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ON OPTIMAL OPERATIONAL SEQUENCE OF COMPONENTS IN A WARM STANDBY SYSTEM

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Abstract

We consider an *open problem* of obtaining the optimal operational sequence for the 1-out-of-n system with warm standby. Using the virtual age concept and the cumulative exposure model, we show that the components should be activated in accordance with the increasing sequence of their lifetimes. Lifetimes of the components and the system are compared with respect to the stochastic precedence order and its generalization. Only specific cases of this optimal problem were considered in the literature previously.

Keywords: Cumulative exposure model; stochastic precedence order; virtual age; warm standby system

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1. Introduction

As an introductory reasoning, consider first one component that starts operating at $t = 0$. Assume that in the process of production it acquires an initial unobserved resource R; see [6]. For mechanical or electronic items, for instance, it can be a 'distance' between the initial value of the key parameter and the boundary that defines a failure of the component. It is natural to assume that it is a continuous random variable with the cumulative distribution function

$$F(r) = \mathbb{P}(R \leq r).$$

A similar notion of a random resource (hazard potential) was considered in [17]. Suppose that, for each realization of R, the component's remaining resource is monotonically decreasing with time. Therefore, the run-out resource, to be called *wear*, monotonically increases. The wear in $[0, t)$ can be defined as

$$W(t) = \int_0^t w(u)\,\mathrm{d}u, \tag{1.1}$$

where $w(t)$ denotes the rate of wear. Thus, the value of R is an intrinsic property of a manufactured item, whereas the rate $w(t)$ defines the 'depletion' of R in a given environment. The larger rate corresponds to a more severe environment, whereas $w(t) \equiv 1$ can be often

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considered as a baseline one. The failure occurs when the wear $W(t)$ reaches R. Denote the corresponding random time by T. Then

$$\mathbb{P}(T \leq t) \equiv \mathbb{P}(R \leq W(t)) = F(W(t)). \tag{1.2}$$

Therefore, the described survival model can be interpreted in terms of the accelerated life model (ALM); see [1] and [13]. Our reasoning in what follows is based on the ALM (1.2), whereas the discussion above can be considered as a useful interpretation.

In applications, the most common specific case of the described setting is the cumulative exposure model; see [13], which corresponds to the case when the scale transformation in (1.2) is linear, i.e.

$$\mathbb{P}(T \leq t) \equiv \mathbb{P}(R \leq wt) = F(wt). \tag{1.3}$$

Engineering systems, especially those that are used in mission-critical applications such as aerospace, power generation, flight control, and computing, are often designed with redundancies in order to meet the stringent safety and reliability requirements; see [10] and [11]. One of the widely-applied redundancy techniques in various applications is the standby redundancy, when one or several components operate and redundant components serve as the standby spares. In the case of failure of an operating component, a replacement procedure is initiated to activate a standby component and to replace the failed one so that a system continues to operate.

According to its failure characteristics before the activation, a standby component can be categorized as 'hot', 'cold', or 'warm'. A hot standby component works concurrently with the online primary component and thus is ready to take over at any time for fast recovery. In this case, the standby component is fully exposed to the operating stress and is characterized by the same failure rate as the online one. A cold standby component is unpowered and shielded from operation and environmental stresses. As a more general option that, e.g. we can take into account the nonideal standby mode conditions or/and partial loading, a warm standby component is characterized by the failure rate that is smaller than that for the fully operational component; see [9]–[11], [18], and [19]. Obviously, the former two types of loading are the special cases of the warm standby mode.

