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DIFFERENTIAL INCLUSIONS WITH STATE CONSTRAINTS

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1. Introduction

In "Viability Theory", we select trajectories which are viable in the sense that they always satisfy a given constraint. Since the fundamental work of Nagumo [**26**], we know that in order to guarantee existence of viable trajectories, we need to satisfy certain tangential conditions. In the case of differential inclusions and using the modern terminology and notation of tangent cones, this condition takes the form $F(t,x)\cap T_K(x)\neq\emptyset$, where $F(.,.)$ is the orientor field involved in the differential inclusion, K is the viability (constraint) set and $T_K(x)$ is the tangent cone to K at x. Results on the existence of viable solutions for differential inclusions can be found in Aubin–Cellina [**2**] and Papageorgiou [**30, 32**].

Now suppose that the above sufficient tangential condition is no longer satisfied, but we still want a dynamical system to provide "viable trajectories" and be as close as possible to the original one. The natural choice in this case is to replace the original orientor field $F(.,.)$ by its projection onto the tangent cone $T_K(x)$. This way we pass to the so called "projected differential inclusion". It is not difficult to show (see Aubin–Cellina [**2**, Proposition 2, p. 266]), that the "projected differential inclusion" $\dot{x}(t)\in \operatorname{proj}(F(x(t)), T_K(x(t)))$, is in fact equivalent to the differential inclusion $\dot{x}(t)\in F(t,x(t))-N_K(x(t))$, which following Aubin–Cellina [**2**], is called "differential variational inequality".

Differential variational inequalities, appear naturally in several areas of applied mathematics, like in mechanics in the study of elastoplastic systems (see Moreau [**25**]), in economics in the study of planning procedures (see Henry [**19**]) and in control theory in the study of feedback systems (see Aubin–Cellina [**21**]). We should also mention the important recent works of Aubin [**1**], Cellina–Marchi [**7**], Cornet [**10**] and Gamal [**17**].

In this work we examine, mainly in $\mathbb{R}^n$, differential variational inequalities that arise in the above mentioned areas and obtain existence theorems for both convex and nonconvex valued orientor fields.

We also examine the case, where the underlying state space in an infinite dimensional Hilbert space. Then we present a convergence result and study the dependence of the

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trajectories on the initial data. Finally we have an existence result for a random version of the original problem. Parts of our work extend the results of Aubin [1], Aubin–Cellina [2], Castaing [6], Cellina–Marchi [7], Cornet [10] and Henry [19], who imposed more restrictive hypotheses on the data of the problem and considered only autonomous systems. Also our work can be viewed as a perturbed version of Moreau [25]. Gamal in [17] examined the more general infinite dimensional problem, where instead of $N_{K(t)}(x(t))$ we have $\partial f(t, x(t))$, the subdifferential of a normal, convex integrand. So he was forced to introduce several extra hypotheses, that make his results noncomparable to ours.

2. Definitions and notation

Let (Ω, Σ, μ) be a σ-finite measure space and X a finite dimensional Banach space. We will be using the following notation:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{nonempty, closed, (convex)}\}$$

$$P_{k(c)}(X) = \{A \subseteq X\colon \text{nonempty, compact, (convex)}\}.$$

A multifunction $F\colon \Omega \to P_f(X)$ is said to be a measurable, if for every $x \in X, \omega \to d(x, F(\omega)) = \inf\{\|x - z\|\colon z \in F(\omega)\}$ is measurable. This definition is equivalent to saying that there exist $f_n\colon \Omega \to X$ measurable functions such that $F(\omega) = cl\{f_n(\omega)\}_{n \geq 1}$. Furthermore if Σ is μ-complete, then the above definitions are equivalent to saying that $GrF = \{(\omega, x) \in \Omega \times X\colon x \in F(\omega)\} \in \Sigma \times B(X)$, $B(X)$ being the Borel σ-field of X (graph measurability). More on measurable multifunctions can be found in Himmelberg [20] and Wagner [38].

We will say that $F(.)$ is integrably bounded, if it is measurable and $\omega \to |F(\omega)| = \sup\{\|z\|\colon z \in F(\omega)\}$ is an L^1_+-function.

Let $S^p_F = \{f \in L^p(X)\colon f(\omega) \in F(\omega)\ \mu-\text{a.e.}\}$, $1 \leqq p \leqq \infty$. Having S^1_F we can now define a set valued integral for $F(.)$, by setting $\{\int_\Omega F(\omega)\, d\mu(\omega)\colon f \in S^1_F\}$. Clearly $S^1_F \neq \emptyset \Rightarrow \int_\Omega F \neq \emptyset$. Furthermore if $F(.)$ is integrably bounded, then $S^1_F \neq \emptyset$.

If $A \in 2^X \backslash \{\emptyset\}$, by $\sigma(x^*, A)$ $x^* \in X^*$, we will denote the support function of A i.e. $\sigma(x^*, A) = \sup\{(x^*, z)\colon z \in A\}$. Let Y, Z be Hausdorff topological spaces and let $F\colon Y \to 2^Z \backslash \{\emptyset\}$. We say that $F(.)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)), if for all U open in Z, $F^+(U) = \{y \in Y\colon F(y) \subseteq U\}$ (resp. $F^-(U) = \{y \in Y\colon F(y) \cap U \neq \emptyset\}$) is open Y. More on the continuity of multifunctions and their use in the theory of orientor fields can be found in Cesari [8]. If Z is a metric space, on $P_f(Z)$ we can define a generalized metric $h(.,.)$ by setting:

$$h(A, B) = \max\left[\sup(d(a, B), a \in A), \sup(d(b, A), b \in B)\right].$$

This metric is known as the Hausdorff metric. A multifunction $F: Y \to P_f(Z)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from the topological space Y into the metric space $(P_f(Z), h)$. Let $Y = T = [0, b]$. Then a multifunction $F: T \to P_f(Z)$ is said to be absolutely continuous, if it is such from the metric space T into the metric space $(P_f(Z), h)$. In this case there exists $r(.) \in L_+^1$ s.t.

$$|d(x, F(t)) - d(z, F(t')| \leqq \|x - z\| + \int_{t'}^{t} r(s)\, ds$$

(see Moreau [25]). The function $r(.)$ is sometimes called the modulus of absolute continuity of $F(.)$.

