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EXACT RESULTS FOR A FLUID MODEL WITH STATE-DEPENDENT FLOW RATES

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We consider a modulated fluid system with a finite state-space Markov chain J_t as modulating process and general state-dependent net input rates. We derive differential equations for the transient and the stationary distribution of (W_t, J_t), where W_t is the content process, and the corresponding Laplace transforms with respect to time. Moreover, we study the level hitting times of W_t. Our results lead to explicit formulas in the case of two modulating states.

1. INTRODUCTION

We consider a Markov-modulated fluid model with content process $W = \{W_t | t \geq 0\}$ and modulating process $J = \{J_t | t \geq 0\}$, which is a right-continuous irreducible continuous-time Markov chain with state space $\{1, \ldots, n\}$ and rate transition matrix $Q = (q_{ij})$. If $J_t = i$, the state-dependent *net* input rate at time t is $r_i(W_t)$. We suppose that the rate functions $r_i(y)$ are piecewise continuous and $r_i(0) \geq 0$. Let $C \leq \infty$ be the capacity. For each i, the function $r_i(\cdot)$ is either everywhere positive or everywhere negative on $(0, C)$; that is, for each state, the process is either increasing or decreasing. This is done in order to avoid the situation where the

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process remains constant on any positive interval, thereby ensuring that 0 and C can be the only possible atoms of the steady-state distribution. In the case $C < \infty$, we assume that $r_i(C) = 0$ for those indices i for which $r_i(\cdot) > 0$ on $(0, C)$, and $r_i(C) = r_i(C-)$ if $r_i(\cdot) < 0$ on $(0, C)$. Thus, during a time interval in which $J_t = i$, the process W_t follows a deterministic path according to the equation $dW_t/dt = r_i(W_t)$, as long as W_t stays in $(0, C)$. Let $\pi_i(t) = P(J_t = i)$, which is, of course, equal to $[\pi_i(0)e^{Qt}]_i$, and $q_i = -q_{ii}$.

In this article we determine the following:

(a) the transient distribution of (W_t, J_t) [i.e., the function $p(i, y, t) = P(W_t \le y, J_t = i)$]
(b) the stationary distribution of (W_t, J_t), if it exists
(c) the distribution of the hitting time $\tau_x = \inf\{t \ge 0 | W_t = x\}$.

The solutions for (a) and (c) are in terms of systems of differential equations, which sometimes become simpler by considering certain related Laplace transforms (LTs). Because finding the distribution functions in (a) or (c) is tantamount to solving the corresponding differential equations, our results indicate to what extent closed-form solutions are possible. For a two-state modulating Markov chain, we obtain several explicit solutions.

The literature on fluid models focuses on steady-state results in the case of linear increase or decrease under any modulating state and an infinite buffer. Boxma, Perry, and van der Duyn Schouten [4] have presented a detailed discussion of the history of the subject; Kulkarni [12] has given a survey. Elwalid and Mitra [8] considered a system with piecewise constant rates and characterized the stationary distribution of (W_t, J_t). For corresponding nonmodulated storage processes with a pure jump input process and a general release rate see Harrison and Resnick [9] and Brockwell, Resnick, and Tweedie [5]. Non-Markovian fluid models were studied by, among others, Chen and Yao [6], Kella and Whitt [11], and Boxma et al. [4]. These articles establish connections to ordinary queues with instantaneous inputs. Using a general point process approach, Kaspi, Kella, and Perry [10] considered a fluid "production" process with on and off times forming an alternating renewal process and state-dependent increase and decrease rates while the machine is on or off, respectively. Related steady-state results on on/off production processes were given by Perry and Posner [15,16]. Boxma et al. [4] derived several exact results for a fluid system with constant rates governed by a three-state semi-Markov process. In Boxma, Kella, and Perry [3] this model was extended to more than three modulation states, with a general release rate for one state with exponential sojourn time and a linear increase for the others; the stationary distribution of the content is shown to decompose into the stationary distributions of some clearing process and some dam, which are then further analyzed. Stability results for arbitrary nondecreasing input processes and general modulation structure can be found in Asmussen and Kella [2].

