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STOCHASTIC COMPARISONS OF NONHOMOGENEOUS PROCESSES

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The purpose of this article is to describe various conditions on the parameters of pairs of nonhomogeneous Poisson or pure birth processes under which the corresponding epoch times or interepoch intervals are stochastically ordered in various senses. We derive results involving the usual stochastic order, the multivariate hazard rate order, the multivariate likelihood ratio order, as well as the dispersive and the mean residual life orders. A sample of applications involving generalized Yule processes, load-sharing models, and minimal repairs in reliability theory illustrate the usefulness of the new results.

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1. INTRODUCTION AND MOTIVATION

Nonhomogeneous Poisson processes arise naturally in many applications of probability. In reliability theory, the times of repair of an item which is being continuously minimally repaired are the epoch times of a nonhomogeneous Poisson process. In the study of records, the times of the consecutive record values of a sequence of independent and identically distributed nonnegative random variables are the epoch times of a nonhomogeneous Poisson process. Therefore, results which give stochastic comparisons of the epoch times or of the interepoch times of different nonhomogeneous Poisson processes can be useful in reliability theory and in the studies of progressive records.

Roughly speaking, the intensity of a jump of a nonhomogeneous Poisson process at any time t depends only on t, and not on any other information about the past or the present of the process. If the intensity of a jump at any time t depends on t and also on the state (i.e., the number of previous jumps) of the process, but not on any other information about the past or the present of the process, then the resulting process is called nonhomogeneous pure birth process. Recently, Hu and Pan [8] have used nonhomogeneous pure birth processes to model the epochs in which insurance claims occur. Other applications of nonhomogeneous pure birth processes in epidemiology and load sharing are described in Section 5.

The purpose of this article is to describe various conditions on the parameters of pairs of nonhomogeneous Poisson or pure birth processes under which the corresponding epoch or interepoch times are stochastically ordered in various senses.

In Section 2, we give the definitions of nonhomogeneous Poisson and pure birth processes, and we describe a useful presentation of pure birth processes. We also derive in Section 2 some mathematical formulas that are used later in the article. The main results are given in Sections 3 and 4. We obtain a large number of stochastic comparisons, in various senses, of epoch times and interepoch intervals of pairs of nonhomogeneous Poisson and pure birth processes. Some of the results provide stochastic comparisons in various senses of vectors of epoch times and interepoch intervals. It is remarkable that some of the stochastic comparison results actually characterize relationships between the pairs of nonhomogeneous Poisson processes that are compared. Finally, in Section 5, we describe some applications in epidemiology, reliability theory, and load sharing. One particular application that we describe involves the construction of computable upper or lower bounds on various probabilistic quantities of interest.

This article may be contrasted with the work of Shaked and Szekli [21]. The work [21] focused on the usual stochastic ordering of epoch times and interepoch intervals of two point processes, whereas, here, we derive many results that give finer comparisons in other stochastic ordering senses. However, whereas the results of [21] apply to general point processes, here we derive results only for nonhomogeneous Poisson and pure birth processes. The results in [21] were mainly applied to comparisons of replacement and maintenance policies in reliability theory; here, we indicate also some other areas of applications of the new results. In fact, we show that the new results here provide useful bounds in almost any application where nonhomogeneous pure birth processes are used. Finally, the methods of proof of the present article differ from the methods of [21]: Whereas in [21], most of the basic results were essentially proven using coupling, here we use more analytical methods because most of the stochastic orderings that we derive are stronger than the usual stochastic order and the tool of coupling does not suffice for their derivation.

In this article, "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively. Any inverse function that we use is understood to be the right-continuous one.

2. NONHOMOGENEOUS POISSON AND PURE BIRTH PROCESSES

A counting process $\{N(t), t \geq 0\}$ is a *nonhomogeneous Poisson process* with intensity (or rate) function $r \geq 0$ if

(a) $\{N(t), t \geq 0\}$ has the Markov property,
(b) $P\{N(t+\Delta t) = n+1 | N(t) = n\} = r(t)\Delta t + o(\Delta t), n \geq 1$,
(c) $P\{N(t+\Delta t) > n+1 | N(t) = n\} = o(\Delta t), n \geq 1$.

We assume that

$$\int_t^{\infty} r(u)\, du = \infty \quad \text{for all } t \geq 0; \tag{2.1}$$

this ensures that, with probability 1, the process has a jump after any time point t. For convenience, if $r(t_0) = \infty$ for some t_0, then we define $r(t) = \infty$ for $t \geq t_0$.

A nonnegative function r which satisfies (2.1) can be interpreted as the hazard rate function of a lifetime of an item. More explicitly, if r satisfies (2.1) and we define f by

$$f(t) = r(t)\exp\left(-\int_0^t r(u)\, du\right) = r(t)e^{-R(t)}, \quad t \geq 0, \tag{2.2}$$

where $R(t) \equiv \int_0^t r(u)\, du$, then f is a probability density function of a lifetime; in fact, f is the probability density function of the time of the first epoch of the underlying nonhomogeneous Poisson process.

Let $0 \equiv T_0 \leq T_1 \leq T_2 \leq \cdots$ be the epoch times of the nonhomogeneous Poisson process. Denote by f_n the density function of $T_n, n \geq 1$. Then,

$$f_n(t) = f(t)\frac{(R(t))^{n-1}}{(n-1)!}, \quad t \geq 0, n \geq 1; \tag{2.3}$$

this is Eq. (3) in Baxter [4]. Note, in particular, that $f_1 \equiv f$. It is worthwhile to mention that in the monograph by Kamps [9], the definition of the epoch times is extended to the so-called generalized order statistics; various extensions of (2.3) can be found there.

Let $X_n = T_n - T_{n-1}, n \geq 1$, be the interepoch intervals of the nonhomogeneous Poisson process. Denote by g_n the density function of $X_n, n \geq 1$. Then, $g_1 = f$ and

$$g_n(t) = \int_0^\infty r(s)\,\frac{R^{n-2}(s)}{(n-2)!}\,f(s+t)\,ds, \quad t \geq 0, n \geq 2; \tag{2.4}$$

this is Eq. (7) in Baxter [4].

The following extension of the nonhomogeneous Poisson process will also be studied in this article. Let $r_n, n \geq 1$, be nonnegative functions that satisfy (2.1). A counting process $\{N(t), t \geq 0\}$ is a *nonhomogeneous pure birth process* with intensity (or rate) functions $r_n \geq 0$ if

(a) $\{N(t), t \geq 0\}$ has the Markov property,
(b) $P\{N(t+\Delta t) = n+1 | N(t) = n\} = r_n(t)\Delta t + o(\Delta t), n \geq 1$,
(c) $P\{N(t+\Delta t) > n+1 | N(t) = n\} = o(\Delta t), n \geq 1$.

Nonhomogeneous pure birth processes are called “relevation counting processes” in [14], where some applications of them in reliability theory are described. When all the r_n’s are identical, a nonhomogeneous pure birth process reduces to a nonhomogeneous Poisson process.

Let $0 \equiv T_0 \leq T_1 \leq T_2 \leq \cdots$ be the epoch times of the above nonhomogeneous pure birth process and let $X_n = T_n - T_{n-1}, n \geq 1$, be the corresponding interepoch intervals. We will now describe a useful stochastic representation of these epoch and interepoch times. Consider a set of independent absolutely continuous nonnegative random variables $\{Y_n, n \geq 1\}$, with corresponding hazard rate functions $r_n, n \geq 1$. Define, recursively,

$$\hat{T}_1 = Y_1, \tag{2.5}$$

$$\hat{T}_n = [Y_n | Y_n > \hat{T}_{n-1}], \quad n \geq 2, \tag{2.6}$$

where, for any event A, the notation $[Y_n|A]$ stands for the random variable Y_n truncated on A (so, the distribution of $[Y_n|A]$ is the conditional distribution of Y_n given A). Also, define

$$\hat{X}_1 = Y_1,$$

$$\hat{X}_n = [Y_n - \hat{T}_{n-1} | Y_n > \hat{T}_{n-1}], \quad n \geq 2.$$

Then, it is easy to verify that the joint distribution of the T_n’s is the same as the joint distribution of the $\hat{T}_n$’s, and that the joint distribution of the X_n’s is the same as the joint distribution of the $\hat{X}_n$’s.

Using (2.5) and (2.6), it is easy to derive the density functions of the $\hat{T}_n$’s (or, equivalently, of the T_n’s). In order to do that, let k_n denote the density function of $Y_n, n \geq 1$; that is, $k_n(t) = r_n(t)\exp[-\int_0^t r_n(u)\,du], t \geq 0$. Also, let $\overline{K}_n$ denote the

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survival function of $Y_n, n \geq 1$; that is, $\bar{K}_n(t) = \exp[-\int_0^t r_n(u)\,du], t \geq 0$. The probability density functions f_n of the T_n's are then given, recursively, by

$$f_1(t) = k_1(t), \quad t \geq 0, \tag{2.7}$$

$$f_n(t) = k_n(t) \int_0^t \frac{f_{n-1}(u)}{\bar{K}_n(u)}\,du, \quad t \geq 0, n \geq 2. \tag{2.8}$$

Also, the probability density functions g_n of the $\hat{X}_n$'s (or, equivalently, of the X_n's) are then given by

$$g_1(t) = k_1(t), \quad t \geq 0,$$

$$g_n(t) = \int_0^\infty k_n(u+t)\,\frac{f_{n-1}(u)}{\bar{K}_n(u)}\,du, \quad t \geq 0, n \geq 2,$$

where the f_n's are defined in (2.7) and (2.8).

3. STOCHASTIC COMPARISONS OF EPOCH TIMES

3.1. Epoch Times of Nonhomogeneous Poisson Processes

Consider two nonhomogeneous Poisson processes with intensity functions r and s, respectively, and with associated density functions [see Eq. (2.2)] f and g, respectively, associated distribution functions F and G, respectively, and associated cumulative hazard functions R and S, respectively, where $R(t) \equiv \int_0^t r(u)\,du$ and $S(t) \equiv \int_0^t s(u)\,du$. Let the epoch times of the first nonhomogeneous Poisson process be denoted by $T_{1,1} \leq T_{1,2} \leq \cdots$ and let the epoch times of the other nonhomogeneous Poisson process be denoted by $T_{2,1} \leq T_{2,2} \leq \cdots$.