Reliability analysis of the warm standby systems is much more challenging than that for cold and hot standby systems. Indeed, the lifetime of a cold standby system is just the sum of lifetimes of all components; the lifetime of a hot standby system is just the maximum of individual lifetimes, whereas in the warm standby case, a switch of the regimes from the warm standby to the operational mode should be taken into account. In accordance with the linear cumulative exposure model based on the scale transformation (1.3) with $w < 1$, the equivalent lifetime (*virtual age*) of a warm standby component having spent some time in this mode before switching to the active mode is this time reduced by the lifetime deceleration factor w plus the lifetime spent in the active mode afterwards. More general models that are not restricted to the case of a linear scale transformation are usually based on the notion of the 'virtual age'. See, e.g. [4] and [5] for applications of the virtual age concept to regimes switching of the described type.

Remark 1.1. Note that we can arrive at (1.2) formally without employing the notion of resource. Indeed, let the baseline environment be more severe and denote the corresponding lifetime in it by $F(t)$. Thus, the lifetime of a component operating in a milder environment should be longer. Assume that this ordering is in the sense of the usual stochastic ordering, i.e. $F_m(t) < F(t)$, which implies that

$$F_m(t) = F(W(t)),$$

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where $W(0) = 0$ and the time dependent scale transformation function is increasing and $W(t) = t$ for all $t > 0$.

Optimal (in terms of maximizing reliability characteristics of a system) activation sequence for components obviously does not exist in a hot standby system, it is trivial (no difference) for the cold standby systems and is meaningful for general warm standby systems. Only some special cases (see [4] and [19]) for the latter case were considered in the literature. In this paper, we discuss the problem in much more generality and, therefore, under certain assumptions, *solving an open problem* of theoretical reliability.

Before stating the problem formally, for the sake of completeness we give the formal definitions of stochastic orders that will be used in this paper; see [3] and [16].

Definition 1.1. Let X and Y be two continuous nonnegative random variables with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinite, and l_X and l_Y may be 0. Further, let $\bar{F}_X(\cdot)$ and $\bar{F}_Y(\cdot)$ be the survival functions of X and Y, respectively. Then X is said to be smaller than Y in

(i) usual stochastic (st) order, denoted as $X \leq_{\text{st}} Y$, if

$$\bar{F}_X(x) \leq \bar{F}_Y(x) \quad \text{for all } x \in (0, \infty);$$

(ii) hazard rate (hr) order, denoted as $X \leq_{\text{hr}} Y$, if

$$\frac{\bar{F}_Y(x)}{\bar{F}_X(x)} \quad \text{is increasing in } x \in (\min(l_X, l_Y), \infty);$$

(iii) stochastic precedence (sp) order, denoted as $X \leq_{\text{sp}} Y$, if

$$\mathbb{P}(X \leq Y) \geq \tfrac{1}{2}.$$

2. Problem formulation

We want to obtain an optimal sequence of activation of the standby components for a heterogeneous system of n components, with one active component and others in a warm standby mode. We assume that in a standby mode all components are characterized by the same deceleration factor $w < 1$. Generalization to the general case $w(t)$ will be also discussed. Intuitive reasoning based on the notions of resource of the components prompts us to first activate the weakest component, then the weakest from the remaining, and so on. Specific cases in the literature support this intuition. However, the type of stochastic ordering for the components and other assumptions of the model are crucial for our discussion.

Denote the lifetimes of the components of the system in the active (operational) regime by T_i, $i = 1, 2, \ldots, n$. Assume that they are ordered in some, nonspecified for now, stochastic sense, i.e.

$$T_1 \leq T_2 \leq \cdots \leq T_n. \tag{2.1}$$

If the operating component fails, the next operable one (that did not fail in the warm standby mode) is activated, and so on. The question is to define a sequence of activation for standby components that will maximize the lifetime of the whole system (in some stochastic sense). Some important specific cases were studied in [4] and [19], as follows.

- The hazard rate ordering was considered for the lifetimes of two components. Then it was proved that one should start with the weaker in this sense component, which results in the maximum expected lifetime of a system.

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- For the 1-out-of-n system, only the specific case of exponentially distributed lifetimes and linear model (1.3) was considered. Then, under the assumption of the hazard rate ordering, it was proved that if activation starts with the *weakest component*, and the next weakest is chosen from the remaining components, and so on, reliability of the system will be maximal in the sense of the usual stochastic order.