2. THE DISTRIBUTION OF (W_t, J_t)

We first consider $p(i, y, t)$. Let us assume for a moment that the initial distribution is absolutely continuous. Then the measure $P(W_t \in dy, J_t = j)$ has a density for any t and j. By the standard arguments for the forward equation,

$$p(j, y, t + \varepsilon) = (1 - q_j \varepsilon) p(j, y + r_j(y)\varepsilon, t) + \sum_{i \neq j} q_{ij} \varepsilon p(i, y, t) + o(\varepsilon), \quad j = 1, \ldots, n. \tag{2.1}$$

It follows from (2.1) that the limit

$$\lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} [p(j, y, t + \varepsilon) - p(j, y, t)] - r_j(y) \frac{1}{r_j(y)\varepsilon} [p(j, y + r_j(y)\varepsilon, t) - p(j, y, t)] \right)$$

exists. Since the partial derivative with respect to y exists (being the density of $P(W_t \in dt, J_t = j)$), the same is true for that with respect to t, so that (2.1) yields the following system of n linear partial differential equations:

$$\frac{\partial}{\partial t} p(j, y, t) - r_j(y) \frac{\partial}{\partial y} p(j, y, t) = -q_j p(j, y, t) + \sum_{i \neq j} q_{ij} p(i, y, t), = \sum_i q_{ij} p(i, y, t), \quad j = 1, \ldots, n. \tag{2.2}$$

Equation (2.2) generalizes Theorem 1 of Kulkarni [12]. The pertinent boundary conditions are

$$\begin{aligned} p(j, 0, t) &= 0 && \text{if } r_j(y) > 0 \text{ on } (0, C), \\ p(j, C, t) &= \pi_j(t) && \text{if } r_j(y) < 0 \text{ on } (0, C). \end{aligned} \tag{2.3}$$

Moreover, $p(j, y, 0)$ is given by the initial distribution of (W_0, J_0). The partial differential equation (2.2) is difficult to solve even in the case of constant rates, as pointed out by Kulkarni [12]. However, by taking LTs with respect to t and considering them as functions of the state variable y, (2.2) can be reduced to a system of ordinary differential equations. Let $f_j(y) = \int_0^\infty e^{-st} p(j, y, t)\, dt$ (with $s > 0$ fixed). In terms of these LTs, (2.2) becomes

$$s f_j(y) - p(j, y, 0) - r_j(y) f_j'(y) = \sum_i q_{ij} f_i(y), \quad j = 1, \ldots, n. \tag{2.4}$$

The solution of (2.4) is uniquely determined subject to the n boundary conditions:

$$\begin{aligned} f_j(0) &= 0 && \text{if } r_j(y) > 0 \text{ on } (0, C), \\ f_j(C) &= \int_0^\infty e^{-st} \pi_j(t)\, dt && \text{if } r_j(y) < 0 \text{ on } (0, C). \end{aligned} \tag{2.5}$$

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It is easy to see that the integral in (2.5) is equal to the jth component of $\pi_i(0) \times (Q - sI)^{-1}$, where I is the $n \times n$ identity matrix.

The distribution of (W_0, J_0) occurs explicitly in (2.4) in the form of $p(j, y, 0)$. Any initial distribution can be obtained as a weak limit of absolutely continuous ones (i.e., those for which $P(W_0 \in dy, J_0 = j)$ has a density for any j). It is thus clear that (2.4) also holds for an arbitrary initial distribution. The most important case is, of course, that of constant initial values; that is, $(W_0, J_0) \equiv (y_0, i_0)$. Then, we have

$$p(j, y, 0) = \begin{cases} 1 & \text{if } y \geq y_0, j = i_0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Next, we assume that (W_t, J_t) is stationary. In this case, $(\partial/\partial t)p(j, y, t) \equiv 0$, and setting $p_j(y) = p(j, y, 0)$, we obtain from (2.4)

$$-r_j(y)p_j'(y) = \sum_i q_{ij} p_i(y), \quad j = 1, \ldots, n \tag{2.7}$$

subject to

$$\begin{aligned} p_j(0) &= 0 && \text{if } r_j(y) > 0 \text{ on } (0, C), \\ p_j(C) &= \lim_{t\to\infty} \pi_j(t) && \text{if } r_j(y) < 0 \text{ on } (0, C). \end{aligned} \tag{2.8}$$

Note that $\lim_{t\to\infty} \pi_j(t) = \pi_j$, where the n-component row vector $\pi = (\pi_i)$ is the unique normalized solution of $\pi Q = 0$.

All solutions of this first-order system (and similarly those of (2.4)) are linear combinations of n linearly independent functions, and the coefficients are given by the boundary conditions. Because there is no general way to determine n fundamental solutions, a more explicit determination of the functions $p_j(\cdot)$ seems to be possible only for certain functions $r_j(\cdot)$.