In this subsection, we derive some results which stochastically compare, in several senses, vectors of $T_{1,j}$'s with vectors of $T_{2,j}$'s.

For the sake of completion, we start by giving conditions under which the epoch times of the first process are smaller than the epoch times of the other process in the *usual stochastic order* sense. Recall that a random variable or vector $\boldsymbol{X}$ is smaller in the usual stochastic order than the random variable or vector $\boldsymbol{Y}$ (of the same dimension) if

$$E\phi(\boldsymbol{X}) \leq E\phi(\boldsymbol{Y})$$

for all increasing functions ϕ for which the above expectations exist. This relationship is usually denoted by $\boldsymbol{X} \leq_{\text{st}} \boldsymbol{Y}$. If the distribution functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $F_{\boldsymbol{X}}$ and $F_{\boldsymbol{Y}}$, respectively, then this relation will sometimes be denoted by $F_{\boldsymbol{X}} \leq_{\text{st}} F_{\boldsymbol{Y}}$. The following result is essentially not new (see Remark 3.2).

THEOREM 3.1: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier. Then, $F \leq_{\text{st}} G$ if and only if*

$$(T_{1,1}, T_{1,2}, \ldots, T_{1,n}) \leq_{\text{st}} (T_{2,1}, T_{2,2}, \ldots, T_{2,n}), \quad n \geq 1. \tag{3.1}$$

Remark 3.2: Note that $F \le_{\rm st} G$ in Theorem 3.1 is equivalent to $R(t) \ge S(t)$, $t \ge 0$. Thus, Theorem 3.1 is essentially the same as Proposition 3.9 in [21]. Roughly speaking, inequality (3.1) for $n = \infty$ is denoted in [21] as $N_1 \ge_{\rm st-\mathcal{D}} N_2$, where N_1 and N_2 are the underlying nonhomogeneous Poisson processes.

We now proceed to comparisons in the sense of the hazard rate order. The next result gives conditions under which the epoch times of the first process are smaller than the epoch times of the other process in the *multivariate hazard rate order*. Shaked and Shanthikumar have given a few equivalent definitions of this order in different articles; see references in [20, p. 148]. For the purpose of this article, we will use a modification of the definition given in Theorem 4.D.1 of [20] (thus, incidentally, correcting a typographical error there). Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional nonnegative random vectors with multivariate conditional hazard rate functions $\eta_{\cdot|\cdot}(\cdot|\cdot)$ and $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ as defined in Theorem 4.C.2 of [20]. For any vector $\boldsymbol{t} = (t_1, t_2, \dots, t_n)$ and any subset $I = \{i_1, i_2, \dots, i_k\} \subseteq \{1,2,\dots,n\}$, let $\boldsymbol{t}_I$ denote the vector $(t_{i_1}, t_{i_2}, \dots, t_{i_k})$. Let $\boldsymbol{0}$ and $\boldsymbol{e}$ denote respectively vectors of 0's and 1's; the dimensions of $\boldsymbol{0}$ and $\boldsymbol{e}$ can be determined from the context. When the multivariate conditional hazard rate functions of two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ satisfy

$$\eta_{i|I\cup J}(u|\boldsymbol{s}_{I\cup J}) \ge \lambda_{i|I}(u|\boldsymbol{t}_I) \quad \text{whenever } I \cap J = \emptyset,\ \boldsymbol{0} \le \boldsymbol{s}_I \le \boldsymbol{t}_I \le u\boldsymbol{e}, \text{ and } \boldsymbol{0} \le \boldsymbol{s}_J \le u\boldsymbol{e}, \tag{3.2}$$

where $i \in \overline{I \cup J}$ (this denotes the complement of $I \cup J$ in $\{1,2,\dots,n\}$) and $\boldsymbol{s}_{I\cup J}$ and $\boldsymbol{t}_I$ are possible realizations of the underlying random vectors, we say that $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ in the multivariate hazard rate order and we denote it by $\boldsymbol{X} \le_{\rm hr} \boldsymbol{Y}$. If the distribution functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $F_{\boldsymbol{X}}$ and $F_{\boldsymbol{Y}}$, respectively, then this relation will sometimes be denoted by $F_{\boldsymbol{X}} \le_{\rm hr} F_{\boldsymbol{Y}}$. In the univariate case, the relation $F_X \le_{\rm hr} F_Y$ is equivalent to the requirement that $\bar{F}_Y/\bar{F}_X$ is an increasing function, and if the corresponding hazard rate functions r_X and r_Y exist, then the relation $F_X \le_{\rm hr} F_Y$ is equivalent to the requirement that $r_X \ge r_Y$. It is known (see [20, Thms. 4.C.1 and 4.D.1) that $\boldsymbol{X} \le_{\rm hr} \boldsymbol{Y} \Rightarrow \boldsymbol{X} \le_{\rm st} \boldsymbol{Y}$. Thus, the next result gives a stronger conclusion than Theorem 3.1, but under a stronger assumption.

THEOREM 3.3: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier. Then, $F \le_{hr} G$ if and only if*

$$(T_{1,1}, T_{1,2}, \dots, T_{1,n}) \le_{\rm hr} (T_{2,1}, T_{2,2}, \dots, T_{2,n}), \quad n \ge 1.$$

PROOF: Fix an $n \ge 1$. Let $\eta_{\cdot|\cdot}(\cdot|\cdot)$ be the multivariate conditional hazard rate functions associated with $(T_{1,1}, T_{1,2}, \dots, T_{1,n})$ and let $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ be the multivariate conditional hazard rate functions associated with $(T_{2,1}, T_{2,2}, \dots, T_{2,n})$.

First, let us obtain an explicit expression for $\lambda_{i|I}(u|\boldsymbol{t}_I)$ in (3.2). Since $T_{2,1} \le T_{2,2} \le \cdots \le T_{2,n}$ a.s., it follows that $\boldsymbol{t}_I$ in (3.2) can be a realization ("history") of observations up to time u only if I is of the form $I = \{1,2,\dots,m\}$ for some $m \ge 1$, or $I = \emptyset$ (i.e., $m = 0$). Then, we have

$$\lambda_{i|I}(u|\boldsymbol{t}_I) = \begin{cases} s(u) & \text{if } i = m+1 \\ 0 & \text{if } i > m+1, \end{cases}$$

where $I = \{1,2,\ldots,m\}$; here, s is the hazard rate function associated with G.

Next, let us obtain an explicit expression for $\eta_{i|I\cup J}(u|\boldsymbol{s}_{I\cup J})$ in (3.2). Since $T_{1,1} \le T_{1,2} \le \cdots \le T_{1,n}$ a.s., we see that when $I = \{1,2,\ldots,m\}$, $\boldsymbol{s}_{I\cup J}$ in (3.2) can be a realization of observations up to time u only if J is of the form $J = \{m+1, m+2,\ldots,k\}$ for some $k \ge m+1$, or $J = \emptyset$ (i.e., $k = m$). Then, we have

$$\eta_{i|I\cup J}(u|\boldsymbol{s}_{I\cup J}) = \begin{cases} r(u) & \text{if } i = k+1 \\ 0 & \text{if } i > k+1, \end{cases}$$

where $I = \{1,2,\ldots,m\}$ and $J = \{m+1, m+2,\ldots,k\}$; here, r is the hazard rate function associated with F.

Suppose that $F \le_{\text{hr}} G$. Since i in (3.2) must satisfy $i \in \overline{I\cup J}$ (i.e., $i > k$), we see that if $k > m$, then

$$\eta_{i|I\cup J}(u|\boldsymbol{s}_{I\cup J}) = r(u) \ge 0 = \lambda_{i|I}(u|\boldsymbol{t}_I) \quad \text{if } i = k+1,$$
$$\eta_{i|I\cup J}(u|\boldsymbol{s}_{I\cup J}) = 0 = \lambda_{i|I}(u|\boldsymbol{t}_I) \quad \text{if } i > k+1;$$

thus, (3.2) holds. If $k = m$ (i.e., $J = \emptyset$), then, using $F \le_{\text{hr}} G$, we get

$$\eta_{i|I\cup J}(u|\boldsymbol{s}_{i\cup J}) = r(u) \ge s(u) = \lambda_{i|I}(u|\boldsymbol{t}_I) \quad \text{if } i = k+1,$$
$$\eta_{i|I\cup J}(u|\boldsymbol{s}_{I\cup J}) = 0 = \lambda_{i|I}(u|\boldsymbol{t}_I) \quad \text{if } i > k+1;$$

thus, (3.2) holds in this case too.

The necessity part follows from (3.2) with $i = 1$ (then, $I = J = \emptyset$). ■

The order $\le_{\text{hr}}$ is not closed under marginalization (although it is closed under the dynamic conditional marginalization described in [19]). Thus, it does not follow from Theorem 3.3 that $T_{1,n} \le_{\text{hr}} T_{2,n}$ for $n \ge 2$ under the conditions stated there. However, in the following result, it is shown that this indeed happens to be the case.

THEOREM 3.4: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier. Then, $F \le_{\text{hr}} G$ if and only if $T_{1,n} \le_{\text{hr}} T_{2,n}$ for all $n \ge 1$.*

PROOF: Recall that r and s denote the hazard rate functions of F and G, respectively, and that the corresponding cumulative hazard functions R and S are defined by $R(t) \equiv \int_0^t r(u)\,du$ and $S(t) \equiv \int_0^t s(u)\,du$, respectively. The survival function $\bar{F}_{1,n}$ of $T_{1,n}$ is given by

$$\bar{F}_{1,n}(t) = P(T_{1,n} > t) = \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!} e^{-R(t)} = \bar{\Gamma}_n(R(t)), \quad t \ge 0, \tag{3.3}$$

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where $\bar{\Gamma}_n$ is the survival function of the gamma distribution with scale parameter 1 and shape parameter n; see, for example, Gupta and Kirmani [7] or Kochar [12]. The corresponding density function $f_{1,n}$ is given by

$$f_{1,n}(t) = \gamma_n(R(t))r(t), \quad t \geq 0,$$

where γ_n is the density function associated with $\bar{\Gamma}_n$. The corresponding hazard rate function $r_{F_{1,n}}$ is given by

$$r_{F_{1,n}}(t) = r_{\Gamma_n}(R(t))r(t), \quad t \geq 0,$$

where r_{Γ_n} is the hazard rate function associated with $\bar{\Gamma}_n$. Similarly,

$$r_{F_{2,n}}(t) = r_{\Gamma_n}(S(t))s(t), \quad t \geq 0.$$

If $F \leq_{\text{hr}} G$, then

$$r_{F_{1,n}}(t) = r_{\Gamma_n}(R(t))r(t) \geq r_{\Gamma_n}(S(t))s(t) = r_{F_{2,n}}(t), \quad t \geq 0,$$

where the inequality follows from $r(t) \geq s(t)$, $R(t) \geq S(t)$, and the fact that the hazard rate function of the gamma distribution described is increasing.

