

## A NOTE ON BRØNDSTED'S FIXED POINT THEOREM

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### Abstract

We show that for the case of uniformly convex Banach spaces, the conditions of Brøndsted's fixed point theorem can be relaxed.

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### 1. Introduction and main theorem

The object of this short note is a fixed point theorem by Arne Brøndsted. Let us formulate this theorem.

Let  $(X, \|\cdot\|)$  be a Banach space and let  $M \subset X$  be a closed set. We denote the closed unit ball by  $B = \{x \in X \mid \|x\| \leq 1\}$ . Assume that

$$M \cap B = \emptyset. \quad (1.1)$$

Consider a mapping  $T : M \rightarrow M$  that maps each  $x \in M$  in the direction of the ball: if  $Tx \neq x$ , then there exists  $t > 1$  such that

$$x + t(Tx - x) \in B. \quad (1.2)$$

**THEOREM 1.1** (Brøndsted [2]). *In addition to the assumptions above, suppose that*

$$\inf\{\|x\| \mid x \in M\} > 1. \quad (1.3)$$

*Then the mapping  $T$  has a fixed point.*

Observe that condition (1.3) is stronger than condition (1.1) only if  $\dim X = \infty$ .

To prove Theorem 1.1, Brøndsted endows the set  $M$  with a partial order in the following way.

**DEFINITION 1.2.** If  $x, y \in M$ , we write  $x \leq y$  provided either  $x = y$  or there exists  $t > 1$  such that

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$$x + t(y - x) \in B.$$

The second possibility can equivalently be formulated as follows: there exist  $\tilde{t} > 1$  and  $a \in X, \|a\| = 1$ , such that  $x + \tilde{t}(y - x) = a$  and  $x + t(y - x) \notin B$  for all  $t < \tilde{t}$ .

Equation (1.2) takes the form

$$x \leq Tx \quad \text{for all } x \in M. \tag{1.4}$$

Then Brøndsted observes that this partial order is finer than that of the Caristi type [3] and, by some of his other results [1], the fixed point exists.

Our aim is to show that for the class of uniformly convex Banach spaces  $X$ , Theorem 1.1 remains valid even in the critical case when condition (1.3) is replaced by (1.1). This does not follow from Brøndsted’s original method. We recall a definition.

**DEFINITION 1.3.** A Banach space  $(X, \|\cdot\|)$  is said to be uniformly convex if for any  $\sigma > 0$ , there exists  $\gamma > 0$  such that if  $\|x\| = \|y\| = 1$  and  $\|x - y\| \geq \sigma$ , then  $\|x + y\| \leq 2 - \gamma$ .

For example, the space  $L^p, p \in (1, \infty)$ , is uniformly convex. Similarly,  $\ell_p$  is uniformly convex. Each uniformly convex Banach space is reflexive and a Hilbert space is uniformly convex (see [4] and references therein).

We now state our main result.

**THEOREM 1.4.** Assume that  $X$  is a uniformly convex Banach space. If the mapping  $T$  satisfies condition (1.4) and condition (1.1) is fulfilled, then  $T$  has a fixed point.

**EXAMPLE 1.5.** For the space  $X$ , take  $\ell_p, 1 < p < \infty$ . For each  $n \in \mathbb{N}$ , define

$$M_n = \{\mathbf{x} = \{x_k\} \in \ell_p \mid x_n \geq 1 + 1/n\}, \quad M = \bigcup_{n \in \mathbb{N}} M_n.$$

It is not hard to show that the set  $M$  is closed and  $M \cap B = \emptyset$ . A sequence

$$\mathbf{x}_j = (0, \dots, 0, 1 + 1/j, 0, \dots) \quad (\text{where } 1 + 1/j \text{ stands at the } j\text{th place}),$$

belongs to  $M$  and  $\|\mathbf{x}_j\| \rightarrow 1$  as  $j \rightarrow \infty$ . Thus, the set  $M$  satisfies the hypotheses of Theorem 1.4, but not those of Theorem 1.1.

