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STRUCTURE OF THE KERNELS ASSOCIATED WITH INVARIANT SUBSPACES OF THE BERGMAN SHIFT

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In this article we consider index 1 invariant subspaces M of the operator of multiplication by $\zeta(z) = z$, M_ζ, on the Bergman space $L_a^2(\mathbb{D})$ of the unit disc $\mathbb{D}$. It turns out that there is a positive sesquianalytic kernel l_λ defined on $\mathbb{D} \times \mathbb{D}$ which determines M uniquely. Here we study the boundary behaviour and some of the basic properties of the kernel l_λ. Among other things, we show that if the lower zero set of M, $\underline{Z}(M)$, is nonempty, the kernel l_λ for fixed $\lambda \in \mathbb{D}$ has a meromorphic continuation across $\mathbb{T} \backslash \underline{Z}(M)$, where $\mathbb{T}$ is the unit circle. Furthermore we consider some special types of kernels l_λ and by studying their structure we obtain information for the invariant subspaces related to them. Lastly, and after introducing the general vector valued setting, we discuss some analogous results for the case of $\bigoplus_{i=1}^{m} L_a^2(\mathbb{D})$, where m is a positive integer.

1. INTRODUCTION

Let k be a positive sesquianalytic kernel on $\mathbb{D}$; that is for each $\lambda \in \mathbb{D}$ the function k_λ is an analytic function on $\mathbb{D}$ such that $\sum_{i,j=1}^{n} a_i \overline{a}_j k_{\lambda_i}(\lambda_j) \geqslant 0$ for all $n \in \mathbb{N}$, $a_i \in \mathbb{C}$, $\lambda_j \in \mathbb{D}$, $i, j \in \{1 \dots n\}$. It is well known that every positive sesquianalytic kernel k on $\mathbb{D}$ is the reproducing kernel for a unique Hilbert space $\mathcal{H}(k)$ of analytic functions on $\mathbb{D}$ (see [4]). In particular, if $\langle , \rangle_{\mathcal{H}(k)}$ denotes the Hilbert space inner product, $f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}(k)}$ for every $f \in \mathcal{H}(k), \lambda \in \mathbb{D}$.

Now denote by M_ζ the multiplication operator associated with the identity function $\zeta(z) = z$, $z \in \mathbb{D}$. Let also $\operatorname{Lat}(M_\zeta, \mathcal{H}(k))$ be the lattice of the invariant subspaces of $(M_\zeta, \mathcal{H}(k))$. Set $M \ominus \zeta M \equiv M \cap (\zeta M)^\perp$ and define the *index* of M to be the dimension of $M \ominus \zeta M$. That is $\operatorname{ind} M = \dim M \ominus \zeta M$. Furthermore for a subset S of $\mathcal{H}(k)$ write $[S]$ for the smallest invariant subspace which contains all of S. For a single nonzero function $f \in \mathcal{H}(k)$ write $[f]$ for $[\{f\}]$. Such invariant subspaces are called cyclic and a

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function $f \in \mathcal{H}(k)$ such that $[f] = \mathcal{H}$ is called a *cyclic vector* in $\mathcal{H}(k)$. In this article we are interested in the case of the classical Bergman space on the unit disc $\mathbb{D}$; that is the space $L_a^2(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ that are square integrable with respect to the Lebesque area measure on $\mathbb{D}$. Suppose that $M \in \mathrm{Lat}(M_\zeta, L_a^2(\mathbb{D}))$, ind $M = 1$, and that G is a unit vector in $M \ominus \zeta M$. Since $M = [G]$ (see [3]) it is elementary to show that M/G is the closure of the analytic polynomials in $L_a^2(|G|^2 d\mathcal{A})$, where $\mathcal{A}$ is the normalised Lebesque measure on the unit disc. Moreover, it is not hard to see that the point evaluations are bounded on M/G, and hence M/G has a reproducing kernel which we denote by k_λ^G. If $k_\lambda(z)$ denotes the reproducing kernel for the Bergman space and if P_M denotes the projection onto M, then it is elementary to show that $k_\lambda^G(z) = (P_M k_\lambda(z))/(\overline{G(\lambda)}G(z))$. It is well known (see Theorem 9.5 [7]) that there is a positive sesquianalytic kernel l_λ^M defined on $\mathbb{D} \times \mathbb{D}$ such that

$$(\star) \qquad \frac{P_M k_\lambda(z)}{\overline{G(\lambda)}G(z)} = \left(1 - \overline{\lambda} z l_\lambda^M(z)\right) k_\lambda(z),$$

and (see [5] or for a more general result see Remark 3.3) that l_λ^M determines the invariant subspace M uniquely. We call l_λ^M the *associated kernel* for M and if there is no ambiguity we exclude the superscript in l_λ^M.

Since the kernel l_λ^M defines the subspace M uniquely, it seems natural to ask about the structural properties of l_λ^M and relate them to common properties of the functions in M. In this article we study the boundary behaviour of l_λ^M and some of its properties. Moreover we consider the structure of certain types of kernels l_λ^M and we provide information for the structure of the cyclic vectors in M.

Now we introduce the appropriate notation and some definitions which are essential for the development of this article. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and $\mathbb{D}_e = \mathbb{C}_\infty \backslash \mathbb{D}$, where $\mathbb{D}$ is the unit disc. Denote by $\mathbb{T}$ the unit circle and given a function $f \in L_a^2(\mathbb{D})$, define its lower zero set, $\underline{Z}(f)$, to consist of all actual zeros of f inside $\mathbb{D}$, and all points λ on $\mathbb{T}$ for which $\liminf_{z \to \lambda,\, z \in \mathbb{D}} |f(z)| = 0$. Extend this notion to collection of functions S in $L_a^2(\mathbb{D})$ by declaring $\underline{Z}(S) \equiv \cap\{\underline{Z}(f) : f \in S\}$.

Definition 1.1: Suppose that E denotes an open arc or a union of open arcs on $\mathbb{T}$. We say that a meromorphic function f on $\mathbb{D}$ has a *meromorphic continuation in* $\mathbb{D}_e$ *across* E, if there is a neighbourhood V of E and a meromorphic function F defined on $\mathbb{D}_e \cup V$ such that $F(z) = f(z)$ for every $z \in V \cap \mathbb{D}$.