The necessity part follows from the fact that F is the distribution function of $T_{1,1}$ and G is the distribution function of $T_{2,1}$. ■

Next, we proceed to comparisons in the likelihood ratio order sense. The following result (Thm. 3.6) gives conditions under which the epoch times of the first process are smaller than the epoch times of the other process in the *multivariate likelihood ratio order*. This order is defined as follows (see, e.g., [20, Sect. 4.E]). Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors with density functions $f_{\boldsymbol{X}}$ and $f_{\boldsymbol{Y}}$, respectively. If ($\wedge$ and $\vee$ denote respectively the minimum and the maximum operations)

$$\begin{aligned} f_{\boldsymbol{X}}(x_1 \wedge y_1, x_2 \wedge y_2, \dots, x_n \wedge y_n) f_{\boldsymbol{Y}}(x_1 \vee y_1, x_2 \vee y_2, \dots, x_n \vee y_n) \\ \geq f_{\boldsymbol{X}}(x_1, x_2, \dots, x_n) f_{\boldsymbol{Y}}(y_1, y_2, \dots, y_n) \end{aligned}$$

for all $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ in $\mathbb{R}^n$, then we denote $\boldsymbol{X} \leq_{\text{lr}} \boldsymbol{Y}$. If the distribution functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $F_{\boldsymbol{X}}$ and $F_{\boldsymbol{Y}}$, respectively, then this relation will sometimes be denoted by $F_{\boldsymbol{X}} \leq_{\text{lr}} F_{\boldsymbol{Y}}$. In the univariate case, the relation $F_X \leq_{\text{lr}} F_Y$ is equivalent to the requirement that f_Y/f_X is an increasing function. It is known (see [20, Thm. 4.E.4]) that $\boldsymbol{X} \leq_{\text{lr}} \boldsymbol{Y} \Rightarrow \boldsymbol{X} \leq_{\text{hr}} \boldsymbol{Y}$. Thus, Theorem 3.6 gives a stronger conclusion than Theorem 3.3, but under the additional assumption (3.4). The following lemma is used in the proof of Theorem 3.6. The proof of the lemma is straightforward and is therefore omitted.

Lemma 3.5: *Let F and G be two distribution functions with associated hazard rate functions r and s. If $F \leq_{\text{hr}} G$ and if*

$$\frac{s(t)}{r(t)} \text{ is increasing in } t \geq 0, \tag{3.4}$$

then $F \leq_{\text{lr}} G$.

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THEOREM 3.6: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier. If $F \le_{\rm hr} G$ and if the associated hazard rate functions r and s satisfy (3.4), then*

$$(T_{1,1}, T_{1,2}, \ldots, T_{1,n}) \le_{\rm lr} (T_{2,1}, T_{2,2}, \ldots, T_{2,n}), \quad n \ge 1.$$

PROOF: First, note that by Lemma 3.5, we have that $F \le_{\rm lr} G$. Thus, the stated result is true for $n = 1$. So, let $n \ge 2$. The density function of $(T_{1,1}, T_{1,2}, \ldots, T_{1,n})$ is given by

$$h_{1,n}(x_1, x_2, \ldots, x_n) = r(x_1)r(x_2) \cdots r(x_{n-1})f(x_n) \quad \text{for } x_1 \le x_2 \le \cdots \le x_n,$$

where f is the density function associated with F. Similarly, the density function of $(T_{2,1} T_{2,2}, \ldots, T_{2,n})$ is given by

$$h_{2,n}(x_1, x_2, \ldots, x_n) = s(x_1)s(x_2) \cdots s(x_{n-1})g(x_n) \quad \text{for } x_1 \le x_2 \le \cdots \le x_n,$$

where g is the density function associated with G.

Consider now $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ such that $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$. We want to prove that

$$\begin{aligned} &r(x_1 \wedge y_1)r(x_2 \wedge y_2) \cdots r(x_{n-1} \wedge y_{n-1})f(x_n \wedge y_n) \\ &\quad \times s(x_1 \vee y_1)s(x_2 \vee y_2) \cdots s(x_{n-1} \vee y_{n-1})g(x_n \vee y_n) \\ &\quad \ge r(x_1)r(x_2) \cdots r(x_{n-1})f(x_n)s(y_1)s(y_2) \cdots s(y_{n-1})g(y_n). \end{aligned} \tag{3.5}$$

Let $E = \{i \le n-1 : x_i \ge y_i\}$. Then, (3.5) reduces to

$$\left(\prod_{i \in E} r(y_i)s(x_i)\right) f(x_n \wedge y_n)g(x_n \vee y_n) \ge \left(\prod_{i \in E} r(x_i)s(y_i)\right) f(x_n)g(y_n),$$

and this follows from (3.4) and $F \le_{\rm lr} G$. ■

Since the order $\le_{\rm lr}$ is closed under marginalization (see [20, Thm. 4.E.3(b)]), we get, as a corollary, that if $F \le_{\rm hr} G$ and if (3.4) holds, then $T_{1,n} \le_{\rm lr} T_{2,n}$ for all $n \ge 1$. The following theorem is a variation of this corollary. When one compares the following theorem to the above-stated corollary, it should be noted that (3.4) implies (3.6); see Remark 3.8. It should also be noted that $F \le_{\rm lr} G$ implies $F \le_{\rm hr} G$.

THEOREM 3.7: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier. If $F \le_{\rm lr} G$ and if the cumulative hazard functions R and S, defined by $R(t) \equiv \int_0^t r(u)\,du$ and $S(t) \equiv \int_0^t s(u)\,du$, respectively, satisfy*

$$\frac{S(t)}{R(t)} \text{ is increasing in } t \ge 0, \tag{3.6}$$

then

$$T_{1,n} \le_{\rm lr} T_{2,n}, \quad n \ge 1. \tag{3.7}$$

Conversely, if (3.7) holds, then $F \le_{\rm lr} G$ and (3.6) holds.

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PROOF: By (2.3), the density function of $T_{1,n}$ is given by

$$f_{1,n}(t) = f(t)\,\frac{(R(t))^{n-1}}{(n-1)!}, \quad t \geq 0, n \geq 1,$$

where f is the density function associated with F, and the density function of $T_{2,n}$ is given by

$$f_{2,n}(t) = g(t)\,\frac{(S(t))^{n-1}}{(n-1)!}, \quad t \geq 0, n \geq 1,$$

where g is the density function associated with G. Thus,

$$\frac{f_{2,n}(t)}{f_{1,n}(t)} = \frac{g(t)}{f(t)}\left(\frac{S(t)}{R(t)}\right)^{n-1}.$$

Now, if $F \leq_{\text{lr}} G$ and (3.6) holds, then $f_{2,n}/f_{1,n}$ is increasing and we get (3.7).

Conversely, suppose that (3.7) holds. Applying (3.7) with $n = 1$, we obtain $F \leq_{\text{lr}} G$. In order to obtain (3.6), denote $H \equiv S/R$, $h \equiv g/f$, and $h_n \equiv f_{2,n}/g_{2,n}$. First, suppose that the h_n's are differentiable. Then, when $n \geq 2$, we have

$$h_n'(t) = H^{n-2}(t)[h'(t)H(t) + (n-1)h(t)H'(t)], \quad t \geq 0.$$

If H is not increasing, then $H'(t_0) < 0$ for some t_0. Therefore, for a large enough n, we have that $h_n'(t_0) < 0$, and this contradicts (3.7). If the h_n's are not differentiable, then the above argument can be easily modified to obtain the same result. ■

Remark 3.8: Sengupta and Deshpande [18] and Rowell and Siegrist [16] have shown that (3.4) $\Rightarrow$ (3.6) (in fact, they treated (3.4) and (3.6) as notions of relative aging of two life distributions). Thus, the assumptions in Theorem 3.7 are weaker than the assumptions in Theorem 3.6. It is of interest to note that (3.4) does not imply that $F \leq_{\text{lr}} G$. In fact, (3.4) does not even imply that $F \leq_{\text{hr}} G$. In order to see this, let r be a decreasing hazard rate function such that $r(0+) > 1$ [e.g., $r(t) = t^{-1}$], and let $s(t) \equiv 1$ (i.e., the hazard rate function of a standard exponential random variable). Then, (3.4) holds, but $r(t)$ is not larger than or equal to $s(t)$ for all $t > 0$.

Remark 3.9: It is also of interest to note that $F \leq_{\text{lr}} G$ does not imply (3.6). In order to see it, let F be the uniform distribution on [0,1] and let G be the gamma(2) distribution. Then,

$$\frac{g(t)}{f(t)} = \begin{cases} te^{-t}, & 0 \leq t \leq 1 \\ \infty, & t > 1, \end{cases}$$

and this is increasing in t (i.e., $F \leq_{\text{lr}} G$). However, the corresponding $S(t)/R(t)$ is positive when $0 \leq t \leq 1$, and it is 0 when $t > 1$. Therefore, (3.6) does not hold.

