MODELLING DEPENDENCE IN INSURANCE CLAIMS PROCESSES WITH LÉVY COPULAS

BY

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ABSTRACT

In this paper we investigate the potential of Lévy copulas as a tool for modelling dependence between compound Poisson processes and their applications in insurance. We analyse characteristics regarding the dependence in frequency and dependence in severity allowed by various Lévy copula models. Through the introduction of new Lévy copulas and comparison with the Clayton Lévy copula, we show that Lévy copulas allow for a great range of dependence structures.

Procedures for analysing the fit of Lévy copula models are illustrated by fitting a number of Lévy copulas to a set of real data from Swiss workers compensation insurance. How to assess the fit of these models with respect to the dependence structure exhibited by the dataset is also discussed.

Finally, we provide a decomposition of the trivariate compound Poisson process and discuss how trivariate Lévy copulas model dependence in this multivariate setting.

KEYWORDS

Lévy copula, Dependence, Compound Poisson process, Insurance, Real data.

1. INTRODUCTION

In a non-life insurance company, an event may give rise to claims of different types. Such events range from a work-related accident resulting in claims for medical costs and allowance costs, to a natural peril causing losses in motor and home classes of business. Furthermore, dependence in claims processes can have an impact on both frequency (claim counts) and severity (claim amounts). This has direct implications on pricing, reserving and capital allocation of an insurance company (Embrechts et al., 2002; Denuit et al., 2005; McNeil et al., 2005). It is also highly relevant for solvency purposes and in risk based capital regulatory systems such as Solvency II.

A natural and standard choice for modelling insurance claims processes is the compound Poisson process (e.g., Bowers et al., 1997; Mikosch, 2009;

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Asmussen and Albrecher, 2010). In a multivariate setting, dependence between multiple compound Poisson processes (loosely interpreted as classes of business) can be intuitively represented in a *common shock* representation (Lindskog and McNeil (2003); see also Yuen and Wang (2002)). In such a representation, classes of business (potentially) share 'common shocks' — claims occurring at the same time in two or more different classes according to an identical arrival process. Furthermore, dependence between the sizes of the claims occurring simultaneously can be modelled with distributional copulas. This approach has a number of advantages. Firstly, the common shock model allows for detailed and separate specification of dependence in frequency and dependence in severity. In addition, as the model is specified upon a continuous time (Markov) stochastic process, the model also allows for the consideration of dependence over alternative time horizons in an internally consistent manner.

Unfortunately, due to its flexibility, the common shock model becomes increasingly parameter intensive as the number of dimensions increases. For example, the case of four classes of business can require the specification of up to fifteen independent Poisson arrival processes (because of jumps that can be common to two, three, or four classes), six bivariate distributional copulas, four trivariate distributional copulas, one quadvariate distributional copula and twenty-four jump size distributions.

An alternative approach is to apply a distributional copula directly to the *aggregate* claims of each class at a chosen time horizon, creating a multivariate distribution of aggregate claim amounts (see, for example, McNeil et al., 2005; Bargès et al., 2009). Similarly, a distributional copula may be applied to the aggregate *number* of claims over a chosen time horizon, (see, for example, Bäuerle and Grübel, 2005; Genest and Nešlehová, 2007). The model is then reduced to modelling dependence between random variables for a given time horizon. This approach possesses a number of benefits, including relative parsimony in model specification, and in particular with the facilitation of a "*bottom-up approach to multivariate model building*" whereby models are built by combining the information of a class of business (i.e. the marginals), with that of the dependence structure across classes (McNeil et al., 2005, p. 185). This is in contrast to a common shock based approach where models are built from common shock events. The focus here is on the classes of business, rather than the common shock events. Unfortunately, as the distributional copula for aggregate claims will depend on the chosen time horizon, in general it is not possible to infer the copula for a different time horizon (to consider the risks faced by an insurance company over 1, 2 or 5 years, for instance, Fosker et al., 2010, p. 8). This approach also requires sufficient data for the aggregate claim amounts in each class of business for the chosen time horizon. For example, if only a single year of data is available, then using a time horizon of one year would allow for only 1 data point for fitting a distributional copula. This also results in an inefficient use of data where individual accident information is known.

In contrast to the two methods discussed above, Lévy copulas provide a new method which bridges the benefits of the common shock and distributional copula approaches. Under this approach, dependence is introduced via a multivariate function (the Lévy copula) which couples the marginal tail integrals of the compound Poisson processes for each class of business into a multivariate tail integral which completely specifies the desired multivariate (dependent) compound Poisson processes model. In a nutshell, the tail integral of a compound Poisson process (related to its Lévy measure) represents the expected number of losses over a threshold (the argument of the function) over one unit of time; refer to the following section for a formal definition. Such a representation combines the advantages of the common shock model and the distributional copula approach, by being parsimonious, facilitating a bottom-up approach, allowing changes of time horizon (time consistency), and by being efficient in the way it uses available data.

Lévy copulas were introduced in a series of publications by Tankov (2003), Cont and Tankov (2004) and Kallsen and Tankov (2006). In applications, the Clayton Lévy copula have been used to model the dependence between compound Poisson processes firstly to estimate ruin probabilities for an insurance company with multiple classes of business (Bregman and Klüppelberg, 2005). Optimal investment and reinsurance problems for a multiline insurer under a Lévy copula framework was studied in Bäuerle and Blatter (2011). In the closely related area of operational risk modelling, applications of Lévy copulas between operational loss cells is discussed in Böcker and Klüppelberg (2008), Biagini and Ulmer (2009) and Böcker and Klüppelberg (2010). On the statistical front, a maximum likelihood scheme for fitting Lévy copulas to data is provided in Esmaeili and Klüppelberg (2010a), who focus in particular on fitting a Clayton Lévy copula. Additional theoretical developments in more general Lévy process settings can also be found in Barndorff-Nielsen and Lindner (2007), Bäuerle et al. (2008) and Eder and Klüppelberg (2009).

In this paper, we first focus on a careful review of the concept of Lévy copula and shed some light on how this function is generating dependence between compound Poisson processes. To date, there has been limited consideration of the properties enabled by specific Lévy copula models in applications, with the notable exception being the Clayon Lévy copula. Section 3 develops new Lévy copula models and illustrates how their dependence structures can be compared. It is illustrated how many of the special properties of the Clayton Lévy copula may not hold in general. Furthermore, it is also important from a practical point of view to consider alternative Lévy copula models so as to provide additional flexibility in the type of dependence available to the modeller. This allows for a better understanding of the actuarial applications of Lévy copulas and illustrates the range of dependence structures enabled by them, in particular with respect to the impact of different models on the dependence in frequency and/or severity. Section 4 provides a modelling example using a set of worker's compensation claims and the newly developed Lévy copulas. The issue of model selection is also discussed, as the fit of different Lévy copulas to the data is compared. Finally, as insurance companies normally run more than two (possibly dependent) classes of business, dependence beyond a bivariate setting is of particular relevance. This is investigated in Section 5, where the similarities and differences between the bivariate and trivariate cases are highlighted. Such a development is of interest as common jumps can then occur between any sub-set of the considered processes.

2. Dependence between compound Poisson processes

This section provides an introduction to Lévy copulas and their implications on dependence between compound Poisson processes. Note that whilst compound Poisson processes in general can have jumps in both positive and negative directions, compound Poisson processes with only positive jumps are considered for the purpose of insurance claims modelling. Hence, only *positive Lévy copulas* are addressed in this paper.

2.1. Lévy copulas and compound Poisson processes

Consider a bivariate compound Poisson process $\{S_1(t), S_2(t)\}$, for instance, to model two dependent classes of business; see also Sato (1999, Theorem 4.3) for a comprehensive definition of a multivariate compound Poisson process. It is known that $\{S_1(t), S_2(t)\}$ can be decomposed into unique (superscript $\perp$) and common (superscript $\parallel$) jumps, so that

$$\begin{cases} S_1(t) = S_1^{\perp}(t) + S_1^{\parallel}(t) \\ S_2(t) = S_2^{\perp}(t) + S_2^{\parallel}(t), \end{cases} \tag{2.1}$$

where $S_1^{\perp}(t)$ and $S_2^{\perp}(t)$ are independent compound Poisson processes and where $S_1^{\parallel}(t)$ and $S_2^{\parallel}(t)$ are dependent compound Poisson processes whose jumps (the 'common shocks') occur at the same time (Lindskog and McNeil, 2003; Esmaeili and Klüppelberg, 2010a). In general, the jump size distributions of $S_i^{\perp}(t)$ and $S_i^{\parallel}(t)$ are not identical. However, the jump size distribution of $S_i(t)$ will be a mixture of the jump size distributions for $S_i^{\perp}(t)$ and $S_i^{\parallel}(t)$ (see, for example, Mikosch, 2009, Proposition 3.3.4).

Let us now introduce more formally the concept of 'tail integral'. The tail integral of a Lévy process measures its expected number of jumps (above a certain threshold) per unit of time. In the (less general) case of a compound Poisson process $S_i(t)$, $i=1,2$, the tail integral boils down to

$$U_i(x) = \begin{cases} \lambda_i \bar{F}_i(x), & x \in (0,\infty) \\ \infty & x = 0. \end{cases} \tag{2.2}$$

where $\bar{F}_i(x)$ is the survival function for the jump size of $S_i(t)$. Furthermore, the joint tail integral of a bivariate compound Poisson process $\{S_1(t), S_2(t)\}$ is given by

$$U(x_1,x_2) = \begin{cases} \lambda^{\parallel}\bar{F}(x_1,x_2), & (x_1,x_2) \in (0,\infty)^2 \\ U_1(x_1) & x_1 \in (0,\infty), x_2 = 0 \\ U_2(x_2) & x_1 = 0, x_2 \in (0,\infty) \\ \infty & (x_1,x_2) = (0,0). \end{cases} \tag{2.3}$$

where $\lambda^{\parallel}$ is the Poisson parameter for the (common) jumps in $S_1^{\parallel}(t)$ and $S_2^{\parallel}(t)$ and $\bar{F}(x_1, x_2)$ is the joint survival function for the sizes of the common jumps. A formal definition of the tail integral for a Lévy process with positive jumps is given in Appendix A.1.

A Lévy copula $\mathfrak{C}$ couples the marginal tail integrals to the joint tail integral so that

$$\mathfrak{C}(U_1(x_1), U_2(x_2)) = U(x_1,x_2); \tag{2.4}$$

see Appendix A.2 for a formal definition of positive Lévy copulas. The mechanism is strikingly similar to the one with which distributional copulas couple the marginal distribution functions to the multivariate distribution function and is formalised in what is described in Cont and Tankov (2004) as "a reformulation of Sklar's theorem for tail integrals and Lévy copulas."

Theorem 2.1. *(Sklar's theorem for Lévy copulas, Tankov, 2003) If U is a tail integral with margins $U_1(\cdot), \ldots, U_d(\cdot)$, then there exists a Lévy copula $\mathfrak{C}$ such that*

$$U(x_1, \ldots, x_d) = \mathfrak{C}(U_1(x_1), \ldots, U_d(x_d)). \tag{2.5}$$

If $U_1(\cdot), \ldots, U_d(\cdot)$ are continuous on $[0,\infty]$ then this Lévy copula is unique. Otherwise, it is unique on the product of the ranges of the marginal tail integrals.

The converse is also true. If $\mathfrak{C}$ is a Lévy copula and $U_1(\cdot), \ldots, U_d(\cdot)$ are marginal tail integrals, then (2.5) *defines a multidimensional tail integral.*

On one hand, a common shock approach would require the separate modelling of the Poisson parameters and jump size distributions of $S_1^{\perp}(t)$, $S_2^{\perp}(t)$, $S_1^{\parallel}(t)$ and $S_2^{\parallel}(t)$, as well as the dependence structure of the jump sizes of $S_1^{\parallel}(t)$ and $S_2^{\parallel}(t)$. On the other hand, if the Lévy copula is known, only the Poisson parameters and jump size distributions for $S_1(t)$ and $S_2(t)$ (which are directly observable) need to be specified. This is because the decomposition of $S_1(t)$ and $S_2(t)$ into unique and common components as shown in (2.1) stems directly from the Lévy copula (Böcker and Klüppelberg, 2008), as summarised in the following lemma.