Now take any nonempty closed set  $M \subset X$  with  $M \cap B = \emptyset$  in a uniformly convex Banach space  $X$  and let  $f : B \rightarrow B$  be a mapping. Construct  $T$  as follows. Take

$$\mathbf{x} \in M, \quad \mathbf{y} = f\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)$$

and let

$$\lambda_0 = \min\{\lambda \in [0, 1] \mid \lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in M\}.$$

It is clear that  $\lambda_0 > 0$ . Define  $T\mathbf{x} = \lambda_0\mathbf{x} + (1 - \lambda_0)\mathbf{y}$ . We obviously obtain  $\mathbf{x} \leq T\mathbf{x}$  and  $T$  has a fixed point. Since we assume only  $M \cap B = \emptyset$ , this fact follows from Theorem 1.4 and it does not follow from Theorem 1.1.

## 2. Proof of Theorem 1.4

The scheme of the proof is quite standard by itself. It is clear that a maximal element of the set  $M$  provides a fixed point. To prove that the maximal element exists, we check the conditions of Zorn's lemma. This argument and the technique developed below make it possible to give a direct proof of Theorem 1.1 as well.

**PROPOSITION 2.1.** *Suppose that vectors  $a, x \in X$  have the properties*

$$\|(1-t)a + tx\| > 1 \quad \text{for all } t \in (0, 1), \quad \|x\| > 1 \text{ and } \|a\| = 1.$$

*Then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that inequality  $\|x\| \leq 1 + \delta$  implies  $\|x - a\| \leq \varepsilon$ .*

This proposition has a 'physical' interpretation. Let  $x$  be a light source placed away from the ball  $B$ , where  $\|x\| > 1$ . According to the proposition, the diameter of the light spot on the ball tends to zero as  $x$  approaches the ball, that is,  $\|x\| \rightarrow 1$ .

Here the uniform convexity of the norm is essential: such a feature fails for the norm  $\|(p, q)\| = \max\{|p|, |q|\}$  in  $\mathbb{R}^2$ .

**PROOF.** Assume the opposite: there exist  $\varepsilon > 0$  and sequences  $a_n, x_n$ , with

$$\|x_n\| > 1, \quad \|a_n\| = 1, \quad \|x_n\| \rightarrow 1 \quad \text{and} \quad \|(1-t)a_n + tx_n\| > 1, \quad (2.1)$$

such that

$$\|x_n - a_n\| > \varepsilon.$$

Consequently, for all sufficiently large  $n$ , the estimate

$$\|x_n - a_n\| = \left\| x_n - \frac{x_n}{\|x_n\|} + \frac{x_n}{\|x_n\|} - a_n \right\| \leq \alpha_n + \left\| a_n - \frac{x_n}{\|x_n\|} \right\|,$$

where

$$\alpha_n = \|x_n\| \left( 1 - \frac{1}{\|x_n\|} \right) \rightarrow 0,$$

implies

$$\left\| a_n - \frac{x_n}{\|x_n\|} \right\| \geq \varepsilon/2.$$

Substituting  $t = 1/2$  in (2.1),

$$\|a_n + x_n\| > 2. \quad (2.2)$$

The inequality

$$\left\| a_n + \frac{x_n}{\|x_n\|} \right\| > 2 - \alpha_n$$

follows from (2.2) in the same way as above. This contradicts the hypothesis of uniform convexity of the space  $X$ . The proposition is proved.  $\square$

Let  $C \subset M$  be a chain and put  $\rho = \inf\{\|u\| \mid u \in C\}$  where  $\rho \geq 1$ . The inclusion  $x \in C$  implies that  $\|x\| > 1$  provided  $\rho = 1$  and  $\|x\| \geq \rho$  provided  $\rho > 1$ .

For any  $x \in C$ , define a set

$$K_x(\rho) = \{y \in M \mid \|y\| \geq \rho, x \leq y\}.$$

The sets  $K_x(\rho)$  are nonvoid:  $x \in K_x(\rho)$  and

$$x_1 \leq x_2 \implies K_{x_2}(\rho) \subset K_{x_1}(\rho). \tag{2.3}$$

**LEMMA 2.2.** *The sets  $K_x(\rho)$  are closed.*

**PROOF.** Indeed, let a convergent sequence  $\{y_k\}$  belong to  $K_x(\rho)$  and  $y_k \rightarrow y \in M$ . This means that there are sequences  $\{\beta_k\} \subset (0, 1)$  and  $\{a_k\} \subset X$  with  $\|a_k\| = 1$ , such that

$$y_k = \beta_k a_k + (1 - \beta_k)x.$$

The sequence  $\{\beta_k\}$  contains a convergent subsequence; we keep the same notation for this subsequence, say  $\beta_k \rightarrow \beta$ . If  $\beta = 0$ , then  $\|\beta_k a_k\| \rightarrow 0$  and  $y = x \in K_\rho(x)$ . If  $\beta \neq 0$ , put

$$a = \frac{1}{\beta}y + \left(1 - \frac{1}{\beta}\right)x$$

so that

$$a_k = \frac{1}{\beta_k}y_k + \left(1 - \frac{1}{\beta_k}\right)x \rightarrow a.$$