Now we will introduce a mode of set convergence, which is in general different from the convergence in the Hausdorff metric (or pseudometric). So let Z be a metric space and let $\{A_n\}_{n \geqq 1} \subseteq 2^Z \setminus \{\emptyset\}$. We set:

$$\varliminf_{n \to \infty} A_n = \{x \in Z: \varlimsup d(x, A_n) = 0\}$$

$$\varlimsup_{n \to \infty} A_n = \{x \in Z: \varliminf d(x, A_n) = 0\}.$$

Both sets are always closed and may be empty. We will say that the A_n's converge to A in the Kuratowski sense, denoted by $A_n \xrightarrow{K} A$, if and only if $\varlimsup A_n = A = \varliminf A_n$. For more details about this mode of convergence and its relation to the convergence in the Hausdorff metric, we refer to Salinetti–Wets [36].

In connection with the above mode of set convergence, the following equicontinuity concept is useful. If Y, Z are metric spaces, a family $\{F_i\}_{i \in I}$ of multi-functions from Y into $P_f(Z)$ is said to be equi-h^*-u.s.c. at $x \in Y$, if for every $\varepsilon > 0$, there exists $\delta(\varepsilon, x) > 0$ s.t. for $y \in Y$ for which $d_Y(y, x) < \delta \Rightarrow h^*(F_i(y), F_i(x)) = \sup\{d(z, F_i(x)): z \in F_i(y)\} < \varepsilon$, for all $i \in I$. We will say that the family $\{F_i(.)\}_{i \in I}$ is equi-h^*-u.s.c., if it is equi-h^*-u.s.c. at every $x \in Y$.

Finally let K be a nonempty subset of a Banach space Y and let $y \in \bar{K}$. We define the (Bouligand) tangent cone to K at y, to be the set

$$T_K(y) = \left\{ v \in Y: \varliminf_{h \to 0^+} \frac{1}{h} \inf_{k \in K} \|y + hv - k\| = 0 \right\}.$$

This is a closed but not necessarily convex cone. If K is convex, then $T_K(y)$ is convex and coincides with the tangent cone introduced by Clarke [9] (in fact we need only have that K is locally convex at y in order for $T_K(y)$ to be convex and equal to the Clarke tangent cone). In this case, the negative polar cone of $T_K(y)$, is called the "normal cone to K at y" and is denoted by $N_K(y)$ i.e. $N_K(y) = \{y^* \in Y^*: (y^*, p) \leqq 0$, for all $p \in T_K(y)\}$. It is not difficult to check that $N_K(y) = \{y^* \in Y^*: (y^*, y) = \sigma(y^*, K)\} = \partial \delta_K(y)$, where $\delta_K(.)$ is the indicator function of K and ∂ denotes the subdifferential in the sense of convex analysis (see Rockafellar [35]).

3. Existence theorems

Let $T=[0,b]$ be a closed, bounded interval in $\mathbb{R}_+$ and let $X=\mathbb{R}^n$. We will study the following differential inclusion:

$$\left\{\begin{array}{l} -\dot{x}(t)\in N_{K(t)}(x(t))+F(t,x(t)) \text{ a.e.} \\ \quad x(0)=x_0 \end{array}\right\}. \tag{$*$}$$

By a solution of $(*)$, we understand an absolutely continuous function $x\colon T\to R^n$ s.t. $x(0)=x_0$ and for which there exists $f\in S^1_{F(.,x(.))}$ s.t. $-\dot{x}(t)\in N_{K(t)}(x(t))+f(t)$ a.e.

We will start with an existence result in which the set valued perturbation $F(.,.)$ is assumed to be nonconvex valued. Cellina–Marchi [7] and Gamal [17] also examined the case of nonconvex perturbations and assumed that the perturbation is h-continuous in both variables t and x. Here we go even further in this direction and assume that $F(t,.)$ is only l.s.c. Lower semicontinuous fields arise often in applications and in particular in control theory. Namely if $f(t,.,u)$ $u\in U$, is continuous, then $F(t,x)=U\{f(t,x,u)\colon u\in U\}$ is l.s.c. in x (see Aubin–Cellina [2, p. 47]). Also the feedback control multifunctions $C(t,x)=\{u\in U(t,x)\colon f(t,x,u)\in T(x)\}$ are l.s.c. (see Aubin–Cellina [2, p. 49]). Moreover in connection with the "bang-bang" and "maximum" principles we are interested in the multifunction $\overline{\text{ext}}\,F(t,x)$. If $F(t,.)$ is h-continuous and compact valued, then $\overline{\text{ext}}\,F(t,.)$ is l.s.c. (see Papageorgiou [33]). Finally in the problem of regulation of control systems (i.e. finding controls that produce viable trajectories of $\dot{x}(t)=f(x(t),u(t))$, $x(0)=x_0$, $u(t)\in U, x(t)\in K$), we deal with the feedback multifunction $C(x)=\{u\in U\colon f(x,u)\in T_K(x)=\text{tangent cone to the viability domain } K \text{ at } x\}$, which is an example of a l.s.c. (see Aubin–Cellina [2, pp. 49 and 239]) orientor field (see Aubin–Cellina [2, pp. 49 and 239]). So it is important to have an existence result for nonconvex, l.s.c. perturbations. Our result extends those of Bressan [3], Kaczynski–Olech [21], Lojasiewicz [24] and Papageorgiou [27, 29], where no viability constraints were present i.e. $K(t)=X\Rightarrow N_{K(t)}(x)=\{0\}$ for all $x\in X$.