Lemma 2.2. *Common jumps in $S_1^{\parallel}(t)$ and $S_2^{\parallel}(t)$ arrive at a rate*

$$\lambda^{\parallel} = \mathfrak{C}(\lambda_1, \lambda_2), \tag{2.6}$$

whereas the sizes of these common jumps have joint survival function

$$\bar{F}^{\parallel}(x_1, x_2) = \frac{1}{\lambda^{\parallel}}\,\mathfrak{C}(\lambda_1 \bar{F}_1(x_1), \lambda_2 \bar{F}_2(x_2)), \tag{2.7}$$

and marginal survival functions

$$\begin{cases} \bar{F}_1^{\parallel}(x) = \dfrac{1}{\lambda^{\parallel}}\,\mathfrak{C}(\lambda_1 \bar{F}_1(x), \lambda_2), \text{ and} \\ \bar{F}_2^{\parallel}(x) = \dfrac{1}{\lambda^{\parallel}}\,\mathfrak{C}(\lambda_1, \lambda_2 \bar{F}_2(x)). \end{cases} \tag{2.8}$$

Unique jumps in $S_i^{\perp}(t)$, $i = 1, 2$, arrive at rates

$$\lambda_i^{\perp} = \lambda_i - \lambda^{\parallel}, \; i = 1, 2, \tag{2.9}$$

whereas their sizes are distributed with survival functions

$$\bar{F}_i^{\perp}(x) = \frac{1}{\lambda_i^{\perp}}\left(\lambda_i \bar{F}_i(x) - \lambda^{\parallel} \bar{F}_i^{\parallel}(x)\right), \; i = 1, 2. \tag{2.10}$$

In general, the distributions of the sizes of common jumps and unique jumps in each compound Poisson process distributions will not be identical. However, Lemma 2.3 provides conditions which must be satisfied by a bivariate Lévy copula to allow for identically distributed unique and common jump sizes in each compound Poisson process.

Lemma 2.3. *(Identically distributed unique and common jump sizes) A bivariate compound Poisson process with Lévy copula $\mathfrak{C}$ satisfying*

$$\begin{cases} \bar{F}_1(x)\,\mathfrak{C}(\lambda_1, \lambda_2) = \mathfrak{C}(\lambda_1 \bar{F}_1(x), \lambda_2), \\ \bar{F}_2(x)\,\mathfrak{C}(\lambda_1, \lambda_2) = \mathfrak{C}(\lambda_1, \lambda_2 \bar{F}_2(x)), \end{cases} \tag{2.11}$$

has unique and common jump sizes in a given compound Poisson process which are identically distributed (and are identical to the marginal jump size distribution of the process).

Proof. If the jump size distributions of the common jumps are equivalent to the marginal jump size distributions of the process, then (2.11) follows from a rearrangement of (2.8). In addition, (2.10) further implies that

$$\begin{aligned} F_i^{\perp}(x) &= \frac{1}{\lambda_i - \lambda^{\parallel}}\,(\lambda_i \bar{F}_i(x) - \lambda^{\parallel} \bar{F}_i(x)) \\ &= \bar{F}_i(x), \text{ for } i = 1, 2, \end{aligned} \tag{2.12}$$

as required. □

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2.2. Changes of time horizon

If the Lévy copula for a time horizon of one unit of time is given by $\mathfrak{C}$, then the Lévy copula for a time horizon of length T is expressed as

$$\mathfrak{C}_T(u_1, \ldots, u_d) = T\mathfrak{C}\left(\frac{u_1}{T}, \ldots, \frac{u_d}{T}\right). \tag{2.13}$$

This result is due to the properties of the tail integral and Lévy processes (Barndorff-Nielsen and Lindner, 2007, Equation 13). This shows how a Lévy copula approach allows for easy changes of time horizon, in contrast to the distributional copula approach. Interestingly, Lévy copulas and the distributional copula of the aggregate claim amounts at time T are related via the following asymptotic relation,

$$\mathfrak{C}(u_1, \ldots, u_d) = \lim_{T \to 0^+} \frac{1}{T} C_T(u_1 T, \ldots, u_d T), \tag{2.14}$$

where $C_T(\cdot, \ldots, \cdot)$ is the time dependent distributional copula for an increment of time length T (Kallsen and Tankov, 2006). Whilst the Lévy copula of the process can be interpreted as the distributional copula of the aggregate claims amount for small T, in general $C_T(\cdot, \ldots, \cdot)$ cannot be inferred from $\mathfrak{C}$.

2.3. Constructing positive Lévy copulas

In this section we present two methods for constructing Lévy copulas, due to Tankov (2003), Cont and Tankov (2004) and Kallsen and Tankov (2006) (see also Bäuerle and Blatter, 2011). In Method 2.1, Lévy copulas are derived from a multivariate Lévy process using Sklar's theorem for Lévy copulas. Note, however, that there are only a limited number of multivariate Lévy processes from which Lévy copulas can be derived. As an alternative, Method 2.2 allows for the construction of Archimedean families of Lévy copulas.

Method 2.1. *Consider a d-dimensional spectrally positive Lévy process with continuous marginal tail integrals. A positive Lévy copula $\mathfrak{C}$ can be constructed as*

$$\mathfrak{C}(u_1, \ldots, u_d) = U\left(U_1^{-1}(u_1), \ldots, U_d^{-1}(u_d)\right), \tag{2.15}$$

where $U(\cdot, \ldots, \cdot)$ is the multivariate tail integral of the multivariate Lévy process and $U_1(\cdot), \ldots, U_d(\cdot)$ are the marginal tail integrals.

Remark 2.1. *If the marginal tail integrals are not continuous then a Lévy copula can still be constructed from (2.15) by an extension procedure, see Tankov (2003) and Kallsen and Tankov (2006).*

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Method 2.2. *For a function* $\phi : [0,\infty] \to [0,\infty]$ *with* $\phi(0) = \infty$ *and* $\phi(\infty) = 0$ *and a defined inverse* $\phi^{-1}(\cdot)$,

$$\mathfrak{C}(u_1, \ldots, u_d) = \phi^{-1}\big(\phi(u_1) + \ldots + \phi(u_d)\big), \tag{2.16}$$

where the inverse must satisfy

$$(-1)^k (\phi^{-1})^{(k)}(z) > 0, \ \ for \ \ z > 0, \ \ k = 1, \ldots, d, \tag{2.17}$$

and $(\phi^{-1})^{(k)}(z)$ *denotes the k-th derivative of the inverse of* $\phi(\cdot)$ *with respect to z.*

Remark 2.2. *When constructing Archimedean distributional copulas, special care is needed in defining the inverse of the generator. The case of Lévy copulas is easier. Archimedean generators of Lévy copulas have a domain of* $[0,\infty]$ *and a range of* $[0,\infty]$*, so there is no need for a "pseudo-inverse" (Nelsen, 1999).*

2.4. Fundamental Lévy copulas

For an independent multivariate compound Poisson process, the tail integral of the multivariate process is expressed as

$$U(x_1, \ldots, x_d) = U_1(x_1) I_{\{x_2 = \ldots = x_d = 0\}} + \ldots + U_d(x_d) I_{\{x_1 = \ldots = x_{d-1} = 0\}}; \tag{2.18}$$

see Bregman and Klüppelberg (2005). This means that the tail integral of an independent d-dimensional Lévy process is equal to 0 except for the cases where it is equal to the marginal tail integral. As a consequence, the independence Lévy copula is given by

$$\mathfrak{C}_\perp(u_1, \ldots, u_d) = u_1 I_{\{u_2 = \ldots = u_d = \infty\}} + \ldots + u_d I_{\{u_1 = \ldots = u_{d-1} = \infty\}}, \tag{2.19}$$

where the indicator functions are now changed since a marginal tail integral evaluated at 0 is equal to ∞ by definition.

The comonotonic Lévy copula is derived in a multivariate setting as

$$\mathfrak{C}_\parallel(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d); \tag{2.20}$$

see Cont and Tankov (2004). This implies that the tail integral of a completely dependent d-dimensional Lévy process is given by the smallest of the marginal tail integrals.

Remark 2.3. *Comonotonicity in a multivariate compound Poisson process means that all jumps in one process are functions of the jumps in the other. However, unless* *the rates of jumps in the marginal compound Poisson processes are equal, there will always exist unique jumps in the multivariate compound Poisson process with a comonotonic Lévy copula, so that all arrival processes are not necessarily identical. This stems from the discontinuity at 0 of the tail integral of a compound Poisson process.*

3. Comparisons and illustrations of bivariate Lévy copulas

The current applied literature on Lévy copulas places considerable emphasis on the properties and application of the Clayton Lévy copula. The purpose of this section is to illustrate that Lévy copulas allow for a richer range of dependence structures by developing new models and by comparing their main features. After introducing the pure common shock Lévy copula, an analysis of the Clayton Lévy copula (Tankov 2003) is included for comparison purposes. Two other new Lévy copulas are also introduced, one that fits well the data set that is considered in this paper (see Section 4), and another one that allows for negative dependence in severity. Throughout this section, the dependence structures induced by the different models will be compared by examination of their 'Lévy copula density',

$$\mathfrak{c}(u_1, u_2) = \frac{\partial^2 \mathfrak{C}(u_1, u_2)}{\partial u_1 \, \partial u_2}, \tag{3.1}$$

where $u_i = U_i(x_i)$, $i = 1, 2$. The volume under the density on $[0, \lambda_1] \times [0, \lambda_2]$ is the expected number of common jumps per unit time,

$$\int_0^{\lambda_2} \int_0^{\lambda_1} \mathfrak{c}(u_1, u_2) du_1 du_2 = \lambda^{\parallel}. \tag{3.2}$$

Since this is constant (for given λ_1 and λ_2), the relative repartition of the density on $[0, \lambda_1] \times [0, \lambda_2]$ is informative of the dependence structure. First, note that small u_1 and u_2 indicate larger jump sizes (and vice versa), because the expected number of jumps will be higher as the argument of the tail integral is lower. Thus, a relative higher density at small u_1 and u_2 will indicate a propensity for common jumps of large sizes in both components (and vice versa). Similarly, if more density is present at small u_1 and large u_2, common jumps of large sizes in the first component will have a higher propensity to occur with common jumps of small sizes in the second component, and vice versa.

We consider in the rest of this section the following illustration scheme. Assuming $\lambda_1 = 100$ and $\lambda_2 = 100$, Lévy copula densities and the distributional copula of common jump sizes are compared under three possible values for the expected number of common jumps $\lambda^{\parallel} = 30, 60, 90$. The purpose of this exercise is to demonstrate the range of dependence structures available by using different Lévy copulas, while holding the dependence in frequency constant.

3.1. Pure common shock Lévy copula

Lemma 2.2 showed how the Lévy copula affects both dependence in frequency and dependence in severity in a bivariate compound Poisson process. However, it is sometimes desirable to assume independence between common jump sizes, and use a model which allows for dependence in the frequency only. We refer to such a dependence structure as a ‘pure common shock model’ (not to be confused with a process consisting of only common jumps; see (2.20)). The corresponding Lévy copula representation is given in Definition 3.1.