It is worthwhile to note that whenever a meromorphic continuation exists, it is unique. Additionally we assume that $\mathbb{T} \backslash \underline{Z}(M)$ is nonempty.

In our first result (see Corollary 2.7) we show that the kernel function l_λ^M for fixed $\lambda \in \mathbb{D}$ has a meromorphic continuation across $\mathbb{T} \backslash \underline{Z}(M)$. Our next result (see Theorem 2.8) considers the boundary behaviour of the kernel $l_\lambda(z)$. Particularly we show,

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THEOREM 1.2. *Let V be an open subset of $\mathbb{C}$ such that $V \cap \mathbb{T} \neq \emptyset$. If there is a nonzero element of M which extends to be analytic in V, then the kernel l_λ^M satisfies the following boundary conditions:*

(i) $\lim_{\lambda \to \zeta} (1 - |\lambda|^2 l_\lambda(\lambda)) |G(\lambda)|^2 = 1$ *for every* $\zeta \in V \cap \mathbb{T}$;

(ii) $\lim_{\lambda \to \zeta} \frac{\partial}{\partial z} \Big[(1 - \overline{\lambda} z l_\lambda(z)) \overline{G(\lambda)} G(z) \Big] \Big|_{z=\lambda} = 0$ *for every* $\zeta \in V \cap \mathbb{T}$,

where G is a unit cyclic vector in M.

Moreover, in Theorem 2.8 we prove,

THEOREM 1.3. *Suppose that a given positive kernel $l_\lambda(z)$ on $\mathbb{D} \times \mathbb{D}$ is the kernel function which appears in the expression for the reproducing kernel of M/G for some nonzero index 1 invariant subspace M with $\mathbb{T} \backslash \underline{Z}(M) \neq \emptyset$. Then G is the unique (up to a constant unit multiple) solution of*

(i) $\lim_{\lambda \to \zeta} (1 - |\lambda|^2 l_\lambda(\lambda)) |G(\lambda)|^2 = 1$ *for every* $\zeta \in \mathbb{T} \backslash \underline{Z}(M)$;

(ii) $\lim_{\lambda \to \zeta} \frac{\partial}{\partial z} \Big[(1 - \overline{\lambda} z l_\lambda(z)) \overline{G(\lambda)} G(z) \Big] \Big|_{z=\lambda} = 0$ *for every* $\zeta \in \mathbb{T} \backslash \underline{Z}(M)$.

Furthermore we consider certain types of kernels $l_\lambda^M(z)$ and we obtain information for M and its cyclic vectors. In Theorem 2.13 we show,

THEOREM 1.4. *If there is a constant $c \in (0,1)$ such that $\overline{\lim}_{\lambda \to \zeta,\, \lambda \in \mathbb{D}} l_\lambda(\lambda) \leqslant c < 1$ for every $\zeta \in \mathbb{T}$, then any unit cyclic vector G factors as $G(z) = B(z)F(z)$, where B is a Blaschke product, which is a finite product of interpolating Blaschke products, and $F(z)$ is an outer function which is bounded above and below.*

Our next result considers rotationally invariant sesquianalytic kernels on $\mathbb{D} \times \mathbb{D}$, and in Theorem 2.17 we prove

THEOREM 1.5. *Suppose that $l_\lambda(z)$ is a rotationally invariant sesquianalytic kernel on $\mathbb{D} \times \mathbb{D}$. Then the following holds: $k_\lambda^G(z) = (1 - \overline{\lambda} z l_\lambda(z)) k_\lambda(z)$ is a reproducing kernel for M/G, where $M \in \operatorname{Lat}(M_\zeta, L_a^2(\mathbb{D}))$, $\operatorname{ind} M = 1$ with G a unit cyclic vector in M, if and only if $l_\lambda(z) = k/(k+1)$ for some $k \in \mathbb{Z}_+ \cup \{0\}$.*

In the last section we set $\mathcal{D}$ to be a separable Hilbert space and $\mathcal{H}(k)$ to be a Hilbert space with reproducing kernel k. We consider $\mathcal{H}(k, \mathcal{D})$, which is the space of $\mathcal{D}$-valued $\mathcal{H}(k)$ functions. After discussing this general setting, one easily concludes that an analogous result as in ($\star$) holds. In this sense, and in contrast with the above theorem, where $\mathcal{D}$ is $\mathbb{C}$, we show that if k is the Bergman kernel and if $\mathcal{D}$ is $\mathbb{C}^2$, then for all $c \in [0,1)$, $l_\lambda(z)$=c is an associated kernel for some index 1 invariant subspace of $\bigoplus_{i=1}^{2} L_a^2(\mathbb{D})$ (see Theorem 3.5).

2. Some properties and the boundary behaviour of the kernels

In this section we let $M \in \operatorname{Lat}\big(M_\zeta, L_a^2(\mathbb{D})\big)$, $\operatorname{ind} M = 1$ and $\mathbb{T}\backslash \underline{Z}(M) \neq \emptyset$ and with G we denote any unit cyclic vector in M. First we show that the kernel l_λ^M has a meromorphic continuation in $\mathbb{D}_e$ across $\mathbb{T}\backslash \underline{Z}(M)$, and then we study its boundary behaviour. In order to show our main results we need the following facts. The first of them is due to Hedenmalm (see [**6**, lemma 1.4]).

Lemma 2.1. *$\mathbb{T}\backslash \underline{Z}(M) \neq \emptyset$ if and only if M contains a (nonzero) function which extends to be analytic in a neighbourhood V of a point $\zeta_o \in \mathbb{T}$.*

The main idea of the following lemma is due to Aleman and Richter (see [**1**, Lemma 3.1]).

Lemma 2.2. *Let V be an open subset of $\mathbb{C}$ such that $V \cap \mathbb{T} \neq \emptyset$. If there is a nonzero element f of M which extends to be analytic in V, then any cyclic vector G in M and every $g \in M^\perp$ have a meromorphic continuation in $\mathbb{D}_e$ across $V \cap \mathbb{T}$.*

Proof: Suppose that $g \in M^\perp$ and fix a point $\zeta_o \in V \cap \mathbb{T}$. We use standard Duality Theory, see [**2**, Section 5], to find an analytic function Ψ with $\int_{\mathbb{D}} |\Psi'|^2 \, d\mathcal{A} < \infty$ such that $\big(z\Psi(z)\big)' = g(z)$, $z \in \mathbb{D}$.