Theorem 3.1. *If $K\colon T\to P_{fc}(X)$ is an absolutely continuous multifunction with modulus $r(.)\in L^1_+$ and $F\colon T\times X\to P_f(X)$ is another multifunction such that*

(1) *$(t,x)\to F(t,x)$ is measurable,*

(2) *for every $t\in T$, $x\to F(t,x)$ is l.s.c.,*

(3) *$|F(t,x)|\leqq\phi_1(t)\|x\|+\phi_2(t)$ a.e., with $\phi_1(.)$, $\phi_2(.)\in L^1_+$,*

then $(*)$ admits a solution.

Proof. Let us start by obtaining an *a priori* estimate for the solutions of $(*)$. So let $x(.)$ be a solution. From Moreau [25] (see also Daures [11]), we know that:

$$\|\dot{x}(t)\|\leqq r(t)+\phi_1(t)\|x\|+\phi_2(t) \text{ a.e.}$$

$$\Rightarrow\|x(t)\|\leqq\|x_0\|+\int_0^t r(s)\,ds+\int_0^t\phi_2(s)\,ds+\int_0^t\phi_1(s)\|x(s)\|\,ds.$$

Invoking Gronwall's inequality we get that:

$$\|x(t)\| \leqq [\|x_0\| + \|r\|_1 + \|\phi_2\|_1] \exp \|\phi_1\|_1 = M.$$

Let $\phi(t) = \phi_1(t)M + \phi_2(t)$, $\phi(.) \in L^1_+$. Define $\hat{F}: T \times X \to P_f(X)$ by:

$$\hat{F}(t,x) = \begin{cases} F(t,x) & \text{if } \|x\| \leqq M \\ F\left(t, \dfrac{Mx}{\|x\|}\right) & \text{if } \|x\| > M. \end{cases}$$

Let $p_M: X \to \overline{B(0,M)}$ be the M-radial retraction. We know that $p_M(.)$ is continuous. Hence since $\hat{F}(t,x) = F(t, p_M(x))$, we deduce that $(t,x) \to \hat{F}(t,x)$ is measurable and $x \to \hat{F}(t,x)$ is l.s.c. Furthermore note that $|\hat{F}(t,x)| \leqq \phi(t)$ a.e.

We will consider the following evolution inclusion:

$$\left\{ \begin{array}{l} -\dot{x}(t) \in N_{K(t)}(x(t)) + \hat{F}(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \end{array} \right\}. \tag{$*$}'$$

We will obtain a solution for this problem and then we will show that this solution also solves the original problem $(*)$.

Let $W \subseteq C(T,X)$ be the following set:

$$W = \{y \in C(T,X): y(t) = x_0 + \int_0^t u(s)\,ds, \quad t \in T, \|u(t)\| \leqq r(t) + \phi(t)\} \text{ a.e.}$$

It is easy to check that W is closed and through a straightforward application of the Arzela–Ascoli theorem, we have that W is compact in $C(T,X)$. Next let $R: W \to P_f(L^1(X))$ be the multifunction defined by:

$$R(y) = S^1_{\hat{F}(.,y(.))}$$

We claim that $R(.)$ is l.s.c. Let $y_n \to y$ in W. Then from Theorem 4.1. of [31], we have that:

$$S^1_{\underline{\lim}\,\hat{F}(.,y_n(.))} \subseteq \underline{\lim}\, S^1_{\hat{F}(.,y_n(.))}.$$

Since $\hat{F}(t,.)$ is l.s.c., for all $t \in T$ we have:

$$\hat{F}(t, y(t)) \subseteq \underline{\lim}\, \hat{F}(t, y_n(t))$$

$$\Rightarrow S^1_{\hat{F}(.,y(.))} \subseteq \underline{\lim}\, S^1_{\hat{F}(.,y_n(.))}$$

$$\Rightarrow R(y) \subseteq \underline{\lim}\, R(y_n).$$

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So from Delahaye–Denel [14], we conclude that $R(.)$ is l.s.c. as claimed. Apply Fryszkowski's selection theorem [16], to find $k: W \to L^1(X)$ continuous s.t. for all $y \in W$, $k(y) \in R(y)$. Now for $y \in W$, consider the following evolution inclusion:

$$\left\{ \begin{array}{l} -\dot{x}(y)(t) \in N_{K(t)}(x(y)(t)) + k(y)(t) \text{ a.e.} \\ x(y)(0) = x_0 \end{array} \right\}. \qquad (*)''$$

From Daures [11] and Moreau [25], we know that $(*)''$ has a unique solution $x(y)(.) \in W$. We claim that the map $s: y \to x(y)(.)$ from W into itself is continuous. To prove this, since W is compact, it suffices to show that Grs is closed in $W \times W$. So let $(y_n, x_n) \to (y, x) \in W \times W$ in $C(T, X) \times C(T, X)$. From the Dunford–Pettis theorem (see Dunford–Schwartz [15, Theorem 9, p. 292]), we know that $\overline{\{\dot{x}_n\}}^w_{n \geqq 1}$ is sequentially w-compact in $L^1(X)$ and so we may assume that $\dot{x}_n \xrightarrow{w} v = \dot{x}$ in $L^1(X)$.