Definition 3.1. *Pure common shock Lévy copula)*
The pure common shock Lévy copula is given by

$$\mathfrak{C}_\delta(u_1,u_2) = \delta(u_1 \wedge \lambda_1)(u_2 \wedge \lambda_2) + [u_1 - \delta\lambda_2(u_1 \wedge \lambda_1)]\, I_{\{u_2 = \infty\}} + [u_2 - \delta\lambda_1(u_2 \wedge \lambda_2)]\, I_{\{u_1 = \infty\}}, \tag{3.3}$$

$$\text{for} \quad 0 \le \delta \le \min\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right).$$

where λ_1 and λ_2 are the Poisson parameters for the bivariate compound Poisson process, and where δ is a parameter which will determine the intensity of the common jumps, since

$$\lambda^{\parallel} = \mathfrak{C}_\delta(\lambda_1,\lambda_2) = \delta\lambda_1\lambda_2. \tag{3.4}$$

Lemma 3.1. *The pure common shock Lévy copula* (3.3) *satisfies the necessary conditions of a positive Lévy copula (see Appendix A.2).*

Proof. The positive Lévy copula is clearly increasing in each component u_1 and u_2, satisfies $\mathfrak{C}_\delta(0,u_2) = \mathfrak{C}_\delta(u_1,0) = 0$ and has margins $\mathfrak{C}_\delta(\infty,u_2) = u_2$ and $\mathfrak{C}_\delta(u_1,\infty) = u_1$. For all (a_1,a_2), $(b_1,b_2) \in [0,\infty)^2$, and with $a_1 \le b_1$ and $a_2 \le b_2$,

$$\begin{aligned}&\mathfrak{C}_\delta(b_1,b_2) - \mathfrak{C}_\delta(a_1,b_2) - \mathfrak{C}_\delta(b_1,a_2) + \mathfrak{C}_\delta(a_1,a_2)\\ &\quad = \delta\big[(b_2 \wedge \lambda_2) - (a_2 \wedge \lambda_2)\big]\big[(b_1 \wedge \lambda_1) - (a_1 \wedge \lambda_1)\big] \ge 0,\end{aligned} \tag{3.5}$$

and for the case $b_1 = \infty$, $b_2 \in [0,\infty)$ and $(a_1,a_2) \in [0,\infty)^2$,

$$\begin{aligned}&\mathfrak{C}_\delta(b_1,b_2) - \mathfrak{C}_\delta(a_1,b_2) - \mathfrak{C}_\delta(b_1,a_2) + \mathfrak{C}_\delta(a_1,a_2)\\ &\quad = b_2 - a_2 + \delta(a_1 \wedge \lambda_1)[(a_2 \wedge \lambda_2) - (b_2 \wedge \lambda_2)] \ge 0,\end{aligned} \tag{3.6}$$

since $\delta(a_1 \wedge \lambda_1) \leq 1$ due to the restriction on δ. All other cases are proven in a similar way. □

Note that the upper bound on the Lévy copula parameter δ in (3.3) is necessary as a result of $\lambda_i^{\perp} \geq 0$ and (2.9), so that

$$\lambda^{\parallel} \leq \min(\lambda_1, \lambda_2). \tag{3.7}$$

The case of $\delta = 0$ leads to the independence Lévy copula (2.19).

Lemma 3.2. *A bivariate compound Poisson process with dependence specified by the pure common shock Lévy copula* (3.3) *with non-zero δ has independent and identically distributed common and independent jump sizes within one process, and independent common jump sizes in both processes.*

Proof. This Lévy copula satisfies

$$\mathfrak{C}_\delta(\lambda_1 \bar{F}_1(x), \lambda_2) = \delta \lambda_1 \bar{F}_1(x) \lambda_2 = \bar{F}_1(x) \mathfrak{C}_\delta(\lambda_1, \lambda_2), \tag{3.8}$$

and similarly for the second argument. It follows by Lemma 2.3 that the resulting common and independent jump sizes within one process are independent and identically distributed. Finally, application of (2.7) gives

$$\bar{F}^{\parallel}(x_1, x_2) = \frac{1}{\delta_1 \lambda_1 \lambda_2} \mathfrak{C}_\delta(\lambda_1 \bar{F}_1(x_1), \lambda_2 \bar{F}(x_2)), \tag{3.9}$$

$$= \bar{F}_1(x_1) \bar{F}_2(x_2), \tag{3.10}$$

indicating independence. □

For $(u_1, u_2) \in [0, \lambda_1] \times [0, \lambda_2]$ the Lévy copula density for the pure common shock Lévy copula is simply given by the parameter δ. A plot of this density would then display a flat plane at that level, which indicates no prevalence of certain jump sizes over others for given jump sizes in other processes. As the dependence in frequency increases, the height of the plane above 0 also increases.

3.2. Clayton Lévy copula

The bivariate Clayton (positive) Lévy copula, introduced in Cont and Tankov (2004), is given by

$$\mathfrak{C}_\delta(u_1, u_2) = \left(u_1^{-\delta} + u_2^{-\delta}\right)^{-\frac{1}{\delta}}, \text{ for } \delta > 0. \tag{3.11}$$

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As $\delta \to 0$, the Clayton Lévy copula (3.11) tends to the independence Lévy copula, while as $\delta \to \infty$, (3.11) tends to the comonotonic Lévy copula.

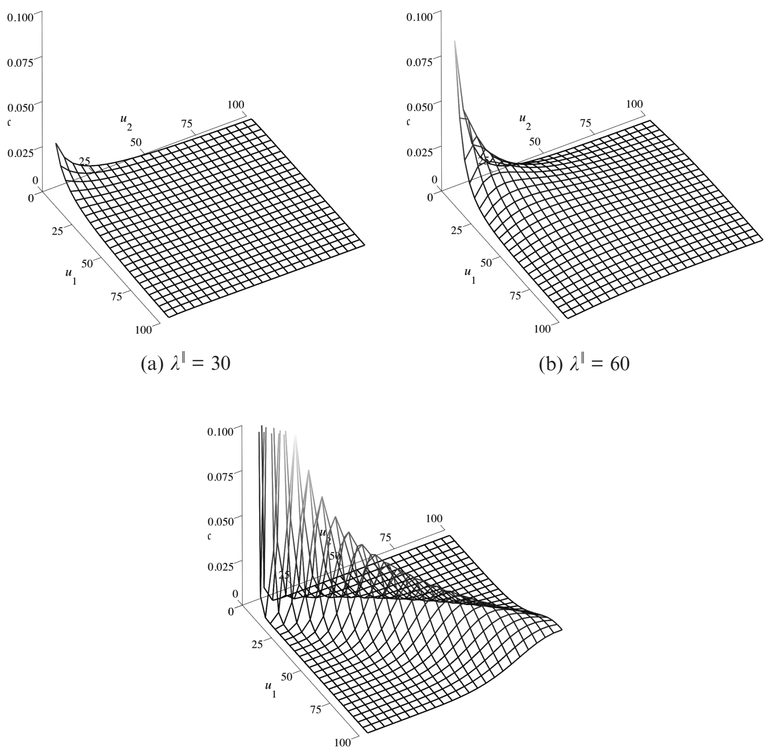
A particular property of the Clayton Lévy copula is that the survival copula of the sizes of common jumps is the Clayton distributional copula, that is,

$$\hat{C}(a_1, a_2) = \left(a_1^{-\delta} + a_2^{-\delta} - 1\right)^{-\frac{1}{\delta}}; \tag{3.12}$$

see Bregman and Klüppelberg (2005). The Lévy copula densities for the three scenarios of $\lambda^{\parallel}$ are shown in Figure 1. In contrast to the case of the pure common shock Lévy copula, the Clayton Lévy copula density is not a flat plane, reflecting dependence in the sizes of the common jumps. Additionally, as the dependence in frequency is increased, the intensity of common jumps is more prevalent at larger sizes, since the density is increasingly concentrated at small values of u_1 and u_2.

The Clayton Lévy copula is a homogeneous function of order one, that is,

$$\mathfrak{C}_\delta(\alpha u_1, \alpha u_2) = \alpha \mathfrak{C}_\delta(u_1, u_2). \tag{3.13}$$



(a) $\lambda^{\parallel} = 30$

(b) $\lambda^{\parallel} = 60$

(c) $\lambda^{\parallel} = 90$

FIGURE 1: Clayton Lévy copula densities for $\lambda_1 = 100$ and $\lambda_2 = 100$.

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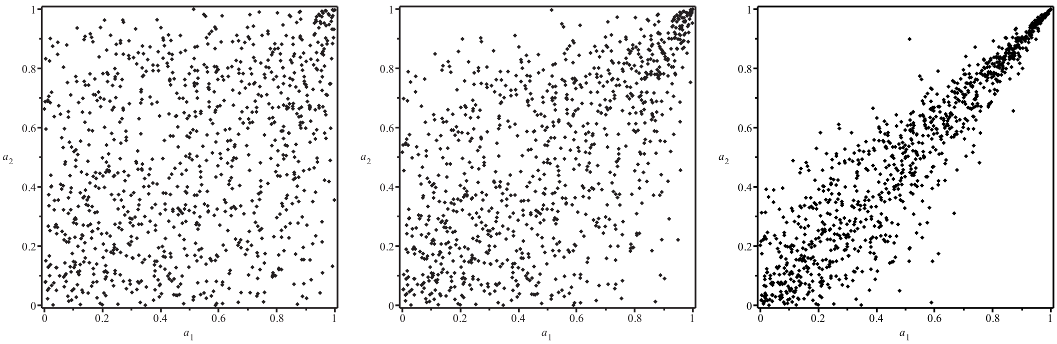

(a) $\lambda^{\parallel} = 30$ (b) $\lambda^{\parallel} = 60$ (c) $\lambda^{\parallel} = 90$

FIGURE 2: Simulations from the distributional copula of common jump sizes under the Clayton Lévy copula for $\lambda_1 = 100$ and $\lambda_2 = 100$.

An important consequence of (3.13) is that a Clayton Lévy copula for a new time horizon of length T is unchanged from the original so that

$$\mathfrak{C}_{\delta,T}(u_1,u_2) = \mathfrak{C}_{\delta}(u_1,u_2), \text{ for } T > 0. \tag{3.14}$$

To study dependence in severity, Figure 2 shows scatterplots of 1000 simulations from the distributional copula of the sizes of common jumps. Under the Clayton Lévy copula, the survival copula of the common jumps sizes is a Clayton distributional copula. The distributional copula of the sizes of common jumps, $C(\cdot,\cdot)$ is then derived from the survival copula $\hat{C}(\cdot,\cdot)$ using the relationship

$$C(a_1, a_2) = a_1 + a_2 - 1 + \hat{C}(1 - a_1, 1 - a_2); \tag{3.15}$$

see Nelsen (1999). As the dependence in frequency increases, the dependence in the sizes of common jumps is increasingly evident in the right-tail. That is, the prevalence of common jumps of relatively large sizes in both component increases, which further confirms our deductions from the Clayton Lévy densities in Figure 1.

3.3. Archimedean model I

In this section we introduce Archimedean model I, constructed using Method 2.2. Archimedean model I is an extension of a Lévy copula introduced in Chapter 5 of Cont and Tankov (2004).

Definition 3.2. *(Archimedean model I)*

$$\mathfrak{C}_{\delta}(u_1,u_2) = \frac{1}{\delta}\ln\left(\frac{1-e^{-\delta(u_1+u_2)}}{e^{-\delta u_1} - 2e^{-\delta(u_1+u_2)} + e^{-\delta u_2}}\right), \textit{ for } \delta > 0, \tag{3.16}$$

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is a bivariate Archimedean positive Lévy copula with Archimedean generator

$$\phi(z) = \frac{e^{-\delta z}}{1 - e^{-\delta z}}. \tag{3.17}$$

In contrast to the Clayton Lévy copula, Archimedean model I does not tend to the independence Lévy copula as $\delta \to 0$. Instead, the degree of dependence enabled under Archimedean model I is restricted as

$$\lim_{\delta \to 0} \mathfrak{C}_{\delta}(u_1, u_2) = \frac{u_1 u_2}{u_1 + u_2}. \tag{3.18}$$

This means that as $\delta \to 0$, Archimedean model I tends to a Clayton Lévy copula with a parameter of 1. Archimedean model I tends to the comonotonic Lévy copula as $\delta \to \infty$.

