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Existence results for diffuse interface models describing phase separation and damage*

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In this paper, we analytically investigate multi-component Cahn–Hilliard and Allen–Cahn systems which are coupled with elasticity and uni-directional damage processes. The free energy of the system is of the form $\int_\Omega \frac{1}{2}\Gamma\nabla c : \nabla c + \frac{1}{2}|\nabla z|^2 + W^{\mathrm{ch}}(c) + W^{\mathrm{el}}(e,c,z)\,\mathrm{d}x$ with a polynomial or logarithmic chemical energy density W^{ch}, an inhomogeneous elastic energy density W^{el} and a quadratic structure of the gradient of damage variable z. For the corresponding elastic Cahn–Hilliard and Allen–Cahn systems coupled with uni-directional damage processes, we present an appropriate notion of weak solutions and prove existence results based on certain regularization methods and a higher integrability result for strain e.

Key words: Cahn–Hilliard systems; Allen–Cahn systems; Phase separation; Damage; Elliptic-parabolic systems; Energetic solution; Weak solution; Doubly nonlinear differential inclusions; Existence results; Rate-dependent systems; Logarithmic-free energy

1 Introduction

Phase separation and damage are common phenomena in many fields, including material sciences, biology and chemical reactions. Such microstructural processes take place to reduce the total free energy, which may include bulk chemical energy, interfacial energy and elastic strain energy.

The knowledge of the mechanisms inducing phase separation and damage processes is very important for technological applications, as for instance in the area of micro-electronics, due to the ongoing miniaturization. The materials used in this area are typically alloys consisting of mixtures of several components (cf. [26]).

Phase separation and damage processes are usually described by two separate models in the mathematical literature. To describe phase separation processes for alloys, phase-field models of Cahn–Hilliard and Allen–Cahn type coupled with elasticity are well adapted. On the other hand, damage processes for standard materials are often modelled as unilateral processes within a gradient-theory [19]. A phase-field approach which describes *both* phase separation and damage processes in a unifying model has been recently introduced in [27].

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The main objective of this work is to prove under general assumptions existence results for multi-component systems where Cahn–Hilliard as well as Allen–Cahn equations are *coupled* with rate-dependent damage differential inclusions for elastic materials. We are interested in free energies of the system which may contain a chemical energy of *logarithmic* or *polynomial type*, an *inhomogeneous* elastic energy and a *quadratic term* of the gradient of the damage variable. To this end, we establish some regularization methods which enable us to show existence results for gradient terms $|\nabla z|^p$ of the damage variable z in the free energy even if the assumption $p > n$ (n space dimension) is dropped. In contrast to [27,30], now the physical meaningful term $|\nabla z|^2$ can be treated (cf. [18,19]). In addition, we also provide a higher integrability result for strain tensor. As a consequence, the chemical-free energy may also have a logarithmic structure such that we are not restricted to polynomial growth as in [27]. We focus on the modelling of rate-dependent damage processes, but we would like to mention that the results of this work can be extended to rate-independent systems (i.e. the dissipation potential is homogeneous of degree one) by some modifications. In the following, we will introduce the model formally.

The elastic material we want to consider in this work is an N component alloy occupying a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. To account for phase separation, deformation and damage processes in *one* model, a state at a fixed time point is described by a triple (u, c, z), where $u : \Omega \to \mathbb{R}^n$ denotes the deformation, $c : \Omega \to \mathbb{R}^N$ denotes the vector of chemical concentrations and $z : \Omega \to \mathbb{R}$ denotes the damage variable. The meaning of the variables and its governing evolutionary process is explained more explicitly below.

The mixture of the alloy is described by a phase field vector $c = (c_1, \dots, c_N)$, where element c_k for $k = 1, \dots, N$ denotes the concentration of component k. Therefore, we will restrict the state space for c to the physically meaningful condition $\sum_{j=1}^N c_j = 1$ in Ω. The constraint $c_k > 0$, $k = 1, \dots, N$, in Ω is also used for logarithmic chemical potentials (see below).