Let $z \in S_K^\infty$. Then we have:

$$(-\dot{x}_n(t) - k(y_n)(t), z(t) - x_n(t)) \leqq 0 \text{ a.e.}$$

$$\Rightarrow \int_A (-\dot{x}_n(s) - k(y_n)(s), z(s) - x_n(s))\, ds \leqq 0, \quad A \subseteq T \text{ measurable.}$$

Recalling that $k(y_n) \xrightarrow{s} k(y)$ in $L^1(X)$, by passing to the limit as $n \to \infty$ in the above inequality, we get that:

$$\int_A (-\dot{x}(s) - k(y)(s), z(s) - x(s))\, ds \leqq 0.$$

Since $A \subseteq T$ measurable was arbitrary, we deduce that:

$$(-\dot{x}(t) - k(y)(t), z(t) - x(t)) \leqq 0 \text{ a.e.}$$

From Lemma 1.1 of Hiai–Umegaki [18], we know that $K(.)$ admits a Castaing representation from elements in S_K^∞. So we conclude that:

$$(-\dot{x}(t) - k(y)(t), x(t)) = \sigma(-\dot{x}(t) - k(y)(t), K(t)) \text{ a.e.}$$

$$\Rightarrow -\dot{x}(t) \in N_{K(t)}(x(t)) + k(y)(t) \text{ a.e.}$$

$$\Rightarrow x = s(y) \text{ and so } s(.) \text{ is indeed continuous.}$$

Apply Schauder's fixed point theorem to find $y \in W$ s.t. $y = x(y)$. Clearly this solves $(*)'$. Then using the definition of $\hat{F}(.,.)$ and Gronwall's inequality, we have:

$$-\dot{y}(t) \in N_{K(t)}(y(t)) + \hat{F}(t, y(t)) \text{ a.e.}$$

$$y(0)=x_0$$

$$\Rightarrow \|y(t)\| \leqq \|x_0\| + \int_0^t r(s)\,ds + \int_0^t \phi_2(s)\,ds + \int_0^t \phi_1(s)\|x(s)\|\,ds$$

$$\Rightarrow \|y(t)\| \leqq M$$

$$\Rightarrow \hat{F}(t,y(t)) = F(t,y(t)) \text{ and so } y(.) \text{ solves } (*). \qquad \square$$

Next we will consider the existence of a solution of $(*)$, when the set valued perturbation is convex valued. Our result extends earlier ones by Aubin–Cellina **[2]**, Cornet **[10]**, Daures **[11]** and Moreau **[25]**.

Theorem 3.2. *If $K: T \to P_{fc}(X)$ is an absolutely continuous multifunction with modulus $r(.) \in L^1_+$ and $F: T \times X \to P_{fc}(X)$ is a multifunction s.t.*

(1) *$t \to F(t,x)$ is measurable*

(2) *$x \to F(t,x)$ is u.s.c.*

(3) $|F(t,x)| \leqq \phi_1(t)\|x\| + \phi_2(t)$ a.e. *with* $\phi_1(.), \phi_2(.) \in L^1_+$

then $(*)$ admits a solution.

Proof. As in the proof of Theorem 3.1, we can get an *a priori* estimate for the solutions of $(*)$, namely that $\|x(t)\| \leqq [\|x_0\| + \|r\|_1 + \|\phi_2\|_1] \exp \|\phi_1\|_1 = M$. As before, we define the modified orientor field $\hat{F}: T \times X \to P_{fc}(X)$. Again $\hat{F}(.,.)$ has the same measurability and continuity properties as F and $|\hat{F}(t,x)| \leqq \phi(t) = \phi_1(t)M + \phi_2(t)$ a.e. Once more we consider the modified evolution equation $(*)'$. From Lemma 3 of DeBlasi–Myjak **[13]**, we know that we can find $G_n: T \times X \to P_{kc}(X)$ s.t. $\ldots \subseteq G_{n+1}(t,x) \subseteq G_n(t,x) \subseteq \ldots$ and $h(G_n(t,x), \tilde{F}(t,x)) \to 0$ as $n \to \infty$ where $\tilde{F}: T \times X \to P_{kc}(X)$ is the multifunction postulated by the lemma (namely $\tilde{F}(t,x) \subseteq \hat{F}(t,x)$ and if $u: T \to X$ and $v: T \to X$ are measurable and $v(t) \in \hat{F}(t,u(t))$ a.e., then $v(t) \in \tilde{F}(t,u(t))$ a.e.). Also from Remark 4.1. of DeBlasi **[12]**, we know that we can have $G_n(t,.)$ to be locally h-Lipschitz, while $|G_n(t,x)| \leqq \phi(t)+1$ a.e.

Now consider the following approximating problems for $n \geqq 1$:

$$-\dot{x}(t) \in N(x(t)) + G_n(t,x(t)) \quad x(0)=x_0$$

From Theorem 3.1 we know that for every $n \geqq 1$, the above problem has a solution. Let $\{x_n\}_{n \geqq 1}$ be this sequence of solutions. These functions live in

$$W' = \{y \in C(T,X): y(t) = x_0 + \int_0^t \dot{y}(s)\,ds, \|\dot{y}(t)\| \leqq r(t) + \phi(t) + 2 \text{ a.e.}\}$$

and this set by the Arzela–Ascoli theorem is compact in $C(T,X)$. So by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C(T, X)$. Then exploiting the local Lipschitzness of $G_n(t,.)$ and the fact that $h(G_n(t,x), \tilde{F}(t,x)) \to 0$ as $n \to \infty$, we get $G_n(t, x_n(t)) \xrightarrow{h} \tilde{F}(t, x(t))$ a.e.

Since for all $n \geqq 1$, $\|\dot{x}_n(t)\| \leqq r(t) + \phi(t) + 2$ a.e., through a simple application of the Dunford–Pettis theorem, we can see that $\{\dot{x}_n(.)\}_{n \geq 1}$ is sequentially w-compact in $L^1(X)$. So by passing to a subsequence if necessary, we may assume that $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^1(X)$. Let $g_n \in S^1_{G_n(.,x_n(.))}$ s.t.