Before we close this subsection, it is worthwhile to mention that Gupta and Kirmani [7] showed that if F and G are distribution functions associated with two nonhomogeneous Poisson processes as described earlier, then $F \leq_{\text{c}} [\leq_{*}, \leq_{\text{su}}]\; G$ if,

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and only if, $T_{1,n} \leq_{c} [\leq_{*}, \leq_{su}]\, T_{2,n}$, where $\leq_c$, $\leq_*$, and $\leq_{su}$ are the transform orders described in [20, Sect. 3.C]. Using the idea of their proof, we also obtain the following result. Recall that two univariate random variables X and Y, with distribution functions F and G, respectively, are said to be ordered in the dispersive order (denoted by $X \leq_{disp} Y$ or $F \leq_{disp} G$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 \leq \alpha \leq \beta \leq 1$ (see [20, Sect. 2.B]). See also Proposition 4.3 in Section 4 for a simple condition which implies $F \leq_{disp} G$.

THEOREM 3.10: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier. Then, $F \leq_{disp} G$ if and only if $T_{1,n} \leq_{disp} T_{2,n}$ for all $n \geq 1$.*

PROOF: Fix an $n \geq 1$, and denote by $F_{1,n}$ and $F_{2,n}$ the distribution functions of $T_{1,n}$ and $T_{2,n}$, respectively. Recall from (3.3) that

$$F_{1,n}(t) = \psi_n(F(t)) \quad \text{and} \quad F_{2,n}(t) = \psi_n(G(t)),$$

where $\psi_n(u) \equiv \Gamma_n(-\log(1-u))$, $u \in [0,1]$. Therefore,

$$F_{2,n}^{-1}(F_{1,n}(t)) - t = (\psi_n(G))^{-1}(\psi_n(F(t))) - t = G^{-1}(F(t)) - t, \quad t \geq 0.$$

Thus, from (2.B.6) in [20], it is seen that $F \leq_{disp} G$ if and only if $T_{1,n} \leq_{disp} T_{2,n}$. ■

3.2. Epoch Times of Nonhomogeneous Pure Birth Processes

In this subsection, we derive stochastic comparison results of epoch times of two nonhomogeneous pure birth processes. So, consider two such processes, indexed by $i = 1,2$, parameterized by the sets $\{r_{i,n}, n \geq 1\}$ of hazard rate functions that satisfy (2.1). The corresponding epoch times will be denoted by $0 \equiv T_{i,0} \leq T_{i,1} \leq T_{i,2} \leq \cdots$. In the sequel, we will use the representation described in (2.5) and (2.6); that is, let $\{Y_{i,n}, n \geq 1\}$, $i = 1,2$, be two sets of independent absolutely continuous nonnegative random variables, where $Y_{i,n}$ has the hazard rate function $r_{i,n}$. If we define

$$\hat{T}_{i,1} = Y_{i,1}, \tag{3.8}$$

$$\hat{T}_{i,n} = [Y_{i,n} | Y_{i,n} > \hat{T}_{i,n-1}], \quad n \geq 2, \tag{3.9}$$

then, for $i = 1,2$, the joint distribution of the $T_{i,n}$'s is the same as the joint distribution of the $\hat{T}_{i,n}$'s.

The first result, which we include for the sake of completion, gives conditions under which the epoch times of the two nonhomogeneous pure birth processes are ordered according to the usual stochastic order. This result may be compared with Theorem 3.1.

THEOREM 3.11: *Let the $T_{i,n}$'s be the epoch times of the two nonhomogeneous pure birth processes parameterized by the sets $\{r_{i,n}, n \geq 1\}$ of hazard rate functions. Let $\{Y_{i,n}, n \geq 1\}$, $i = 1,2$, be two sets of independent absolutely continuous nonnegative*

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random variables, where $Y_{i,n}$ has the hazard rate function $r_{i,n}$. If $Y_{1,1} \le_{\mathrm{st}} Y_{2,1}$ and if $Y_{1,j} \le_{\mathrm{hr}} Y_{2,j}$ for $j \ge 2$, then

$$(T_{1,1}, T_{1,2}, \ldots, T_{1,n}) \le_{\mathrm{st}} (T_{2,1}, T_{2,2}, \ldots, T_{2,n}), \quad n \ge 1. \tag{3.10}$$

Proof: We will show that $(\hat{T}_{1,1}, \hat{T}_{1,2}, \ldots, \hat{T}_{1,n}) \le_{\mathrm{st}} (\hat{T}_{2,1}, \hat{T}_{2,2}, \ldots, \hat{T}_{2,n})$, where the $\hat{T}_{i,n}$'s are defined in (3.8)and (3.9). The result then follows from the fact that the joint distribution of the $T_{i,n}$'s is the same as the joint distribution of the $\hat{T}_{i,n}$'s. Let $\bar{K}_{i,n}$ denote the survival function of $Y_{i,n}$; that is, $\bar{K}_{i,n}(t) = \exp[-\int_0^t r_{i,n}(u)\,du]$, $t \ge 0$. We will apply Theorem 4.B.4 in [20]. Note that for $j \ge 2$, we have

$$[\hat{T}_{1,j} | \hat{T}_{1,1} = t_1, \hat{T}_{1,2} = t_2, \ldots, \hat{T}_{1,j-1} = t_{j-1}] = [Y_{1,j} | Y_{1,j} > t_{j-1}],$$

and this is stochastically increasing in t_{j-1} (see [20, Thm. 1.A.11]). Therefore, $(\hat{T}_{1,1}, \hat{T}_{1,2}, \ldots, \hat{T}_{1,n})$ is CIS (conditionally increasing in sequence; see [20, p. 117]). Next, note that

$$\begin{aligned}
&[\hat{T}_{1,j} | \hat{T}_{1,1} = t_1, \hat{T}_{1,2} = t_2, \ldots, \hat{T}_{1,j-1} = t_{j-1}] \\
&\quad = [Y_{1,j} | Y_{1,j} > t_{j-1}] \\
&\quad \le_{\mathrm{st}} [Y_{2,j} | Y_{2,j} > t_{j-1}] \\
&\quad = [\hat{T}_{2,j} | \hat{T}_{2,1} = t_1, \hat{T}_{2,2} = t_2, \ldots, \hat{T}_{2,j-1} = t_{j-1}],
\end{aligned}$$

where the inequality, which is equivalent to

$$\frac{\bar{K}_{1,j}(u)}{\bar{K}_{1,j}(t_{j-1})} \le \frac{\bar{K}_{2,j}(u)}{\bar{K}_{2,j}(t_{j-1})}, \quad u \ge t_{j-1},$$

follows from $Y_{1,j} \le_{\mathrm{hr}} Y_{2,j}$. Thus, (3.10) follows from Theorem 4.B.4 in [20]. ■

Using some general ideas from Shaked and Szekli [21], it is possible to construct an alternative, although lengthier, proof of Theorem 3.11.

Proceeding now to comparisons in the sense of the hazard rate stochastic order, we first have the following result which may be compared with Theorem 3.3.

Theorem 3.12: *Let $T_{i,n}$ and $Y_{i,n}$ be as in Theorem* 3.11. *If $Y_{1,j} \le_{\mathrm{hr}} Y_{2,j}$ for $j \ge 1$, then $(T_{1,1}, T_{1,2}, \ldots, T_{1,n}) \le_{\mathrm{hr}} (T_{2,1}, T_{2,2}, \ldots, T_{2,n})$ for all $n \ge 1$.*

Proof: The proof is similar to the proof of Theorem 3.3. Fix an $n \ge 1$. Let $\eta_{\cdot|\cdot}(\cdot|\cdot)$ be the multivariate conditional hazard rate functions associated with $(T_{1,1}, T_{1,2}, \ldots, T_{1,n})$ and let $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ be the multivariate conditional hazard rate functions associated with $(T_{2,1}, T_{2,2}, \ldots, T_{2,n})$.

In order to obtain an explicit expression for $\lambda_{i|I}(u|\boldsymbol{t}_I)$ in (3.2), we first note, as in the proof of Theorem 3.3, that I must be of the form $I = \{1, 2, \ldots, m\}$ for some m. Then, we have

$$\lambda_{i|I}(u|\boldsymbol{t}_I) = \begin{cases} r_{2,m+1}(u) & \text{if } i = m+1 \\ 0 & \text{if } i > m+1, \end{cases}$$

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where $I = \{1,2,..,m\}$. Similarly, in $\eta_{i|I\cup J}(u|\mathbf{s}_{I\cup J})$ in (3.2), we must have $I = \{1,2,\ldots,m\}$ and $J = \{m+1, m+2,\ldots,k\}$ for some $k \geq m$. Then, we have

$$\eta_{i|I\cup J}(u|\mathbf{s}_{I\cup J}) = \begin{cases} r_{1,k+1}(u) & \text{if } i = k+1 \\ 0 & \text{if } i > k+1, \end{cases}$$

where $I = \{1,2,\ldots,m\}$ and $J = \{m+1, m+2,\ldots,k\}$. The rest of the proof follows the lines of the proof of Theorem 3.3. ■

Finally, we obtain a comparison result in the sense of the multivariate likelihood ratio order. The following result extends Theorem 3.6 to nonhomogeneous pure birth processes. At a first glance, condition (3.11) in the following theorem looks restrictive; however, in many applications (see Sect. 5), the hazard rate functions $r_{1,1}, r_{1,2},\ldots$ are proportional, and the hazard rate functions $r_{2,1}, r_{2,2},\ldots$ are also proportional, and then (3.11) can often be verified.

THEOREM 3.13: *Let $T_{i,n}$ and $Y_{i,n}$ be as in Theorem* 3.11. *If $Y_{1,j} \leq_{\text{hr}} Y_{2,j}$, if $r_{2,j}/r_{1,j}$ is increasing*, *and if*

$$r_{2,j+1}(t) - r_{2,j}(t) \geq r_{1,j+1}(t) - r_{1,j}(t), \quad t \geq 0, \tag{3.11}$$

for $j \geq 1$, then $(T_{1,1}, T_{1,2},\ldots,T_{1,n}) \leq_{\text{lr}} (T_{2,1}, T_{2,2},\ldots,T_{2,n})$ for all $n \geq 1$.

PROOF: In this proof, we denote by $\bar{K}_{i,n}$ and $k_{i,n}$ the survival and the density functions of $Y_{i,n}$, respectively; that is, $\bar{K}_{i,n}(t) = \exp[-\int_0^t r_{i,n}(u)\,du]$ and $k_{i,n}(t) = r_{i,n}(t)\exp[-\int_0^t r_{i,n}(u)\,du]$, $t \geq 0$.

