_n]$. Their ideals are, respectively, the reduction of*

$$(D_A^{(X)} F_i \ \ |A| < n_i, \ 1 \le i \le m)$$

and the reduction of

$$(D_A^{(X)} s_i \ \ |A| < q_i, \ 1 \le i \le e)$$

which both coincide.

Proof. Denote by $\mathcal{R}$ the ridge of C. Let us blow up the origin. The strict transforms of C and $\mathcal{R}$ are the tautological line bundles over $\operatorname{Proj} k[X]/\mathcal{I}$ and $\operatorname{Proj} k[X]/\mathcal{J}$, respectively. So the strict transforms of the Hilbert–Samuel strata are empty or line bundles over Hilbert–Samuel strata of $\operatorname{Proj} k[X]/\mathcal{I}$ and $\operatorname{Proj} k[X]/\mathcal{J}$, respectively: these Hilbert–Samuel strata are subcones of $\operatorname{Spec} k[X_1, \ldots, X_n]$ and by Remark 3.3, they coincide. The equalities of the ideals are consequences of Theorem 3.1. □

PROPOSITION 3.5. *Let k be a field, and let $\mathcal{I} \subset k[X_1, \ldots, X_n]$ be an ideal generated by homogeneous polynomials; we set $C := V(\mathcal{I})$. Let $J, \mathcal{J}$ be, respectively, the ideals of the directrix and of the ridge of $\mathcal{I}$. The Hilbert–Samuel function is constant on C if and only if $\mathcal{I}_{red} = \mathcal{J}_{red} = J$.*

Proof. By Proposition 3.4 above, it is enough to prove the statement when $\mathcal{I} = \mathcal{J}$.

Let us note that, when $\operatorname{char}(k) = 0$, additive polynomials are homogeneous of degree 1. Hence, $J = \mathcal{I}$ and C is the intersection of hyperplanes of $\operatorname{Spec} k[X_1, \ldots, X_n]$, which implies the proposition in this easy case.

Now we consider the case $\operatorname{char}(k) = p > 0$. As seen above (*viz.,* before stating Proposition 3.2), there exist variables $(Z_1, \ldots, Z_e; W_1, \ldots, W_d)$ in $k[X_1, \ldots, X_n]$ $(d + e = n)$ such that $\mathcal{I} = (\sigma_1, \ldots, \sigma_e)$, where σ_i are homogeneous additive polynomials of degrees q_i with $q_1 \le q_2 \le \cdots \le q_e$ and

$$\sigma_i = Z_i^{q_i} + t_i(Z_{i+1}, \ldots, Z_e, W_1, \ldots, W_d).$$

When $\mathcal{I}_{red} = J$, by definition of the directrix, after translations on $(Z_1, \ldots, Z_e)$ if necessary, we get $C_{red} = V(Z_1, \ldots, Z_e)$. Furthermore, for any point $x \in C$, we have $\nu_x^*(C, \mathbb{A}^n) = (q_1, \ldots, q_e, \infty, \infty, \ldots)$, where $\mathbb{A}_k^n := \operatorname{Spec} k[Z_1, \ldots, Z_e; W_1, \ldots, W_d]$ (Definition 2.3(2)). This implies that C is normally flat over C_{red} [6, Th. 3.2(2)], which is equivalent to the constancy of the Hilbert–Samuel function on C [6, Th. 3.3]. This proves the converse implication in the proposition.

Let us prove the direct implication. We are in the very extreme case where, in the corresponding presentation (3.6), $e = m, F_i = s_i, 1 \le i \le e$. Then, by Theorem 3.1, $\bigcap_{i=1}^m \operatorname{HS}_{C_i, x} = \operatorname{HS}_{C, x} = C_{red}$. Consider the additive group subscheme $D \subset \mathbb{A}^n$ defined by the equations $\sigma_i^{q_e/q_i}, 1 \le i \le e$. Then D is flat over $\mathbb{A}^d = \operatorname{Spec} k[W_1, \ldots, W_d]$. Note that $D_{red} = C_{red}$.