$$-\dot{x}_n(t) \in N_{K(t)}(x_n(t)) + g_n(t) \text{ a.e.}$$

Again, thanks to the Dunford–Pettis theorem, we may assume that $g_n \xrightarrow{w} f$ in $L^1(X)$. Invoking Mazur's lemma, we can find $z_k \in \operatorname{conv} \bigcup_{n \geqq k} g_n$ s.t. $z_k \xrightarrow{s} f$ in $L^1(X)$. By passing to a subsequence if necessary, we may assume that $z_k(t) \to f(t)$ a.e. Since $h(G_n(t, x_n(t)), \tilde{F}(t, x(t))) \to 0$ as $n \to \infty$, a.e., for $t \in T \backslash N'$, $\lambda(N') = 0$ and $\varepsilon > 0$, we can find $n_0(\varepsilon, t) > 0$ s.t. for $n \geqq n_0$, we have:

$$G_n(t, x_n(t)) \subseteq \tilde{F}(t, x(t)) + \varepsilon B_1 \ (B_1 = \text{unit ball in } X).$$

$$\Rightarrow f(t) \in \overline{\operatorname{conv}} \bigcup_{n \geqq n_0} g_n(t) \subseteq F(t, x(t)) + \varepsilon B_1 \text{ a.e.}$$

Let $\varepsilon \downarrow 0$. We get $f(t) \in \tilde{F}(t, x(t))$ a.e. $\Rightarrow f(t) \in \hat{F}(t, x(t))$ a.e. $\Rightarrow f \in S^1_{F(.,x(.))}$. Let $\hat{x}(.)$ be the unique solution of

$$-\dot{z}(t) \in N_{K(t)}(z(t)) + f(t) \text{ a.e.}$$

$$z(0) = x_0$$

(see Daures **[11]**). We need to show that $x = \hat{x}$. From the monotonicity of the subdifferential operator, we have:

$$(\dot{x}_n(t) - \dot{\hat{x}}(t), x_n(t) - \hat{x}(t)) \leqq (g_n(t) - f(t), x_n(t) - \hat{x}(t)) \text{ a.e.}$$

Integrating both sides, we get that:

$$\tfrac{1}{2}\|x_n(t) - \hat{x}(t)\|^2 \leqq \int_0^t (g_n(s) - f(s), x_n(s) - \hat{x}(s))\, ds$$

$$= \int_0^t (g_n(s) - f(s), x_n(s) - x(s))\, ds + \int_0^t (g_n(s) - f(s), x(s) - \hat{x}(s))\, ds \to 0$$

(since $g_n \xrightarrow{w} f$ in $L^1(X)$ and $x_n \to x$ in $C(T, X)$).

$$\Rightarrow x_n \to \hat{x} \quad \text{in} \quad C(T, X)$$

$$\Rightarrow x = \hat{x}$$

$$\Rightarrow x(.) \text{ is the desired solution of } (*). \qquad \square$$

Finally we have an existence result for the case where the perturbation is convex but not necessarily closed valued. So we have:

Theorem 3.3 *If $K: T \to P_{fc}(X)$ is absolutely continuous with modulus $r(.) \in L^1_+$ and $F: T \times X \to 2^X \setminus \{\emptyset\}$ is a multifunction s.t.*

(1) *$(t, x) \to F(t, x)$ is graph measurable*

(2) *for all $(t, x) \in T \times X$, $F(t, x)$ is convex with int $F(t, x) \neq \emptyset$*

(3) *for every $t \in T$, $x \to \overline{F(t, x)}$ is h-continuous. then $(*)$ admits a solution.*

Proof. As in the proof of Theorem 5 of [**34**], we can find $f: T \times X \to X$ Caratheodory function (i.e. measurable in t, continuous in x) s.t. for every $(t, x) \in T \times X$, $f(t, x) \in F(t, x)$. Then consider the following evolution inclusion:

$$\left\{\begin{matrix} -\dot{x}(t) \in N_{K(t)}(x(t)) + f(t, x(t)) \\ x(0) = x_0 \end{matrix}\right\} \qquad (\hat{*})$$

From Theorem 3.2, we know that $(\hat{*})$ has a solution $x(.)$. Clearly $x(.)$ also solves $(*)$. $\square$

4. Hilbert space case

In this section we see what are the necessary modifications in the hypotheses, in order to accommodate the case where the state space is an infinite dimensional separable Hilbert space.

For that purpose we consider the evolution inclusion $(*)$, with a convex valued perturbation $F(t, x)$.

So assume that X is a separable Hilbert space.

Theorem 4.1. *If $K: T \to P_{fc}(X)$ is absolutely continuous, with modulus $r(.) \in L^1_+$ and $F: T \times X \to P_{fc}(X)$ is a multifunction s.t.*

(1) *$(t, x) \to F(t, x)$ is measurable*

(2) *for every $t \in T$, $F(t, .)$ has a sequentially closed graph in $X_w \times X_w$, where X_w is the Hilbert space X with the weak topology*

(3) *$|F(t, x)| \leq \phi(t)$ a.e., with $\phi(.) \in L^1_+$. then $(*)$ admits a solution.*

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Proof. As before we consider the set $W \subseteq C(T,X)$ defined by $W = \{y \in C(T,X): y(t) = x_0 + \int_0^t u(s)\,ds,\ t \in T,\ \|u(t)\| \leqq r(t) + \phi(t) \text{ a.e.}\}$

It is easy to see that W is strongly equicontinuous, hence *a fortiori* weakly equicontinuous. Also since for every $t \in T$, $W(t) = \{y(t): y \in W\}$ is bounded, from Alaoglu's theorem it is relatively w-compact. So from the Arzela–Ascoli theorem, W is relatively compact in $C(T,X_w)$. Let $y_a \in W$, $y_a \to y$ in $C(T,X_w)$. Then $y_a(t) = x_0 + \int_0^t u_a(s)\,ds$, $t \in T$, $\|u_a(t)\| \leqq r(t) + \phi(t)$ a.e. From the Dunford–Pettis theorem we may assume that $u_a \xrightarrow{w} a$ in $L^1(X) \Rightarrow y_a(t) \xrightarrow{w} x_0 + \int_0^t u(s)\,ds$, $t \in T \Rightarrow y(t) = x_0 + \int_0^t u(s)\,ds$, $t \in T$ with $\|u(t)\| \leqq r(t) + \phi(t)$ a.e. $\Rightarrow W$ is closed in $C(T,X_w)$, hence compact. Finally note that the $C(T,X_w)$-topology on W is meterizable since it is equal to the topology of pointwise convergence on a countable dense subset of $T = [0,b]$.