An easy calculation with power series leads to:

$$\lim_{r\to 1^-} \int_0^{2\pi} h(r\zeta)\overline{\Psi(r\zeta)}\frac{|d\zeta|}{2\pi} = \int_{\mathbb{D}} h(z)\overline{\big(z\Psi(z)\big)'}d\mathcal{A}, \quad h \in L_a^2(\mathbb{D}). \tag{2.3}$$

Without loss of generality we assume $f(\zeta_o) \neq 0$. Indeed, if $f(\zeta_o) = 0$, one shows easily that $\big[f/(z-\zeta_o)\big] = [f]$.

By (2.3), for λ in some $\mathbb{D}$-neighbourhood of ζ_o, we have

$$0 = \Big\langle \frac{f}{1-\overline{\lambda}z}, g\Big\rangle \quad \Big(\text{since } \frac{f}{1-\overline{\lambda}z} \in [f] \subseteq M\Big)$$
$$0 = \Big\langle \frac{f - f(1/\overline{\lambda})}{1-\overline{\lambda}z}, g\Big\rangle + f(1/\overline{\lambda})\Big\langle \frac{1}{1-\overline{\lambda}z}, g\Big\rangle.$$

Note that $1/(1-\overline{\lambda}z)$ is the Szegö kernel, thus

$$0 = \Big\langle \frac{f - f(1/\overline{\lambda})}{1-\overline{\lambda}z}, g\Big\rangle + f(1/\overline{\lambda})\,\overline{\Psi(\lambda)}.$$

Hence,

$$f(1/\overline{\lambda})\overline{\Psi(\lambda)} = -\int_{\mathbb{D}} \frac{f - f(1/\overline{\lambda})}{1-\overline{\lambda}z}\overline{g(z)}d\mathcal{A}. \tag{2.4}$$

For $\lambda \in V$, write

$$\begin{aligned} F(\lambda) &= -\frac{1}{f(1/\overline{\lambda})}\int_{\mathbb{D}} \frac{f - f(1/\overline{\lambda})}{1-\overline{\lambda}z}\overline{g(z)}d\mathcal{A} \\ &= \frac{1}{\overline{\lambda}f(1/\overline{\lambda})}\int_{\mathbb{D}} \frac{f - f(1/\overline{\lambda})}{z - \frac{1}{\overline{\lambda}}}\overline{g(z)}d\mathcal{A}. \end{aligned} \tag{2.5}$$

Since f extends to be analytic in a neighbourhood V of ζ_o, then (2.5) implies that F has a meromorphic continuation in $\mathbb{D}_e$ across $V \cap \mathbb{D}$. Since $F = \overline{\Psi}$ on $V \cap \mathbb{D}$ and since $g(z) = \big(z\Psi(z)\big)'$, $z \in \mathbb{D}$, then g also has a meromorphic continuation in $\mathbb{D}_e$ across $V \cap \mathbb{T}$. Now observe that any cyclic vector G of M is an element of $(\zeta M)^{\perp}$. Since $\operatorname{ind} M = 1$, it is easy to see that

$$(2.6) \qquad (\zeta M)^{\perp} = \begin{cases} M^{\perp} \vee \{1\} & \text{if } 0 \notin Z(M) \\ M^{\perp} \vee \{z^{\rho}\} & \text{if } 0 \in Z(M), \text{ with multiplicity } \rho, \end{cases}$$

where for a set S, $\vee\{S\}$ denotes the closed linear span of S. Hence, G has also a meromorphic continuation in $\mathbb{D}_e$ across $V \cap \mathbb{T}$. ∎

Considering the above, it is not hard to show the following

Corollary 2.7. *The kernel function l_{λ}^{M}, for fixed $\lambda \in \mathbb{D}$, has a meromorphic continuation in $\mathbb{D}_e$ across $\mathbb{T}\backslash \underline{Z}(M)$.*

Proof: Write $P_M = Id - P_{M^{\perp}}$, where Id is the identity operator on $L_a^2(\mathbb{D})$, and use $(\star)$ to get $l_{\lambda}(z) = \Big(1 - \big(1 - (1-\overline{\lambda}z)^2 P_{M^{\perp}} k_{\lambda}(z)\big)/\big(G(z)\overline{G(\lambda)}\big)\Big)1/\overline{\lambda}z$.

Now observe that $P_{M^{\perp}} k_{\lambda}(z)$ is a function in $M^{\perp}$; thus the result follows from Lemma 2.2 and Lemma 2.1. ∎

Next we consider the boundary behaviour of the associated kernel l_{λ}, and we relate it with the cyclic vectors in M.

Theorem 2.8. *Let V be an open subset of $\mathbb{C}$ such that $V \cap \mathbb{T} \neq \emptyset$. If there is a nonzero element f of M which extends to be analytic in V, then the kernel l_{λ} satisfies the following boundary conditions:*

(i) $\lim\limits_{\lambda \to \zeta} \big(1 - |\lambda|^2 l_{\lambda}(\lambda)\big)\big|G(\lambda)\big|^2 = 1$ *for every* $\zeta \in V \cap \mathbb{T}$;

(ii) $\lim\limits_{\lambda \to \zeta} \dfrac{\partial}{\partial z}\Big[\big(1 - \overline{\lambda} z l_{\lambda}(z)\big)\overline{G(\lambda)}G(z)\Big]\Big|_{z=\lambda} = 0$ *for every* $\zeta \in V \cap \mathbb{T}$.

In the case where G is continuous on $\overline{\mathbb{D}}$, condition (i) was proved in [**7**, Theorem 9.8]. It is also worthwhile to observe that by Lemma 2.1 the above conditions hold on $\mathbb{T}\backslash \underline{Z}(M)$.