As D is a cone, Theorem 3.1 applied to D implies that $\mathrm{HS}_{D,x} = D_{red}$. In other terms, it can be assumed that $q_1 = \cdots = q_e$. In this case, we may replace the σ_i by some $\tau_i = Z_i^{q_e} + r_i$, $r_i \in k[W_1, \ldots, W_d]$ for $1 \le i \le e$ by performing a linear change of coordinates. Since $\mathrm{HS}_{D,x} = D_{red}$ and since the natural morphism $\eta \colon D \longrightarrow \mathbb{A}^d$ is dominant, each polynomial τ_i has a prime factor of order q_e as an element of $K[Z_i]$, where $K = \mathrm{Frac}(k[W_1, \ldots, W_d])$. Therefore, $r_i \in K^{q_e}$. Now $k[W_1, \ldots, W_d]$ is integrally closed, so $r_i \in k[W_1, \ldots, W_d]^{q_e}$. Up to an affine $k[W_1, \ldots, W_d]$-linear change of coordinates on $\mathbb{A}^n$, we thus have $\tau_i = Z_i^{q_e}$, $1 \le i \le e$, so the conclusion holds. $\qquad\square$

§4. Proof of the Theorem

Recall that our goal is to prove that the Hilbert–Samuel function is locally constant on a Noetherian scheme $\mathcal{X}$ such that $\mathcal{O}_{\mathcal{X},x}$ is excellent for every $x \in \mathcal{X}$ if and only if its reduction $\mathcal{X}_{red}$ is regular and $\mathcal{X}$ is normally flat along $\mathcal{X}_{red}$. By [6, Th. 3.3] (see Remark 2.5), we have only need to prove the following: if the Hilbert–Samuel function is constant on $\mathcal{X}$, then $\mathcal{X}_{red}$ is everywhere regular.

LEMMA 4.1. *Let $\mathcal{X} = \mathrm{Spec}\, A$ with A a catenary Noetherian local ring, and let $x \in \mathcal{X}$ be the closed point. Assume that the Hilbert–Samuel function is constant on $\mathcal{X}$. Let $\mathrm{Dir}_x(\mathcal{X}) \subset \mathrm{Rid}_x(\mathcal{X}) \subset T_x(\mathcal{X})$ be respectively the directrix and the ridge of the tangent cone of $\mathcal{X}$ at x, embedded in the Zariski tangent space $T_x(\mathcal{X})$ (Definition 2.8 and the following comments). Then, we have*

$$\mathrm{Dir}_x(\mathcal{X}) = (\mathrm{Rid}_x(\mathcal{X}))_{red}.$$

Proof. When $\dim \mathcal{X} = 0$, $\mathrm{Dir}_x(\mathcal{X}) = \mathrm{Rid}_x(\mathcal{X})_{red}$ is the origin in $T_x(\mathcal{X})$. From now on, we assume

$$\dim \mathcal{X} \ge 1. \qquad\qquad\qquad\square$$

Let us note the following remark that we will use later on.

REMARK 4.2. Let $\mathcal{X}' \longrightarrow \mathcal{X}$ be the blowing up centered at x (with $\dim \mathcal{X} \ge 1$). The common value of the Hilbert–Samuel function at the generic points of the irreducible components of $\mathcal{X}$ does not change: it is $H_{\mathcal{X}}(x)$. Let $x' \in \mathcal{X}'$ map to x. By specialization [6, Th. 2.33(1)], $H_{\mathcal{X}'}(x') \ge H_{\mathcal{X}}(x)$ and by [11, Th. 3.8, p. II.28], $H_{\mathcal{X}'}(x') \le H_{\mathcal{X}}(x)$:

$$\text{every point } x' \text{ above } x \text{ is near to } x.$$

LEMMA 4.3. *Let k be a field, and let $\mathcal{I} \subset R := k[X_1, \ldots, X_n]$ be a homogeneous ideal. Let $\mathcal{J}$ be the ideal of its ridge, and let J be the ideal of its directrix. Assume*

$$J \ne \mathcal{J}_{red}.$$

Let $x \in \mathrm{Spec}(R/\mathcal{I})$ be the origin, and let $\mathcal{X}' \longrightarrow \mathrm{Spec}(R/\mathcal{I})$ be the blowing up along x. There exists $x' \in \mathcal{X}'$ above x such that