First, note that by Lemma 3.5, we have $Y_{1,j} \leq_{\text{lr}} Y_{2,j}, j \geq 1$; thus, the stated result is obvious for $n = 1$. So, let $n \geq 2$. For $i = 1,2$, the density function $h_{i,n}$ of $(T_{i,1}, T_{i,2},\ldots,T_{i,n})$ is given by

$$h_{i,n}(x_1, x_2,\ldots,x_n) = \prod_{j=1}^{n-1} \frac{k_{i,j}(x_j)}{\bar{K}_{i,j+1}(x_j)}\, k_{i,n}(x_n) \quad \text{for } x_1 \leq x_2 \leq \cdots \leq x_n.$$

Note that condition (3.11) can be written as

$$\frac{\bar{K}_{2,j}(t)\bar{K}_{1,j+1}(t)}{\bar{K}_{2,j+1}(t)\bar{K}_{1,j}(t)} \text{ is increasing in } t \geq 0. \tag{3.12}$$

Consider now $(x_1, x_2,\ldots,x_n)$ and $(y_1, y_2,\ldots,y_n)$ such that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. We want to prove that

$$\begin{aligned} &\prod_{j=1}^{n-1} \frac{k_{1,j}(x_j \wedge y_j)}{\bar{K}_{1,j+1}(x_j \wedge y_j)}\, k_{1,n}(x_n \wedge y_n) \prod_{j=1}^{n-1} \frac{k_{2,j}(x_i \vee y_j)}{\bar{K}_{2,j+1}(x_j \vee y_j)}\, k_{2,n}(x_n \vee y_n) \\ &\quad \geq \prod_{j=1}^{n-1} \frac{k_{1,j}(x_j)}{\bar{K}_{1,j+1}(x_j)}\, k_{1,n}(x_n) \prod_{j=1}^{n-1} \frac{k_{2,j}(y_j)}{\bar{K}_{2,j+1}(y_j)}\, k_{2,n}(y_n). \end{aligned} \tag{3.13}$$

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Let $E = \{j \le n-1: x_j \ge y_j\}$. Then, (3.13) reduces to

$$\left(\prod_{j\in E} r_{1,j}(y_j)\,\frac{\bar{K}_{1,j}(y_j)}{\bar{K}_{1,j+1}(y_j)}\,r_{2,j}(x_j)\,\frac{\bar{K}_{2,j}(x_j)}{\bar{K}_{2,j+1}(x_j)}\right)k_{1,n}(x_n\wedge y_n)k_{2,n}(x_n\vee y_n)$$
$$\ge \left(\prod_{j\in E} r_{1,j}(x_j)\,\frac{\bar{K}_{1,j}(x_j)}{\bar{K}_{2,j+1}(x_j)}\,r_{2,j}(y_j)\,\frac{\bar{K}_{2,j}(y_j)}{\bar{K}_{2,j+1}(y_j)}\right)k_{1,n}(x_n)k_{2,n}(y_n),$$

and this follows from the monotonicity of $r_{2,j}/r_{1,j}$, from (3.12), and from $Y_{1,j} \le_{\text{lr}} Y_{2,j}$. ■

4. STOCHASTIC COMPARISONS OF INTEREPOCH INTERVALS

4.1. Interepoch Intervals of Nonhomogeneous Poisson Processes

As in Section 3.1, consider two nonhomogeneous Poisson processes with intensity functions r and s. Denote the associated density functions [see Eq. (2.2)] by f and g and the associated distribution functions by F and G. Finally, let the associated cumulative hazard functions be denoted by R and S; that is, $R(t) \equiv \int_0^t r(u)\,du$ and $S(t) \equiv \int_0^t s(u)\,du$, $t \ge 0$. Let $T_{i,n}$ be as defined in Section 3.1. The interepoch intervals will be denoted by $X_{i,n} = T_{i,n} - T_{i,n-1}$, $n \ge 1$, with $T_{i,0} \equiv 0$, $i = 1,2$.

In this subsection, we derive some results which stochastically compare vectors of $X_{1,j}$'s with vectors of $X_{2,j}$'s.

First, for the sake of completion, we devote some space to a discussion on comparisons of interepoch intervals in the sense of the usual stochastic order. The following result is essentially a restatement of Proposition 3.10 of Shaked and Szekli [21].

THEOREM 4.1: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier. If $F \le_{\text{disp}} G$, then*

$$(X_{1,1}, X_{1,2}, \dots, X_{1,n}) \le_{\text{st}} (X_{2,1}, X_{2,2}, \dots, X_{2,n}), \quad n \ge 1. \tag{4.1}$$

Roughly speaking, inequality (4.1) for $n = \infty$ is denoted in [21] as $N_1 \ge_{\text{st-}\infty} N_2$, where N_1 and N_2 are the underlying nonhomogeneous Poisson processes.

A similar result worth mentioning is the following. It can be proven using Theorem 2.7 of Shaked and Szekli [21].

THEOREM 4.2: *Let r and s be intensity functions associated with two nonhomogeneous Poisson processes as described earlier. If*

$$r(u) \ge s(u+x), \quad u \ge 0, x \ge 0, \tag{4.2}$$

then

$$(X_{1,1}, X_{1,2}, \dots, X_{1,n}) \le_{\text{st}} (X_{2,1}, X_{2,2}, \dots, X_{2,n}), \quad n \ge 1.$$

Note that (4.2) holds if $F \le_{\text{hr}} G$ and if r or s is decreasing [i.e., F or G is DFR (decreasing failure rate)]. Thus, Theorem 4.2 is a stronger result than Theorem 8 of

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Gupta and Kirmani [7] or Theorem 4.4 of Kochar [11]. In fact, we have the following relationship among the conditions of Theorems 4.1 and 4.2.

PROPOSITION 4.3: *Let F and G be two distribution functions with respective hazard rate functions r and s. If* (4.2) *holds, then* $F \le_{\text{disp}} G$.

PROOF: Condition (4.2) implies that $r(u) \ge s(u)$; that is, $F \le_{\text{hr}} G$. This, in turn, implies $F \le_{\text{st}} G$ and, therefore, $F^{-1}(\alpha) \le G^{-1}(\alpha)$ for all $\alpha \in (0,1)$.

Now, (4.2) therefore gives $r(F^{-1}(\alpha)) \ge s(G^{-1}(\alpha))$ for all $\alpha \in (0,1)$, which is equivalent to $F \le_{\text{disp}} G$ by (2.B.8) in [20]. ■

From Proposition 4.3, it is seen that Theorem 4.2 follows from Theorem 4.1. Proposition 4.3 also strengthens a result of Bartoszewicz [3] and Bagai and Kochar [1], which is stated as Theorem 2.B.13(a) in [20]. This is so because if $F \le_{\text{hr}} G$ and if r or s is decreasing, then (4.2) holds.

Condition (4.2) defines what can be called a "shifted hazard rate order" in the spirit of Shanthikumar and Yao [22], who defined a "shifted likelihood ratio order." However, it should be noted that whereas (4.2) is the same as $X \le_{\text{hr}} [Y - x \mid Y > x]$ for all $x \ge 0$, where X and Y have the hazard rates functions r and s, respectively, the condition of Shanthikumar and Yao is the same as $[X - x \mid X > x] \le_{\text{lr}} Y$ for all $x \ge 0$. See also [13].

We now proceed to a comparison of the interepoch intervals in the sense of the hazard rate order.

THEOREM 4.4: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier, with corresponding hazard rate functions r and s. If* $F \le_{\text{hr}} G$ *and if* $\bar F$ *and* $\bar G$ *are logconvex (i.e.,* DFR*), and if (3.4) holds, then* $X_{1,n} \le_{\text{hr}} X_{2,n}$ *for each* $n \ge 1$.

PROOF: For the purpose of this proof, we denote F by F_1, G by F_2, r by r_1, s by r_2, and the cumulative hazard functions are denoted by R_i; that is, $R_i(t) = \int_0^t r_i(u)\,du$, $i = 1,2$. Let $\bar G_{i,n}$ denote the survival function of $X_{i,n}, i = 1,2$. The stated result is obvious for $n = 1$, so let us fix an $n \ge 2$. Then, from (2.4), we obtain

$$\bar G_{i,n}(t) = \int_0^\infty r_i(s)\,\frac{R_i^{n-2}(s)}{(n-2)!}\,\bar F_i(s+t)\,ds, \quad t \ge 0,\ i \in \{1,2\}. \tag{4.3}$$

Condition (3.4) means that

$$r_i(t) \text{ is } \mathrm{TP}_2 \text{ (totally positive of order 2) in } (i,t)$$

(a nonnegative function h of two variables, x and y, say, is called TP_2 if $h(x',y)/h(x,y)$ is increasing in y whenever $x \le x'$). Condition (3.4) also implies that $R_2(t)/R_1(t)$ is increasing in $t \ge 0$; that is, $R_i(t)$ is TP_2 in (i,t). Since a product of TP_2 kernels is TP_2, we get that

$$r_i(t)\,\frac{R_i^{n-2}(t)}{(n-2)!} \text{ is } \mathrm{TP}_2 \text{ in } (i,t).$$

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The assumption $F_1 \leq_{\rm hr} F_2$ implies that

$$\bar{F}_i(s+t) \text{ is } \mathrm{TP}_2 \text{ in } (i,s) \text{ and in } (i,t).$$

Finally, the logconvexity of $\bar{F}_1$ and of $\bar{F}_2$ means that

$$\bar{F}_i(s+t) \text{ is } \mathrm{TP}_2 \text{ in } (s,t).$$

Thus, by Theorem 5.1 of Karlin [10, p. 123], we get that $\bar{G}_{i,n}(t)$ is TP_2 in (i,t); that is, $X_{1,n} \leq_{\rm hr} X_{2,n}$. ■

Next, we discuss the likelihood ratio order. The following result gives conditions under which the interepoch intervals of the two processes are comparable in the multivariate likelihood ratio order.