To ensure the existence of the stationary distribution, let us assume that the state space of (W, J) is of the form $S = I \times \{1, \ldots, n\}$ for some interval I which may contain one or both of its end points, but does not have to, and that (W, J) is irreducible with respect to S. Note that $\int_x^y du/r_j(u)$ $(-\int_y^x du/r_j(u))$ is the time it takes to get from x to y when in state j for the case $r_j(\cdot) > 0$ and $y \geq x$ ($r_j(\cdot) < 0$ and $y \leq x$). Thus, for the irreducibility, it is sufficient to assume that all these integrals are finite for all $x, y \in I$. Now, using the method of uniformization, let $\lambda \geq \max_i q_i$ and denote

$$p_{ij} = \begin{cases} \dfrac{q_{ij}}{\lambda}, & i \neq j \\[2ex] 1 - \dfrac{q_i}{\lambda}, & i = j. \end{cases}$$

Then, we can view the modulating Markov chain as a discrete-time Markov chain with transition matrix $P = (p_{ij})$ embedded at arrival epochs of an independent Pois-

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son process with rate λ. It is evident that the transition kernel of the state of our fluid process embedded right before state changes of the uniformized Markov chain has the following representation:

$$P_{i,x}(j,(y,\infty)) = p_{ij} \exp\left\{-\lambda \int_x^y \frac{du}{r_j(u)}\right\} \qquad \text{for } y \geq x \tag{2.9}$$

when $r_j(\cdot) > 0$ and

$$P_{i,x}(j,[0,y)) = p_{ij} \exp\left\{-\lambda \int_y^x \frac{du}{-r_j(u)}\right\} \quad \text{for } y \leq x \tag{2.10}$$

when $r_j(\cdot) < 0$.

This discrete-time Markov chain is (strong) Feller and, thus, when $C < \infty$, the conditions for Theorem 12.0.1 of [13, p. 286] are satisfied and the existence of a stationary probability measure is assured. Therefore (PASTA), the existence of a stationary probability measure for the continuous-time process is also assured. Using a geometric trial argument, it can also be argued that the continuous-time fluid process is regenerative with finite mean regeneration epochs (e.g., returns to state (i, x) for $0 < x < C$), which implies the existence of a limiting/stationary/ergodic distribution.

3. A SPECIAL TWO-STATE CASE

Let $n = 2$ and assume that there are two constant input rates, which we may, without restriction of generality, assume to be $c_1 = c > 0$ and $c_2 = 0$. Furthermore, we suppose that fluid is released at the rate $r(y)$, where $r(y) \in (0, c)$, $0 < y < C$, depends on the state but not on the modulation. Thus, for the net inflow, we have $r_1(y) = c - r(y)$ and $r_2(y) = -r(y)$. Set $\lambda = q_{21}$ and $\mu = q_{12}$. Under the initial condition $(W_0, J_0) = (y_0,1)$, (2.4) becomes

$$\begin{aligned} sf_2(y) - 1_{[y_0,\infty)}(y) - r(y) f_2'(y) &= -\lambda f_2(y) + \mu f_1(y), \\ sf_1(y) + (c - r(y)) f_1'(y) &= -\mu f_1(y) + \lambda f_2(y). \end{aligned} \tag{3.1}$$

It follows from (3.1) that f_1 is of the form

$$f_1(y) = \begin{cases} d_1(s)h_1(y,s) + d_2(s)h_2(y,s) & \text{if } y \leq y_0 \\ d_3(s)h_1(y,s) + d_4(s)h_2(y,s) + \dfrac{\lambda}{s(s+\lambda+\mu)} & \text{if } y \geq y_0, \end{cases} \tag{3.2}$$

where $h_1(\cdot, s)$ and $h_1(\cdot, s)$ are linearly independent solutions of the second-order linear differential equation

$$\begin{aligned} r(y)(c - r(y))h''(y) + [r(y)(2s + \lambda + \mu - r'(y)) - c(s + \lambda)]h'(y) \\ - s(s + \lambda + \mu)h(y) = 0. \end{aligned} \tag{3.3}$$

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The coefficients $d_i(s), i = 1,2,3,4$, can be determined from four conditions:

1. the continuity of $f_1(y)$ at $y = y_0$
2. the differentiability of $f_1(y)$ at $y = y_0$
3. $f_1(0) = 0$
4. $f_2(C) = \int_0^\infty e^{-st}\pi_2(t)\,dt$.

Note that

$$\pi_2(t) = \begin{cases} \left(1 + \left(\dfrac{\lambda}{\mu}\right)e^{-(\lambda+\mu)t}\right)\dfrac{\mu}{\lambda+\mu} & \text{if } J_0 = 2 \\ (1 - e^{-(\lambda+\mu)t})\,\dfrac{\mu}{\lambda+\mu} & \text{if } J_0 = 1, \end{cases}$$

so that the right-hand side in condition 4 becomes

$$\int_0^\infty e^{-st}\pi_2(t)\,dt = \begin{cases} \dfrac{s+\mu}{s(s+\lambda+\mu)} & \text{if } J_0 = 2 \\ \dfrac{\mu}{s(s+\lambda+\mu)} & \text{if } J_0 = 1. \end{cases}$$

Thus, at least in cases in which a fundamental system of (3.3) is known, the LTs f_1 and f_2 are explicitly computable. The solution under the initial condition $(W_0, J_0) = (y_0, 2)$ is obtained similarly.