Proof: Fix a point $\zeta_o \in V \cap \mathbb{T}$. By $(\star)$,

$$1 - \frac{P_{M^{\perp}} k_{\lambda}(z)}{k_{\lambda}(z)} = \big(1 - \overline{\lambda} z l_{\lambda}(z)\big)\overline{G(\lambda)}G(z),$$

and

$$\frac{\partial}{\partial z}\left(1 - \frac{P_{M^{\perp}} k_{\lambda}(z)}{k_{\lambda}(z)}\right) = \frac{P_{M^{\perp}} k_{\lambda}(z)}{(k_{\lambda}(z))^2}\frac{\partial}{\partial z}k_{\lambda}(z) - \frac{\dfrac{\partial}{\partial z}P_{M^{\perp}} k_{\lambda}(z)}{k_{\lambda}(z)}.$$

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Since

$$\lim_{\lambda\to\zeta_o}\frac{1}{k_\lambda(\lambda)}=\lim_{\lambda\to\zeta_o}\left(1-|\lambda|^2\right)^2=0,$$

$$\lim_{\lambda\to\zeta_o}\left.\frac{\frac{\partial}{\partial z}k_\lambda(z)}{(k_\lambda(z))^2}\right|_{z=\lambda}=\lim_{\lambda\to\zeta_o}-2\lambda\left(1-|\lambda|^2\right)=0,$$

to conclude the proof of the theorem, it is enough to show that $P_{M^\perp}k_\lambda(\lambda)$ and $\frac{\partial}{\partial z}P_{M^\perp}k_\lambda(z)\Big|_{z=\lambda}$ are bounded as λ approaches ζ_o, $\lambda\in V\cap\mathbb{D}$. The proof depends on Lemma 2.2 and more precisely on equation (2.5).

In what follows, C is a positive constant depending on V and it may vary depending on the estimates. We denote by $\|\cdot\|$ the norm in $L^2_a(\mathbb{D})$. We write equation (2.5) as

$$F(\lambda)=\frac{1}{\overline{\lambda}}\frac{1}{f(1/\overline{\lambda})}\int_{V\cap\mathbb{D}}\frac{f-f(1/\overline{\lambda})}{z-1/\overline{\lambda}}\overline{g(z)}\,d\mathcal{A}+\frac{1}{\overline{\lambda}}\frac{1}{f(1/\overline{\lambda})}\int_{\mathbb{D}\setminus V}\frac{f-f(1/\overline{\lambda})}{z-1/\overline{\lambda}}\overline{g(z)}\,d\mathcal{A},$$

where $g\in M^\perp$, $\lambda\in V$ and $g(z)=\left(z\Psi(z)\right)'$, $z\in\mathbb{D}$. In addition, there is a compactly contained neighbourhood V' of ζ_o inside V where we may suppose that f has no zeros. Suppose that U_1 is any neighbourhood of ζ_o which is compactly supported in V' and U_2 is another neighbourhood of ζ_o which is compactly supported in U_1. If $(1/\overline{\lambda})\in U_2$, then

$$\int_{U_1\cap\mathbb{D}}\left|\frac{f-f(1/\overline{\lambda})}{z-1/\overline{\lambda}}\right|\left|\overline{g(z)}\right|d\mathcal{A}\leqslant C\|g\|,$$

because $\left(f(z)-f\left((1/\overline{\lambda})\right)\right)/\left(z-(1/\overline{\lambda})\right)$ is uniformly bounded for $z\in U_1$ and for $(1/\overline{\lambda})\in U_2$. Furthermore,

$$\int_{\mathbb{D}\setminus U_1}\left|\frac{f-f(1/\overline{\lambda})}{z-1/\overline{\lambda}}\right|\left|\overline{g(z)}\right|d\mathcal{A}\leqslant C\|g\|,$$

because $\left|z-(1/\overline{\lambda})\right|$ is bounded away from 0 and $\left|f(1/\overline{\lambda})\right|$ remains bounded.

Suppose that $U=\left\{\lambda\in\mathbb{C}:(1/\overline{\lambda})\in U_2\right\}$. Then $\left|F(\lambda)\right|\leqslant C\|g\|$ for all $\lambda\in U$. Since $F(z)=\overline{\Psi(z)}$ for $z\in\mathbb{D}\cap V'$, $\left|z\Psi(z)\right|\leqslant C\|g\|$ for every $z\in\mathbb{D}\cap U$. We now apply the Cauchy integral formula for the derivatives and we obtain

$$\left|g(z)\right|\leqslant C\|g\|,\ z\in\mathbb{D}\cap U,\tag{2.9}$$

$$\left|g'(z)\right|\leqslant C\|g\|,\ z\in\mathbb{D}\cap U,\tag{2.10}$$

where C depends only on the neighbourhood V and the function f, but not on $g\in M^\perp$.

If we choose g to be $P_{M^\perp}k_\lambda$, considering $P_{M^\perp}k_\lambda$ as a function of z, the proof is complete. Indeed, $\left|P_{M^\perp}k_\lambda(\lambda)\right|=\left\|P_{M^\perp}k_\lambda\right\|^2$, $\lambda\in\mathbb{D}$, and hence by (2.9), $\left\|P_{M^\perp}k_\lambda\right\|\leqslant C$ for every $\lambda\in\mathbb{D}\cap U$.

In view of (2.9) and (2.10), $\left|P_{M^\perp}k_\lambda(\lambda)\right|\leqslant C$ and $\left|\frac{\partial}{\partial z}P_{M^\perp}k_\lambda(z)\Big|_{z=\lambda}\right|\leqslant C$, for $\lambda\in\mathbb{D}\cap U$, with C depending only on the neighbourhood V and the function f. That is

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the end the proof, since ζ_o is an arbitrary point in $\mathbb{T} \cap V$ and U is a neighbourhood of ζ_o. ∎

LEMMA 2.11. *Suppose that $r_\lambda(z)$ is a positive definite sesquianalytic kernel on $\mathbb{D} \times \mathbb{D}$ and I is an open set on $\mathbb{T}$ such that $\lim_{\lambda \to \zeta} r_\lambda(\lambda)$ is defined for $\zeta \in I$ and*

$$(\alpha) \qquad r_\zeta(\zeta) < 1 \quad \text{for every } \zeta \in I,$$

$$(\beta) \qquad \lim_{\lambda \to \zeta} \left[\frac{\partial}{\partial z} (1 - \overline{\lambda} z r_\lambda(z)) \Big|_{z=\lambda} \right] \quad \text{exists for every } \zeta \in I.$$

Then there is at most one analytic function G (modulo a unit constant multiple) on $\mathbb{D}$ such that

$$(\mathrm{A}) \qquad \lim_{\lambda \to \zeta} \left(1 - |\lambda|^2 r_\lambda(\lambda)\right) \left|G(\lambda)\right|^2 = 1 \quad \zeta \in I,$$

$$(\mathrm{B}) \qquad \lim_{\lambda \to \zeta} \frac{\partial}{\partial z} \left[\left(1 - \overline{\lambda} z r_\lambda(z)\right) \overline{G(\lambda)} G(z) \right] \Big|_{z=\lambda} = 0 \quad \zeta \in I.$$