Our aim is to consider this problem in more generality for arbitrary lifetime distributions which is a challenging problem. Arguably, the choice of stochastic ordering in the previous works was a barrier to obtaining more general results. In what follows, we use the stochastic precedence order, which is *natural* in many reliability settings and, in spite of this, not sufficiently explored in the literature so far.

The problem we consider is based on the definition of the warm standby mode via the general model (1.2) or its specific case (1.3). It should be noted that this is an assumption itself (note that all previous specific studies of reliability of the warm standby systems relied on these or similar expressions). However, in order to consider switching from one regime to another, one must have a stochastic model for that. The virtual age concept based on the ALM in (1.2) and (1.3) is well established in the literature as a way to deal with this.

3. Two components

Let us consider first the system with two components with lifetimes in an operational mode ordered as $T_1 < T_2$ in some stochastic sense to be defined below. Thus, T_i, $i = 1, 2$, denotes the time to failure of the component i. Let $Z \equiv T_2 - T_1$, and let t_i be the realizations of T_i, $i = 1, 2$, and $z = t_2 - t_1$ be the corresponding realization of Z. Then

$$\mathbb{P}(Z \geq 0) = \mathbb{P}(T_2 \geq T_1)$$

defines the probability of the event $T_2 \geq T_1$. Denote by Y_{12} the lifetime of a system when the first component is activated first and by Y_{21} when the second is activated first, and y_{12} and y_{21} the corresponding realization. We will show later that under given assumptions

$$z \geq 0 \quad \Longrightarrow \quad y_{12} - y_{21} \geq 0,$$

which, as each realization of Z corresponds to the realization of $Y_{12} - Y_{21}$, implies that

$$Z \geq 0 \quad \Longrightarrow \quad Y_{12} - Y_{21} \geq 0.$$

Thus, specifically, if $P(Z \geq 0) \geq \frac{1}{2}$ then

$$\mathbb{P}((Y_{12} - Y_{21}) \geq 0) \geq \tfrac{1}{2},$$

which, in fact, is also the definition of the stochastic precedence (sp) order for the components $\mathbb{P}(Z \geq 0) \geq \frac{1}{2}$ and for the variants of the system $\mathbb{P}((Y_{12} - Y_{21}) \geq 0) \geq \frac{1}{2}$ (see [3] and [7]), i.e.

$$T_2 \geq_{\mathrm{sp}} T_1 \quad \Longrightarrow \quad Y_{12} \geq_{\mathrm{sp}} Y_{21}.$$

Thus, the stochastic precedence order for two random variables $X >_{sp} Y$ means that $\mathbb{P}(X \geq Y) \geq \frac{1}{2}$ and it seems to be *natural* in *some* reliability settings, e.g. for stress-strength reliability modeling; see [7]. It is also consistent for the current problem, as the components and the variants of the system will be ordered only in the sense of this order. Note that for the independent random variables, the stochastic precedence order is weaker than the usual stochastic order; see [3]. On the other hand, comparison with the ordering of expectations depends on parameters involved; see [7].

In spite of its obvious attractiveness the stochastic precedence order has attracted much less attention in the literature and only a few papers are devoted to it; see [3] and [7]. However, it may be the most natural one in some reliability settings (e.g. stress/strength problems). In fact, it was even suggested in [7] to call it (at least at some instances) the stress-strength order, which naturally compares two random variables as in structural reliability. For recent advances, see [12] and [15]. Further, to avoid any confusion caused by the different definitions of the stochastic precedence order (see [3]), and as we are considering lifetimes of engineering items, we assume that the corresponding distribution functions are absolutely continuous.

We will first prove the following result.