Example 1: $r(y) = y$, $c = 1, W_0 \in (0,1)$. If $C \geq 1$, we have

$$W_t = \begin{cases} W_0 e^{-t} & \text{if } J_0 = 2 \\ 1 - (1 - W_0)e^{-t} & \text{if } J_0 = 1 \end{cases}$$

for values of t smaller than the first jump time of J. If $C < 1$, after hitting level C the process W_t stays there until J_t changes from 1 to 2. Note that levels 0 and 1 cannot be attained. After some algebra, (3.3) yields

$$y(1-y)h''(y) + [y(2s+\lambda+\mu-1) - s - \lambda]h'(y) - s(s+\lambda+\mu)h(y) = 0. \tag{3.4}$$

Two linearly independent solutions of (3.11) are

$$h_1(y,s) = y^{1+s+\lambda}F(1+\lambda, -\mu+1, 2+s+\lambda, y), \tag{3.5}$$

$$h_2(y,s) = F(-s-\lambda-\mu, -s, -s+\lambda, y), \tag{3.6}$$

where

$$F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{x^n}{n!}$$

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is the Gauss hypergeometric function (see, e.g., Abramowitz and Stegun [1, Sect. 15]). Thus, $f_1(\cdot)$ and $f_2(\cdot)$ are linear combinations of the functions in (3.5) and (3.6) on $(0, y_0]$ and on $[y_0, C)$ with coefficients determined by conditions 1–4.

Example 2: $r(y) \equiv 1 < c$. Then, (3.1) is a linear differential equation with constant coefficients:

$$(c-1)h''(y) + [(2-c)s + (1-c)\lambda + \mu]h'(y) - s(s+\lambda+\mu)h(y) = 0. \tag{3.7}$$

Let $d = c - 1 > 0$. As linearly independent solutions of (3.7), we can take

$$h_1(y,s) = e^{g_+(s)y}, \qquad h_2(y,s) = e^{g_-(s)y}, \tag{3.8}$$

where

$$g_\pm(s) = \frac{1}{2d}\left(\pm\left[((1-d)s - d\lambda + \mu)^2 + ds(s+\lambda+\mu)\right]^{1/2} - (1-d)s + d\lambda - \mu\right). \tag{3.9}$$

Note that $g_+(s) > 0 > g_-(s)$. The coefficients $d_i(s)$ are now easily obtained. As an example, let $C = \infty$ and $J_0 = 2$. Then, $d_3(s) \equiv 0$, because otherwise, f_1 would be unbounded on (y_0, ∞), which is impossible. For d_1, d_2, and d_4, we have the equations

$$\begin{aligned} d_1(s)h_1(y_0,s) + d_2(s)h_2(y_0,s) &= d_3(s)h_1(y_0,s) + \frac{\lambda}{s(s+\lambda+\mu)}, \\ d_1(s)g_+(s)h_1(y_0,s) + d_2(s)g_-(s)h_2(y_0,s) &= d_3(s)g_+(s)h_1(y_0,s), \\ d_1(s)h_1(0,s) + d_2(s)h_2(0,s) &= 0. \end{aligned} \tag{3.10}$$

The solution of (3.10) is

$$d_1(s) = -d_2(s) = \frac{\lambda g_-(s)}{s(s+\lambda+\mu)(g_-(s) - g_+(s))h_2(y_0,s)},$$

$$d_4(s) = \frac{g_+(s)h_1(y_0,s) - g_-(s)h_2(y_0,s)}{g_-(s)h_1(y_0,s)}\, d_1(s).$$

Thus,

$$f_1(y) = \begin{cases} \dfrac{\lambda g_-(s)[e^{g_+(s)y} - e^{g_-(s)y}]}{s(s+\lambda+\mu)(g_-(s) - g_+(s))e^{g_-(s)y_0}} & \text{if } y \le y_0 \\[2ex] d_3(s)h_1(y,s) + d_4(s)h_2(y,s) + \dfrac{\lambda}{s(s+\lambda+\mu)} & \text{if } y \ge y_0. \end{cases} \tag{3.11}$$