$$H_{\mathcal{X}'}(x') < H_{\mathrm{Spec}(R/\mathcal{I})}(x).$$

Proof. By [12, lemme 5.2.2], every point in $\mathcal{X}'$ is on the strict transform of the ridge and is near to x as a point of the ridge. Furthermore, a point x' near to x is on the strict transforms of $\mathcal{Y}_1, \ldots, \mathcal{Y}_e$, the hypersurfaces of equations σ_i of degrees $q_1, \ldots, q_e$, $q_1 \le \cdots \le q_e$, $\mathcal{J} = (\sigma_1, \ldots, \sigma_e)$ and near to x for all $\mathcal{Y}_i$.

Let us define, as in the proof of Proposition 3.5, $\tau_i = Z_i^{q_e} + r_i$, $r_i \in k[W_1, \ldots, W_c]$, where $\text{Vect}_k(X_1, \ldots, X_n) = \text{Vect}_k(Z_1, \ldots, Z_e, W_1, \ldots, W_c))$ is a renaming of variables after performing a k-linear change of variables. As $J \neq \mathcal{J}_{red}$, there is at least one r_{i_0} which is not a q_e-th power. So, there is a point $x' \in \pi^{-1}(x)$ on the strict transform of the (unique) prime factor of $\tau_{i_0} = Z_{i_0}^{q_e} + r_{i_0}$ which is not near to x for the hypersurface of equation τ_{i_0}. Hence, it is not near to x for some $\mathcal{Y}_i$: this point x' is not near to x for $\mathcal{X}$.   □

End of the proof of Lemma 4.1. From now on, we suppose that $\text{Dir}_x(\mathcal{X}) \subsetneq \text{Rid}_x(\mathcal{X})_{red}$ and we will deduce a contradiction. By going to the completion, we may suppose that $\mathcal{O}_{\mathcal{X},x}$ is the quotient of a regular ring $\mathcal{O}_{\mathcal{Z},x}$ of residue field k. Let $(u, y) := (u_1, \ldots, u_d, y_1, \ldots, y_r)$ be a regular system of parameters of $\mathcal{O}_{\mathcal{Z},x}$ such that the initial forms of (y) (with respect to the maximal ideal $\mathfrak{m}$ at x) are a standard basis for the ideal of the directrix of $\mathcal{X}$ at x embedded in $\text{Spec}\, k[U, Y]$, where $U_i := \text{in}_{\mathfrak{m}}(u_i)$ and $Y_j := \text{in}_{\mathfrak{m}}(y_j)$, for $i \in \{1, \ldots, d\}, j \in \{1, \ldots, r\}$. Let $(f_1, \ldots, f_m)$ be a standard basis of $\mathcal{I} \subset \mathcal{O}_{\mathcal{Z},x}$ with respect to (u, y). Let $\mathcal{Z}' \longrightarrow \mathcal{Z}$ be the blowing up along x. Denote by $\mathcal{X}'$ the strict transform of $\mathcal{X}$ and recall that $\mathcal{I} \subset \mathcal{O}_{\mathcal{Z},x}$ is the ideal of $\mathcal{X}$. The fiber above x is canonically isomorphic to the Proj of the tangent cone of $\mathcal{X}$ at x.

$$\text{Let } x' \text{ be a point in this fiber not near to } x \text{ for the tangent cone.} \qquad (4.1)$$

By Lemma 4.3, x' exists. Let $\mathcal{I}' \subset \mathcal{O}_{\mathcal{Z}',x'}$ be the strict transform of $\mathcal{I}$, and let $t \in \mathcal{O}_{\mathcal{Z}',x'}$ be a generator of the exceptional ideal. By (4.1), there exists a standard basis $(g_1, \ldots, g_{m'}) \in (\mathcal{O}_{\mathcal{Z}',x'}/(t))^{m'}$ of $\mathcal{I}' \bmod (t)$ with $\nu_{x'}^*(g_1, \ldots, g_{m'}) <_{lex} \nu_x^*(\mathcal{I})$ (notation (2.1)); this implies $\nu_{x'}^*(\mathcal{I}') <_{lex} \nu_x^*(\mathcal{I})$: x' is not near to x, this is a contradiction with Remark 4.2.