Theorem 3.1. *Let the following stochastic precedence order hold for the two component system described above:*

$$T_2 \geq_{\text{sp}} T_1.$$

Then the corresponding order of components achieves the maximum lifetime of a system in the sense of the stochastic precedence order, i.e. $Y_{12} \geq_{\text{sp}} Y_{21}$.

Proof. Let the event $\{T_1 < T_2\}$ occur, and t_i be the realization of T_i, $i = 1, 2$ ($t_1 < t_2$). If the first component starts first then the corresponding realization of a lifetime of a system in accordance with the linear cumulative exposure model (1.3) with $w < 1$ for a milder regime is

$$Y_{12} = t_1 + (t_2 - wt_1) = t_2 + (1 - w)t_1 > t_2, \tag{3.1}$$

where wt_1 is the virtual (equivalent) age of the second component just after switching into activation (from a warm standby mode) and, therefore, the remaining lifetime in this realization is $(t_2 - wt_1)$.

Now let the second (better) component start first. We have two specific cases.

Case I: $\alpha t_1 < t_2$*, (where* $\alpha = 1/w$*).* This means that the first component (in a warm standby mode) will fail before the active second component (in realizations). Note that as t_1 is the age of the first component at failure (in an active mode), in accordance with the model, αt_1 is the age of the first component at failure if it operates all the time in the warm standby mode. Thus, the lifetime of a system in this case is just $Y_{21} = t_2$.

Case II: $t_2 < \alpha t_1$. This means (in realizations) that the active second component fails before the warm standby one and that the switching should be performed at t_2. Then the lifetime of a system in this realization is the sum

$$Y_{21} = t_2 + \frac{\alpha t_1 - t_2}{\alpha} = t_1 + t_2(1 - w), \tag{3.2}$$

where $(\alpha t_1 - t_2)$ is the time that the first component should operate (after t_2), if it were operating in the warm standby mode. However, it was switched to the active mode and this time should be recalculated as $(\alpha t_1 - t_2)/\alpha$.

Thus, we must compare (3.1) with (3.2). Then

$$t_2 - wt_1 > t_2(1 - w),$$

which holds as $t_1 < t_2$, meaning that $Y_{12} \geq Y_{21}$. The above reasoning means that the event $\{T_1 \leq T_2\}$ implies the event $\{Y_{12} \geq Y_{21}\}$ and, accordingly, $\mathbb{P}(Y_{12} \geq Y_{21}) \geq \mathbb{P}(T_1 \leq_{\text{sp}} T_2) \geq \frac{1}{2}$, completing the proof. □

Thus, it is most beneficial to activate at first the first component with a smaller lifetime *in each realization*.

Remark 3.1. As the virtual age concept is well defined for a general model ((1.1) and (1.2)) and the function $W(t)$ is monotonically increasing (therefore, the inverse function exists), Theorem 3.1 can be generalized to this case. Indeed, we compare the relations that correspond to (3.1) and (3.2) in this case. Relationship (3.1) becomes

$$t_1 + (t_2 - W(t_1)),$$

whereas (3.2) can now be written as

$$t_2 + W(W^{-1}(t_1) - t_2), \tag{3.3}$$

where W^{-1} denotes the inverse function which exists due to the monotonicity of $W(t)$. Assume additionally that $W(t)$ is concave, i.e. $W''(t) = w'(t) \leq 0$, which means that the rate of wear in (1.1) is decreasing (nonincreasing). Then we can proceed with (3.3), which results in the inequalities

$$t_2 + W(W^{-1}(t_1) - t_2) \leq t_2 + t_1 - W(t_2) \leq t_1 + t_2 - W(t_1).$$

The first one obviously follows from our *sufficient condition* $w'(t) \leq 0$, whereas the second follows from monotonicity of $W(t)$ and $t_1 \leq t_2$. It seems that the assumption of concavity is essential for the stochastic precedence order in this case as it is easy to see via the counterexample ($W(t) = t^2$) that the corresponding ordering for the system does not always hold.