PROOF: Suppose that $G_i, i = 1,2$ are two analytic functions on $\mathbb{D}$ which satisfy (A) and (B). From (α) and (A) we conclude that $\left|G_i(\zeta)\right| = \lim_{\lambda \to \zeta} \left|G_i(\lambda)\right|$ exists on I, and $\left|G_1(\zeta)\right| = \left|G_2(\zeta)\right|$, $\zeta \in I$. We apply $(\alpha), (\beta)$ and (A), on (B) and we get that

$$\left(\frac{G_i'}{G_i}(\zeta) = \lim_{\lambda \to \zeta} \frac{G_i'(\lambda)}{G_i(\lambda)} \right)$$

exists on I, and

$$\lim_{\lambda \to \zeta} \left[\left(\frac{\partial}{\partial z} \left(1 - \overline{\lambda} z r_\lambda(z)\right) \right) \left|G_i(\lambda)\right|^2 + \frac{G_i'(\lambda)}{G_i(\lambda)} \right] = 0, \ \zeta \in I.$$

Now by setting $H_i(z) \equiv \left(G_i'(z)\right) / \left(G_i(z)\right)$, $z \in \mathbb{D}$, $i = 1,2$, we conclude easily that $H_1(\zeta) = H_2(\zeta)$, $\zeta \in I$. Since H_i, $i = 1,2$, are meromorphic in $\mathbb{D}$, $H_1(z) = H_2(z)$, for every $z \in \mathbb{D}$. Now choose an open simply connected region U in $\mathbb{D}$ such that $U \cap Z(G_i) \cap Z(G_i') = \emptyset$, $i = 1,2$. In U, since H_i is the logarithmic derivative of G_i, $\left[\log G_1(z)\right]' = \left[\log G_2(z)\right]'$, $z \in U$. This leads to $G_1(z) = kG_2(z)$, $z \in U, k \in \mathbb{C}$ with $|k| = 1$, hence $G_1(z) = G_2(z)$ for all $z \in \mathbb{D}$, modulo a unit constant multiple. ∎

Now it is not hard to show the following theorem.

THEOREM 2.12. *Suppose that a given positive kernel $l_\lambda(z)$ on $\mathbb{D} \times \mathbb{D}$ is the kernel function which appears in the expression for the reproducing kernel of M/G for some nonzero index 1 invariant subspace M with $\mathbb{T} \backslash \underline{Z}(M) \neq \emptyset$. Then a unit cyclic vector G is the unique (modulo a unit constant multiple) solution of (A) and (B) in the above lemma with $I = \mathbb{T} \backslash \underline{Z}(M)$.*

PROOF: Since there is a cyclic vector G such that $k_\lambda^G(z) = \big(1 - \overline{\lambda} z l_\lambda(z)\big)k_\lambda(z)$ and $\mathbb{T}\backslash \underline{Z}(M) \neq \emptyset$, in light of Theorem 2.8, it is not hard to prove that $l_\lambda(z)$ satisfies (α) and (β) of the above lemma; hence G satisfies (A) and (B). The above lemma concludes the proof. ∎

In the rest of this section by studying certain types of kernels $l_\lambda^M(z)$ we obtain information for M and its cyclic vectors.

THEOREM 2.13. *If there is a constant $c \in (0,1)$ such that $\overline{\lim\limits_{\lambda\to\zeta,\lambda\in\mathbb{D}}} l_\lambda(\lambda) \leqslant c < 1$ for every $\zeta \in \mathbb{T}$, then any cyclic vector G factors as $G(z) = B(z)F(z)$, where B is a Blaschke product, which is a finite product of interpolating Blaschke products, and $F(z)$ is an outer function which is bounded above and below.*

PROOF: We define the linear transformation $T : L_a^2 \mapsto M/G$ on the finite linear combinations of the reproducing kernels of $L_a^2(\mathbb{D})$ by $Tk_\lambda(z) = k_\lambda^G(z) = \big(1 - \overline{\lambda} z l_\lambda(z)\big)k_\lambda(z)$, $\lambda, z \in \mathbb{D}$. Note that we have T densely defined with dense range (since finite linear combinations of kernels k_λ are dense in L_a^2, and finite linear combinations of kernels k_λ^G are dense in M/G). Consequently, since $\overline{\lambda} z l_\lambda(z) k_\lambda(z)$ is positive definite, $\left\|T\sum\limits_{i=1}^{n} a_i k_{\lambda_i}\right\| \leqslant \left\|\sum\limits_{i=1}^{n} a_i k_{\lambda_i}\right\|$ for every $n \in \mathbb{N}$ and $\lambda_i \in \mathbb{D}$, $a_i \in \mathbb{C}$, $1 \leqslant i \leqslant n$, and hence $\|T\| \leqslant 1$.

If we write $\overline{\lambda} z l_\lambda(z) = \sum\limits_{n\geqslant 0} \overline{f_n(\lambda)} f_n(z)$ for some H^∞ functions f_n, $n \in \mathbb{N}$, and if we use the hypothesis and the subharmonicity of $\sum\limits_{n\in\mathbb{N}} \big|f_n(z)\big|^2$, we obtain $\sum\limits_{n\in\mathbb{N}} \big|f_n(z)\big|^2 \leqslant c < 1$ for every $z \in \mathbb{D}$. Since $\|f_n\|_\infty \leqslant 1$ for every $n \in \mathbb{N}$, it is clear that $M_{f_n}: L_a^2(\mathbb{D}) \mapsto L_a^2(\mathbb{D})$ is bounded. Furthermore, one has $M_{f_n}^* k_\lambda = \overline{f_n(\lambda)} k_\lambda$, $\lambda \in \mathbb{D}$. By setting $L = \sum\limits_{n\geqslant 0} M_{f_n} M_{f_n}^*$ and considering the above, it is elementary to show that the adjoint of T, namely T^*, is the inclusion map and that $T^*T = I - L$. Furthermore, if $h \in L_a^2(\mathbb{D})$,

$$\|L\| = \sup_{\|h\|=1} \sum_{n\geqslant 0} < M_{f_n}^* h, M_{f_n}^* h > \leqslant \sup_{\|h\|=1} \int_{\mathbb{D}} \sum_{n\geqslant 0} |f_n|^2 \, |h|^2 dA \leqslant c < 1.$$

This implies that T^*T is invertible, and thus T is bounded below. Additionally, $cl\,range\,T = M/G$, and now using well known results from Functional Analysis, it is not hard to conclude that T is invertible. Since T^* is also invertible, it is possible to choose some positive numbers c_1, c_2, such that $c_1\|g/G\|_G \leqslant \big\|T^*(g/G)\big\| \leqslant c_2\|g/G\|_G$ for every $g \in M$. Moreover, and since T^* is the inclusion map, by taking $g = pG$ with p an arbitrary analytic polynomial in $\mathbb{D}$ we get

$$c_1\|pG\| \leqslant \|p\| \leqslant c_2\|pG\|. \tag{2.14}$$

The next argument shows that $G \in H^\infty$.