LEMMA 4.4. *Let $\mathcal{X} = \text{Spec}\, R/\mathcal{I}$, with R an excellent regular local ring, and let $x \in \mathcal{X}$ be the closed point, $\mathcal{Z} := \text{Spec}\, R$. Assume that the Hilbert–Samuel function is constant on $\mathcal{X}$. Then, for any adapted choice of variables, the characteristic polyhedron at x is empty (Definition 2.9).*

Proof. Let us first precise what is an *adapted choice of variables*: we mean a system $(u) := (u_1, \ldots, u_d) \in \mathcal{O}_{\mathcal{Z},x}^d$ which can be extended to a regular system of parameters $(u, y) = (u_1, \ldots, u_d, y_1, \ldots, y_r)$ of R such that $(\text{in}_{\mathfrak{m}}(y_1), \ldots, \text{in}_{\mathfrak{m}}(y_r))$ are equations of the directrix of $\mathcal{X}$ at x, where $\mathfrak{m} \subset R$ is the unique maximal ideal. The statement is that for any such system (u), $\Delta(\mathcal{I}; u) = \emptyset$ where $\mathcal{I} \subset \mathcal{O}_{\mathcal{Z},x}$ is the ideal of $\mathcal{X}$.

Suppose the statement is wrong. Thus, we can find a regular system of parameters $(u, y) = (u_1, \ldots, u_d, y_1, \ldots, y_r)$ and a standard basis $f = (f_1, \ldots, f_m)$ at x of the ideal $\mathcal{I}$ such that the vertices of the first face of $\Delta(f; u; y)$ (Definition 2.10) are *prepared* (see comments right after Definition 2.9). This implies that they are vertices of the characteristic polyhedron $\Delta(\mathcal{I}; u)$. Let $\pi_1 \colon \mathcal{Z}' \longrightarrow \mathcal{Z}$ be the blowing up along x, and let x' be the point of parameters $(u', y') := (u_1, u_2/u_1, \ldots, u_d/u_1, y_1/u_1, \ldots, y_r/u_1)$ (origin of the u_1-*chart*). Set $\delta := \delta(\Delta(f; u; y)) = \delta(\Delta(\mathcal{I}; u))$ (Definition 2.10) to be the modulus of the vertices of the first face.   □

As $\text{in}_{\mathfrak{m}}(f_i) \in k(x)[\text{in}_{\mathfrak{m}}(y_1), \ldots, \text{in}_{\mathfrak{m}}(y_r)]$, $1 \leq i \leq m$, where $k := R/\mathfrak{m}$, we have $\delta > 1$. The usual computations (e.g., analogous to [9, Proof of Prop. 4.15, (4.4)]) show that the smallest first coordinate of points of $\Delta(f'; u'; y')$ is $\delta - 1 > 0$. Moreover, a point v' with first coordinate $\delta - 1$ is obtained by the affine transformation

$$(a_1, a_2, \ldots, a_d) \mapsto (a_1 + \cdots + a_d - 1, a_2, \ldots, a_d) = (\delta - 1, a_2, \ldots, a_d),$$

where $v := (a_1, a_2, \ldots, a_d)$ is a point of the first face of $\Delta(f; u; y)$. Note that if v' is a vertex of $\Delta(f', u', y')$, then v is a vertex of the first face of $\Delta(f, u, y)$.

For $j \in \{1, \ldots, m\}$ and v a vertex, we write the initial form of f_j at v (Definition 2.10) as

$$\mathrm{in}_v(f_j) = F_j(Y_1, \ldots, Y_r) + \sum_{0 \le |A| \le m_j - 1} \lambda_A Y^A U^{(m_j - |A|)v},$$

where $F_j \in k[Y_1, \ldots, Y_r]$ is homogeneous of degree $m_j := \mathrm{ord}_{\mathfrak{m}}(f_j)$. If the corresponding point $v' := (\delta - 1, a_2, \ldots, a_d) \in \Delta(f'; u'; y')$ after the blowing up is a vertex, we have

$$
\begin{aligned}
\mathrm{in}_{v'}(f'_j) = & \\
F_j(Y') + & \sum_{0 \le |A| \le m_j - 1} \lambda_A Y'^A U_1'^{(m_j - |A|)(|v| - 1)} U_2'^{(m_j - |A|)a_2} \cdots U_d'^{(m_j - |A|)a_d} = \\
F_j(Y') + & \sum_{0 \le |A| \le m_j - 1} \lambda_A Y'^A U_1'^{(m_j - |A|)(\delta - 1)} U_2'^{(m_j - |A|)a_2} \cdots U_d'^{(m_j - |A|)a_d}.
\end{aligned}
\tag{4.2}
$$