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4. THE STATIONARY DISTRIBUTION IN THE TWO-STATE CASE

Now, let us turn to the two-state case with general rates $r_1(y) > 0 > r_2(y), y \in (0, C)$, and derive the stationary distribution of (W_t, J_t). Assume that $C < \infty$. Equations (2.7) and (2.8) yield

$$\begin{aligned} r_1(y)p_1'(y) &= -\mu p_1(y) + \lambda p_2(y), \\ r_2(y)p_2'(y) &= -\lambda p_2(y) + \mu p_1(y), \end{aligned} \tag{4.1}$$

subject to

$$p_1(0) = 0, \qquad p_2(C) = \frac{\mu}{\lambda + \mu}. \tag{4.2}$$

Assume that $r_1(y)$ and $r_2(y)$ are continuously differentiable on $(0, C)$. Then, system (4.1) can be transformed into

$$0 = r_1(y)r_2(y)p_1''(y) + [r_1'(y)r_2(y) + \mu r_2(y) + \lambda r_1(y)]p_1'(y), \tag{4.3}$$

$$p_2(y) = \frac{\mu}{\lambda} p_1(y) + \frac{r_1(y)}{\lambda} p_1'(y). \tag{4.4}$$

It follows from (4.3) that $p_1'(y) = C_1 e^{g(y)}$, where C_1 is some constant and $g(y)$ is an antiderivative of $-[\lambda r_1(y) + \mu r_2(y) + r_1'(y)r_2(y)]/r_1(y)r_2(y)$, given by

$$g(y) = -\int_{y_0}^{y} \frac{\lambda r_1(u) + \mu r_2(u) + r_1'(u)r_2(u)}{r_1(u)r_2(u)}\, du$$

$(y_0 \in (0, C))$ up to some additive constant. Since $p_1(0) = 0$, we have

$$p_1(y) = C_1 \int_0^y e^{g(x)}\, dx,$$

provided the integral is finite. The constant C_1 can now be computed from $p_2(C) = \mu/(\lambda + \mu)$. From (4.4), we obtain

$$C_1 = \frac{\lambda\mu}{\lambda + \mu}\left(\mu \int_0^C e^{g(x)}\, dx + r_1(C-)e^{g(C)}\right)^{-1}.$$

Hence,

$$\begin{aligned} p_1(y) &= \frac{\lambda\mu}{\lambda + \mu}\left(\mu \int_0^C e^{g(x)}\, dx + r_1(C-)e^{g(C)}\right)^{-1} \int_0^y e^{g(x)}\, dx, \\ p_2(y) &= \frac{\mu}{\lambda + \mu}\left(\mu \int_0^C e^{g(x)}\, dx + r_1(C-)e^{g(C)}\right)^{-1} \left[\mu \int_0^y e^{g(x)}\, dx + r_1(y)e^{g(y)}\right]. \end{aligned}$$

Let W^* be a random variable for which $P(W^* \le y) = \lim_{t\to\infty} P(W_t \le y)$, $y \ge 0$. Then,

$$P(W^* \leq y) = p_1(y) + p_2(y)$$
$$= \frac{\mu}{\lambda+\mu} \quad \frac{(\lambda+\mu)\int_0^y e^{g(x)}\,dx + r_1(y)e^{g(y)}}{\mu\int_0^C e^{g(x)}\,dx + r_1(C-)e^{g(C)}}, \quad 0 < y < C. \quad \textbf{(4.5)}$$

If $\int_0^y e^{g(x)}\,dx = \infty$ for some $y \in (0, C)$, one can modify the rates and, for example, consider $\tilde{r}_1(y) = r_1(y) + \varepsilon$ and $\tilde{r}_2(y) = r_2(y) - \varepsilon$ and then let $\varepsilon \downarrow 0$ in (4.5). This approach is used in the following.

Example 3: Let us combine linear and constant release rates and take $r_1(y) = B_1 - A_1 y$ and $r_2(y) = -(A_2 y + B_2)$, $y \in (0, C)$, where $A_1 \in \mathbb{R}\backslash\{0\}$, $A_2 > 0$, $B_1 > 0$, and $B_2 > 0$. Recall that we assume that $r_1(y) > 0$ for all $y \in (0, C)$ to ensure that there is a positive net inflow while the system is in state 1. Therefore, we may, without restriction of generality, suppose that $A_1 C \leq B_1$. We can take

$$g(y) = -\int_0^y \frac{\lambda r_1(u) + \mu r_2(u) + r_1'(u) r_2(u)}{r_1(u) r_2(u)}\,du = \int_0^y \left(\frac{\lambda}{A_2 u + B_2} - \frac{\mu - A_2}{B_1 - A_1 u}\right) du$$
$$= \frac{\lambda}{A_2}\log\left(1 + \frac{A_2 x}{B_2}\right) + \frac{\mu - A_2}{A_1}\log\left(1 - \frac{A_1 x}{B_1}\right).$$