CLAIM. The cyclic vectors G are multipliers of $L_a^2(\mathbb{D})$ and hence elements in H^∞.

If we choose $\{p_n\}_{n\in\mathbb{N}}$ to be a sequence of analytic polynomials converging in $L^2_a(\mathbb{D})$ to f, we can get at least pointwise convergence of $\{p_nG\}_{n\in\mathbb{N}}$ to fG. We use Fatou's lemma and (2.14) to get

$$\|fG\| \leqslant \varliminf_{n\to\infty} \|p_nG\| \leqslant \varliminf_{n\to\infty} 1/c_1\|p_n\| = 1/c_1\|f\|,$$

which implies that G is a multiplier of $L^2_a(\mathbb{D})$, that is equivalent of G being an H^∞ function.

The above claim implies that G factors as $G(z) = k\Phi(z)F(z)$ where k is a constant and Φ, F are H^∞ inner and H^∞ outer functions respectively.

In what follows, k and k' denote positive numbers which may vary at each step of the proof depending on the estimates.

If $|F(z)| \leqslant k$, $z \in \mathbb{D}$, then $\|pG\| \leqslant k\|p\Phi\|$, and by (2.14)

$$\|p\| \leqslant k\|p\Phi\| \quad \text{for every analytic polynomial } p \text{ in } \mathbb{D}. \tag{2.15}$$

A result due to McDonald and Sundberg, see [**10**, Proposition 22], forces Φ to be a Blaschke product B and in fact, a finite product of interpolating Blaschke products (see also Horowitz [**8**, p. 202]).

To complete the proof it remains to show that $1/F$ is an element in H^∞. To this end, if $h \in H^2$ and since by Beurling's Theorem F is a cyclic element in H^2, then there exists a sequence of analytic polynomials in $\mathbb{D}$, $\{p_n\}_{n\in\mathbb{N}}$, such that $p_nF \to h$ in H^2 norm, and hence in $L^2_a(\mathbb{D})$ norm. Particularly, $p_n(z) \to (h(z))/(F(z))$ pointwise in $\mathbb{D}$. Since $\Phi \in H^\infty$, $\lim_{n\to\infty} \|p_nG - h\Phi\| = 0$. We put everything together and we use Fatou's lemma to get

$$\|h/F\| \leqslant \varliminf_{n\to\infty} \|p_n\| \leqslant k \varliminf_{n\to\infty} \|p_nG\| = k\|h\Phi\|.$$

Thus, choosing h to be an analytic polynomial p in $\mathbb{D}$, we get

$$\|p/F\| \leqslant k\|p\Phi\| \leqslant k'\|p\|.$$

A similar argument as in the claim shows that $1/F \in H^\infty$. ∎

Lemma 2.16. *Suppose that $M \in \operatorname{Lat}(M_\zeta, L^2_a(\mathbb{D}))$, $\operatorname{ind} M = 1$ and $l_\lambda(z)$ is the associated kernel for M. If in addition $l_\lambda(z)$ is rotationally invariant; that is $l_{\lambda\cdot\zeta}(z\cdot\zeta) = l_\lambda(z)$ for every $\lambda, z \in \mathbb{D}$, $\zeta \in \mathbb{T}$, then any unit cyclic vector G in M, is of the form $G(z) = c_kz^k$ for some $k \in \mathbb{Z}_+ \cup \{0\}$, where $|c_k| = \sqrt{k+1}$.*

Proof: The reproducing kernel property of k^G_λ, the fact that the Lebesgue measure on $\mathbb{D}$ is rotationally invariant, and the hypothesis of the lemma, imply that

$$p(\lambda) = \int_{\mathbb{D}} p(z)\overline{k^G_\lambda(z)}|G(z\zeta)|^2 d\mathcal{A}(z)$$

for every analytic polynomial p in $\mathbb{D}$, and every ζ in $\mathbb{T}$. Since M/G is the closure of the analytic polynomials in $L_a^2(|G|^2 d\mathcal{A})$, the above equation implies that $|G(z\zeta)| = |G(z)|$ for $z \in \mathbb{D}$, $\zeta \in \mathbb{T}$. Hence, $G(z\zeta) = c(\zeta)G(z)$ for some function $c(\zeta)$. By taking the derivative with respect to ζ we see that $zG'(z) = \alpha G(z)$ for every $z \in \mathbb{D}$ and for some constant α.

Furthermore, if we write $G(z) = \sum\limits_{n\geqslant 0} c_n z^n$, $c_n \in \mathbb{C}$, $n \in \mathbb{N}$, and use the above equation, we have $\sum\limits_{n\geqslant 1} nc_n z^n = \sum\limits_{n\geqslant 0} \alpha c_n z^n$, $\alpha, c_n \in \mathbb{C}$, $n \geqslant 0$. This implies that there is a unique $k \in \mathbb{Z}_+ \cup \{0\}$ such that $G(z) = c_k z^k$. Moreover, since $\|G\| = 1$, $|c_k| = \sqrt{k+1}$. ∎

THEOREM 2.17. *Suppose that $l_\lambda(z)$ is a rotationally invariant sesquianalytic kernel on $\mathbb{D}\times\mathbb{D}$. Then the following holds: $k_\lambda^G(z) = \big(1 - \overline{\lambda} z l_\lambda(z)\big)k_\lambda(z)$ is a reproducing kernel for M/G, where $M \in \operatorname{Lat}\big(M_\zeta, L_a^2(\mathbb{D})\big)$, ind $M = 1$, if and only if, $l_\lambda(z) = k/(k+1)$ for some $k \in \mathbb{Z}_+ \cup \{0\}$.*

PROOF: We suppose that $k_\lambda^G(z) = \big(1 - \overline{\lambda} z l_\lambda(z)\big)k_\lambda(z)$ is a rotationally invariant reproducing kernel for M/G. If $\{e_n\}_{n=0}^\infty$ is an orthonormal basis of M/G, then $k_\lambda^G(z) = \sum\limits_{n\geqslant 0} \overline{e_n(\lambda)} e_n(z)$, and for every $n \in \mathbb{N}$, $e_n(z) = a_n z^n$, $n \in \mathbb{N}$ for some $a_n \in \mathbb{C}$.