Further, since $(\mathrm{in}_v(f_j))_{j \in \{1, \ldots, m\}}$ is prepared, so is $(\mathrm{in}_{v'}(f'_j))_{j \in \{1, \ldots, m\}}$. In particular, v' is prepared, that is, v' is a vertex of $\Delta(\mathcal{I}'; u')$ where $\mathcal{I}'$ is the strict transform of $\mathcal{I}$. By Remark 4.2, $V(y', u_1') \subset \mathcal{X}'$ is permissible at x', id est $\delta - 1 \ge 1$.

CLAIM 4.5. *If $\delta - 1 = 1$, we claim that the ideal*

$$J_\zeta \subset \mathrm{gr}_\zeta \mathcal{O}_{\mathcal{Z}', \zeta} \cong \mathrm{Fr}\left(\frac{\mathcal{O}_{\mathcal{Z}', \zeta}}{(u_1', y')} \right)[U_1', Y_1', \ldots, Y_r']$$

of the directrix at the generic point ζ of $V(y', u_1')$ is generated by the initial forms $(U_1', Y_1', \ldots, Y_r')$ of the elements $(u_1' = u_1, y_1', \ldots, y_r')$ (abuse of notations with (4.2)).

Proof. For $1 \le j \le m$, the initial forms are (with the obvious abuse of notation)

$$\mathrm{in}_\zeta(f_j) = F_j(Y_1', \ldots, Y_r') + \sum_{0 \le |A| \le m_j - 1} F_{j,A} Y'^A U_1'^{(m_j - |A|)(\delta - 1)} \tag{4.3}$$

$$\in \mathcal{O}_{\mathcal{Z}', x'} / (u_1', y')[U_1', Y_1', \ldots, Y_r'],$$

by identifying $\mathcal{O}_{\mathcal{Z}', x'} / (u_1', y')$ with the polynomial ring $k[\overline{u}_2', \ldots, \overline{u}_d']_{(\overline{u}_2', \ldots, \overline{u}_d')}$, where $\overline{u}_j' = u_j'$ mod (u_1', y'), $2 \le j \le d$, we get $F_{j,A} \in k[\overline{u}_2', \ldots, \overline{u}_d']$.

By [14, Lem. 1.9], $(\mathrm{in}_\zeta(f_1), \ldots, \mathrm{in}_\zeta(f_m))$ generate in $k(\zeta)[U_1', Y_1', \ldots, Y_r']$ the ideal of the tangent cone of $\mathcal{X}'$ at ζ. The field extension $k(x) \longrightarrow k(\zeta) = k(x)(\overline{u_2'}, \ldots, \overline{u_d'})$ is separable, so by [6, Lem. 2.10, p.18], the ideal of the directrix of $(F_1(Y_1', \ldots, Y_r'), \ldots, F_m(Y_1', \ldots, Y_r'))$ in $k(\zeta)[U_1', Y_1', \ldots, Y_r']$ is $(Y_1', \ldots, Y_r')$. Assume that the ideal $J_\zeta \subset k(\zeta)[U_1', Y_1', \ldots, Y_r']$ has codimension $\le r$. Then, we have $J_\zeta = (Z_1', \ldots, Z_r')$, with

$$Z_i' = Y_i' + \lambda_i U_1', \ \lambda_i \in k(\zeta), \ 1 \le i \le r. \tag{4.4}$$

Furthermore, we have, for $1 \le j \le m$,

$$\mathrm{in}_\zeta(f_j) = F_j(Y_1' + \lambda_1 U_1', \ldots, Y_r' + \lambda_r U_1') \in \mathcal{O}_{\mathcal{Z}', x'} / (u_1', y')[U_1', Y_1', \ldots, Y_r']. \tag{4.5}$$