Now, (4.5) yields, for $y \in (0, C)$,

$$P(W^* \leq y) = \frac{\mu}{\lambda+\mu}$$
$$\times \frac{(\lambda+\mu)\int_0^y (A_2 x + B_2)^{\lambda/A_2}(B_1 - A_1 x)^{(\mu - A_2)/A_1}\,dx + (A_2 y + B_2)^{\lambda/A_2}(B_1 - A_1 y)^{(\mu + A_1 - A_2)/A_1}}{\mu\int_0^C (A_2 x + B_2)^{\lambda/A_2}(B_1 - A_1 x)^{(\mu - A_2)/A_1}\,dx + (A_2 C + B_2)^{\lambda/A_2}(B_1 - A_1 C)^{(\mu + A_1 - A_2)/A_1}}. \quad \textbf{(4.6)}$$

Let us consider a few important special cases.

1. Let $B_1 = c$, $A_1 > 0$, and $B_2 \to 0$. This yields the model of Section 3 for linear but possibly unequal rates: $r_1(y) = c - A_1 y$ and $r_2(y) = -A_2 y$. We obtain the asymptotic distribution from (4.6):

$$P(W^* \leq y) = \frac{\mu}{\lambda+\mu}$$
$$\times \frac{(\lambda+\mu)\int_0^y x^{\lambda/A_2}\left(\frac{c}{A_1} - x\right)^{(\mu - A_2)/A_1} dx + A_1 y^{\lambda/A_2}\left(\frac{c}{A_1} - y\right)^{(\mu + A_1 - A_2)/A_1}}{\mu\int_0^C x^{\lambda/A_2}\left(\frac{c}{A_1} - x\right)^{((\mu - A_2)/A_1) - 1} dx + A_1 C^{\lambda/A_2}\left(\frac{c}{A_1} - C\right)^{(\mu + A_1 - A_2)/A_1}}. \quad \textbf{(4.7)}$$

If, additionally, $A_1 = A_2 = A$, (4.7) slightly simplifies to

$$P(W^* \le y) = \frac{\mu}{\lambda+\mu} \times \frac{(\lambda+\mu)\int_0^y x^{\lambda/A}\left(\frac{c}{A}-x\right)^{(\mu/A)-1} + Ay^{\lambda/A}\left(\frac{c}{A}-y\right)^{\mu/A}}{\mu\int_0^C x^{\lambda/A}\left(\frac{c}{A}-x\right)^{(\mu/A)-1} dx + AC^{\lambda/A}\left(\frac{c}{A}-C\right)^{\mu/A}}. \tag{4.8}$$

2. Let $A_1 < 0$, $B_1 \to 0$, and $B_2 \to 0$. In this case, $r_1(y) = |A_1|y, r_2(y) = -A_2 y$. It is easy to conclude from (4.6) that the limiting distribution has positive mass on $(0, C)$ if and only if $\rho = (\lambda/A_2) - (\mu/|A_1|) + (A_2/|A_1|) + 1 > 0$, and, in this case, is of the form $P(W^* \le y) = \gamma(y/C)^\rho, y \in (0, C)$, where

$$\gamma = \frac{\frac{1}{\rho} + \frac{|A_1|}{\lambda+\mu}}{\frac{1}{\rho} + \frac{|A_1|}{\mu}}.$$

3. For constant release rates, say $r_1(y) = B_1$ and $r_2(y) = -B_2$, the stationary distribution can also be obtained from (4.6) by letting $A_1 \to 0$ and $A_2 \to 0$, but it is easier to start again from (4.5) noting that in this case,

$$g(y) = \int_0^y \frac{\lambda B_1 - \mu B_2}{B_1 B_2}\, du = \sigma y,$$

where $\sigma = (\lambda/B_2) - (\mu/B_1)$, so that a short calculation yields

$$P(W^* \le y) = \frac{\frac{\lambda(B_1+B_2)}{(\lambda+\mu)B_2}\, e^{\sigma y} - 1}{\frac{\lambda B_1}{\mu B_2}\, e^{\sigma C} - 1}, \quad y \in (0, C). \tag{4.9}$$

5. HITTING TIMES

We now consider the hitting times $\tau_x = \inf\{t > 0 | W_t = x\}$. Note that τ_x can be infinite with positive probability. Let $C < \infty$. We want to compute the Laplace–Stieltjes transform (LST) $\phi_i(a, x, s) = E(e^{-s\tau_x} | W_0 = a, J_0 = i)$.