Even though Archimedean model I is not a homogeneous function of order one, the time scaled Lévy copula is derived by a simple adjustment of the parameter δ. If $\mathfrak{C}_{\delta}$ is an Archimedean model I Lévy copula defined for a time horizon equal to one unit of time, then the equivalent Lévy copula for a time horizon of length T is given by

$$\mathfrak{C}_{\delta, T}(u_1, u_2) = \mathfrak{C}_{\frac{\delta}{T}}(u_1, u_2). \tag{3.19}$$

This is a very convenient result as it means that dependence in multivariate compound Poisson processes may be time scaled with a simple change of parameter for the Lévy copula. This property is a result of the Archimedean generator $\phi(\cdot)$ being a function of δz.

The survival copula of the sizes of common jumps is then derived as

$$\hat{C}(a_1, a_2) = \frac{\ln\left(\frac{1 - h_2(a_1,\lambda_1,\lambda_2,\delta)h_2(a_2,\lambda_2,\lambda_1,\delta)}{h_2(a_1,\lambda_1,\lambda_2,\delta) + h_2(a_2,\lambda_2,\lambda_1,\delta) - 2h_2(a_1,\lambda_1,\lambda_2,\delta)h_2(a_2,\lambda_2,\lambda_1,\delta)}\right)}{\ln\left(\frac{1 - e^{-\delta(\lambda_1+\lambda_2)}}{e^{-\delta\lambda_1} + e^{-\delta\lambda_2} - 2e^{-\delta(\lambda_1+\lambda_2)}}\right)}, \tag{3.20}$$

where

$$h_2(\xi,\lambda_1,\lambda_2,\delta) = \frac{1 - e^{\xi \ln\left(\frac{1 - e^{-\delta(\lambda_1+\lambda_2)}}{e^{-\delta\lambda_1} - 2e^{-\delta(\lambda_1+\lambda_2)} + e^{-\delta\lambda_2}}\right) - \delta\lambda_2}}{e^{-\delta\lambda_2} - 2e^{\xi \ln\left(\frac{1 - e^{-\delta(\lambda_1+\lambda_2)}}{e^{-\delta\lambda_1} - 2e^{-\delta(\lambda_1+\lambda_2)} + e^{-\delta\lambda_2}}\right) - \delta\lambda_2} + e^{\xi \ln\left(\frac{1 - e^{-\delta(\lambda_1+\lambda_2)}}{e^{-\delta\lambda_1} - 2e^{-\delta(\lambda_1+\lambda_2)} + e^{-\delta\lambda_2}}\right)}}. \tag{3.21}$$

Notice that in contrast to the case of the Clayton Lévy copula, the survival copula of the sizes of common jumps is dependent on the values of λ_1 and λ_2.

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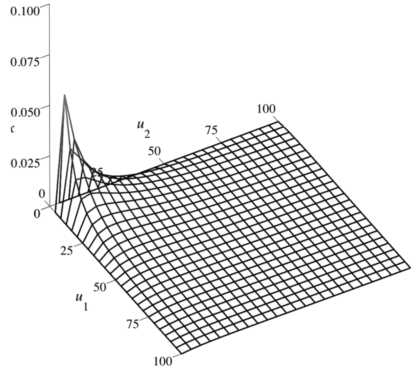
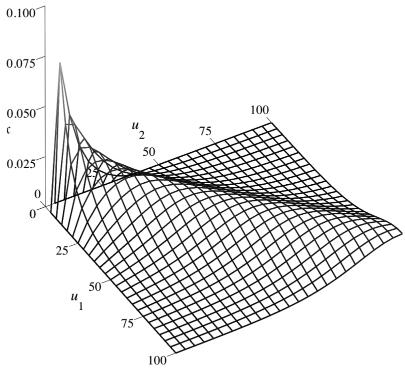
N/A

(a) $\lambda^{\parallel} = 30$



(b) $\lambda^{\parallel} = 60$



(c) $\lambda^{\parallel} = 90$

FIGURE 3: Archimedean model I densities for $\lambda_1 = 100$ and $\lambda_2 = 100$.

Additionally, it does not bear any resemblance to any commonly known bivariate Archimedean distributional copulas (see, for instance, Nelsen, 1999).

Due to the restriction indicated by (3.18), a dependence in frequency of $\lambda^{\parallel} = 30$ cannot be produced by this Lévy copula, which explains why there is no density for that case in Figure 3. For the cases of $\lambda^{\parallel} = 60$ and $\lambda^{\parallel} = 90$, Archimedean model I models positive jump dependence as the Lévy copula density is concentrated at those values where $u_1 = u_2$; not dissimilar to the Clayton Lévy copula. However, there is a notable difference in the way that the Lévy copula density changes with changes in $\lambda^{\parallel}$ compared to the Clayton Lévy copula.

As is observed in the Lévy copula densities in Figure 3, there is a lack of significant change in the distributional copula of common jump sizes, shown in Figure 4. However, as dependence in frequency increases, the dependence in sizes of common jumps becomes stronger and is also positive and predominantly in the right tail.

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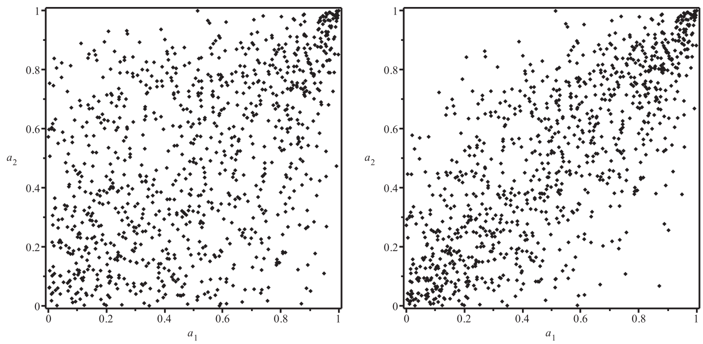
N/A



(a) $\lambda^{\parallel} = 30$ (b) $\lambda^{\parallel} = 60$ (c) $\lambda^{\parallel} = 90$

FIGURE 4: Simulations from the distributional copula of common jump sizes under Archimedean model I for $\lambda_1 = 100$ and $\lambda_2 = 100$.

3.4. Archimedean model II

Both previous Lévy copulas model positive dependence in both frequency and severity. Although only positive dependence in frequency is possible under a Lévy copula model (since $\lambda^{\parallel} \geq 0$), we present here a Lévy copula which allows for both negative dependence in severity, as well as dependence in the left-tail.

Definition 3.3. *(Archimedean model II)*

$$\mathfrak{C}_{\delta}(u_1, u_2) = \ln\left(\left(\left(e^{u_1} - 1\right)^{-\delta} + \left(e^{u_2} - 1\right)^{-\delta}\right)^{-\frac{1}{\delta}} + 1\right), \quad for \quad \delta > 0, \tag{3.22}$$

is a bivariate positive Lévy copula with Archimedean generator

$$\phi(z) = \left(e^{z} - 1\right)^{-\delta}. \tag{3.23}$$

Similar to the Clayton case, as $\delta \to 0$, Archimedean model II tends to the independence Lévy copula. As $\delta \to \infty$, Archimedean model II tends to the comonotonic Lévy copula.

The Lévy copula for a time horizon of length T, expressed in terms of an Archimedean model II Lévy copula defined for a time horizon of length one, is derived as

$$\mathfrak{C}_{\delta,T}(u_1, u_2) = T\ln\left(\left(\left(e^{\frac{u_1}{T}} - 1\right)^{-\delta} + \left(e^{\frac{u_2}{T}} - 1\right)^{-\delta}\right)^{-\frac{1}{\delta}} + 1\right). \tag{3.24}$$

Clearly, Archimedean model II is not a homogeneous function of order one, nor does it exhibit the same time scaling property as Archimedean model I.

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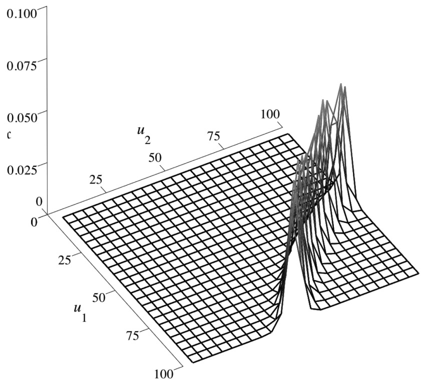
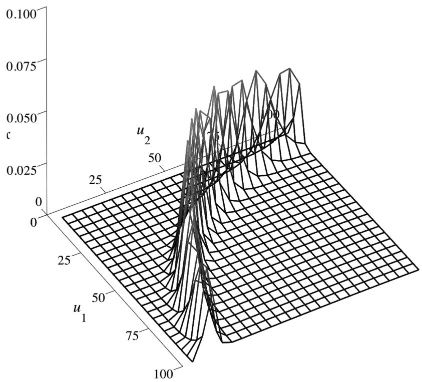
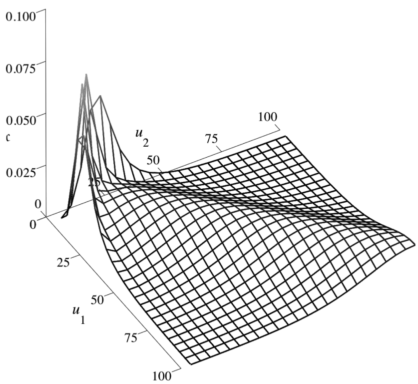


(a) $\lambda^{\|} = 30$

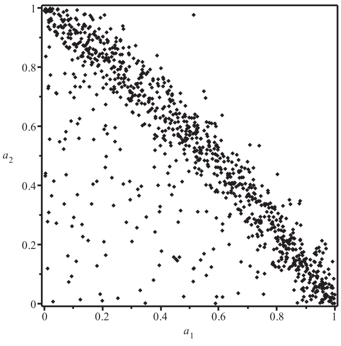


(b) $\lambda^{\|} = 60$

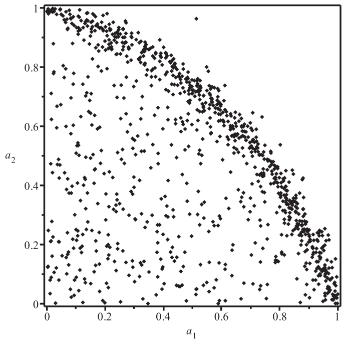


(c) $\lambda^{\|} = 90$

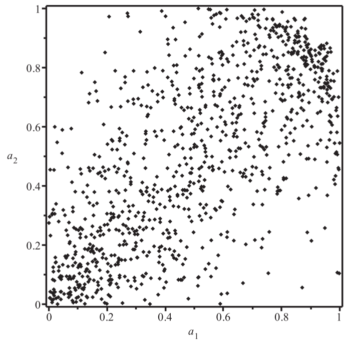
FIGURE 5: Archimedean model II densities for $\lambda_1 = 100$ and $\lambda_2 = 100$.



(a) $\lambda^{\|} = 30$



(b) $\lambda^{\|} = 60$



(c) $\lambda^{\|} = 90$

FIGURE 6: Simulations from the distributional copula of common jump sizes under Archimedean model II for $\lambda_1 = 100$ and $\lambda_2 = 100$.

As illustrated in Figure 5, the dependence structure enabled under Archimedean model II is clearly distinct from those enabled by the Clayton Lévy copula and Archimedean model I. The fact that this density has mass at points where u_1 is small and u_2 is large, and vice versa, suggests negative dependence in the sizes of the common jumps. This is confirmed in Figure 6. Clearly, Archimedean model II allows for negative dependence in the sizes of common jumps. In addition to this, as the dependence in frequency increases ($\lambda^{\parallel}$ increases), the sizes of common jumps tend to positive dependence. On the other hand, as $\lambda^{\parallel}$ decreases, the dependence in severity becomes increasingly negatively dependent.

4. Modelling Example: Application to Swiss Workers Compensation Claimsc

In this section Lévy copulas are used to model dependence in a real set of data provided by SUVA ("Schweizerische Unfallversicherungsanstalt"). SUVA is a body incorporated under Swiss public law which provides accident and occupational disease compensation insurance to around 2 million employed and unemployed people in Switzerland (almost a third of Swiss residents).