THEOREM 4.5: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier, with corresponding density functions f and g and with corresponding hazard rate functions r and s. If $F \leq_{\rm hr} G$, if f and/or g are logconvex, if r and/or s are logconvex, and if (3.4) holds, then*

$$(X_{1,1}, X_{1,2}, \ldots, X_{1,n}) \leq_{\rm lr} (X_{2,1}, X_{2,2}, \ldots, X_{2,n}), \quad n \geq 1.$$

PROOF: First, note that by Lemma 3.5, we have $F \leq_{\rm lr} G$.

We will give the proof when f and r are logconvex; the proofs of the other cases are similar. Note that the logconvexity of f and r implies that f and r are positive over $(0,\infty)$. The result is obvious for $n = 1$; thus, let us fix an $n \geq 2$. The density function $l_{1,n}$ of $(X_{1,1}, X_{1,2}, \ldots, X_{1,n})$ is given by

$$l_{1,n}(x_1, \ldots, x_n) = \prod_{j=1}^{n-1} r(x_1 + \cdots + x_j) f(x_1 + \cdots + x_n), \quad x_k \geq 0, k = 1, \ldots, n. \tag{4.4}$$

The density function $l_{2,n}$ of $(X_{2,1}, X_{2,2}, \ldots, X_{2,n})$ is given by

$$l_{2,n}(x_1, \ldots, x_n) = \prod_{j=1}^{n-1} s(x_1 + \cdots + x_j) g(x_1 + \cdots + x_n), \quad x_k \geq 0, k = 1, \ldots, n. \tag{4.5}$$

The logconvexity of f implies that

$$\begin{aligned} f(x_1 \vee y_1 + \cdots + x_n \vee y_n) f(x_1 \wedge y_1 + \cdots + x_n \wedge y_n) \\ \geq f(x_1 + \cdots + x_n) f(y_1 + \cdots + y_n) \end{aligned} \tag{4.6}$$

for all $x_k \geq 0$ and $y_k \geq 0$, $k = 1, \ldots, n$. Similarly, the logconvexity of r implies, for $j = 1, \ldots, n-1$, that

$$\begin{aligned} r(x_1 \vee y_1 + \cdots + x_j \vee y_j) r(x_1 \wedge y_1 + \cdots + x_j \wedge y_j) \\ \geq r(x_1 + \cdots + x_j) r(y_1 + \cdots + y_j) \end{aligned} \tag{4.7}$$

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for all $x_k \geq 0$ and $y_k \geq 0$, $k = 1, \ldots, j$. Therefore,

$$
\begin{aligned}
l_{1,n}&(x_1 \wedge y_1, \ldots, x_n \wedge y_n) l_{2,n}(x_1 \vee y_1, \ldots, x_n \vee y_n) \\
&= \prod_{j=1}^{n-1} r(x_1 \wedge y_1 + \cdots + x_j \wedge y_j) f(x_1 \wedge y_1 + \cdots + x_n \wedge y_n) \\
&\quad \times \prod_{j=1}^{n-1} s(x_1 \vee y_1 + \cdots + x_j \vee y_j) g(x_1 \vee y_1 + \cdots + x_n \vee y_n) \\
&\geq \prod_{j=1}^{n-1} \frac{s(x_1 \vee y_1 + \cdots + x_j \vee y_j)}{r(x_1 \vee y_1 + \cdots + x_j \vee y_j)} r(x_1 + \cdots + x_j) r(y_1 + \cdots + y_j) \\
&\quad \times \frac{g(x_1 \vee y_1 + \cdots + x_n \vee y_n)}{f(x_1 \vee y_1 + \cdots + x_n \vee y_n)} f(x_1 + \cdots + x_n) f(y_1 + \cdots + y_n) \\
&\geq \prod_{j=1}^{n-1} \frac{s(y_1 + \cdots + y_j)}{r(y_1 + \cdots + y_j)} r(x_1 + \cdots + x_j) r(y_1 + \cdots + y_j) \\
&\quad \times \frac{g(y_1 + \cdots + y_n)}{f(y_1 + \cdots + y_n)} f(x_1 + \cdots + x_n) f(y_1 + \cdots + y_n) \\
&= l_{1,n}(x_1, \ldots, x_n) l_{2,n}(y_1, \ldots, y_n),
\end{aligned}
$$

where the first inequality follows from (4.6) and (4.7), and the second inequality follows from (3.4) and from $F \leq_{\mathrm{lr}} G$ (i.e., g/f is increasing). This gives the stated result. ■

Remark 4.6: In light of the conditions in Theorem 4.5, the following question is of interest. Let f be a density function of a nonnegative random variable, and let r be the corresponding hazard rate function. Does the logconvexity of f imply the logconvexity of r, and vice versa? It turns out that neither is the case. First, consider the hazard rate function

$$r(t) = e^t, \quad t \geq 0.$$

Here, $\log r$ is linear, so it is logconvex. The corresponding density function is given by

$$f(t) = e^{1+t-e^t}, \quad t \geq 0,$$

and a computation of the second derivative shows that f here is strictly logconcave (see [14]), and thus it is not logconvex. In order to see that "f is logconvex" does not imply that "r is logconvex," consider the hazard rate function

$$r(t) = \frac{1}{t+3} - \frac{1}{(t+3)^2} = \frac{t+2}{(t+3)^2}, \quad t \geq 0.$$

This is indeed a hazard rate function since it is nonnegative, and it integrates to ∞. A straightforward computation shows that $(d^2/dt^2)\log r(t) < 0$ for $0 < t < 2\sqrt{2} - 2$. Therefore, r is not logconvex. The corresponding density function is

$$f(t) = \frac{3(t+2)}{(t+3)^3} \exp\left\{\frac{t}{3(t+3)}\right\}, \quad t \geq 0.$$

A straightforward computation gives

$$\frac{d^2}{dt^2} \log f(t) = \frac{2t^3 + 10t^2 + 13t + 1}{(t+3)^3(t+2)^2},$$

and this is positive for all $t \geq 0$. Thus, f is logconvex.

Since the multivariate likelihood ratio order is closed under marginalization (see [20, Thm. 4.E.3(b)]), we get the following result as a corollary of Theorem 4.5.

Corollary 4.7: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier, with corresponding density functions f and g and with corresponding hazard rate functions r and s. If* $F \leq_{\text{hr}} G$, *if f and/or g are logconvex, if r and/or s are logconvex, and if (3.4) holds, then* $X_{1,n} \leq_{\text{lr}} X_{2,n}$ *for all* $n \geq 1$.

The next result gives different conditions under which the interepoch intervals of the two processes are comparable in the likelihood ratio order.

Theorem 4.8: *Let f and g be density functions associated with two nonhomogeneous Poisson processes as described earlier, with corresponding hazard rate functions r and s. If* $F \leq_{\text{hr}} G$, *if f and g are logconvex, and if (3.4) holds, then* $X_{1,n} \leq_{\text{lr}} X_{2,n}$ *for each* $n \geq 1$.

Proof: First, note that by Lemma 3.5, we have $F \leq_{\text{lr}} G$.

As in the proof of Theorem 4.4, for the purpose of the present proof we denote f by f_1, g by f_2, r by r_1, s by r_2, and the cumulative hazard functions are denoted by R_i, $i = 1,2$. Let $g_{i,n}$ denote the density function of $X_{i,n}$, $i = 1,2$. The stated result is obvious for $n = 1$, so let us fix an $n \geq 2$. From (2.4), we obtain

$$g_{i,n}(t) = \int_0^\infty r_i(s) \frac{R_i^{n-2}(s)}{(n-2)!} f_i(s+t)\, ds, \quad t \geq 0,\ i = 1,2.$$

As in the proof of Theorem 4.4, we have that

$$r_i(t) \frac{R_i^{n-2}(t)}{(n-2)!} \text{ is } \text{TP}_2 \text{ in } (i,t).$$

The assumption $F_1 \leq_{\text{lr}} F_2$ implies that

$$f_i(s+t) \text{ is } \text{TP}_2 \text{ in } (i,s) \text{ and in } (i,t).$$

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Finally, the logconvexity of f_1 and of f_2 means that

$$f_i(s+t) \text{ is } \mathrm{TP}_2 \text{ in } (s,t).$$

Thus, by Theorem 5.1 of Karlin [10, p. 123], we get that $g_{i,n}(t)$ is TP_2 in (i,t); that is, $X_{1,n} \le_{\mathrm{lr}} X_{2,n}$. ■

We close this subsection with a comparison result in the mean residual life order. Recall that two univariate random variables X and Y, with distribution functions F and G, respectively, are said to be ordered in the mean residual life order (denoted by $X \le_{\mathrm{mrl}} Y$ or $F \le_{\mathrm{mrl}} G$) if $E[X-t|X>t] \le E[Y-t|Y>t]$ for all t's for which these conditional expectations are defined. Recall also that a distribution function F is said to be IMRL (increasing mean residual life) if the mean residual life at time t, defined as earlier, or, alternatively, as $(\int_t^\infty \bar{F}(u)\,du)/\bar{F}(t)$, is increasing in t, for which the ratio is well defined.