4. *n* components

Consider the 1-out-of-n components warm standby system. It is a coherent system meaning that each component is relevant and its structure function is monotone. It is well known (see [2]) that in this case improving the reliability of any of the components will improve the reliability of a system, which means the usual stochastic order both on the level of components and of the system. On the other hand, it can also be easily seen that increasing the mean lifetime of a component does not necessarily lead to an increase in the mean lifetime of a system. Similarly, if we decrease the failure rate of a component, then it does not always imply that the system failure rate will also decrease. This means that the result is sensitive to the employed type of stochastic ordering. The following results contribute to our discussion of the stochastic precedence order.

Lemma 4.1. *If the lifetime of a component in a coherent system is improved in the sense of the stochastic precedence order, then the lifetime of the coherent system will also be improved in the same sense.*

Proof. Denote a lifetime of a coherent system of $(n+1)$ components by $\tau = \tau(T_1, T_2, \dots, T_n, T)$ where for notational convenience, the lifetime of the $(n+1)$th component is denoted just by T. Let us replace this component with another one with lifetime T^*, whereas all other lifetimes stay the same and denote the system lifetime $\tau^* = \tau(T_1, T_2, \dots, T_n, T^*)$. For convenience, we will call the defined systems τ and τ^*, respectively. Since τ^* is the same as τ except T is replaced by T^*, the set of all minimal path sets for both systems will be the same (for a given system, the minimal path set is a set of minimum number of components whose functioning ensures the functioning of the system). Let $\{P_1, P_2, \dots, P_m\}$ be the set of all minimal path sets for both systems. Further, let T_{P_i} denote the lifetime of the minimal path set P_i for $i = 1, 2, \dots, m$.

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For $1 \le k \le m$ and $\{j_1, j_2, \dots, j_k\} \subseteq \{1, 2, \dots, m\}$, let $\{P_{j_1}, P_{j_2}, \dots, P_{j_k}\} \subseteq \{P_1, P_2, \dots, P_m\}$ be the set of minimal path sets that contain the component T (for convenience we denote the component and its lifetime by the same letter). Similarly, let $\{P^*_{j_1}, P^*_{j_2}, \dots, P^*_{j_k}\} \subseteq \{P_1, P_2, \dots, P_m\}$ be the set of minimal path sets that contain the component T^*. Note that, for $1 \le r \le k$, $T_{P_{j_r}}$ and $T_{P^*_{j_r}}$ may not be the same even though $P_{j_r} \equiv P^*_{j_r}$. In fact, for $1 \le r \le k$,

$$T_{P_{j_r}} = \min\{S_r, T\}, \qquad T_{P^*_{j_r}} = \min\{S_r, T^*\},$$

where $S_r = \min_{l \in P_{j_r}}\{T_l\} = \min_{l \in P^*_{j_r}}\{T_l\}$.

As before, denote by the lower case letters the realizations of the corresponding random variables. Let us assume that $t \le t^*$, meaning that the realization of the replaced component is larger than that for the initial component. Then, for $1 \le r \le k$,

$$t_{P_{j_r}} = \min\{s_r, t\} \le \min\{s_r, t^*\} = t_{P^*_{j_r}},$$

which implies that

$$\max\{t_{P_{j_1}}, t_{P_{j_2}}, \dots, t_{P_{j_k}}\} \le \max\{t_{P^*_{j_1}}, t_{P^*_{j_2}}, \dots, t_{P^*_{j_k}}\}. \tag{4.1}$$