If an alloy is cooled down below a critical temperature then usually there occur spinodal decomposition and coarsening phenomena. Well established models for describing that such effects are Cahn–Hilliard and Allen–Cahn equations, which describe mass preserving and mass non-preserving phase separation in solids (cf. [1, 9, 10, 25, 28]) for modelling aspects. Analytical investigations of Cahn–Hilliard equations can be found in [4, 7, 11, 20–22] and that for Allen–Cahn equations in [6, 12, 13]. The essential difference between these two equations is that the Cahn–Hilliard equation is a fourth order parabolic evolutionary equation expressible as a H^{-1} gradient flow of free energy with respect to c, whereas the Allen–Cahn equation is a second order parabolic equation arising from an L^2 gradient flow. More precisely,

$$\begin{aligned} &\text{Allen–Cahn:} && \partial_t c = -\mathbb{M}\big(-\operatorname{div}(\Gamma \nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u), c, z)\big), \\ &\text{Cahn–Hilliard:} && \partial_t c = \operatorname{div}\big(\mathbb{M}\nabla\big(-\operatorname{div}(\Gamma \nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u), c, z)\big)\big). \end{aligned} \tag{1}$$

Here W^{ch} denotes the chemical energy density, W^{el} is the elastic energy density, $\mathbb{M}$ is the mobility matrix satisfying $\sum_{l=1}^N \mathbb{M}_{kl} = 0$ for all $k = 1, \dots, N$ and Γ is the gradient energy tensor, which is a fourth order symmetric and positive definite tensor, mapping matrices from $\mathbb{R}^{N \times n}$ into itself.

In this work, W^{ch} may be a chemical energy density of polynomial type, i.e. $W^{\mathrm{ch}}(c) = W^{\mathrm{ch,pol}}(c)$, or of logarithmic type, i.e. $W^{\mathrm{ch}}(c) = W^{\mathrm{ch,log}}(c)$ (see (A8)). Note that phase separation only arises if matrix A in (A8) is non-positive definite, since the first term in (A8) is convex.

Elastic behaviour is modelled by a deformation variable u so that each material point $x \in \Omega$ from the reference configuration is located at $x + u(x)$. We use the assumption that strain e is sufficiently small so that we can work with the linearized strain tensor given by $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$. In this work, we will neglect inertia effects $\rho\ddot{u}$ and volume forces l. Therefore, the momentum balance equation $\mathrm{div}(\sigma) + l = \rho\ddot{u}$ from continuum mechanics becomes a quasi-static force equation, i.e.

$$\mathrm{div}(\sigma) = 0. \tag{2}$$

The stress tensor σ is defined by $W^{\mathrm{el}}_{,e}$, i.e. as a derivative of elastic energy with respect to strain.

Analytical results for multi-component Cahn–Hilliard equations coupled with elastic deformations can be found in [20] whereas Allen–Cahn systems with elasticity are, for instance, studied in [5]. Finite element error estimates of Cahn–Hilliard equations with logarithmic-free energies and concentration-dependent mobilities are derived in [2]. Recent numerical results for Cahn–Hilliard and Allen–Cahn equations can be found in [3]. It turns out that different elastic moduli of phases in the mixture influence the rate of coarsening and the morphology of phases decisively [14]. Numerical investigations of elastic Cahn–Hilliard systems are conducted in [23].

The damage process we want to consider in this paper is uni-directional, i.e. it can only increase in time and the material is not able to heal itself. The phase field variable z satisfying $0 \leqslant z \leqslant 1$ is interpreted as damage in a way that $z(x) = 1$ stands for a non-damaged, and $z(x) = 0$ for a maximally damaged material point $x \in \Omega$. We assume that the damage in our model is *not complete*, which means that a maximal damaged part has still elastic properties. These constraints lead to a differential inclusion formulation for the evolution of z which relates the derivative of the energy dissipation of the system depending on the rate of damage with the derivative of free energy with respect to z. More precisely, we consider the doubly nonlinear differential inclusion (cf. [19]),

$$0 \in \partial\rho(\partial_t z) - \Delta z + W^{\mathrm{el}}_{,z}(e(u), z) + \partial I_{[0,\infty)}(z). \tag{3}$$

The energy dissipation density due to damage progression is given by ρ where we assume the structure

$$\rho(\dot{z}) = -\alpha\dot{z} + \frac{\beta}{2}|\dot{z}|^2 + I_{(-\infty,0]}(\dot{z})$$

with $\alpha, \beta > 0$. Because of the quadratic term $\frac{\beta}{2}|\dot{z}|^2$, the damage evolution is called rate-dependent whereas $\beta = 0$ would correspond to rate-independent systems. See [15,29,31,32] for analytical results on rate-independent damage and numerical experiments (without phase separation). We also refer to [8,17] for further analytical investigations of damage models. In comparison to [27] we use a gradient-of-damage theory with the Laplacian Δz in (3) instead of a p-Laplacian $\mathrm{div}(|\nabla z|^{p-2}\nabla z)$ with $p > n$.