Take any valuation w on $k(\zeta)[U_1', Y_1', \ldots, Y_r']$ obtained by giving positive weights on $U_1', \overline{u}_2', \ldots, \overline{u}_d'$ and weight 1 on the Y_i such that $w(Z_i') = 1$, for $1 \le i \le r$, and $\mathrm{in}_w(Z_i') \ne Y_i' \in \mathrm{gr}_w k(\zeta)[U_1', Y_1', \ldots, Y_r']$ for at least one i, and such that, for all i, $1 \le i \le r$, $\mathrm{in}_w(Z_i') =$

$Y_i' + \mu_i U_1' \overline{u}_2'^{\,a(i,2)} \cdots \overline{u}_d'^{\,a(i,d)}$ with $\mu_i \in k$, $a(i,j) \in \mathbb{Z}$, $2 \le j \le d$. By [15, Cor. 4.1.1, p. 286], for at least one j, $1 \le j \le m$, $\mathrm{in}_w(F_j) \ne F_j(Y_1', \ldots, Y_r')$: by (4.5), the exponents $a(i,j)$ are all in $\mathbb{N}$. By taking all the possible w, in (4.4), for all $i, 1 \le i \le r$, we get $\lambda_i \in \mathcal{O}_{\mathcal{Z}',x'}/(u_1', y')$.

We can find $(z_1, \ldots, z_r) \in \mathcal{O}_{\mathcal{Z}',x'}^r$ with $\mathrm{in}_\zeta(z_i) = Z_i'$, and $\Delta(f', u', z)$ has only vertices with first coordinate > 1. This contradicts the fact that all vertices of $\Delta(f', u', y')$ of abscissa $\delta - 1 = 1$ are prepared. We arrived to a contradiction which proves the claim. $\qquad\square$

End of the proof of Lemma 4.4. Assume $\delta - 1 = 1$, By Claim 4.5 above, the initial forms $(U_1', Y_1', \ldots, Y_r')$ generate the ideal of the directrix of $\mathcal{X}'$ at ζ. By Lemma 4.1, $(U_1', Y_1', \ldots, Y_r')$ are the equations of the reduced ridge at ζ: let

$$\pi_2 \colon \mathcal{X}'' \longrightarrow \mathcal{X}'$$

be the blowing up of $\mathcal{X}'$ along $D := V(y', u_1')$. In $\mathcal{X}''$, there is no point near to ζ, as the Hilbert–Samuel function is constant on $\mathcal{X}'$. This contradicts Remark 4.2 applied to $\zeta, \mathcal{X}'$.

All this leads to $\delta - 1 > 1$.

Consider the point $x'' \in \pi_2^{-1}(D) \subset \mathcal{X}''$ of parameters

$$(u'', y'') := (u_1', u_2', \ldots, u_d', y_1'/u_1', \ldots, y_r'/u_1').$$

Let $(f'') = (f_1'', \ldots, f_m'')$ be the strict transforms of (f'). Notice that $\Delta(f'', u'', y'')$ is obtained from $\Delta(f', u', y')$ by applying the affine transformation

$$(a_1', a_2', \ldots, a_d') \mapsto (a_1' - 1, a_2', \ldots, a_d').$$

In particular, $(a_1' - 1, a_2', \ldots, a_d')$ is a vertex of $\Delta(f'', u'', y'')$ if and only if $(a_1', a_2', \ldots, a_d')$ is one of $\Delta(f', u', y')$. For a vertex $v'' := (\delta - 2, a_2, \ldots, a_d)$ of $\Delta(f'', u'', y'')$ with smallest first coordinate (arising from a vertex $v := (a_1, \ldots, a_d)$ of the first face of $\Delta(f, u, y)$) and $j \in \{1, \ldots, m\}$, we have

$$\mathrm{in}_{v''}(f_j'') = F_j(Y'') + \sum_{0 \le |A| \le m_j - 1} \lambda_A Y''^A U_1''^{(m_j - |A|)(\delta - 2)} \prod_{\ell=2}^d U_\ell''^{(m_j - |A|)a_\ell}.$$

As v and $v' := (\delta - 1, a_2, \ldots, a_d) \in \Delta(f', u', y')$ are prepared vertices, so is v''. Hence, $v'' \in \Delta(\mathcal{I}'', u'')$ is a vertex of the characteristic polyhedron, where $\mathcal{I}''$ denotes the strict transform of $\mathcal{I}'$. This provides that $D' := V(y'', u_1'') \subset \mathcal{X}''$ is permissible at x'' (Remark 4.2) and thus $\delta - 2 \ge 1$.