4.1. Data analysis

The dataset used in this modelling example is a random sample of 5% of the claims from class 41A, relating to the construction sector, from accident year 1999. The sample size of the dataset is 2326. Each claim is divided into two claim classes. The first class relates to medical costs whilst the second corresponds to daily allowance costs. Importantly, claims have been subject to 3 years of development. That is, claims data is at 2002 year end.

Dependence in frequency between medical claims and daily allowance claims is evident by the existence of 1089 accidents which resulted in a claim in both classes. That is, there were 1089 common claims. Table 1 breaks down the claim numbers in terms of unique and common claims for each class.

TABLE 1

Number of unique and common claim payments in each class

		Allowance		
		Claim	No claim	Total
Medical	Claim	1089	1160	2249
	No claim	10	67	77
	Total	1099	1227	2326

Swiss law requires all worker's compensation accidents to be reported to SUVA even if the accident does not result in a claim payment. There are a total of 67 reported accidents with a claim size of 0 Swiss francs (CHF) in both medical and daily allowance classes.

Our modelling approach is to let $S_1(t)$ be a compound Poisson process for medical claims and $S_2(t)$ be a compound Poisson process for daily allowance claims. We let $\mathfrak{C}_\delta$ denote the Lévy copula with parameter δ specifying the dependence between the two processes.

The jump sizes for each process are reflected in positive claim amounts and the model will assume that a claim payment of 0 does not reflect a jump in the compound Poisson process. This means that the jump size distributions of $S_1(t)$ and $S_2(t)$ do not have masses at 0. As such, the 67 accidents which resulted in no losses in either class will be ignored, as the model is concerned with those accidents that resulted in claims only, leaving a total of 2259 accidents to which the bivariate compound Poisson process will be fitted. As a result of the thinning property of the compound Poisson process (see, for instance, Esmaeili and Klüppelberg, 2010a), removing these data points does not change the assumption of a bivariate compound Poisson process.

TABLE 2

SUMMARY STATISTICS FOR CLAIM SIZES IN EACH CLASS

Statistic	Medical claim sizes	Allowance claim
Mean	1 492.77	6 760.32
Standard deviation	5 764.39	17 890.12
Skewness	8.88	6.35
Kurtosis	105.03	52.03
Minimum	15	26
Median	249	1 763
Maximum	97 506	186 850

Table 2 shows summary statistics for the claim sizes in each class, where accidents without claims have been removed. Sample kurtoses of 105.03 and 52.03 for medical claim sizes and daily allowance claim sizes respectively, shown in Table 2, suggest a heavy tailed claim size distribution for both classes.

Dependence in frequency is evident by the presence of 1089 claims common to both classes. Dependence in the severity of these claims is evident in Figure 7, showing scatter plots of the claim sizes, the logarithm of claim sizes and the empirical copula (whole and upper-right quadrant), respectively. Figure 7 suggests positive dependence in medical and daily allowance claim sizes for the 1089 common claims. Furthermore, this dependence appears to be right-tailed. That is, there is stronger positive dependence amongst larger claim sizes as opposed to smaller claim sizes.

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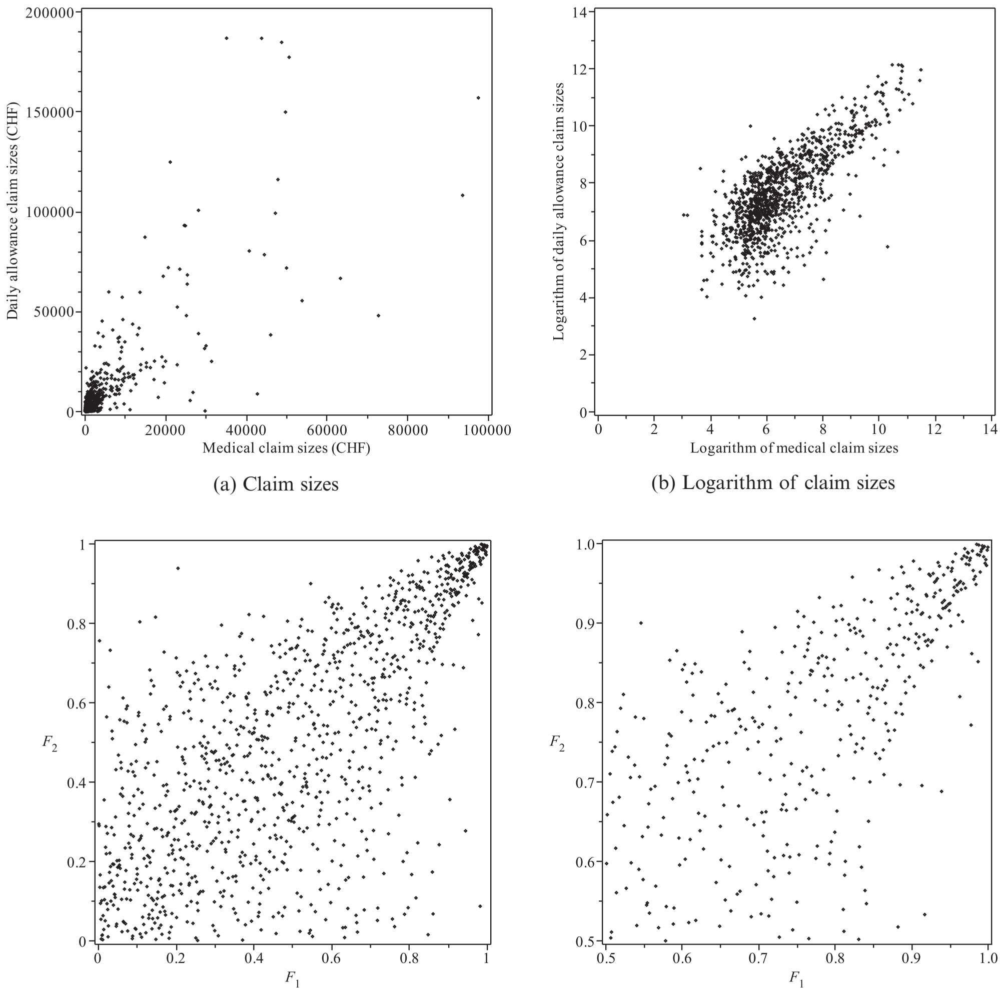

(a) Claim sizes

(b) Logarithm of claim sizes

(c) Empirical copula

(d) Upper-right quadrant of the empirical copula

FIGURE 7: Scatterplots for common medical and daily allowance claims.

4.2. Parameter estimation

In fitting a bivariate compound Poisson process, the Poisson parameters, marginal jump size distribution parameters and Lévy copula parameters are estimated simultaneously. The fit will depend on the choice of marginal jump distributions $F_1(x)$ and $F_2(x)$ with parameters θ_1 and θ_2 respectively, and the choice of a Lévy copula $\mathfrak{C}_\delta$ with parameter(s) δ.

A maximum likelihood estimation method for a bivariate compound Poisson process requires the following observation scheme (Esmaeili and Klüppelberg, 2010a). Let n be the total number of claims (jumps) occurring in a

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time interval of length T. The number of jumps in each class is n_1 and n_2. The number of claims common to both classes is $n^{\parallel}$, and the number of claims unique to each class is expressed as $n_1^{\perp}$ and $n_2^{\perp}$ respectively. The jump sizes in the first and second components are denoted by $x_1^{\perp}, \ldots, x_{n_1^{\perp}}^{\perp}$ and $y_1^{\perp}, \ldots, y_{n_2^{\perp}}^{\perp}$ respectively, while the sizes of the observed common jumps in both components are denoted by $(x^{\parallel}_1, y^{\parallel}_1), \ldots, (x^{\parallel}_{n^{\parallel}}, y^{\parallel}_{n^{\parallel}})$.

Maximising the full likelihood function can become numerically intensive for large datasets. Furthermore, as the full likelihood function is not the same under different Lévy copulas, this maximisation must be performed for each of the Lévy copula candidates in order to select one. To address this issue, one can use a method analogous to the inference functions for margins ("IFM") method (Joe, 1997, Chapter 10.1) in order to heuristically select a Lévy copula model; see also Esmaeili and Klüppelberg (2010b). This relies on the following representation of the log-likelihood function for the bivariate compound Poisson process.

$$\begin{aligned} l(\delta, \lambda_1, \lambda_2, \theta_1, \theta_2) = {} & n_1 \ln \lambda_1 - \lambda_1 T + \sum_{i=1}^{n_1} \ln f_1(x_i; \theta_1) + n_2 \ln \lambda_2 - \lambda_2 T + \sum_{i=1}^{n_2} \ln f_2(y_i; \theta_2) \\ & + \sum_{i=1}^{n_1^{\perp}} \ln \left(1 - \frac{\partial}{\partial u_1} \mathfrak{C}_{\delta}(u_1, \lambda_2) \bigg|_{u_1 = \lambda_1 \bar{F}_1(x_i^{\perp}; \theta_1)} \right) \\ & + \sum_{i=1}^{n_2^{\perp}} \ln \left(1 - \frac{\partial}{\partial u_2} \mathfrak{C}_{\delta}(\lambda_1, u_2) \bigg|_{u_2 = \lambda_2 \bar{F}_2(y_i^{\perp}; \theta_2)} \right) + \mathfrak{C}_{\delta}(\lambda_1, \lambda_2) T \\ & + \sum_{i=1}^{n^{\parallel}} \ln \frac{\partial^2}{\partial u_1 \partial u_2} \mathfrak{C}_{\delta}(u_1, u_2) \bigg|_{u_1 = \lambda_1 \bar{F}_1(x_i^{\parallel}; \theta_1),\, u_2 = \lambda_2 \bar{F}_2(y_i^{\parallel}; \theta_2)}, \end{aligned} \tag{4.1}$$

assuming the existence of $\frac{\partial^2}{\partial u_1 \partial u_2} \mathfrak{C}_{\delta}(u_1, u_2)$ for all $(u_1, u_2) \in (0, \lambda_1) \times (0, \lambda_2)$. Using an IFM approach involves the estimation of parameters λ_1, λ_2, θ_1 and θ_2 firstly without consideration of the dependence structure between the two compound Poisson processes. That is, these parameters are estimated by maximisation of the log-likelihood

$$\begin{aligned} & l^*(\lambda_1, \lambda_2, \theta_1, \theta_2) \\ & = n_1 \ln \lambda_1 - \lambda_1 T + \sum_{i=1}^{n_1} \ln f_1(x_i; \theta_1) + n_2 \ln \lambda_2 - \lambda_2 T + \sum_{i=1}^{n_2} \ln f_2(y_i; \theta_2), \end{aligned} \tag{4.2}$$

producing parameter estimates $(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\theta}_1, \tilde{\theta}_2)$. This is where the choice of the marginal jump size distributions occurs. Then, using the parameters estimated above (of the best marginal models), different Lévy copulas are fit to the data by estimating their parameter(s) δ through maximisation of $l(\delta, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\theta}_1, \tilde{\theta}_2)$ (keeping the parameters $\tilde{\lambda}_.$ and $\tilde{\theta}_.$ constant). Finally, once the jump size distributions and Lévy copula have been chosen, all parameters can be estimated simultaneously on maximisation of the full likelihood $l(\delta, \lambda_1, \lambda_2, \theta_1, \theta_2)$. This method is less computationally intensive than simultaneous maximisation of all parameters using the full likelihood and trial and error of different Lévy copula models and jump size distributions.

4.3. Maximum likelihood estimation – IFM approach

In fitting the bivariate compound Poisson process, we begin with an IFM approach. Firstly, we choose a time unit of one year so that $T = 1$. We then derive maximum likelihood estimates for λ_1 and λ_2 based on the marginal compound Poisson processes. The estimates for λ_1 and λ_2 are derived as $\tilde{\lambda}_1 = 2249$ and $\tilde{\lambda}_2 = 1099$.