In conclusion, the systems that we would like to consider in this work are governed by (1), (2) and (3) and can be rewritten as

$$\begin{cases} \partial_t c = -\mathscr{S} w & \text{in } \Omega_T, \\ w = \mathbb{P}(-\operatorname{div}(\Gamma \nabla c) + W^{\mathrm{ch}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u), c, z)) & \text{in } \Omega_T, \\ \operatorname{div}(\sigma) = 0 & \text{in } \Omega_T, \\ \partial\rho(\partial_t z) - \Delta z + W^{\mathrm{el}}_{,z}(e(u), z) + \partial I_{[0,\infty)}(z) \ni 0 & \text{in } \Omega_T, \end{cases} \qquad (S_0)$$

where w denotes chemical potential. Here matrix $\mathbb{P}$ denotes the orthogonal projection of $\mathbb{R}^N$ onto the tangent space $T\Sigma = \{x \in \mathbb{R}^N \mid \sum_{k=1}^N x_k = 0\}$ of affine plane $\Sigma := \{x \in \mathbb{R}^N \mid \sum_{l=1}^N x_k = 1\}$. The operator $\mathscr{S}$ determines whether we have an Allen–Cahn- or a Cahn–Hilliard-type diffusion of the system. More precisely,

$$\begin{aligned} &\text{Allen–Cahn:} && \mathscr{S} : L^2(\Omega;\mathbb{R}^N) \to L^2(\Omega;\mathbb{R}^N), && \mathscr{S}(f) := \mathbb{M}f, \\ &\text{Cahn–Hilliard:} && \mathscr{S} : H^1(\Omega;\mathbb{R}^N) \to \left(H^1(\Omega;\mathbb{R}^N)\right)^\star, && \mathscr{S}(f) := \langle \mathbb{M}\nabla f, \nabla \cdot \rangle_{L^2}. \end{aligned} \qquad (4)$$

In the Cahn–Hilliard case, operator $\mathscr{S}$ is invertible when restricted to $\mathscr{S} : Y \to \mathscr{D}$, where spaces Y and $\mathscr{D}$ are defined as follows:

$$\begin{aligned} Y &:= \left\{ c \in H^1(\Omega;\mathbb{R}^N) \mid \int_\Omega c = 0, \ \sum_{k=1}^N c_k = 0 \right\}, \\ \mathscr{D} &:= \Big\{ c^\star \in \left(H^1(\Omega;\mathbb{R}^N)\right)^\star \mid \langle c^\star, c \rangle_{(H^1)^\star \times H^1} = 0 \text{ for all } c = d(x)(1,\ldots,1), \\ &\text{where } d \in H^1(\Omega) \text{ and for all } c = e_k,\ k = 1,\ldots N \Big\} \quad e_k : k\text{th unit function.} \end{aligned} \qquad (5)$$

We need to impose some restrictions on mobility matrix $\mathbb{M}$. We assume that $\mathbb{M}$ is symmetric and positive definite on the tangent space $T\Sigma$. In addition, due to constraint $\sum_{k=1}^N c_k = 1$, $\mathbb{M}$ has to satisfy the property $\sum_{l=1}^N \mathbb{M}_{kl} = 0$ for all $k = 1,\ldots,N$. Note that $\mathbb{M} = \mathbb{M}\,\mathbb{P}$.

We abbreviate $D_T := (0,T) \times D$ and $(\partial\Omega)_T := (0,T) \times \partial\Omega$, where $D \subseteq \partial\Omega$ with $\mathscr{H}^{n-1}(D) > 0$ denotes the Dirichlet boundary. The initial-boundary conditions (IBC) of our systems are summarized as follows:

$$\begin{aligned} c(0) &= c^0 \text{ in } \Omega, & \sigma \cdot \vec{\nu} &= 0 \text{ on } (\partial\Omega)_T \setminus D_T, \\ z(0) &= z^0 \text{ in } \Omega, & \Gamma \nabla c \cdot \vec{\nu} &= 0 \text{ on } (\partial\Omega)_T, \\ u &= b \text{ on } D_T, & \nabla z \cdot \vec{\nu} &= 0 \text{ on } (\partial\Omega)_T \end{aligned} \qquad \text{(IBC)}$$