By the previous lemma there is a $k \in \mathbb{Z}_+ \cup \{0\}$ such that $G(z) = c_k z^k$, $|c_k| = \sqrt{k+1}$. Therefore, $e_n(z) = \sqrt{(n+k+1)/(k+1)}z^n$, $n \in \mathbb{N}$. Hence,

$$\begin{aligned} k_\lambda^G(z) =& \sum_{n\geqslant 0} \frac{n+k+1}{k+1}\overline{\lambda}^n z^n = \frac{1}{k+1}\Big(\sum_{n\geqslant 0}(n+1)\overline{\lambda}^n z^n + k\sum_{n\geqslant 0}\overline{\lambda}^n z^n\Big) \\ =& \frac{1}{k+1}\Big(\frac{1}{(1-\overline{\lambda}z)^2} + \frac{k}{(1-\overline{\lambda}z)}\Big) = \Big(1 - \overline{\lambda}z\Big(\frac{k}{k+1}\Big)\Big)k_\lambda(z). \end{aligned}$$

Thus, $l_\lambda(z) = k/(k+1)$ for some $k \in \mathbb{Z}_+ \cup \{0\}$.

For the converse we suppose that $l_\lambda(z) = k/(k+1)$ for some $k \in \mathbb{Z}_+ \cup \{0\}$. We write

$$\Big(1 - \overline{\lambda}z\Big(\frac{k}{k+1}\Big)\Big)k_\lambda(z) = \sum_{n\geqslant 0}\overline{e_n(\lambda)}e_n(z), \text{ where } e_n(z) = \sqrt{\frac{n+k+1}{k+1}}z^n,\ n \in \mathbb{N}.$$

If $M = \{f \in L_a^2(\mathbb{D}) : f(0) = 0$, where 0 has multiplicity at least k$\}$, then for any unit cyclic vector G in M, $G(z) = c_k z^k$ with $|c_k| = \sqrt{k+1}$. Now it is easy to verify that $\{e_n\}_{n=0}^\infty$ is indeed an orthonormal basis of M/G, which is equivalent of $\big(1 - \overline{\lambda}z(k/(k+1))\big)k_\lambda(z)$ being a reproducing kernel for M/G. ∎

3. Extensions to the vector valued case

For the needs of this section we will use the results and the notation as in [9]. If k is a Bergman type kernel (for the definition we refer to the defining property [9, 0.13]), with $\mathcal{H}(k)$ we denote the associated reproducing Hilbert space. Let also $\mathcal{D}$ be a separable

Hilbert space. Now consider $\mathcal{H}(k)\otimes\mathcal{D}\equiv\mathcal{H}(k,\mathcal{D})$ and think of it as a space of $\mathcal{D}$-valued analytic functions. It is the set of all analytic functions $f:\mathbb{D}\to\mathcal{D}$ such that for all $x\in\mathcal{D}$, the function $f_x(\lambda)=\langle f(\lambda),x\rangle_{\mathcal{D}}$ defines a function in $\mathcal{H}(k)$ with $\|f\|^2=\sum_{n=1}^{\infty}\|f_{e_n}\|^2<\infty$ for some orthonormal basis $\{e_n\}_{n\geqslant 1}$ of $\mathcal{D}$. It can be easily shown that the above expression is independent of the choice of the orthonormal basis. In particular, for $f\in\mathcal{H}(k)$, $x\in\mathcal{D}$, the function $fx:\lambda\to f(\lambda)x$ is in $\mathcal{H}(k,\mathcal{D})$ and $\|fx\|=\|f\|\|x\|_{\mathcal{D}}$. If $f\in\mathcal{H}(k,\mathcal{D})$, $x\in\mathcal{D}$ and $\lambda\in\mathbb{D}$, we have $\langle f(\lambda),x\rangle_{\mathcal{D}}=\langle f,k_\lambda x\rangle_{\mathcal{H}(k,\mathcal{D})}$. There is an obvious identification of the elementary tensors $f\otimes x$ with the functions fx.

An analytic function ϕ on $\mathbb{D}$ is a multiplier of $\mathcal{H}(k)$ if $\phi f\in\mathcal{H}(k)$ for all $f\in\mathcal{H}(k)$. We shall write $\mathcal{M}(k)$ for the collection of all multipliers. A standard use of the closed graph theorem shows that each $\phi\in\mathcal{M}(k)$ defines a bounded linear operator $M_\phi:f\to\phi f$ on $\mathcal{H}(k)$. Thus we define the multiplier norm by $\|\phi\|_M=\|M_\phi\|$. Each scalar valued multiplier $\phi\in\mathcal{M}(k)$ defines also an operator on $\mathcal{H}(k,\mathcal{D})$ of the same norm, and we shall denote this operator by M_ϕ. Again we will say that a subspace $\mathcal{M}$ of $\mathcal{H}(k,\mathcal{D})$ is multiplier invariant, if $M_\phi\mathcal{M}\subseteq\mathcal{M}$ for each $\phi\in\mathcal{M}$. The index of an invariant subspace $M\subseteq\mathcal{H}(k,\mathcal{D})$ is defined in the same way as in the scalar case. Moreover, for the rest of this section we also suppose that $\operatorname{ind}M=1$ and that G is a unit vector in $M\ominus\zeta M$.

Considering the above, it can be easily shown that [**9**, Corollary 0.8] and [**9**, Wandering Subspace Theorem] hold in this general vector valued setting. Here we present the versions of these theorems that we shall use in the sequel.