In fitting the marginal jump size distributions we let X_1 denote the logarithm of medical claim sizes and X_2 denote the logarithm of daily allowance claim sizes. Table 3 shows the maximised log-likelihood for a number of distributions when fit to the logarithm of both medical claim sizes and daily allowance claim sizes. It can be seen that a Gumbel distribution maximises the log-likelihood for the logarithm of medical claim sizes while a Gaussian distribution maximises the log-likelihood for the logarithm of daily allowance claim sizes. The parameter estimates for the Gumbel distribution are $\tilde{a} = 5.1476$ and $\tilde{b} = 1.1048$, while the parameter estimates for the Normal distribution are $\tilde{\mu} = 7.6305$ and $\tilde{\sigma} = 1.4403$.

TABLE 3

MAXIMISED LOG-LIKELIHOOD VALUES FOR FITTING THE LOGARITHM OF CLAIM SIZE DATA

	Maximised log-likelihood	
Distribution	X_1	X_2
Gaussian	–3960.32	–1960.43
Gumbel	–3759.12	–1995.93
Weibull	–4071.53	–2003.26
Cauchy	–4056.66	–2122.89

The final step is to maximise the full likelihood assuming marginal parameter estimates $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{a}$, $\tilde{b}$, $\tilde{\mu}$ and $\tilde{\sigma}$ constant while deriving an estimate for the Lévy copula parameter δ under different Lévy copulas. The maximised log likelihood, parameter estimates for δ and the implied value for $\lambda^{\parallel}$ under the IFM method are shown in Table 4.

The IFM method identifies Archimedean model I as an appropriate Lévy copula for use in the model, based on the maximised value of the likelihood function, while Archimedean model II performs poorly, which could have been expected because of its tendency to model negative dependence in severity.

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TABLE 4

MAXIMISED LOG-LIKELIHOOD AND LÉVY COPULA PARAMETER ESTIMATES UNDER THE IFM METHOD.

Lévy copula	Maximised l	$\tilde{\delta}$	Implied $\tilde{\lambda}^{\parallel}$
Pure common shock	7845.03	0.0004406	1089.00
Clayton	8510.28	2.1459	1003.62
Archimedean model I	8624.06	0.0024983	1079.66
Archimedean model II	4564.90	0.0110101	1099.00

The implied $\tilde{\lambda}^{\parallel}$ is calculated as $\mathfrak{C}_{\tilde{\delta}}(\tilde{\lambda}_1, \tilde{\lambda}_2)$ and indicates the estimated expected number of common jumps per unit of time. A value of 1089 represents the maximum likelihood estimate for the number of common jumps. The pure common shock Lévy copula reproduces this estimate, since it affects the dependence in frequency only.

4.4. Maximum likelihood estimation – full model

In this section fitting results using the full likelihood for all Lévy copulas discussed in this paper are presented (with the exception of Archimedean model II, which performed significantly poorly in comparison to the other candidate Lévy copulas). Additional Archimedean Lévy copulas were also tested but excluded from this analysis due to their relatively poor fit.

TABLE 5

MAXIMISED LOG-LIKELIHOOD AND PARAMETER ESTIMATES OF THE BIVARIATE COMPOUND POISSON PROCESS FOR EACH LÉVY COPULA.

Lévy copula	Maximised l	$\hat{\delta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	Implied $\hat{\lambda}^{\parallel}$
Pure common shock	7845.03	0.0004406 (0.0000093)	2249.00 (47.42)	1099.00 (33.15)	1089.00
Clayton	8536.43	2.2632 (0.0688161)	2176.90 (46.26)	1066.27 (31.68)	984.16
Archimedean model I	8631.27	0.0025358 (0.0001110)	2239.42 (47.20)	1113.32 (32.75)	1093.74

Lévy copula	$\hat{a}$	$\hat{b}$	$\hat{\mu}$	$\hat{\sigma}$
Pure common shock	5.1476 (0.0245)	1.1048 (0.0179)	7.6305 (0.0434)	1.4403 (0.0307)
Clayton	5.1007 (0.0253)	1.1404 (0.0189)	7.7012 (0.0430)	1.5498 (0.0252)
Archimedean model I	5.1294 (0.0237)	1.0785 (0.0170)	7.6792 (0.0400)	1.4051 (0.0236)

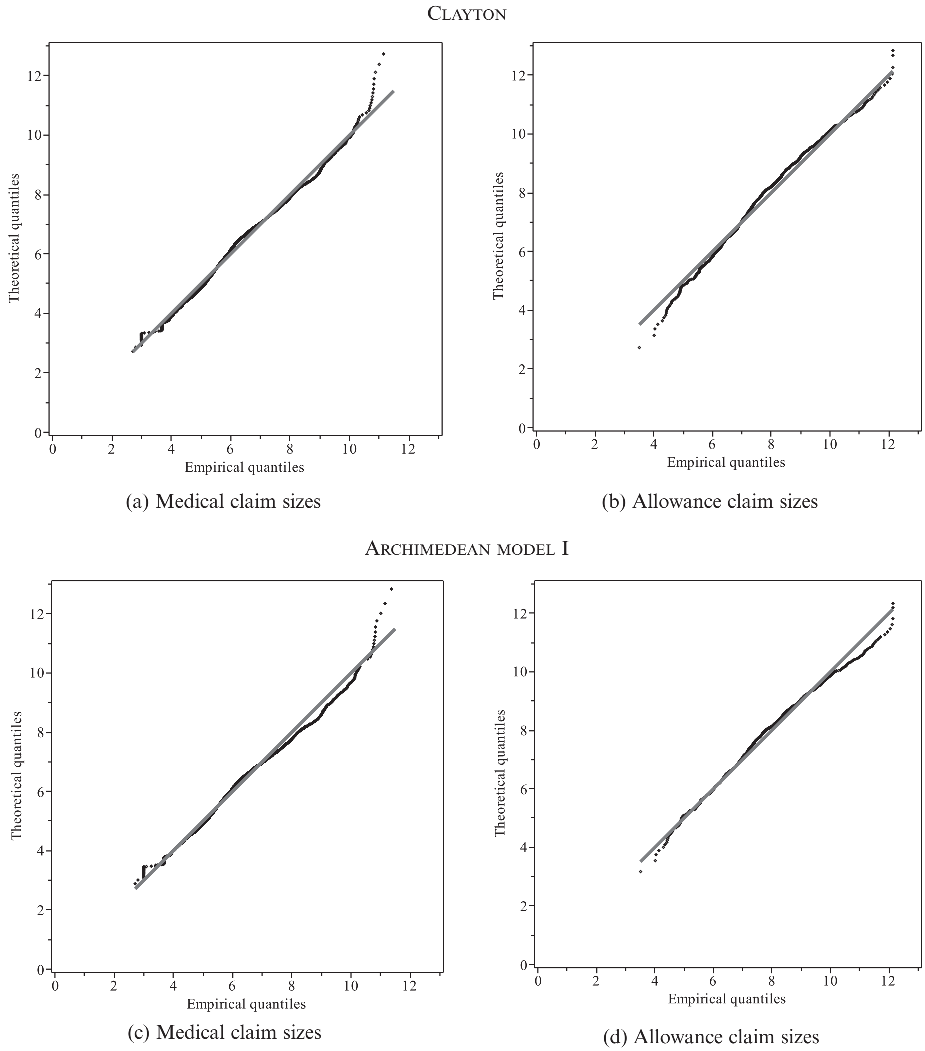

FIGURE 8: Quantile-quantile plots for fitted marginal jump size distributions under Clayton Lévy copula and Archimedean model I

Table 5 shows the maximised log-likelihood and corresponding parameter estimates for each Lévy copula. The standard errors of each parameter estimate are given in parentheses and are calculated as the square roots of the diagonal entries in the inverse Hessian matrix of the log-likelihood function (Klugman et al., 2008, Chapter 15.3). Note that the estimates for δ, the Lévy copula parameter, are not comparable across different Lévy copulas.

As initially suggested from analysis under the IFM method, Archimedean model I maximises the log-likelihood function for the bivariate compound Poisson process. With the exception of the case of the pure common shock Lévy copula, the parameter estimates $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{a}$, $\hat{b}$, $\hat{\mu}$ and $\hat{\sigma}$ differ from those

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produced under the IFM method. Recall from Lemma 2.2 that the Lévy copula affects the distribution of unique jump sizes as well as the distribution of common jump sizes and their dependence structure. As a result of this, the fitting procedure will estimate parameters based on the fit of unique jump sizes and common jump sizes, resulting in different marginal jump size parameters. Also, as the dependence in frequency (via the expected number of common jumps) and the dependence in the severity (of those common jumps) are fit simultaneously, they compete with each other and in doing so, yield different parameter estimates for the marginal Poisson parameters and jump size distribution parameters.

Figure 8 shows quantile-quantile plots for the marginal jump sizes using the fitted parameters from the Clayton Lévy copula and from Archimedean model I. While the parameter estimates for the marginal jump size distributions differ under the IFM method, the quality of fit is still reasonable. Further tests for the goodness-of-fit for the marginal jump size distributions can also be employed (Klugman and Rioux, 2006).

Even though the pure common shock Lévy copula produces the same marginal parameters estimates as under the IFM method, it also produces the lowest value for the maximised log-likelihood. This is because the pure common shock Lévy copula assumes independent jump sizes, which is an invalid assumption as suggested by Figure 7. In addition to this, we will see in the following section that the assumption of identically distributed unique and common jump sizes in each class is incorrect.

4.5. Dependence goodness-of-fit

An initial assessment of the Lévy copula fit would be to compare the fitted dependence in frequency as measured by the implied $\lambda^{\parallel}$. In Table 5 we see that the pure common shock Lévy copula and Archimedean model I produce a good fit for dependence in frequency, as $\lambda^{\parallel}$ is relatively close to the observed number 1089 (in the case of the pure common shock Lévy copula it is equal). However, the dependence in frequency is merely one aspect of the fitted bivariate compound Poisson process than can be assessed.

In order to discuss the fit of the model in terms of dependence in severity (the sizes of common jumps), Figure 9 plots 1089 simulations from the distributional copula of the sizes of common jumps derived under the Clayton Lévy copula and Archimedean model I. The pure common shock Lévy copula only allows for independence in common jump sizes and was consequently omitted from this analysis. On comparison of Figure 9 with the empirical copula of the common claims in Figure 7, the distributional copula under the Clayton Lévy copula appears to offer a better fit for the dependence in the sizes of common jumps. While Archimedean model I offers right-tail positive dependence, its upper-right quadrant does not fit as well as the one of the Clayton Lévy copula. Note that more sophisticated methods can be used in testing the goodness-of-fit of distributional copulas (cf. Genest et al., 2009).