Let $\tau(t_1, t_2, \dots, t_n, t)$ and $\tau(t_1, t_2, \dots, t_n, t^*)$ be the realizations of $\tau(T_1, T_2, \dots, T_n, T)$ and $\tau(T_1, T_2, \dots, T_n, T^*)$, respectively. Then

$$\begin{aligned}
\tau(t_1, t_2, \dots, t_n, t) &= \max\{t_{P_1}, t_{P_2}, \dots, t_{P_m}\} \\
&= \max\Big\{\max_{1 \le r \le k}\{t_{P_{j_r}}\}, \max_{z \in \{1,2,\dots,m\} \setminus \{j_1, j_2, \dots, j_k\}}\{t_{P_z}\}\Big\} \\
&\le \max\Big\{\max_{1 \le r \le k}\{t_{P^*_{j_r}}\}, \max_{z \in \{1,2,\dots,m\} \setminus \{j_1, j_2, \dots, j_k\}}\{t_{P_z}\}\Big\} \\
&= \tau(t_1, t_2, \dots, t_n, t^*),
\end{aligned}$$

where the inequality follows from (4.1). Thus, in realizations,

$$t \le t^* \quad \Longrightarrow \quad \tau(t_1, t_2, \dots, t_n, t) \le \tau(t_1, t_2, \dots, t_n, t^*),$$

which is similar to the results in the previous section; hence,

$$\mathbb{P}(T < T^*) \ge \tfrac{1}{2} \quad \Longrightarrow \quad \mathbb{P}(\tau \le \tau^*) \ge \tfrac{1}{2}. \qquad \square$$

Remark 4.1. The proof of the above lemma can be explained intuitively as follows. Denote by $\phi_y(t_1, t_2, \dots, t_n, t)$ the realization of the state function (0 or 1) of τ at time $y > 0$. Similarly, let $\phi_y(t_1, t_2, \dots, t_n, t^*)$ denote the realization of the state function of τ^* at time $y > 0$ for $t < t^*$. It is clear that $\phi_y(t) = \phi_y(t^*)$ for $y \in [0, t]$ and $y \in [t^*, \infty)$, whereas for $y \in (t, t^*)$, we have $\phi_y(t) \le \phi_y(t^*)$ as the system is coherent and the state function of the $(n+1)$th component has been improved in this interval. Thus, the lifetime of a system with t^* in each realization is longer than that with t if $t < t^*$.

In the following example (suggested by the anonymous reviewer) we illustrate Lemma 4.1. We show that this lemma holds, whereas the improvement of the mean of the component does not result in the improvement of the mean of the system.

Example 4.1. Let us consider a parallel system with lifetime $S = \max\{T_1, T_2\}$, where $T_1 = 1$ almost surely, and T_2 follow an exponential distribution with parameter λ (greater than 1). Suppose that T_2 is replaced by another component with lifetime T_3, where $T_3 = 1$ almost surely. Let $S^* = \max\{T_1, T_3\}$ be the lifetime of the new system. Assume also that T_1, T_2, and T_3 are independent. Note that

$$\mathbb{P}(T_2 \leq T_3) = \mathbb{P}(T_2 \leq 1) = 1 - \mathrm{e}^{-\lambda} \geq \tfrac{1}{2},$$

and, hence, $T_2 \leq_{\text{sp}} T_3$. Now

$$\begin{aligned}
\mathbb{P}(S < S^*) &= \mathbb{P}(\max\{T_1, T_2\} < \max\{T_1, T_3\}) \\
&= \mathbb{P}(\max\{T_1, T_2\} < \max\{T_1, T_3\} | T_1 = 1, T_3 = 1)\mathbb{P}(T_1 = 1)\mathbb{P}(T_3 = 1) \\
&= \mathbb{P}(\max\{1, T_2\} < 1) \\
&= 0.
\end{aligned}$$

Similarly,

$$\mathbb{P}(S > S^*) = \mathbb{P}(\max\{1, T_2\} > 1) = 1 - \mathbb{P}(\max\{1, T_2\} \leq 1) = 1 - \mathbb{P}(T_2 \leq 1) = \mathrm{e}^{-\lambda},$$

and

$$\mathbb{P}(S = S^*) = \mathbb{P}(\max\{1, T_2\} = 1) = \mathbb{P}(T_2 \leq 1) = 1 - \mathrm{e}^{-\lambda}.$$

Thus, $\mathbb{P}(S \leq S^*) = 1 - \mathrm{e}^{-\lambda}$ and $\mathbb{P}(S \geq S^*) = 1$, and both are greater than $\frac{1}{2}$. Hence, both inequalities $S \leq_{\text{sp}} S^*$ and $S \geq_{\text{sp}} S^*$ hold. Further, we note that $\mathbb{E}(S) \geq \mathbb{E}(S^*)$ even though $\mathbb{E}(T_2) < E(T_3)$.