Theorem 3.1. *Let k be any Bergman type kernel and M any index 1 invariant subspace of $\mathcal{H}(k,\mathcal{D})$. Let also G denote a unit vector in $M\ominus\zeta M$. Then there is a positive kernel l_λ on $\mathbb{D}$ such that $\left(P_M\left(k_\lambda(z)x\right)\right)/\left(\langle G(z),G(\lambda)\rangle_{\mathcal{D}}\right)=\left(1-\bar{\lambda}zl_\lambda(z)\right)\left(k_\lambda(z)x\right)$, for all $x\in\mathcal{D}$.*

Theorem 3.2. (Wandering Subspace Theorem) *If k is a Bergman type kernel and M is a multiplier invariant subspace of $\mathcal{H}(k,\mathcal{D})$, then the span of the set $\{\zeta^n f : n\geqslant 0,\ f\in M\ominus\zeta M\}$ is dense in M.*

In the sequel we will apply the above results in the case of the Bergman kernel where $\mathcal{D}=\mathbb{C}^n$, $n=2,3\dots$. Hence, $\mathcal{H}(k,\mathcal{D})=\bigoplus_{i=1}^{n}L^2_a(\mathbb{D})$, $n=2,3\dots$. We consider the operator $M_\zeta:\bigoplus_{i=1}^{n}L^2_a(\mathbb{D})\mapsto\bigoplus_{i=1}^{n}L^2_a(\mathbb{D})$ such that $(f_1,f_2,\dots)\mapsto(zf_1,zf_2,\dots)$. Since $\operatorname{ind}M=1$, the Wandering Subspace Theorem implies that $M=[G]$, where G is a unit vector in $M\ominus\zeta M$. Denote with M/G the closure of the $\mathbb{C}^n$-valued analytic polynomials in $L^2_a\left(\|G(z)\|^2_{\mathbb{C}^n}d\mathcal{A}\right)$, $n=2,3\dots$.

Furthermore, using Theorem 3.1, we conclude that there is a positive sesquianalytic kernel, l^M_λ, defined on $\mathbb{D}\times\mathbb{D}$, which is the reproducing kernel of M/G. The following remark shows that l^M_λ defines the invariant subspace M uniquely.

Remark 3.3. If M_1,M_2 are *index* 1 invariant subspaces of $\bigoplus_{i=1}^{n}L^2_a(\mathbb{D})$, $n=1,2,\dots$, with

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$l_\lambda^{M_1} = l_\lambda^{M_2}$, then $M_1 = M_2$.

Indeed, if G_{M_i} are unit vectors in $M_i \ominus \zeta M_i$, $i = 1,2$, then $M_1/G_{M_1} = M_2/G_{M_2}$, with equality of norms, since the kernel defines the space uniquely. Recall that M_i/G_{M_i} is the closure of the analytic polynomials in $L_a^2\Big(\|G_{M_i}(z)\|_{\mathbb{C}^n}^2 d\mathcal{A}\Big)$, $i = 1,2$, and hence the result follows from [**11**, Theorem 1].

We would like to note that the proofs of Lemma 2.2 and Theorem 2.8 can be easily modified to hold in the case of $\bigoplus_{i=1}^{n} L_a^2(\mathbb{D})$, $n = 2,3,\ldots$. For example, Theorem 2.8 becomes:

Theorem 3.4. *Let V be an open subset of $\mathbb{C}$ such that $V \cap \mathbb{T} \neq \emptyset$. If there is a nonzero element F of M which extends to be analytic in V, then the kernel l_λ^M satisfies the following boundary conditions:*

(i) $\lim_{\lambda\to\zeta}(1 - |\lambda|^2 l_\lambda(\lambda))\|G(\lambda)\|_{\mathbb{C}^n}^2 = 1$ *for every* $\zeta \in V \cap \mathbb{T}$;

(ii) $\lim_{\lambda\to\zeta} \frac{\partial}{\partial z}\Big[(1 - \overline{\lambda} z l_\lambda(z))\langle G(z), G(\lambda)\rangle_{\mathbb{C}^n}\Big]\Big|_{z=\lambda} = 0$ *for every* $\zeta \in V \cap \mathbb{T}$,

where G is a unit cyclic vector in M.

Our last result demonstrates that Theorem 2.17 regarding rotationally invariant kernels, does not extend to the case of $\bigoplus_{i=1}^{2} L_a^2(\mathbb{D})$.

Theorem 3.5. *If k is the Bergman kernel, then for all $c \in [0,1)$, the rotationally invariant kernel $l_\lambda(z)=c$ is an associated kernel for some index 1 invariant subspace of $\bigoplus_{i=1}^{2} L_a^2(\mathbb{D})$.*

Proof: Let $c \in [0,1)$. Let $G(z) = (g_1, g_2)$, where $g_1(z) = \sqrt{\delta}$, $g_2(z) = \sqrt{(1-\delta)(n+1)}z^n$, with $0 < \delta < 1$, $n \in \mathbb{N}$. An elementary calculation shows that

$$\|G\|_{\bigoplus_{i=1}^2 L_a^2(\mathbb{D})} = 1, \text{ and that } \int_{\mathbb{D}} z^k \sum_{i=1}^{2} |g_i|^2 \frac{d\mathcal{A}(z)}{\pi} = 0,\ k > 1. \tag{3.6}$$

Additionally observe that G is continuous on $\overline{\mathbb{D}}$, and by forcing G to satisfy the equation in condition (i) of Theorem 3.4, we get $(1 - |\lambda|^2 l_\lambda(\lambda))\|G(\lambda)\|_{\mathbb{C}^2}^2 = 1$, $\lambda \in \mathbb{T}$. Hence, $\Big(1 - c\big(\delta + (1-\delta)(n+1)\big)\Big) = 1$ and $\delta = 1 - \big(c/n(1-c)\big)$, $c \in [0,1)$, $n \in \mathbb{N}$. Thus, by choosing $n > c/(1-c)$, we get $0 < \delta < 1$, and therefore by (3.6) and the Wandering Subspace Theorem, $l_\lambda(z) = c$ is indeed an associated kernel for $M = [G]$. This concludes the proof. ∎

The above analysis reveals that the study of the structural properties of l_λ^M in the case of $\bigoplus_{i=1}^{n} L_a^2(\mathbb{D})$, $n \geqslant 2$ is of particular interest for the determination of the structure of M. Even though some results from the one dimensional case extend to this case, (compare Theorem 2.8 with Theorem 3.4) some other (compare Theorem 2.17 with Theorem 3.5) do not.

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