and in addition for Cahn–Hilliard systems

$$\mathbb{M}\nabla w \cdot \vec{\nu} = 0 \quad \text{on } (\partial\Omega)_T, \qquad \text{(IBC)}$$

where $\vec{\nu}$ is the unit normal on $\partial\Omega$ pointing outward and b is the boundary value function on the Dirichlet boundary D. The initial values are subject to $0 \leqslant z_0 \leqslant 1$ and $c^0 \in \Sigma \cap \mathbb{R}^N_{>0}$ a.e. in Ω. In the following, we assume that b can be suitably extended to a function on $\overline{\Omega_T}$.

The paper is organized as follows. In Section 2, we introduce an appropriate notion of weak solutions for the system (S_0). To handle the differential inclusion rigorously, we adapt the concept of energetic solutions originally introduced in the context of rate-independent systems (see for instance [29]) to phase separation systems coupled with rate-dependent damage. This approach was first presented in [27]. The main result and their assumptions are stated at the end of Section 2.

In Section 3, we prove existence of weak solutions for the regularization of system (S_0) expressed in classical formulation as

$$\left\{\begin{array}{ll}\partial_t c = -\mathscr{S} w & \text{in } \Omega_T,\\ w = \mathbb{P}(-\mathrm{div}(\Gamma\nabla c) + W^{\mathrm{ch,pol}}_{,c}(c) + W^{\mathrm{el}}_{,c}(e(u),c,z) + \varepsilon\partial_t c) & \text{in } \Omega_T,\\ \mathrm{div}(\sigma) + \varepsilon\mathrm{div}(|\nabla u|^2\nabla u) = 0 & \text{in } \Omega_T,\\ \partial\rho(\partial_t z) - \Delta z - \varepsilon\mathrm{div}(|\nabla z|^{p-2}\nabla z) + W^{\mathrm{el}}_{,z}(e(u),z) + \partial I_{[0,\infty)}(z) \ni 0 & \text{in } \Omega_T,\end{array}\right\} \qquad (S_\varepsilon)$$

where $W^{\mathrm{ch,pol}}$ and W^{el} satisfy certain polynomial growth conditions and $p > n$. The initial-boundary conditions are

$$\text{(IBC) with } (\sigma + \varepsilon|\nabla u|^2\nabla u)\cdot\overrightarrow{\nu} = 0 \text{ instead of } \sigma\cdot\overrightarrow{\nu} = 0. \qquad (\mathrm{IBC}_\varepsilon)$$

It turns out that the weak solutions of the regularized system have the following regularities: $c \in H^1(0,T;L^2(\Omega;\mathbb{R}^N))$, $\nabla u \in L^4(\Omega_T;\mathbb{R}^{n\times n})$ and $\nabla z \in L^p(\Omega_T;\mathbb{R}^n)$ (with $p > n$ as above). These are constructed by adapting the approximation techniques developed in [27].

The limit problem $\varepsilon \searrow 0$ for (S_ε) corresponding to (S_0) with $W^{\mathrm{ch}} = W^{\mathrm{ch,pol}}$ is solved in Section 4. The displacement field u obtained in this process has $H^1(\Omega;\mathbb{R}^n)$-regularity in the first instance. To establish existence results for chemical-free energies of logarithmic type, we prove a higher integrability result for ∇u in Section 5, which is based on some ideas of [16, 20, 22].

Finally, Section 6 is devoted to logarithmic-free energies for concentration c. Following the approach in [20, 22], we use a suitable regularization $W^{\mathrm{ch},\delta}$ with polynomial growth of the logarithmic-free energy density $W^{\mathrm{ch,log}}$ to obtain a solution for (S_0). Using this regularization, the chemical component c_k becomes strictly positive in the limit.

The notation that we will use throughout this paper is collected in the following:

Spaces and sets.