Also let $V \subseteq L^1(X)$ be defined by

$$V = \{h \in L^1(X): \|h(t)\| \leqq r(t) + \phi(t) \text{ a.e.}\}$$

From the Dunford–Pettis theorem (see Dunford–Schwartz [**15**]), we know that V is w-compact in $L^1(X)$ and because X is separable, the weak topology on V is metrizable.

For $h \in V$ consider the following evolution inclusion:

$$\left\{\begin{array}{l} -\dot{x}(t) \in N_{K(t)}(x(t)) + h(t) \text{ a.e.} \\ x(0) = x_0 \end{array}\right\} \tag{$*$}_1$$

From Moreau [**25**], we know that $(*)_1$ has unique solution $x(h) \in W$. Let $s: V \to W$ be the map that, to each $h \in V$, corresponds the unique solution $x(h)$ of $(*)_1$. We claim that this map is continuous from V with the weak $L^1(X)$-topology into W with the $C(T,X_w)$-topology. Since W is compact in $C(T,X_w)$, it suffices to show that Grs is closed in $V \times W$. So let $(h_n,x_n) \in \text{Grs}$, $(h_n,x_n) \to (h,x)$ in $(L^1(X),w) \times C(T,X_w)$. Let $\hat{x} = s(h)$. Then from the monotonicity of the subdifferential, we have:

$$(\dot{x}_n(t) - \dot{\hat{x}}(t), x_n(t) - \hat{x}(t)) \leqq (h(t) - h_n(t), x_n(t) - \hat{x}(t))$$

$$\Rightarrow \tfrac{1}{2}\|x_n(t) - \hat{x}(t)\|^2 \leqq \int_0^t (h(s) - h_n(s), x_n(s) - x(s))\,ds$$

$$+ \int_0^t (h(s) - h_n(s), x(s) - \hat{x}(s))\,ds \to 0$$

$$\Rightarrow x_n \to \hat{x} \quad \text{in} \quad C(T,X)$$

$$\Rightarrow \hat{x} = x$$

$\Rightarrow$Grs is closed and so $s(.)$ is continuous.

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Next let $R: V \to P_{fc}(L^1(X))$ be the multifunction defined by

$$R(h) = S^1_{F(.,s(h)(.))}$$

Clearly $R(h) \subseteq V$ for all $h \in V$. So, to show, that $R(.)$ is u.s.c., it suffices to show that GrR is closed, in $V \times V$ with the relative weak $L^1(X)$-topology. Since the latter is metrizable, we work with sequences. So let $(h_n, f_n) \in \mathrm{GrR}\,(h_n, f_n) \to (h, f)$. Then using Theorem 3.1 of [31], we get that

$$f(t) \in \overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, \{h_n(t)\}_{n \geqq 1} \text{ a.e.}$$

$$\subseteq \overline{\mathrm{conv}}\, w\text{-}\overline{\lim}\, F(t, s(h_n)(t)) \text{ a.e.}$$

$$\subseteq F(t, s(h)(t)) \text{ (hypothesis 2)}$$

$$\Rightarrow f \in S^1_{F(.,s(h)(.))}$$

$$\Rightarrow f \in R(h)$$

$$\Rightarrow \text{GrR is closed and so } R(.) \text{ is u.s.c.}$$

Apply the Kakutani–KyFan fixed point theorem to get $h \in V$ s.t. $h \in R(h)$. Then clearly $s(h)$ is the desired solution of $(*)$. □

Remark. When $K(t) = K \in P_{kc}(X)$, then we can assume that $F(t,.)$ is u.s.c. from X into X. This follows from corollary of Theorem 3.1 in [22]. In this case we may assume that $|F(t,x)| \leqq a(t) + b(t)\|x\|$ a.e. with $a(.)$, $b(.) \in L^1$. In [22], the reader can find some other infinite dimensional results related to the present work.

5. A convergence result

In this section, we will examine the well posedness with respect to the perturbation $F(.,.)$ and the initial data x_0, of the evolution inclusion $(*)$.

So consider the following sequence of evolution inclusions:

$$\left\{\begin{array}{l} -\dot{x}_n(t) \in N_{K(t)}(x_n(t)) + F_n(t, x_n(t)) \text{ a.e.} \\ x_n(0) = z_n \end{array}\right\} \qquad (*)_n$$

and a limit problem

$$\left\{\begin{array}{l} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x(t)) \\ x(0) = x_0 \end{array}\right\} \qquad (*)$$

We have the following well posedness (stability) result. Here $X=\mathbb{R}^n$.

Theorem 5.1. *If $K\colon T\to P_{fc}(X)$ is absolutely continuous with modulus $r(.)\in L_+^\infty$ and $F_n, F\colon T\times X\to P_{fc}(X)$ are multifunctions s.t.*

(1) *for every $n\geqq 1$ and every $x\in X$, $t\to F_n(t,x)$ is measurable*

(2) *for every $t\in T$, $\{F_n(t,.)\}_{n\geqq 1}$ is equi-h^*-u.s.c. and $GrF_n(t,.)\xrightarrow{K}GrF(t,.)$*

(3) *for all $n\geqq 1$, $|F_n(t,x)|\leqq\phi_1(t)\|x\|+\phi_2(t)$ a.e. with $\phi_1(.),\ \phi_2(.)\in L_+^1$*

(4) $z_n\to x_0$

then any sequence of solutions of $(*)_n$, has a subsequence converging uniformly to a solution of $(*)$.