THEOREM 4.9: *Let F and G be distribution functions associated with two nonhomogeneous Poisson processes as described earlier, with corresponding hazard rate functions r and s, respectively. If $F \le_{\mathrm{mrl}} G$, if $\bar{F}$ and $\bar{G}$ are IMRL, and if (3.4) holds, then $X_{1,n} \le_{\mathrm{mrl}} X_{2,n}$ for each $n \ge 1$.*

PROOF: As in the proof of Theorem 4.4, we denote, here, F by F_1, G by F_2, r by r_1, s by r_2, and the cumulative hazard functions are denoted by $R_i, i = 1,2$. The stated result is obvious for $n=1$, so let us fix an $n \ge 2$. The survival function $\bar{G}_{i,n}$ of $X_{i,n}$, $i = 1,2$, is given in (4.3). From Theorem 1.D.3 in [20], it is seen that the stated result is equivalent to

$$\int_t^\infty \bar{G}_{i,n}(x)\,dx \text{ is } \mathrm{TP}_2 \text{ in } (i,t);$$

that is, to

$$\int_{s=0}^\infty r_i(s)\,\frac{R_i^{n-2}(s)}{(n-2)!}\int_{u=s+t}^\infty \bar{F}_i(u)\,du\,ds \text{ is } \mathrm{TP}_2 \text{ in } (i,t). \tag{4.8}$$

Now, from the proof of Theorem 4.4, we know that (3.4) implies that $r_i(s)[R_i^{n-2}(s)/(n-2)!]$ is TP_2 in (i,s). The assumption $F_1 \le_{\mathrm{mrl}} F_2$ means that

$$\int_{u=s+t}^\infty \bar{F}_i(u)\,du \text{ is } \mathrm{TP}_2 \text{ in } (i,s) \text{ and in } (i,t).$$

Finally, the assumption that F_i is IMRL means that

$$\int_{u=s+t}^\infty \bar{F}_i(u)\,du \text{ is } \mathrm{TP}_2 \text{ in } (s,t).$$

Thus, (4.8) follows from Theorem 5.1 in Karlin [10, p. 123]. ■

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4.2. Interepoch Intervals of Nonhomogeneous Pure Birth Processes

In this subsection, as in Section 3.2, we consider two nonhomogeneous pure birth processes, indexed by $i = 1,2$, parameterized by the sets $\{r_{i,n}, n \geq 1\}$ of hazard rate functions that satisfy (2.1). The corresponding epoch times are $0 \equiv T_{i,0} \leq T_{i,1} \leq T_{i,2} \leq \cdots$, and the interepoch intervals are $X_{i,n} = T_{i,n} - T_{i,n-1}, n \geq 1$, with $T_{i,0} \equiv 0$. Let $\{Y_{i,n}, n \geq 1\}$, $i = 1,2$, be two sets of independent absolutely continuous nonnegative random variables, where $Y_{i,n}$ has the hazard rate function $r_{i,n}$. Let $\hat{T}_{i,n}$ be as in (3.8) and (3.9), and define

$$\hat{X}_{i,1} = Y_{i,1}, \tag{4.9}$$

$$\hat{X}_{i,n} = [Y_{i,n} - \hat{T}_{i,n-1} | Y_{i,n} > \hat{T}_{i,n-1}], \quad n \geq 2. \tag{4.10}$$

Then, for $i = 1,2$, the joint distribution of the $X_{i,n}$'s is the same as the joint distribution of the $\hat{X}_{i,n}$'s. Let $\bar{K}_{i,n}$ denote the survival function of $Y_{i,n}$; that is, $\bar{K}_{i,n}(t) = \exp[-\int_0^t r_{i,n}(u)\,du]$. Let $k_{i,n}$ denote the density function of $Y_{i,n}$; that is, $k_{i,n}(t) = r_{i,n}(t)\exp[-\int_0^t r_{i,n}(u)\,du]$, $t \geq 0$. Using (4.9) and (4.10) and extending (4.4), it is easy to see that, for $i = 1,2$, the density function $l_{i,n}$ of $(X_{i,1}, X_{i,2}, \ldots, X_{i,n})$ is given by

$$l_{i,n}(x_1,\ldots,x_n) = \prod_{j=1}^{n-1} r_{i,j}\left(\sum_{l=1}^{j} x_l\right) \frac{\bar{K}_{i,j}\left(\sum_{l=1}^{j} x_l\right)}{\bar{K}_{i,j+1}\left(\sum_{l=1}^{j} x_l\right)} k_{i,n}\left(\sum_{l=1}^{n} x_l\right),$$

$$x_k \geq 0,\ k = 1,\ldots,n. \tag{4.11}$$

In this subsection, we obtain stochastic comparison results, involving interepoch intervals, in the sense of the usual and the multivariate likelihood ratio orders.

The first result gives conditions under which the epoch times of the two nonhomogeneous birth processes are ordered according to the usual stochastic order. This result may be compared with Theorem 4.2.

THEOREM 4.10: *Let $X_{i,n}$ be the interepoch intervals of two nonhomogeneous pure birth processes as described earlier. If $Y_{1,1} \leq_{\text{st}} Y_{2,1}$ and if*

$$r_{1,j}(u) \geq r_{2,j}(u + x), \quad u \geq 0, x \geq 0, j \geq 2, \tag{4.12}$$

then

$$(X_{1,1}, X_{1,2}, \ldots, X_{1,n}) \leq_{\text{st}} (X_{2,1}, X_{2,2}, \ldots, X_{2,n}), \quad n \geq 1. \tag{4.13}$$

PROOF: The result is obvious when $n = 1$. So fix an $n \geq 2$. Let $x'_k \geq x_k \geq 0$, $k = 1,2,\ldots,n$. Now, for $j = 2,\ldots,n$, we have

$$[X_{1,j} | X_{1,1} = x_1, \ldots, X_{1,j-1} = x_{j-1}] = \left[Y_{1,j} - \sum_{k=1}^{j-1} x_k \,\middle|\, Y_{1,j} > \sum_{k=1}^{j-1} x_k\right],$$

$$[X_{2,j} | X_{2,1} = x'_1, \ldots, X_{2,j-1} = x'_{j-1}] = \left[Y_{2,j} - \sum_{k=1}^{j-1} x'_k \,\middle|\, Y_{2,j} > \sum_{k=1}^{j-1} x'_k\right].$$

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Denote $z = \sum_{k=1}^{j-1} x_k$ and $z' = \sum_{k=1}^{j-1} x'_k$. It is not hard to see that (4.12) implies that

$$[Y_{1,j} - z \mid Y_{1,j} > z] \leq_{\text{st}} [Y_{2,j} - z' \mid Y_{2,j} > z'] \tag{4.14}$$

whenever $z' \geq z \geq 0, j \geq 2$ (in fact, (4.12) and (4.14) are equivalent). Thus, the stated result follows from Theorem 4.B.3 in [20]. ■

An alternative proof of Theorem 4.10 can be provided using Theorem 2.7 in [21]; however, the present proof is simpler.

As a corollary of Theorem 4.10 (see a comment following Theorem 4.2), we see that if $Y_{1,1} \leq_{\text{st}} Y_{2,1}$, if $Y_{1,j} \leq_{\text{hr}} Y_{2,j}, j \geq 2$, and if $Y_{1,j}$ or $Y_{2,j}$ are DFR, $j \geq 2$, then (4.13) holds.

The next result gives conditions under which the interepoch intervals are ordered in the multivariate likelihood ratio order.

THEOREM 4.11: *Let $X_{i,n}$ be the interepoch intervals of two nonhomogeneous pure birth processes as described earlier. If $Y_{1,j} \leq_{\text{hr}} Y_{2,j}$, if $r_{2,j}/r_{1,j}$ is increasing, if (3.11) holds, and if $r_{1,j}$ $\overline{K}_{1,j}/\overline{K}_{1,j+1}$,* and *$k_{1,j}$, or $r_{2,j}$, $\overline{K}_{2,j}/\overline{K}_{2,j+1}$, and $k_{2,j}$ are logconvex for all $j \geq 1$, then*

$$(X_{1,1}, X_{1,2}, \ldots, X_{1,n}) \leq_{\text{lr}} (X_{2,1}, X_{2,2}, \ldots, X_{2,n}), \quad n \geq 1.$$

The proof of Theorem 4.11 is a straightforward extension of the proof of Theorem 4.5, using (4.11) rather than (4.4) and (4.5); we omit the details.

It is worth mentioning that $\overline{K}_{i,j}/\overline{K}_{i,j+1}$ is logconvex if and only if $r_{i,j+1} - r_{i,j}$ is increasing.

5. SOME APPLICATIONS

In this section, we describe some applications of the results of Sections 3 and 4. The list of applications that we provide is far from exhaustive and is given only as an indication of the applicability of the mathematical results. In fact, the results of Sections 3 and 4 provide useful computable bounds in almost any area where nonhomogeneous pure birth processes are used.

5.1. Comparisons of Generalized Yule Birth Processes

A Yule (or a linear) birth process is a pure birth process with jump intensity from state n to state $n+1$ of the form $(n+1)\lambda$, where $\lambda > 0$. Let us consider a generalization $N_1 = \{N_1(t), t \geq 0\}$ of the Yule process in which the jump intensity at time t, given that n jumps have occurred already, is of the form $(n+1)\lambda(t)$ (depending on t). Let $N_2 = \{N_2(t), t \geq 0\}$ be another such nonhomogeneous Yule process with jump intensity $(n+1)\eta(t)$.

The generalized Yule process is a nonhomogeneous pure birth process. Thus, if the distribution functions F and G, which are associated with the failure rate functions λ and η of N_1 and N_2 described earlier, satisfy $F \leq_{\text{hr}} G$ (i.e., $\lambda(u) \geq \eta(u)$ for all

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$u \geq 0$), then, by Theorem 3.11 or 3.12, it is seen that by time t, there are stochastically at least as many jumps in N_1 as there are in N_2. This is an intuitively clear result that can also be proven directly. However, if, in addition to $\lambda(u) \geq \eta(u)$ (note that then (3.11) holds too), we also have that η/λ is increasing, then by Theorem 3.13, we obtain the nontrivial fact that the vectors of the first n jumps are ordered in the multivariate likelihood ratio order and sharper inequalities hold (see, e.g., Eq. (5.1)).

If λ and η satisfy (4.12) [i.e., $\lambda(u) \geq \eta(u+x)$ for all $x \geq 0$ and $u \geq 0$], then by Theorem 4.10, all the times between births in N_1 are stochastically smaller than the corresponding times between births in N_2.

A generalized Yule process may model the spread of a disease, where n is the number of infectives and $\lambda(t)$ is the rate in which infectives pass the disease to new individuals at time t; this rate, in general, depends on the calendar time t—for example, it may change with the seasons of the year [2]. Consider now two nonhomogeneous Yule processes, with rates $\lambda(t)$ and $\eta(t)$, which model the spread of a disease under two different health measures that are expected to control the spread. The stochastic inequalities described earlier can direct a health official as to how to fight the spread of a disease if the official can select between the two measures that control the spread with respective rates $\lambda(t)$ and $\eta(t)$.