Let us now specify the ordering in (2.1) as

$$T_1 \leq_{\text{sp}} T_2 \leq_{\text{sp}} \cdots \leq_{\text{sp}} T_n. \tag{4.2}$$

However, the stochastic precedence order is the pairwise order and due to the possible non-transitivity in the sequence (4.2) (see [15]), the lifetime T_n, for instance, is not necessarily greater than some of $T_1, T_2, \ldots, T_{n-1}$. To account for this 'deficiency', let us define the corresponding multivariate generalization that reduces to the sp order for two variables. The *sequential stochastic precedence* (ssp) order for the sequence of independent lifetimes, $T_1, T_2, \ldots, T_n$ is the new order that gives the maximal probability to, e.g. the event $T_1 \leq T_2 \leq \cdots \leq T_n$, as compared with all other permutations of these random variables, i.e.

$$\mathbb{P}_{12\ldots n} \equiv \mathbb{P}(T_1 \leq T_2 \leq \cdots \leq T_n) \geq \mathbb{P}_{\{\boldsymbol{J}\}}, \tag{4.3}$$

where $\mathbb{P}_{\{\boldsymbol{J}\}}$, $\{\boldsymbol{J}\} \equiv \{j_1 j_2 \cdots j_n\}$, denotes the corresponding probability for any other permutation (sequence) $\boldsymbol{J}$. It is easy to see, e.g. that when the lifetimes are distributed exponentially with parameters $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ (pairwise hazard rate order), (4.3) holds. Using the ssp, we can now formulate the following theorem.

Theorem 4.1. *Let the sequential stochastic precedence order (4.3) hold for the* 1*-out-of-n warm standby system described above. Then the corresponding sequence of the activation of components achieves the maximum lifetime of a system in the sense of the stochastic precedence order.*

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Proof. Let the lifetimes of components of our system be ordered in accordance with (4.3). Denote the arbitrary sequenced realizations of these lifetimes by $t_{\{\boldsymbol{J}\}}$. Assume that the system had started operation with the component having the smallest realization from $t_{\{\boldsymbol{J}\}}$. Consider the ith and the $(i+1)$th components in this sequence of realizations, $i = 1, 2, \dots, n-1$. If the realization of the lifetime of the first component from this pair is smaller than that of the second, we do nothing. Otherwise, we change the sequence of activation of these two components. We can do it with all 'nonproperly' sequenced components and eventually arrive at the realizations sequence ascending from the smallest to the largest. Denote by $S_{12\cdots n}$ and $S_{\{\boldsymbol{J}\}}$ the system's lifetimes achieved by the sequences that correspond to notation in (4.3) and by $s_{12\cdots n}$, $s_{\{\boldsymbol{J}\}}$ their realizations, respectively.

The reasoning provided below generalizes the discussion in Theorem 3.1 to the case of n components. In essence, it means that if for the realizations of lifetimes of components $T_1, T_2, \dots, T_n$ ordered in accordance with (4.3), we have $t_1 \leq t_2 \leq \cdots \leq t_n$, then the corresponding realizations for the system's lifetimes are ordered as $s_{12\cdots n} \geq s_{\{\boldsymbol{J}\}}$. On the other hand, it is clear that, as $\mathbb{P}_{12\cdots n}$ is the maximal probability, we have

$$\mathbb{P}(S_{12\cdots n} \geq S_{\{\boldsymbol{J}\}}) = \frac{\mathbb{P}_{12\cdots n}}{\mathbb{P}_{12\cdots n} + \mathbb{P}_{\{\boldsymbol{J}\}}} \geq \frac{1}{2},$$

which means that the system lifetime for the activation sequence defined in (4.3) is longer than any other activation sequence in the sense of the *stochastic precedence order*.