Proof. As before, we have the *a priori* bound

$$M=[\|x_0\|+\|r\|_1+\|\phi_2\|_1]\exp\|\phi_1\|_1$$

Also recall that the solutions of $(*)_n$ and $(*)$ lie in the set

$$w=\left\{y\in C(T,X)\colon y(t)=x+\int_0^t u(s)\,ds,\ t\in T,\ \|x\|\leqq\sup_{n\geqq 1}\|z_n\|,\ \|u(t)\|\leqq\phi(t)\text{ a.e.}\right\}$$

where $\phi(t)=\phi_1(t)M+\phi_2(t)$ a.e. This set, by the Arzela–Ascoli theorem, is compact in $C(T,X)$. So if $\{x_n\}_{n\geqq 1}$ is a sequence of solutions of $(*)_n$, we can find a subsequence (denoted for notational simplicity by the same index) s.t. $x_n\to x$ in $C(T,X)$. Let $f_n\in S^1_{F_n(.,x_n(.))}$ s.t.

$$-\dot{x}_n(t)\in N_{K(t)}(x_n(t))+f_n(t)\text{ a.e.}$$

$$x_n(0)=z_n$$

Once again the Dunford–Pettis theorem allows us to assume that $f_n\xrightarrow{w}f$ in $L^1(X)$. Also from Proposition 2.1 of [**28**], we know that $F_n(t,x_n(t))\xrightarrow{k}F(t,x(t))$ and so as before, through Mazur's lemma, we can get that $f\in S^1_{F(.,x(.))}$. Let $\hat{x}(.)$ be the unique solution of the evolution:

$$-\dot{z}(t)\in N_{K(t)}(z(t))+f(t)\text{ a.e.}$$

$$z(0)=x_0$$

As in the proof of Theorem 3.2, using the monotonicity of the subdifferential and the

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fact that $f_n \xrightarrow{w} f$ in $L^1(X)$ and $x_n \to x$ in $C(T, X)$, we get that $x_n \to \hat{x}$ in $C(T, X) \Rightarrow x = \hat{x}$ and clearly $\hat{x}(.)$ solves $(*)$. □

This leads us to a result concerning the dependence of the solution set on the initial data.

Denote by $S(x_0)$ the solution set of $(*)$.

Theorem 5.2. *If $K: T \to P_{fc}(X)$ is absolutely continuous with modulus $r(.) \in L^\infty_+$ and $F: T \times X \to P_{fc}(X)$ is a multifunction s.t.*

(1) *$t \to F(t, x)$ is measurable*

(2) *$x \to F(t, x)$ is u.s.c.*

(3) *$|F(t, x)| \leqq \phi_1(t)\|x\| + \phi_2(t)$ a.e., with $\phi_1(.)$, $\phi_2(.) \in L^1_+$*

then $x_0 \to S(x_0)$ is an u.s.c. multifunction from X into $P_k(C(T, X))$

Proof. From Theorem 3.2 we know that for every $x_0 \in X$, $S(x_0) \neq \emptyset$. Also it is easy to check that it is closed in $C(T, X)$. Furthermore $S(x_0) \subseteq W'$ (see the proof of Theorem 3.2) and the latter is compact in $C(T, X)$. So $S(x_0) \in P_k(C(T, X))$. In order to show that $S(.)$ is u.s.c., it suffices to show that GrS is closed in $X \times C(T, X)$. So let $(x_0^n, x_n) \in \mathrm{GrS}$, $(x_0^n, x_n) \to (x_0, x)$ in $X \times C(T, X)$. Invoking Theorem 5.1 we get that $(x_0, x) \in \mathrm{GrS} \Rightarrow \mathrm{GrS}$ is closed and so $S(.)$ is u.s.c. □

6. A random evolution inclusion

In this section we consider a version of the original system $(*)$, in which the data depend measurably on a random parameter ω. Such evolutions represent problems that involve some inherent randomness due to ignorance or uncertainties. Random differential systems, have been studied recently by several mathematicians. We refer to the books of Ladde-Lakshmikantham **[23]** and Tsokos–Padgett **[37]** and the references therein.

The evolution inclusion under consideration is the following:

$$\begin{Bmatrix} -\dot{x}(\omega, t) \in N(x(\omega, t)) + f(\omega, t, x(\omega, t)) \\ x(\omega, 0) = x_0(\omega) \end{Bmatrix}. \qquad (**)$$

By a solution of $(**)$ we understand a stochastic process $x(\omega, t)$ with absolutely continuous realizations, satisfying $(**)$ for all $\omega \in \Omega$ and almost all $t \in T$.

Assume that (Ω, Σ, μ) is a complete probability space.

The result presented here extends significantly Theorem 4 of Castaing **[6]**, who studied an unperturbed (i.e. $f = 0$) version of $(**)$ and had more restrictive hypotheses on the measure space and the other data of the problem. Also, our result is related to Theorem 5.3 of **[27]**.

Theorem 6.1. *If $K:\Omega\times T\to P_{kc}(X)$ is a multifunction s.t.*

(1) *for every $t\in T$, $\omega\to K(\omega,t)$ is measurable*

(2) *for every $\omega\in\Omega$, $t\to K(\omega,t)$ is absolutely continuous with modulus $r(.,.)$ where $r(.,.)$ is measurable and for every $\omega\in\Omega$, $r(\omega,.)\in L^1_+$.*

and $f:\Omega\times T\times X\to X$ is a function s.t.

(3) *for every $x\in X$, $(\omega,t)\to f(\omega,t,x)$ is measurable.*

(4) *for every $(\omega,t)\in\Omega\times T$, $x\to f(\omega,t,x)$ is continuous.*

(5) *$\|f(\omega,t,x)\|\leqq\phi_1(\omega,t)\|x\|+\phi_2(\omega,t)$ a.e. on T, for all $\omega\in\Omega$ and with $\phi_1(.,.)$, $\phi_2(.,.)$ both measurable on $\Omega\times T$ and for all $\omega\in\Omega$ $\phi_1(\omega,.)$, $\phi_2(\omega,.)\in L^1_+$.*

then $(**)$ admits a solution.