While these traditional goodness-of-fit approaches are inconclusive, it is also possible to plot the theoretical tail integrals against the empirical tail

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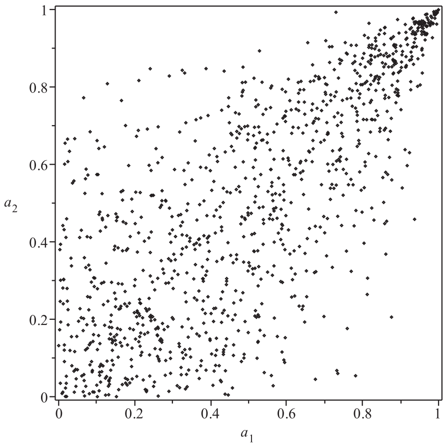
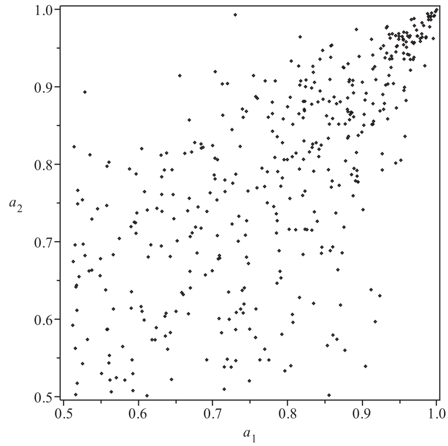
Clayton

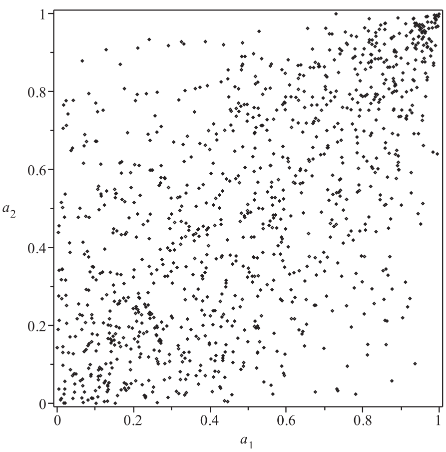


(a) Empirical copula

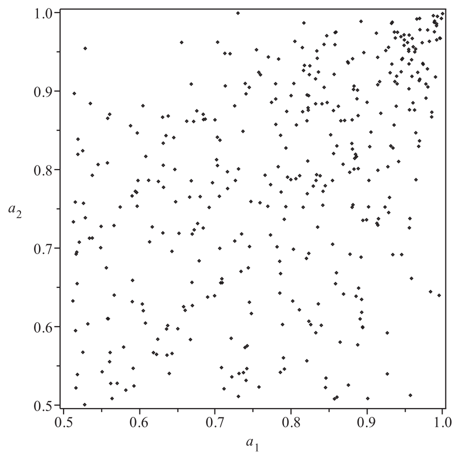


(b) Upper-quadrant of the empirical copula

Archimedean model I



(c) Empirical copula



(d) Upper-quadrant of the empirical copula

Figure 9: Simulations from the distributional copula of common jump sizes under candidate Lévy copula models.

integrals for unique jumps and common jumps in each component, and for each fitted model. The advantage of this approach is that it assesses both fit of the dependence in frequency and severity at the same time. For the common components ($S_i^{\|}(t)$, $i = 1, 2$) the empirical tail integrals are defined as

$$U_{i;n^{\|}}^{\|}(x) = \frac{\text{number of common jumps in component } i \text{ of size} > x}{T}, \text{ for } i = 1, 2, \tag{4.3}$$

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while the empirical tail integrals for unique jumps are defined as

$$U^{\perp}_{i;n_i^{\perp}}(x) = \frac{\text{number of unique jumps in component } i \text{ of size} > x}{T}, \text{ for } i = 1,2. \tag{4.4}$$

Figure 10 shows plots of empirical tail integrals against theoretical fitted tail integrals of unique jumps and common jumps in each component for the Clayton, pure common shock, and the Archimedean model I Lévy copulas (where the tail integral comparisons for unique allowance claims were not plotted due to a small number of data for these types of claims). Note that for each Lévy copula, both curves start at the model and empirical versions

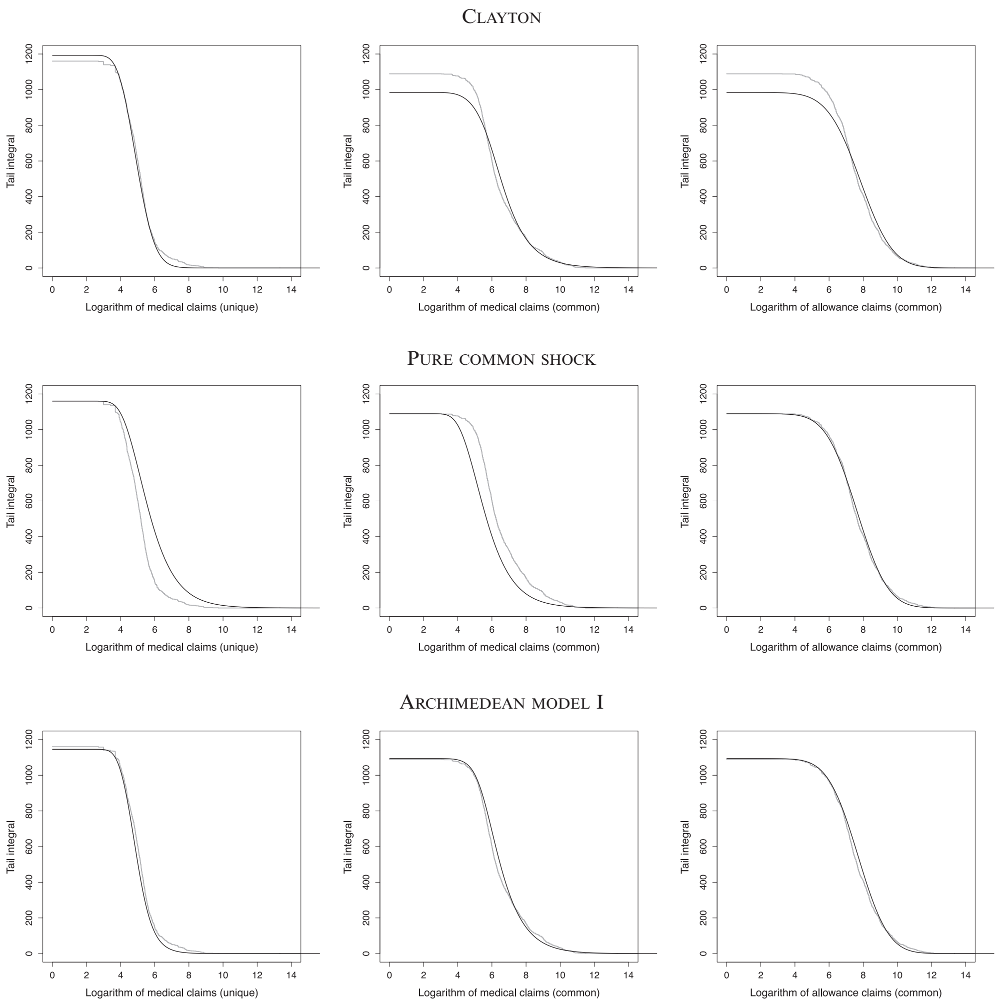

FIGURE 10: Empirical (grey) and theoretical (black) tail integrals for medical (common and unique) and allowance (common) jumps with three candidate Lévy copula models.

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of $\lambda_1^{\perp}$, $\lambda^{\parallel}$ and $\lambda^{\parallel}$, respectively, and are then shaped according to the model and empirical versions of $\bar{F}_1^{\perp}(x)$, $\bar{F}_1^{\parallel}(x)$ and $\bar{F}_2^{\parallel}(x)$ respectively.

Although the Clayton Lévy copula produced a relatively high maximised log-likelihood, we see that the fit of the tail integrals under the Clayton Lévy copula is rather poor. Whilst the pure common shock Lévy copula fits the dependence in frequency perfectly, the jump size distributions for common and unique jumps are not fitted well at all. Finally, it can be seen that Archimedean model I fits the tail integral components rather well in comparison to the other Lévy copulas.

In view of the above, Archimedean model I seems to be the most appropriate choice of Lévy copula for the dependence structure exhibited in the SUVA dataset.

5. Trivariate compound Poisson process

In this section, Lemmas 5.1 and 5.2 extend Lévy copula results for the bivariate compound Poisson process to a trivariate compound Poisson process using a trivariate Lévy copula, adopting a similar approach to the one used in the bivariate case by Esmaeili and Klüppelberg (2010a). Care has to be taken in settings beyond the bivariate case, as 'common jumps' can now be common between two out of the three processes, or between all three processes. In particular, Example 5.1 illustrates the similarities and differences between the bivariate and trivariate cases. Results for the trivariate compound Poisson process can easily be generalised for use with multivariate compound Poisson processes of any dimension.

Lemma 5.1. *(Trivariate compound Poisson process) The constituents of a trivariate compound Poisson process* $\{S_1(t), S_2(t), S_3(t)\}$*, can be expressed as*

$$\begin{cases} S_1(t) = S_1^{\perp}(t) + S_{1;12}^{\parallel}(t) + S_{1;13}^{\parallel}(t) + S_{1;123}^{\parallel}(t), \\ S_2(t) = S_2^{\perp}(t) + S_{2;12}^{\parallel}(t) + S_{2;23}^{\parallel}(t) + S_{2;123}^{\parallel}(t), \\ S_3(t) = S_3^{\perp}(t) + S_{3;13}^{\parallel}(t) + S_{3;23}^{\parallel}(t) + S_{3;123}^{\parallel}(t). \end{cases} \tag{5.1}$$

The compound Poisson processes denoted by $S_{i;ij}^{\parallel}(t)$ *feature an arrival process common with compound Poisson processes* $S_{j;ij}^{\parallel}(t)$*, for* $i, j = 1, 2, 3$ *and* $i < j$ *only. The three compound Poisson processes denoted as* $S_{i;123}^{\parallel}(t)$*, for* $i = 1, 2, 3$ *all feature a common arrival process.*

Proof. Decomposing the tail integral of $S_1(t)$ in terms of the Lévy measure gives,

$$\begin{aligned} U_1(x_1) &= \nu([x_1,\infty)\times\{0\}\times\{0\}) + \lim_{x_3\to 0^+}\nu([x_1,\infty)\times\{0\}\times[x_3,\infty)) \\ &\quad + \lim_{x_2\to 0^+}\nu([x_1,\infty)\times[x_2,\infty)\times\{0\}) + \lim_{x_2,x_3\to 0^+}\nu([x_1,\infty)\times[x_2,\infty)\times[x_3,\infty)) \\ &= U_1^{\perp}(x) + U_{1;12}^{\parallel}(x) + U_{1;13}^{\parallel}(x) + U_{1;123}^{\parallel}(x), \end{aligned} \tag{5.2}$$

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and similarly for $U_2(x)$ and $U_3(x)$. Terms denoted by $U^{\parallel}_{i;ij}(x)$ are tail integrals for compound Poisson processes $S^{\parallel}_{i;ij}(t)$ as in (5.1). It represents the expected number of jumps in $S_i(t)$ which are above x and occur at the same time as jumps of any size in $S_j(t)$ only, for $i, j = 1,2,3$ and $i < j$. Similarly, $U^{\parallel}_{i;123}(x)$ is the tail integral for compound Poisson process $S^{\parallel}_{i;123}(t)$. Since we now have marginal tail integrals broken up into independent components, it is now clear that (5.1) holds. □

Consider now a trivariate compound Poisson process $\{S_1(t), S_2(t), S_3(t)\}$ with Lévy copula $\mathfrak{C}$, and let λ_i, $F_i(x)$, $i = 1,2,3$ denote its (marginal) Poisson parameters and jump size distributions, respectively. As in the bivariate case, it is possible to derive the Poisson parameters and jump size distributions of the compound Poisson processes $S^{\perp}_i(t)$, $S^{\parallel}_{i;ij}(t)$ and $S^{\parallel}_{i;123}(t)$ in terms of the Lévy copula and the marginal tail integrals $U_i(x)$ $i, j = 1,2,3$, $i < j$.