$W^{1,r}(\Omega;\mathbb{R}^n)$	standard Sobolev space,
$W^{1,r}_+(\Omega)$	functions of $W^{1,r}(\Omega)$, which are non-negative almost everywhere,
$W^{1,r}_-(\Omega)$	functions of $W^{1,r}(\Omega)$, which are non-positive almost everywhere,
$W^{1,r}_D(\Omega;\mathbb{R}^n)$	functions of $W^{1,r}(\Omega;\mathbb{R}^n)$, which vanish on $D \subseteq \partial\Omega$ in the sense of traces,
$B_R(A)$	open neighbourhood of $A \subseteq \mathbb{R}^n$ with thickness R,
$Q_R(x_0)$	open cube $\{x \in \mathbb{R}^n \mid \|x - x_0\|_\infty < R\}$,

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$\{f = 0\}$	zero set $\{x \in \overline{\Omega} \mid f(x) = 0 \text{ a.e.}\}$ of function $f \in L^1(\Omega)$ defined up to a set of measure 0 and defined uniquely if $f \in W^{1,p}(\Omega)$ for $p > n$ as $W^{1,p}(\Omega) \hookrightarrow \mathscr{C}^0(\overline{\Omega})$,
Ω_T	set $(0, T) \times \Omega$.

Functions, operations and measures.

$[f]^+$	non-negative part of f, i.e. $\max\{0, f\}$,
I_M	indicator function of subset $M \subseteq X$,
χ_M	characteristic function of subset $M \subseteq X$,
$W_{,e}$	classical derivative of function W with respect to variable e,
$\langle g^\star, f\rangle$	dual pairing of $g^\star \in (W^{1,r}(\Omega;\mathbb{R}^n))^\star$ and $f \in W^{1,r}(\Omega;\mathbb{R}^n)$,
$\partial^{\mathrm{Cl}}E$	generalized Clarke's sub-differential of E,
$\mathrm{d}E$	Gâteaux differential of E,
$p^\star$	Sobolev critical exponent $\frac{np}{n-p}$ for $n > p$,
$\mathrm{diam}(Q)$	diameter of subset $Q \subseteq \mathbb{R}^n$,
$\mathscr{H}^n$	Hausdorff measure of dimension n,
$\mathscr{L}^n$	Lebesgue measure of dimension n.

2 Existence theorem

2.1 Weak formulation

The weak notion, we will derive in this section for the doubly nonlinear differential inclusion occurring in (S_0), is inspired by the concept of energetic solutions for rate-independent systems (see for instance [29]). In the rate-independent setting, the differential inclusion is formulated by a global stability condition and an energy inequality. In [27], we have introduced an approach which uses an energy inequality and a variational inequality to handle rate-dependence coming from the viscosity term $\frac{\beta}{2}|\dot{z}|^2$ in the damage dissipation density function ρ.

The corresponding Gâteaux-differentiable free energy $\tilde{\mathscr{E}} : H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega;\mathbb{R}^N) \times (H^1(\Omega) \cap L^\infty(\Omega)) \to \mathbb{R}$ and the dissipation functional $\tilde{\mathscr{R}} : L^2(\Omega) \to \mathbb{R}$ to system (S_0) are given by

$$\tilde{\mathscr{E}}(u, c, z) := \int_\Omega \frac{1}{2}\Gamma\nabla c : \nabla c + \frac{1}{2}|\nabla z|^2 + W^{\mathrm{ch}}(c) + W^{\mathrm{el}}(e, c, z)\,\mathrm{d}x,$$
$$\tilde{\mathscr{R}}(\dot{z}) := \int_\Omega -\alpha\dot{z} + \frac{\beta}{2}|\dot{z}|^2\,\mathrm{d}x,$$

with viscosity constants $\alpha, \beta > 0$. To account for the constraints of z, we extend functionals $\tilde{\mathscr{E}}$ and $\tilde{\mathscr{R}}$ above for analytical reasons by indicator functions:

$$\mathscr{E}(u, c, z) := \tilde{\mathscr{E}}(u, c, z) + \int_\Omega I_{[0,\infty)}(z)\,\mathrm{d}x, \qquad \mathscr{R}(\dot{z}) := \tilde{\mathscr{R}}(\dot{z}) + \int_\Omega I_{(-\infty,0]}(\dot{z})\,\mathrm{d}x.$$

If we equip space $H^1(\Omega) \cap L^\infty(\Omega)$ with norm $\|\cdot\|_{H^1\cap L^\infty} := \|\cdot\|_{H^1} + \|\cdot\|_{L^\infty}$, the generalized