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The usual arguments yield

$$\phi_i(a,x,s) = (1 - q_i\varepsilon)e^{-s\varepsilon}\phi_i(a + r_i(a)\varepsilon, x, s) + \sum_{j\neq i} q_{ij}\varepsilon e^{-s\varepsilon}\phi_j(a,x,s) + o(\varepsilon). \tag{5.1}$$

It follows that

$$r_i(a)\frac{\partial}{\partial a}\phi_i(a,x,s) = (q_i + s)\phi_i(a,x,s) - \sum_{j\neq i} q_{ij}\phi_j(a,x,s), \quad i = 1,\ldots,n. \tag{5.2}$$

If $a > x$, the boundary conditions are

$$\phi_i(x,x,s) = 1 \quad \text{if } r_i(\cdot) < 0 \text{ on } (0,C), \tag{5.3}$$

$$\frac{\partial}{\partial a}\phi_i(C,x,s) = 0 \quad \text{if } r_i(\cdot) > 0 \text{ on } (0,C). \tag{5.4}$$

For $a < x$, the corresponding conditions are similar.

Let us return to the two-state case of Section 3 to arrive at some explicit results. Set $\varphi(a,x,s) = \phi_1(a,x,s)$ and $\psi(a,x,s) = \phi_2(a,x,s)$. In this example, (5.2) is easily seen to lead to the second-order differential equation for $\psi(\cdot,x,s)$:

$$\begin{aligned} &r(a)(c - r(a))\psi''(a,x,s) \\ &\quad - [(r'(a) + 2s + \lambda + \mu)r(a) - (s + \lambda + r'(a))c]\psi'(a,x,s) \\ &\quad - s(s + \lambda + \mu)\psi(a,x,s) = 0. \end{aligned} \tag{5.5}$$

A prime means differentiation with respect to a. Note the similarity, and the differences, between (3.3) and (5.5). The function φ can be expressed in terms of ψ as follows:

$$\varphi(a,x,s) = \lambda^{-1}[(s + \lambda)\psi(a,x,s) + r(a)\psi'(a,x,s)]. \tag{5.6}$$

We now solve the equations for the special rate functions considered in Section 3.

Example 4: $r(y) = y, c = 1, W_0 \in (0,1)$, and $C \geq 1$. The content will always be in (0,1), never reaching 0 or 1. Solving (5.5), we find that ψ is of the form

$$\psi(a,x,s) = C(x,s)F(s, s + \lambda + \mu, s + \lambda + 1, a) + D(x,s)a^{-s-\lambda}F(-\lambda, \mu, 1 - s - \lambda, a) \tag{5.7}$$

for $a, x \in (0,1)$, where F is the hypergeometric function. The coefficient functions $C(x,s)$ and $D(x,s)$ can be computed from the boundary conditions

$$\psi(x,x,s) = 1, \qquad \varphi'(1,x,s) = 0$$

(φ being given by (5.6)) and can thus be expressed in terms of hypergeometric functions and their derivatives.

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Example 5: $r(y) \equiv 1$ and $c > 1$. Let us consider the case $x = 0$. The general solution of (5.5) is now a linear combination of the two exponential functions $\exp\{e_+(s)a\}$ and $\exp\{e_-(s)a\}$, where

$$e_\pm(s) = \frac{1}{2d}[(1-d)s - d\lambda + \mu \pm ([(1-d)s - d\lambda + \mu]^2 + 4d^2 s(s+\lambda+\mu))^{1/2}].$$

(Recall that $d = c - 1$.) The boundary conditions are

$$\psi(0,0,s) = 1, \qquad \varphi'(C,0,s) = 0.$$

After tedious calculations, we arrive at the exact solutions

$$\varphi(a,0,s) = \lambda^{-1}[\lambda + s + e_-(s)]H(s)\exp\{e_-(s)a\} + \lambda^{-1}[\lambda + s + e_+(s)](1 - H(s))\exp\{e_+(s)a\} \tag{5.8}$$

and

$$\psi(a,0,s) = H(s)\exp\{e_-(s)a\} + (1 - H(s))\exp\{e_+(s)a\}, \tag{5.9}$$

where

$$H(s) = \left[1 - \frac{e_-(s)(\lambda + s + e_-(s))}{e_+(s)(\lambda + s + e_+(s))}\exp\{C(e_-(s) - e_+(s))\}\right]^{-1}. \tag{5.10}$$