As in the rest of the paper, our justification of the proposed procedure is based on considering the corresponding realizations. At each step, it is similar to the case of two components considered in the previous section. The difference to be discussed, however, is that the initial activation time in the case of only two components was 0 and now it is some arbitrary t_a. Let $t_i < t_{i+1}$ and the ith component start first if activated. We emphasize once more the fact that t_i are realizations of T_i, $i = 1, 2, \dots, n$, which are the lifetimes in the activated mode. An event $\alpha t_{i+1} < t_a$ means that both components have failed before the prospective activation and the corresponding comparison is irrelevant. Another possibility is that the ith component fails before the activation whereas the $(i+1)$th does not. In this case, the lifetime of the pair (after activation) is, in accordance with the cumulative exposure model, $(t_{i+1} - wt_a)$. The final possibility is when both of them do not fail before activation. In this case, the lifetime of a pair after activation is (compare with (3.1) corresponding to the $t_a = 0$ case)

$$t_i - wt_a + (t_{i+1} - w(t_i - wt_a)), \tag{4.4}$$

where wt_a is the virtual age of the ith component just after activation and, therefore, its remaining lifetime in this realization is $(t_i - wt_a)$. As the $(i+1)$th component was operating during the time since activation till the failure of the ith component in the warm standby mode, this time should be recalculated to end up with the remaining lifetime of the $(i+1)$th component after its activation as $(t_{i+1} - w(t_i - wt_a))$.

Now let the $(i+1)$th component start first. Reasoning similar to the above results in a smaller (in realizations) lifetime of a pair as compared with the initial sequence. For instance, obviously, the term $(t_{i+1} - wt_a)$, which corresponds to the case when the ith component fails before the activation whereas the $(i+1)$th does not, stays the same. We also have two specific cases for the case when the components do not fail (in the warm standby mode) before t_a (see cases I and II of the previous section). But we can just adjust properly our previous reasoning by considering the remaining lifetimes after activation, which are $t_i - wt_a$ and $t_{i+1} - wt_a$ $(t_i - wt_a < t_{i+1} - wt_a)$, then the reasoning and the comparison with (4.4) will be exactly the same as the comparison of (3.2) with (3.1). □

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Note that, as follows from Lemma 4.1, at each step we are also improving the lifetime of a system with respect to the sp order, which can be important in practice and can be considered as an independent result. However, we eventually need the best solution, which is not guaranteed by the reasoning within the frame of the sp order due to possible nontransitivity. For that we need the ssp order (4.3).

Remark 4.2. Generalization to the model in (1.1) and (1.2) can be performed by reasoning similar to that in Remark 3.1.

5. Concluding remarks

In this paper we have shown that the optimal operational sequence for the 1-out-of-n system with warm standby is when the components are activated in accordance with the increasing sequence of their lifetimes. It turns out that from our reasoning the natural stochastic ordering for comparing lifetimes of the system is the stochastic precedence order and its generalization-for comparing the lifetimes of components.

When the warm standby component is activated, its age should be 'recalculated'. This recalculation is performed using the virtual age concept and the cumulative exposure model.

The proofs were carried out for the linear cumulative exposure model. Generalization to the time-dependent case was also discussed.

Previously, *only specific cases* of the problem were considered in the literature. In [4] and [19], the case of two components was considered and the sequence was justified (in terms of expected lifetimes of a system) for the case when the components were ordered in the sense of the hazard rate ordering. Moreover, the corresponding sequence was justified in [19] for 1-out-of-n system but only for the *exponentially distributed* lifetimes of components.

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