A comparison of a nonhomogeneous Yule process N_1 (with intensities $r_{1,n}(t) = (n+1)\lambda(t)$) with a standard Yule process N_2 (with intensities $r_{2,n}(t) = (n+1)\eta$, independent of t) can provide computable upper or lower bounds on various probabilistic quantities of interest that are associated with N_1. This is based on the fact that the interepoch intervals $X_{2,1}, X_{2,2}, \ldots, X_{2,n}, \ldots$ of the standard Yule process N_2 are independent exponential random variables with rates $\eta, 2\eta, \ldots, n\eta, \ldots$, and the epoch times $T_{2,n}$ are sums of these independent $X_{2,i}$'s. For example, suppose that we have under study a nonhomogeneous Yule process N_1 as above and suppose that $\lambda(t)$ is bounded from below by some constant η (i.e., $\lambda(t) \geq \eta$ for all $t \geq 0$). Define N_2 as the standard Yule process with the associated rate η. Then by Theorem 3.11 or 3.12 and Theorem 4.10, we get $(T_{1,1}, \ldots, T_{1,n}) \leq_{\rm st} (T_{2,1}, \ldots, T_{2,n})$ and $(X_{1,1}, \ldots, X_{1,n}) \leq_{\rm st} (X_{2,1}, \ldots, X_{2,n})$, and, therefore, $E\phi(T_{1,1}, \ldots, T_{1,n}) \leq E\phi(T_{2,1}, \ldots, T_{2,n})$ and $E\phi(X_{1,1}, \ldots, X_{1,n}) \leq E\phi(X_{2,1}, \ldots, X_{2,n})$ for any increasing function ϕ for which the expectations exist. For example, since $ET_{2,n} = \eta^{-1} \sum_{i=1}^{n} i^{-1}$, we can bound $ET_{1,n}$ from above by

$$ET_{1,n} \leq \eta^{-1} \sum_{i=1}^{n} i^{-1}.$$

Another useful example of a bound of this type is based on Theorem 4.1 of Bunge and Nagaraja [6]. The authors give an explicit expression for the expected value of the waiting time until the nth record occurs, when the arrival process is a standard Yule process. If in a particular application the arrival process is a nonhomogeneous (rather than standard) Yule process and its associated rate $\lambda(t)$ is bounded from below, then the explicit expression in Theorem 4.1 of Bunge and Nagaraja [6] provides an upper bound for the expected value of the waiting time until the nth record occurs in that particular application. Another reference in which one can find explicit expressions of probabilistic quantities of interest that are associated with a standard Yule process is the article by Brown et al. [5]. When the rate $\lambda(t)$ that is associated with the nonhomogeneous Yule process is bounded from below or from above, then these explicit expressions can be used as bounds for the corresponding probabilistic quantities that are associated with the nonhomogeneous Yule process. In fact, Brown et al. [5] study a Yule process with immigration (i.e., the intensity is of the form $(n+1)\eta+\theta$); this process can be used to obtain bounds for a nonhomogeneous pure birth process with intensities of the form $r_n(t) = (n+1)\lambda(t) + \mu(t)$.

Consider again the nonhomogeneous Yule process N_1 with intensities $r_{1,n}(t) = (n+1)\lambda(t)$ and the standard Yule process N_2 with intensities $r_{2,n}(t) = (n+1)\eta$ that we described earlier. If, in addition to $\lambda(t) \ge \eta$, we also have that $\lambda(t)$ is decreasing, then $\eta/\lambda(t)$ is increasing, and by Theorem 3.13, we have $(T_{1,1},\ldots,T_{1,n}) \le_{\mathrm{lr}} (T_{2,1},\ldots,T_{2,n})$. Then, for example, we have

$$\begin{aligned} &E[\phi(T_{1,1},\ldots,T_{1,n}) \mid t_j^0 \le T_{1,j} \le t_j^1, j=1,\ldots,n] \\ &\quad \le E[\phi(T_{2,1},\ldots,T_{2,n}) \mid t_j^0 \le T_{2,j} \le t_j^1, j=1,\ldots,n] \end{aligned} \tag{5.1}$$

for all increasing functions ϕ, whenever $t_j^0 < t_j^1, j = 1,\ldots,n$ (see Thm. 4.E.1 in [20]). Such an inequality does not follow, in general, from the weaker condition $(T_{1,1},\ldots,T_{,1n}) \le_{\mathrm{st}} (T_{2,1},\ldots,T_{2,n})$. In the next paragraph, we describe a practical application of (5.1).

When $\lambda(t) \ge \eta$ and $\lambda(t)$ is decreasing, all the conditions of Theorem 4.11 hold. In order to see it, we first note that (3.11) obviously holds because $\lambda(t) \ge \eta$. If $k_{2,j}$ denotes the exponential density with rate $j\eta$, then it is easy to verify that $r_{2,j}$, $\bar{K}_{2,j}/\bar{K}_{2,j+1}$, and $k_{2,j}$ are all logconvex. Thus, from Theorem 4.11, we obtain $(X_{1,1},\ldots,X_{1,n}) \le_{\mathrm{lr}} (X_{2,1},\ldots,X_{2,n})$, where $X_{2,1},\ldots,X_{2,n}$ are independent exponential random variables, as described earlier. This stochastic inequality is useful in situations where benefits are derived during any interepoch time interval (such benefits can be, for instance, the rates in which a working component yields revenue). For example, suppose that the benefit from a realization $(x_1,\ldots,x_n)$ of $(X_{1,1},\ldots,X_{1,n})$ is $\phi(x_1,\ldots,x_n)$, but that the benefits are derived only during an initial period of length t_0 in any interepoch interval. Then, the expected benefit from the first n interepoch intervals of the nonhomogeneous Yule process is $E[\phi(X_{1,1},\ldots,X_{1,n}) \mid X_{1,i} \le t_0, i=1,\ldots,n]$, provided the expectation exists. When ϕ is increasing, this expectation is bounded from above by $E[\phi(X_{2,1},\ldots,X_{2,n}) \mid X_{2,i} \le t_0, i=1,\ldots,n]$; this follows from $(X_{1,1},\ldots,X_{1,n}) \le_{\mathrm{lr}} (X_{2,1},\ldots,X_{2,n})$ and from Theorem 4.E.1 in [20]. The latter expectation is not hard to compute because $X_{2,1},\ldots,X_{2,n}$ are independent exponential random variables.

5.2. Comparisons of Load-Sharing Models

Consider n items that share a load $L_1(t)$ at time t. A common model (see Schechner [17]) is to assume that the failure rate of each item is then $L_1(t)/n$. After i items have

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already failed, each of the remaining $n - i$ items has a load $L_1(t)/(n - i)$. If we denote $r_{1,i}(t) = L_1(t)/(n - i + 1)$, then it is seen that the failure times $T_{1,1} \le T_{1,2} \le \cdots \le T_{1,n}$ are the epoch times of a nonhomogeneous pure birth process. Let L_2 be a second load function shared by n similar items.

If $L_1(u) \ge L_2(u)$ for all $u \ge 0$, then, by Theorem 3.11 or 3.12, it is seen that by time t, there are stochastically at least as many failures in the first model as there are in the second. This is an intuitively clear result that can also be proven directly. However, if, in addition to $L_1(u) \ge L_2(u)$ (note that then (3.11) holds too), we also have that L_2/L_1 is increasing, then, by Theorem 3.13, we obtain the nontrivial fact that the vectors of the n failure times are ordered in the multivariate likelihood ratio order, and sharper inequalities hold.

If $L_1(u) \ge L_2(u + x)$ for all $x \ge 0$ and $u \ge 0$, then, by Theorem 4.10, all the times between failures in the first model are stochastically smaller than the corresponding times between failures in the second model.

If the load L_2 is constant then some probabilistic quantities of interest can be computed explicitly. Thus, when $L_1(t)$ is bounded from below or from above, we can use the load-sharing model associated with a constant L_2 in order to bound some probabilistic quantities of interest involving the model associated with $L_1(t)$. For example, Eq. (4.9) of [15] gives an explicit expression for the mean lifetime of a single member in a load-sharing model with a constant L_2. If $L_1(u) \ge L_2$ for all $u \ge 0$ then, using Theorems 3.11, 3.12, and 4.10, we see that (4.9) of [15] provides an upper bound on the corresponding expectation in the model associated with L_1.

A load-sharing model with a constant L often describes the strength of a bundle of fibers. If the load on the bundle varies with time (e.g., the load may be different during the day than during the night), then the general model, in which L depends on t, applies.

5.3. Comparisons of Benefits Between Times of Minimal Repair

The repair times of an item that is continuously minimally repaired are the epoch times of a nonhomogeneous Poisson process whose intensity function is the hazard rate function of the lifetime distribution of the item, see, for example, Shaked and Szekli [21].

Suppose that an engineer has to decide which of two items that are continuously minimally repaired is to be used. The selected item (which may be, for example, a computer, a car, or an airplane) can then be used until its next failure. If we denote by X a generic intrafailure interval, then it can be assumed that the benefit derived from the item is an increasing function $\phi(X)$ of the length of the interval (see, e.g., a discussion in Shaked and Szekli [21, p. 1093]). If the choice of the engineer is between the ith interval of either of the two items, then Theorems 4.1–4.8 can direct the engineer in his/her choice. For example, if the item is going to be used for the whole duration of the intrafailure interval, then under the conditions of Theorem 4.1 or 4.2, we have $E[\phi(X_{1,i})] \le E[\phi(X_{2,i})]$, and thus the second item is preferable. If the item can be used only after some fixed *burn-in time* x_0, then under the conditions

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of Theorem 4.4, we have $E[\phi(X_{1,i} - x_0)|X_{1,i} > x_0] \le E[\phi(X_{2,i} - x_0)|X_{2,i} > x_0]$, for any fixed x_0, and thus, again, the second item is preferable (by (1.B.5) in [20]). Finally, if the item is going to be used only for a fixed subinterval, $[x_0, x_1]$ say, of the intrafailure interval, then under the conditions of Theorem 4.5 or 4.8, we have $E[\phi(X_{1,i} - x_0)|x_0 < X_{1,i} \le x_1] \le E[\phi(X_{2,i} - x_0)|x_0 < X_{2,i} \le x_1]$, for any fixed $x_0 < x_1$, and thus the second item is preferable (by (1.C.4) in [20]).

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