Lemma 5.2. *Common jumps in $S^{\parallel}_{1;123}(t)$, $S^{\parallel}_{2;123}(t)$ and $S^{\parallel}_{3;123}(t)$ arrive at a rate*

$$\lambda^{\parallel}_{123} = \mathfrak{C}(\lambda_1, \lambda_2, \lambda_3), \tag{5.3}$$

whereas the sizes of these common jumps have survival function

$$\bar{F}^{\parallel}_{123}(x_1, x_2, x_3) = \frac{1}{\lambda^{\parallel}_{123}} \mathfrak{C}(U_1(x_1), U_2(x_2), U_3(x_3)), \tag{5.4}$$

and marginal survival functions

$$\bar{F}^{\parallel}_{1;123}(x) = \frac{1}{\lambda^{\parallel}_{123}} \mathfrak{C}(U_1(x_1), \lambda_2, \lambda_3), \tag{5.5}$$

with $\bar{F}^{\parallel}_{2;123}(x)$ and $\bar{F}^{\parallel}_{3;123}(x)$ derived in a similar way. Jumps which are common between the first two processes, but not all three processes, as in $S^{\parallel}_{1;12}(t)$ and $S^{\parallel}_{2;12}(t)$, arrive at a rate

$$\lambda^{\parallel}_{12} = \mathfrak{C}(\lambda_1, \lambda_2, \infty) - \mathfrak{C}(\lambda_1, \lambda_2, \lambda_3), \tag{5.6}$$

whereas the sizes of these jumps have joint survival function

$$\bar{F}^{\parallel}_{12}(x_1, x_2) = \frac{1}{\lambda^{\parallel}_{12}} \big(\mathfrak{C}(U(x_1), U(x_2), \infty) - \mathfrak{C}(U(x_1), U(x_2), \lambda_3)\big). \tag{5.7}$$

Results for $\lambda^{\parallel}_{13}$, $\lambda^{\parallel}_{23}$, $\bar{F}^{\parallel}_{13}(x_1, x_3)$ and $\bar{F}^{\parallel}_{23}(x_2, x_3)$ are derived in an analogous way. Unique jumps in $S^{\perp}_1(t)$ arrive at rates

$$\lambda^{\perp}_1 = \lambda_1 - \lambda^{\parallel}_{12} - \lambda^{\parallel}_{13} - \lambda^{\parallel}_{123}, \tag{5.8}$$

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whereas their sizes are distributed with survival function

$$\bar{F}_1^{\perp}(x) = \frac{1}{\lambda_1^{\perp}}\left(\lambda_1\,\bar{F}_1(x) - \lambda_{12}^{\parallel}\,\bar{F}_{1;12}^{\parallel}(x) - \lambda_{13}^{\parallel}\,\bar{F}_{1;13}^{\parallel}(x) - \lambda_{123}^{\parallel}\,\bar{F}_{1;123}^{\parallel}(t)\right). \tag{5.9}$$

Results for $\lambda_2^{\perp}$, $\lambda_3^{\perp}$, $\bar{F}_2^{\perp}(x)$ and $\bar{F}_3^{\perp}(x)$ are derived in a similar way.

Proof of Equation (5.3). Consider the expected number of jumps common to all three processes expressed in terms of the joint Lévy measure,

$$\lambda_{123}^{\parallel} = \nu((0,\infty)\times(0,\infty)\times(0,\infty)). \tag{5.10}$$

Using limits and the definition of the multivariate tail integral, and applying Sklar's theorem for Lévy copulas (Theorem 2.1) in a trivariate setting, we have

$$\lambda_{123}^{\parallel} = \lim_{x_1,\,x_2,\,x_3\to 0^+} \mathfrak{C}(U_1(x_1),\,U_2(x_2),\,U_3(x_3)). \tag{5.11}$$

Finally, using the result that $\lim_{x\to 0^+} U_i(x) = \lambda_i$ for $i = 1, 2, 3$, produces the final result for $\lambda_{123}^{\parallel}$. □

Proof of Equation (5.4). Consider the definition of the multidimensional tail integral,

$$U(x_1, x_2, x_3) = \lambda_{123}^{\parallel}\,\bar{F}_{123}^{\parallel}(x_1, x_2, x_3). \tag{5.12}$$

Applying Sklar's theorem for Lévy copulas to the left hand side of the above yields the desired result. □

Proof of Equation (5.6). Expressing $\lambda_{12}^{\parallel}$ in terms of the Lévy measure yields

$$\lambda_{12}^{\parallel} = \lim_{x_1,\,x_2\to 0^+} \nu([x_1,\infty)\times[x_2,\infty)\times\{0\}). \tag{5.13}$$

Then, using the result that

$$\begin{aligned}\nu([x_1,\infty)\times[x_2,\infty)\times\{0\}) &= \nu([x_1,\infty)\times[x_2,\infty)\times[0,\infty)) \\ &\quad - \lim_{x_3\to 0^+} \nu([x_1,\infty)\times[x_2,\infty)\times[x_3,\infty)), \\ &= \mathfrak{C}(U_1(x_1),\,U_2(x_2),\infty) - \mathfrak{C}(U_1(x_1),\,U_2(x_2),\lambda_3), \end{aligned} \tag{5.14}$$

we have,

$$\lambda_{12}^{\parallel} = \lim_{x_1,\,x_2\to 0^+}\left(\mathfrak{C}(U_1(x_1),\,U_2(x_2),\infty) - \mathfrak{C}(U_1(x_1),\,U_2(x_2),\lambda_3)\right). \tag{5.15}$$

Since $\lim_{x \to 0^+} U_i(x) = \lambda_i$ for $i = 1, 2, 3$, the desired result is derived, with the proof of results for λ_{23} and λ_{13} derived similarly. □

Proof of Equation (5.7). Consider the tail integral for jumps common in the first and second components only, expressed in terms of the Lévy measure,

$$\nu([x_1, \infty) \times [x_2, \infty) \times \{0\}) = \lambda_{12}^{\parallel} \bar{F}_{12}^{\parallel}(x_1, x_2). \tag{5.16}$$

Applying (5.14) to the left hand side and dividing both sides by $\lambda_{12}^{\parallel}$ produces the desired result. Results for $\bar{F}_{13}^{\parallel}(x_1, x_3)$ and $\bar{F}_{23}^{\parallel}(x_2, x_3)$ can be proved similarly. □

Proof of Equation (5.8). The Poisson parameters are derived using the result that the Poisson parameters of the right hand side of (5.1) must sum to give the marginal Poisson parameter λ_i, $i = 1, 2, 3$. □

Proof of Equation (5.9). This result is derived from the decomposition of the marginal tail integral

$$\begin{aligned} U_i^{\perp}(x) &= \lambda_i^{\perp} \bar{F}_i^{\perp}(x), \\ &= U_i(x) - U_{i;ij}^{\parallel}(x) - U_{i;ik}^{\parallel}(x) + U_{i;123}^{\parallel}(x), \text{ for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k. \end{aligned} \tag{5.17}$$

□

Example 5.1. *(Trivariate Clayton Lévy copula) We illustrate the above results by considering a trivariate Lévy copula and studying some of its properties. The trivariate Clayton Lévy copula is given by*

$$\mathfrak{C}(u_1, u_2, u_3) = (u_1^{-\delta} + u_2^{-\delta} + u_3^{-\delta})^{-\frac{1}{\delta}}, \ \delta > 0; \tag{5.18}$$

see Tankov (2003). Using (5.3)*, the Poisson parameter for jumps common to all three processes is*

$$\lambda_{123}^{\parallel} = (\lambda_1^{-\delta} + \lambda_2^{-\delta} + \lambda_3^{-\delta})^{-\frac{1}{\delta}}, \tag{5.19}$$

while the joint survival function of these jump sizes is

$$\bar{F}_{123}^{\parallel}(x_1, x_2, x_3) = \frac{1}{\lambda_{123}^{\parallel}} (U_1(x_1)^{-\delta} + U_2(x_2)^{-\delta} + U_3(x_3)^{-\delta})^{-\frac{1}{\delta}}, \tag{5.20}$$

with marginal survival functions

$$\bar{F}_{i;123}^{\parallel}(x) = \left(\frac{(\lambda_i \bar{F}_i(x))^{-\delta} + \lambda_j^{-\delta} + \lambda_k^{-\delta}}{\lambda_1^{-\delta} + \lambda_2^{-\delta} + \lambda_3^{-\delta}} \right)^{-\frac{1}{\delta}}, \text{ for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k. \tag{5.21}$$

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Akin to the bivariate case, the trivariate survival copula of the jump sizes for jumps common to all three processes is a distributional trivariate Clayton copula, so that

$$\bar{F}^{\|}_{123}(x_1, x_2, x_3) = \left[\left(\bar{F}^{\|}_{1;123}(x)\right)^{-\delta} + \left(\bar{F}^{\|}_{2;123}(x)\right)^{-\delta} + \left(\bar{F}^{\|}_{3;123}(x)\right)^{-\delta} - 2\right]^{-\frac{1}{\delta}}. \quad (5.22)$$

However, this is not the case for the jumps common to two processes only, as

$$\begin{aligned}\left(\bar{F}^{\|}_{1;12}(x_1)\right)^{-\delta} + \left(\bar{F}^{\|}_{2;12}(x_2)\right)^{-\delta} &\\ = \left(\frac{1}{\lambda^{\|}_{12}}\right)^{-\delta}\left((U_1(x_1)^{-\delta} + \lambda_2^{-\delta})^{-\frac{1}{\delta}} - (U_1(x_1)^{-\delta} + \lambda_2^{-\delta} + \lambda_3^{-\delta})^{-\frac{1}{\delta}}\right)^{-\delta} &\\ + \left(\frac{1}{\lambda^{\|}_{12}}\right)^{-\delta}\left(\lambda_1^{-\delta} + U_2(x_2)^{-\delta})^{-\frac{1}{\delta}} - (\lambda_i^{-\delta} + U_2(x_2)^{-\delta} + \lambda_3^{-\delta})^{-\frac{1}{\delta}}\right)^{-\delta}, &\end{aligned} \quad (5.23)$$

which is not easily expressed in terms of $\bar{F}^{\|}_{12}(x_1, x_2)$ *and more specifically, is not in the form of the ordinary Clayton copula.*

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APPENDIX

A. Definitions

A.1. *Tail Integral (Tankov, 2003)*

For a d-dimensional Lévy process with positive jumps, the multivariate *tail integral* is defined as the expected number of jumps per unit of time that are above a given level $(x_1, \ldots, x_d)$. The multivariate tail integral is a function $U : [0, \infty]^d \to [0, \infty]$ such that

1. U is equal to zero if one of its arguments is equal to ∞;
2. U is finite everywhere except at zero and $U(0, \ldots, 0) = \infty$;

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3. For all $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in [0,\infty]^d$ and with $a_i \leq b_i$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d+d}\, U(x_{1i_1}, \ldots, x_{di_d}) \geq 0 \tag{A.1}$$

where $x_{j1} = a_j$ and $x_{j2} = b_j$ for all $j = 1, \ldots, d$.

Note that the margins of a tail integral may be derived in a manner similar to that of deriving marginal distribution functions from a multivariate distribution function so that

$$U_i(x) = U(0, \ldots, 0, x, 0, \ldots, 0), \tag{A.2}$$

where x is evaluated at the i-th dimension of $U(\cdot, \cdots, \cdot)$.

A.2. *Lévy copula (Positive Lévy copula, Tankov, 2003)*

For Lévy processes with positive jumps a "positive Lévy copula" is defined to be a function $\mathfrak{C} : [0,\infty]^d \to [0,\infty]$ which satisfies the following:

1. $\mathfrak{C}(u_1, \ldots, u_d)$ is increasing in each component.
2. $\mathfrak{C}(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for any $i = 1, \ldots, d$.
3. Evaluating $\mathfrak{C}$ at ∞ at all components except for the i-th component which is evaluted at u produces margins $\mathfrak{C}_i$, $i = 1, \ldots, d$, which satisfy $\mathfrak{C}_i(u) = u$ for all u in $[0,\infty]$.
4. For all $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in [0,\infty]^d$ and with $a_i \leq b_i$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d}\, \mathfrak{C}(u_{1i_1}, \ldots, u_{di_d}) \geq 0 \tag{A.3}$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j = 1, \ldots, d$.

A.3. *Lévy measure (Chapter 2.8 Sato, 1999)*

The Lévy measure of a multivariate compound Poisson process is given by

$$\nu(A) = \lambda P(A), \text{ for } A \in \mathfrak{B}(\mathbb{R}^d), \tag{A.4}$$

where λ is the Poisson parameter for the arrival of all jumps in the multivariate process, P is the multivariate distribution for the sizes of jumps, and where $\mathfrak{B}$ is the Borel σ-field.The Lévy measure $\nu(A)$ may be interpreted as the expected number of jumps, per unit of time, where the size of the jumps are in A. The tail integral is expressed in terms of the Lévy measure as

$$U(x_1, \ldots, x_d) = \nu([x_1,\infty) \times \cdots \times [x_d,\infty)) \text{ for } \mathbf{x} \in (0,\infty)^d. \tag{A.5}$$

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