Since  $\|a_k\| = 1$  and  $a_k \rightarrow a$ , we have  $\|a\| = 1$ . It follows that

$$y = \beta a + (1 - \beta)x.$$

Since  $y \in M$ , the parameter  $\beta$  cannot be equal to 1. The lemma is proved. □

**LEMMA 2.3.** *Suppose that  $z \in K_x(\rho)$  with  $x \in C$ . If  $\rho > 1$ , then*

$$\|z - x\| \leq (\|x\| - \rho) \frac{\|x\| + 1}{\rho - 1}.$$

*If  $\rho = 1$ , then for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that*

$$\|x\| \leq 1 + \delta \implies \|z - x\| \leq \varepsilon.$$

**PROOF.** *The case  $\rho > 1$ . The formula*

$$x + t(z - x) = a, \quad \text{where } \|a\| = 1, t > 1 \text{ and } \|x\|, \|z\| \geq \rho > 1, \tag{2.4}$$

*implies  $z = (a + (t - 1)x)/t$  and*

$$\rho \leq \|z\| \leq \frac{1}{t} + \frac{t - 1}{t} \|x\|, \quad \frac{1}{t} \leq \frac{\|x\| - \rho}{\|x\| - 1}.$$

Using (2.4) again,

$$\|z - x\| = \frac{1}{t} \|a - x\| \leq \frac{1}{t} (1 + \|x\|).$$

The case  $\rho = 1$ . The condition of the lemma that  $z \in K_x(1)$  means

$$z = \tau a + (1 - \tau)x, \quad \text{where } \|x\|, \|z\| > 1, \|a\| = 1 \text{ and } \tau \in (0, 1).$$

Therefore, the assertion of the lemma follows from Proposition 2.1 and the formulae

$$z - x = \tau(a - x), \quad \|z - x\| \leq \|x - a\|. \quad \square$$

**LEMMA 2.4.** For any  $\varepsilon > 0$ , there exists  $\tilde{x} \in C$  such that

$$C \ni x \geq \tilde{x} \implies \text{diam } K_x(\rho) \leq \varepsilon.$$

**PROOF.** The case  $\rho > 1$ . By definition of the number  $\rho$ , for any  $\varepsilon > 0$ , there is an element  $\tilde{x} \in C$  such that

$$\|\tilde{x}\| \leq \varepsilon + \rho.$$

Take any elements  $z_1, z_2 \in K_{\tilde{x}}$  and apply Lemma 2.3 for each summand on the right side of the inequality

$$\|z_1 - z_2\| \leq \|z_1 - \tilde{x}\| + \|z_2 - \tilde{x}\|. \quad (2.5)$$

Observe also that (2.3) implies

$$\tilde{x} \leq x \in C \implies \text{diam } K_x(\rho) \leq \text{diam } K_{\tilde{x}}(\rho). \quad (2.6)$$

The case  $\rho = 1$ . Fix  $\varepsilon > 0$ . By Lemma 2.3, there exists  $\delta > 0$  such that if  $\tilde{x} \in C$  and  $\|\tilde{x}\| \leq 1 + \delta$ , then for any  $z \in K_{\tilde{x}}(1)$ , one has  $\|\tilde{x} - z\| \leq \varepsilon$ . By definition of the number  $\rho$ , such an element  $\tilde{x} \in C$  exists. Thus, (2.5), (2.6) remain valid.

The lemma is proved.  $\square$

**PROOF OF THEOREM 1.4.** Therefore, we have a nested family of closed sets  $K_x(\rho)$  whose diameters tend to zero. By a well-known theorem, their intersection is not empty and consists of a single point:

$$\bigcap_{x \in C} K_x(\rho) = \{m\}.$$

The point  $m \in M$  is an upper bound for  $C$ . Indeed, for any  $x \in C$ , we have  $m \in K_x(\rho)$  and thus  $x \leq m$ . Theorem 1.4 is proved.  $\square$

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