Proof. Denote by $CS(K(\omega,.))$ the set of continuous selectors of $K(\omega,.)$, $\omega\in\Omega$. From Michael's selection theorem, we know that for all $\omega\in\Omega$, $CS(K(\omega,.))\neq\emptyset$. Define $L:\Omega\to P_f(C(T,X))$ by $L(\omega)=CS(K(\omega,.))$. Then we have:

$$\begin{aligned}L(\omega)&=\{y\in C(T,X): y(t)\in K(\omega,t)\quad\text{for all}\quad t\in T\}\\&=\{y\in C(T,X): d(y(t),K(\omega,t))=0\quad\text{for all}\quad t\in T\}.\end{aligned}$$

Set $u(\omega,t,y)=d(y(t),K(\omega,t))$. Since $\omega\to K(\omega,t)$ is measurable, we have that $\omega\to d(y(t),K(\omega,t))$ is measurable$\Rightarrow\omega\to u(\omega,t,y)$ is measurable. Also let $(t_n,y_n)\to(t,y)$ in $T\times C(T,X)$. Then we have:

$$|d(y_n(t),K(\omega,t_n))-d(y(t),K(\omega,t))|\leqq\|y_n(t)-y(t)\|+h(K(\omega,t_n),K(\omega,t))$$

$$\Rightarrow|u(\omega,t_n,y_n)-u(\omega,t,y)|=|d(y_n(t),K(\omega,t_n))-d(y(t),K(\omega,t))|\to 0$$

$$\Rightarrow(t,y)\to u(\omega,t,y)\text{ is continuous on }T\times C(T,X).$$

Hence $(\omega,t,y)\to u(\omega,t,y)$ is a Caratheodory function and so it is jointly measurable (see Castaing–Valadier [5, Lemma III-14]). So if $\{t_n\}_{n\geqq 1}$ is a dense subset of T, then

$$(\omega,y)\to\sup_{n\geqq 1}u(\omega,t_n,y)=v(\omega,y)$$

is measurable. Therefore:

$$\mathrm{GrL}=\{(\omega,y)\in\Omega\times C(T,X): v(\omega,y)=0\}\in\Sigma\times B(C(T,X)).$$

$$\Rightarrow L(.)\text{ is measurable.}$$

For fixed $\omega \in \Omega$ and $z \in C(T, X)$, consider the following evolution inclusion:

$$\left\{\begin{array}{l} -\dot{x}(t) \in N_{K(\omega,t)}(x(t)) + f(\omega, t, z(t)) \text{ a.e.} \\ x(0) = x_0(\omega) \end{array}\right\}. \qquad (***)$$

From Daures [11] (see also Theorem 3.2), we know that (***) has a unique solution. Let $h: \Omega \times C(T, X) \to C(T, X)$ be the map which to each $(\omega, z) \in \Omega \times C(T, X)$ associates this unique solution of (***). From Theorem 4 of Castaing [6] and the lemma on p. 511 of [34], we get that $\omega \to h(\omega, z)$ is measurable. Also from Brezis [4] we know that if $z_n \to z$ in $C(T, X)$, then we have:

$$\|h(\omega, z_n)(t) - h(\omega, z)(t)\| \leqq \int_0^t \|f(\omega, s, z_n(s)) - f(\omega, s, z(s))\| \, ds \to 0$$

$$\Rightarrow h(\omega, z_n) \to h(\omega, z) \quad \text{in} \quad C(T, X) \quad \text{for all} \quad \omega \in \Omega.$$

$$\Rightarrow z \to h(\omega, z) \text{ is continuous.}$$

$$\Rightarrow h(.,.) \text{ is Caratheodory, hence jointly measurable.}$$

Let $\omega \in \Omega$ and define $W(\omega) \subseteq C(T, X)$ by:

$$W(\omega) = \{y \in C(T, X): y(t) = x_0(\omega) + \int_0^t v(s)\, ds,\ t \in T,\ \|v(t)\| \leqq r(\omega, t) + \phi(\omega, t) \text{ a.e.}\}$$

where

$$\phi(\omega, t) = [\|x_0(\omega)\| + \|r(\omega, .)\|_1 + \|\phi_2(\omega, .)\|_1] \exp(\|\phi_1(\omega, .)\|_1).$$

Then from the Arzela–Ascoli, theorem, we have that for every $\omega \in \Omega$, $W(\omega)$ is compact in $C(T, X)$. Note also that for every $(\omega, z) \in \mathrm{GrL}$, $h(\omega, z) \in W(\omega)$. So, if we fix $\omega \in \Omega$ and apply Schauder's fixed point theorem, we get $z \in L(\omega)$ s.t. $z = h(\omega, z)$. Then let

$$P(\omega) = \{z \in L(\omega): z = h(\omega, z)\}$$

$$\Rightarrow \mathrm{GrP} = \{(\omega, z) \in \Omega \times C(T, X): z = h(\omega, z)\} \cap \mathrm{GrL} \in \Sigma \times B(C(T, X)).$$

So we can apply Aumann's selection theorem to find $p: \Omega \to C(T, X)$ measurable s.t. for all $\omega \in \Omega$, $p(\omega) \in P(\omega)$. Then if we set $x(\omega, t) = p(\omega)(t)$, from the lemma in [34], we know that $x(.,.)$ is a stochastic process with absolutely continuous realizations s.t. $x(\omega, t) \in K(\omega, t)$ for all $(\omega, t) \in \Omega \times T$ and

$$-\dot{x}(\omega, t) \in N_{K(\omega,t)}(x(\omega, t)) + f(\omega, t, x(\omega, t)) \text{ a.e.}$$

for all $\omega \in \Omega$

$$x(\omega,0)=x_0(\omega),$$

$$\Rightarrow x(.,.) \text{ is the desired random solution of } (**). \qquad \square$$

Remark. Of course a more interesting and useful stochastic model is the one driven by white noise, i.e. the perturbation term is $f(\omega,t,x(t))\,dw(t)$ where $w(.)$ is an $\mathbb{R}^n$-valued Brownian motion. The techniques in this case are different and some progress in this direction has already been made by the author.

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