Let ζ' be the generic point of D'. If we assume $\delta - 2 = 1$, then the analogous arguments as in the proof of Claim 4.5 provide that the ideal of the directrix at the generic point ζ' is generated by the initial forms $(U_1'', Y_1'', \ldots, Y_r'')$ of the elements $(u_1'', y_1'', \ldots, y_r'')$. Furthermore, $(U_1'', Y_1'', \ldots, Y_r'')$ are the equations of the reduced ridge at ζ' by Lemma 4.1. After blowing up $\mathcal{X}''$ with center D', there exists no point near to ζ', as the Hilbert–Samuel function is constant on $\mathcal{X}''$. Again, this is a contradiction to Remark 4.2 (applied for ζ' and $\mathcal{X}''$).

In conclusion, we have $\delta - 2 > 1$. An induction on δ leads to a contradiction and therefore $\Delta(\mathcal{I}; u) = \emptyset$.

End of the Theorem's proof. Let $\mathcal{X} = \mathrm{Spec}\, A$ be an affine scheme, let A be an excellent local ring, and let $x \in \mathcal{X}$ be the closed point. Assume that the Hilbert–Samuel function is constant on $\mathcal{X}$. By Lemma 4.1, the directrix at x coincides with the reduced ridge at x. This is hypothesis (*) of [10, Prop. 4.1].

Suppose $A = R/I$ for some excellent regular local ring R. The characteristic polyhedron is empty by Lemma 4.4. By [10, Prop. 4.1], there exist a standard basis $f := (f_1, \ldots, f_m) \in R^m$ for the ideal of $\mathcal{X}$ at x and a regular system of parameters (u, y) of R expressing the empty characteristic polyhedron, that is, such that $\Delta(f; u; y) = \emptyset$. By Lemma 4.4, $V(y)$ is permissible for $\mathcal{X}$ at x. Since the reduced ridge coincides with the directrix, the blowing up along $V(y)$ has no point near to x. In a neighborhood of x, $V(y)$ is the Hilbert–Samuel stratum of $\mathcal{X}$ which is $\mathcal{X}_{red}$: $V(y) = \mathcal{X}_{red}$. This ends the proof in this case.

If $\mathcal{X}$ is not embedded in a regular scheme, the completion $\widehat{A}$ of the local ring A at its maximal ideal is the quotient of a regular ring R. By the argument above, there exist regular parameters (y) in R such that $V(y) \subset \mathrm{Spec}(R)$ is the Hilbert–Samuel stratum of $\widehat{\mathcal{X}} = \mathrm{Spec}\,\widehat{A}$. By [6, Lem. 2.37(2)], $V(y)$ is the preimage of $\mathcal{X}$'s Hilbert–Samuel stratum which is $\mathcal{X}_{red}$, since A is excellent. By [6, Lem. 2.37(2)], $\mathcal{X}_{red}$ is regular at x.

REMARK 4.6. There exist excellent schemes $\mathcal{X}$ with $\mathcal{X}_{red}$ regular and with a nonconstant Hilbert–Samuel function, even if $\mathcal{X}$ is a complete intersection.

Look at this example: $\mathcal{X} \subset \mathrm{Spec}\,k[X_1, X_2, X_3]$ with ideal

$$\mathcal{I} = (X_1^2 + X_2 X_3^2, X_2^2).$$

We have $\mathcal{X}_{red} = V(X_1, X_2)$. On $\mathcal{X}$, the Hilbert–Samuel function takes different values at the origin and at the generic point.

Here is a different argument for this: the characteristic polyhedron $\Delta(\mathcal{I}; X_3)$ is not empty. By Lemma 4.4, the Hilbert–Samuel function of $\mathcal{X}$ cannot be constant.

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