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sub-differential $\partial_z^{\mathrm{Cl}}\mathscr{E}$ at point $(u,c,z) \in H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega;\mathbb{R}^N) \times (H^1(\Omega) \cap L^\infty(\Omega))$ is

$$\partial_z^{\mathrm{Cl}}\mathscr{E}(u,c,z) = \left\{ \mathrm{d}_z\tilde{\mathscr{E}}(u,c,z) + r \in (H^1(\Omega) \cap L^\infty(\Omega))^\star \,\middle|\, r \in \partial I_{H^1_+(\Omega)\cap L^\infty(\Omega)}(z) \right\}. \tag{6}$$

The inclusion $L^1(\Omega) \subset (H^1(\Omega) \cap L^\infty(\Omega))^\star$ will be later used for the construction of a specific sub-gradient. Using property (6), the differential inclusion in (S_0) can be rewritten in a weaker form as

$$0 \in \partial_z^{\mathrm{Cl}}\mathscr{E}(u(t),c(t),z(t)) + \partial_{\dot{z}}\mathscr{R}(\dot{z}(t)).$$

The analytical basis for the formulation of a weak solution is the following proposition (a proof of the related result can be found in [27]).

Proposition 2.1 *Let $(u,c,w,z) \in \mathscr{C}^2(\overline{\Omega_T};\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$ be a smooth solution satisfying (1) and (2) with the initial-boundary conditions. Then the following two conditions are equivalent:*

(i) $0 \in \partial_z^{\mathrm{Cl}}\mathscr{E}(u(t),c(t),z(t)) + \partial_{\dot{z}}\mathscr{R}(\dot{z}(t))$ *for all* $t \in [0,T]$,

(ii) *the energy inequality*

$$\mathscr{E}(u(t),c(t),z(t)) + \int_0^t \langle \mathrm{d}_{\dot{z}}\tilde{\mathscr{R}}(\partial_t z), \partial_t z\rangle \,\mathrm{d}s + \int_0^t \langle \mathscr{S}w(s), w(s)\rangle \mathrm{d}s \leqslant \mathscr{E}(u(0),c(0),z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s$$

for all $0 \leqslant t \leqslant T$ and the variational inequality

$$0 \leqslant \left\langle \mathrm{d}_z\tilde{\mathscr{E}}(u(t),c(t),z(t)) + r(t) + \mathrm{d}_{\dot{z}}\tilde{\mathscr{R}}(\partial_t z(t)), \zeta \right\rangle$$

for all $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ and $r(t) \in \partial I_{H^1_+(\Omega)\cap L^\infty(\Omega)}(z(t))$ and for all $0 \leqslant t \leqslant T$.

If one of the two conditions holds then the following energy balance equation is satisfied:

$$\mathscr{E}(u(t),c(t),z(t)) + \int_0^t \langle \mathrm{d}_{\dot{z}}\tilde{\mathscr{R}}(\partial_t z), \partial_t z\rangle \,\mathrm{d}s + \int_0^t \langle \mathscr{S}w(s), w(s)\rangle \mathrm{d}s = \mathscr{E}(u(0),c(0),z(0)) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s.$$

Remarks for Proposition 2.1. In contrast to [27], the energy inequality in (ii) compares the energy at the beginning $s = 0$ with the energy at an arbitrary time $s = t$ instead of $s = t_1$ with $s = t_2$ for $0 \leqslant t_1 < t_2 \leqslant T$.

Applying the chain rule on the right-hand side of

$$\mathscr{E}(u(t),c(t),z(t)) - \mathscr{E}(u(0),c(0),z(0)) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{E}}(u(s),c(s),z(s))\,\mathrm{d}s$$

and using (1) and (2) as well as the variational inequality in (ii), the '$\geqslant$'-part of the energy balance can be shown.

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We will see that in our approach the mathematical analysis of (S_0) requires several ε-regularization terms (see (S_ε)) to establish energy and variational inequality for differential inclusion and to handle logarithmic-free energy. A transition to $\varepsilon \searrow 0$ will finally give us a solution of the limit problem (S_0).