In the case of infinite capacity, it turns out that the LSTs $\varphi(a,0,\cdot)$ and $\psi(a,0,\cdot)$ can be inverted in closed form. If $C = \infty$, we have $H(s) \equiv 1$ and, thus,

$$\varphi(a,0,s) = \lambda^{-1}[\lambda + s + e_-(s)]\exp\{e_-(s)a\}, \tag{5.11}$$

$$\psi(a,0,s) = \exp\{e_-(s)a\}. \tag{5.12}$$

Let $p = s + \lambda + c^{-1}(\mu - \lambda)$. After some algebraic manipulations, these LTs can be written as

$$\psi(a,0,s) = \exp\left\{-a(s+\lambda) + \frac{ca}{2(c-1)}(p - [p^2 - 4\lambda\mu c^{-2}(c-1)]^{1/2})\right\}, \tag{5.13}$$

$$\varphi(a,0,s) = \frac{c}{2\lambda(c-1)}(p - [p^2 - 4\lambda\mu c^{-2}(c-1)]^{1/2})\psi(a,0,s). \tag{5.14}$$

By (5.13), ψ is of the form

$$\psi(a,0,s) = e^{-a\lambda}e^{-as} + e^{-a\lambda}e^{-as}g(s+b),$$

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where $b = \lambda + c^{-1}(\mu - \lambda)$ and

$$g(p) = \exp\left\{ \frac{ca}{2(c-1)} \left(p - [p^2 - 4\lambda\mu c^{-2}(c-1)]^{1/2}\right) \right\} - 1, \quad p \in [0,\infty).$$

However, $g(p)$ is the LT of the function

$$f(t) = \frac{ca}{2(c-1)} \left(\frac{4\lambda\mu c^{-2}(c-1)}{t^2 + ca(c-1)^{-1}t} \right)^{1/2} \times I_1([4\lambda\mu c^{-2}(c-1)(t^2 + ca(c-1)^{-1}t)]^{1/2}), \quad t \in (0,\infty)$$

(see, e.g., [14, p. 263, formula (5.122)]), where, of course,

$$I_1(x) = \sum_{i=0}^{\infty} \frac{(x/2)^{i+2}}{(i!)^2}$$

is the modified Bessel function of the first kind and of order 1. Hence, $\psi(a,0,\cdot)$ is the LST of the subprobability measure which has an atom of weight $e^{-\lambda a}$ at a and density $c^{-a\lambda}e^{-b(t-a)}f(t-a)$ on (a,∞). The density is thus given by

$$f(t|a,0) = \frac{(\lambda\mu)^{1/2} a e^{-\lambda t - c^{-1}(\mu-\lambda)(t-a)}}{[(t-a)(a+(c-1)t)]^{1/2}} \times I_1(2c^{-1}(\lambda\mu)^{1/2}[((t-1)a + (c-1)t)]^{1/2}), \quad t \in (a,\infty).$$

This measure is a probability if and only if $P(\tau_0 < \infty | W_0 = a, J_0 = 2) = 1$, which holds if and only if $(c-1)\lambda \le \mu$. Indeed, it follows from (5.13) that

$$P(\tau_0 < \infty | W_0 = a, J_0 = 2) = \psi(a,0,0) = \begin{cases} e^{-a(\lambda - (c-1)^{-1}\mu)} & \text{if } (c-1)\lambda > \mu \\ 1 & \text{if } (c-1)\lambda \le \mu. \end{cases}$$

Finally, consider the case $a = x = 0$. Given $W_0 = 0$ and $J_0 = 1$, the hitting time τ_0 is the length of the time interval from leaving level 0 until returning to it. Note that $\varphi(0,0,s) \equiv 1$, so that, by (5.14),

$$E(e^{-s\tau_0} | W_0 = 0, J_0 = 1) = \varphi(0,0,s) = \frac{c}{2\lambda(c-1)} \left(p - [p^2 - 4\lambda\mu c^{-2}(c-1)]^{1/2}\right) \tag{5.15}$$

with p defined as above. Now, note that $p \mapsto p - [p^2 - \alpha^2]^{1/2}$, $\alpha > 0$, is the LT of $t \mapsto \alpha t^{-1} I_1(\alpha t)$ [14, p. 232, formula (3.46)]. Thus, the inverse of the LT (5.15) (i.e., the density of a busy period) is given by

$$t \mapsto \left(\frac{\mu}{(c-1)\lambda} \right)^{1/2} t^{-1} e^{-[\lambda + c^{-1}(\mu-\lambda)]t} I_1(2c^{-1}[(c-1)\lambda\mu]^{1/2} t), \quad t \in (0,\infty).$$

The busy period is almost surely finite if and only if $(c-1)\lambda \leq \mu$. Note that this distribution coincides with that of a busy period for a standard $M/M/1$ queue with arrival and service intensities $c^{-1}(c-1)\lambda$ and $c^{-1}\mu$, respectively (see [7, Sect. II.2.2]).

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