Proposition 2.1 can also be formulated for the regularized system (S_ε) with the regularized energy

$$\tilde{\mathscr{E}}_\varepsilon(u,c,z) := \int_\Omega \frac{1}{2}\Gamma\nabla c : \nabla c + \frac{1}{2}|\nabla z|^2 + W^{\mathrm{ch,pol}}(c) + W^{\mathrm{el}}(e,c,z) + \frac{\varepsilon}{4}|\nabla u|^4 + \frac{\varepsilon}{p}|\nabla z|^p\,\mathrm{d}x,$$

$$\mathscr{E}_\varepsilon(u,c,z) := \tilde{\mathscr{E}}_\varepsilon(u,c,z) + \int_\Omega I_{[0,\infty)}(z)\,\mathrm{d}x,$$

and the initial-boundary conditions $(\mathrm{IBC}_\varepsilon)$. Notice that $\mathbb{P}\partial_t c = \partial_t c$ because $\partial_t c(t,x) \in T\Sigma$.

We can now give a weak notion of (S_ε) and (S_0). (The energy densities $W^{\mathrm{ch,pol}}$ and W^{el} will satisfy some polynomial growth conditions, which are specified in the next section.)

Definition 2.2 (Weak solution for the regularized system (S_ε)) We call a quadruple $q = (u,c,w,z)$ a weak solution of the regularized system (S_ε) with the initial-boundary conditions $(\mathrm{IBC}_\varepsilon)$ if the following properties are satisfied:

(i) The components of q are in the following spaces:

$$u \in L^\infty(0,T;W^{1,4}(\Omega;\mathbb{R}^n)),\ u|_{D_T} = b|_{D_T},$$
$$c \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^N)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^N)),\ c(0)=c^0,\ c \in \Sigma \text{ a.e. in } \Omega_T,$$
$$z \in L^\infty(0,T;W^{1,p}_+(\Omega)) \cap H^1(0,T;L^2(\Omega)),\ z(0)=z^0,\ \partial_t z \leqslant 0,$$

and

$$w \in L^2(0,T;H^1(\Omega;\mathbb{R}^N)) \qquad \text{for C–H systems}$$
$$w \in L^2(\Omega_T;\mathbb{R}^N) \qquad \text{for A–C systems.}$$

(ii) For all $\zeta \in H^1(\Omega;\mathbb{R}^N)$ and for a.e. $t \in [0,T]$:

$$\int_\Omega \partial_t c(t)\cdot\zeta\,\mathrm{d}x = \begin{cases} \int_\Omega \mathbb{M}\nabla w(t) : \nabla\zeta\,\mathrm{d}x & \text{for C–H systems} \\ \int_\Omega \mathbb{M}w(t)\cdot\zeta\,\mathrm{d}x & \text{for A–H systems.} \end{cases} \tag{7}$$

(iii) For all $\zeta \in H^1(\Omega;\mathbb{R}^N)$ and for a.e. $t \in [0,T]$:

$$\int_\Omega w(t)\cdot\zeta\,\mathrm{d}x = \int_\Omega \mathbb{P}\Gamma\nabla c(t) : \nabla\zeta + \mathbb{P}W^{\mathrm{ch,pol}}_{,c}(c(t))\cdot\zeta\,\mathrm{d}x + \int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u(t)),c(t),z(t))\cdot\zeta + \varepsilon\partial_t c(t)\cdot\zeta\,\mathrm{d}x. \tag{8}$$

(iv) For all $\zeta \in W^{1,4}_D(\Omega;\mathbb{R}^n)$ and for a.e. $t \in [0,T]$:

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)),c(t),z(t)) : e(\zeta) + \varepsilon|\nabla u(t)|^2\nabla u(t) : \nabla\zeta\,\mathrm{d}x = 0. \tag{9}$$

(v) For all $\zeta \in W^{1,p}_{-}(\Omega)$ and for a.e. $t \in [0,T]$:

$$\int_\Omega (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t)\cdot\nabla\zeta + \big(W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)\big) - \alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x \geqslant -\langle r(t), \zeta\rangle, \tag{10}$$

where $r(t) \in (W^{1,p}(\Omega))^\star$ satisfies $\langle r(t), z(t) - \zeta\rangle \geqslant 0$ for all $\zeta \in W^{1,p}_{+}(\Omega)$.

(vi) Energy inequality for a.e. $t \in [0,T]$:

$$\begin{aligned}\mathscr{E}_\varepsilon(u(t), c(t), z(t)) - \mathscr{E}_\varepsilon(u^0, c^0, z^0) + \int_\Omega \alpha(z^0 - z(t))\,\mathrm{d}x \\ + \int_{\Omega_t} \beta|\partial_t z|^2 + \varepsilon|\partial_t c|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle \mathscr{S} w(s), w(s)\rangle\,\mathrm{d}s \\ \leqslant \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u), c, z) : e(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_{\Omega_t} |\nabla u|^2 \nabla u : \nabla\partial_t b\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{11}$$

where u^0 is the unique minimizer of $\mathscr{E}_\varepsilon(\cdot, c^0, z^0)$ in $W^{1,4}(\Omega;\mathbb{R}^n)$ with trace $u^0|_D = b(0)|_D$.

With the help of operator $\mathscr{S}$, the diffusion equation (7) can also be written as

$$\int_\Omega \partial_t c(t)\cdot\zeta\,\mathrm{d}x = -\langle \mathscr{S} w(t), \zeta\rangle,$$

which will be used in the following.

Definition 2.3 (Weak solution for the limit system (S_0)) A quadruple $q = (u, c, w, z)$ is called a weak solution of system (S_0) with the initial-boundary conditions if the following properties are satisfied:

(i) The components of q are in the following spaces:

$$\begin{aligned}& u \in L^\infty(0,T; H^1(\Omega;\mathbb{R}^n)),\ u|_{D_T} = b|_{D_T}, \\ & c \in L^\infty(0,T; H^1(\Omega;\mathbb{R}^N)),\ c \in \Sigma \text{ a.e. in } \Omega_T, \\ & z \in L^\infty(0,T; H^1_+(\Omega)) \cap H^1(0,T; L^2(\Omega)),\ z(0) = z^0,\ \partial_t z \leqslant 0\end{aligned}$$

and

$$\begin{aligned}& w \in L^2(0,T; H^1(\Omega;\mathbb{R}^N)) && \text{for C–H systems} \\ & w \in L^2(\Omega_T;\mathbb{R}^N) && \text{for A–C systems.}\end{aligned}$$

(ii) For all $\zeta \in L^2(0,T; H^1(\Omega;\mathbb{R}^N))$ with $\partial_t\zeta \in L^2(\Omega_T;\mathbb{R}^N)$ and $\zeta(T) = 0$:

$$\int_{\Omega_T} (c - c^0)\cdot\partial_t\zeta\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \mathscr{S} w, \zeta\rangle\,\mathrm{d}t.$$

(iii) For all $\zeta \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ and for a.e. $t \in [0,T]$:

$$\int_\Omega w(t)\cdot\zeta\,\mathrm{d}x = \int_\Omega \mathbb{P}\Gamma\nabla c(t) : \nabla\zeta + \mathbb{P}W^{\mathrm{ch}}_{,c}(c(t))\cdot\zeta\,\mathrm{d}x + \int_\Omega \mathbb{P}W^{\mathrm{el}}_{,c}(e(u(t)),c(t),z(t))\cdot\zeta\,\mathrm{d}x.$$

(iv) For all $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$ and for a.e. $t \in [0,T]$:

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)),c(t),z(t)) : e(\zeta)\,\mathrm{d}x = 0.$$

(v) For all $\zeta \in H^1_-(\Omega)\cap L^\infty(\Omega)$ and for a.e. $t \in [0,T]$:

$$\int_\Omega \nabla z(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)),c(t),z(t)) - \alpha + \beta(\partial_t z(t)))\zeta\,\mathrm{d}x \geqslant -\langle r(t),\zeta\rangle,$$

where $r(t) \in (H^1(\Omega)\cap L^\infty(\Omega))^\star$ satisfies $\langle r(t), z(t)-\zeta\rangle \geqslant 0$ for all $\zeta \in H^1_+(\Omega)\cap L^\infty(\Omega)$.

(vi) Energy inequality for a.e. $t \in [0,T]$:

$$\mathscr{E}(u(t),c(t),z(t)) + \int_\Omega \alpha(z^0 - z(t))\,\mathrm{d}x + \int_{\Omega_t}\beta|\partial_t z|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle \mathscr{S}w(s), w(s)\rangle\,\mathrm{d}s \leqslant \mathscr{E}(u^0,c^0,z^0) + \int_{\Omega_t} W^{\mathrm{el}}_{,e}(e(u),c,z) : e(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s,$$

where u^0 is the unique minimizer of $\mathscr{E}(\cdot,c^0,z^0)$ in $H^1(\Omega;\mathbb{R}^n)$ with trace $u^0|_D = b(0)|_D$.

Note that both notions of weak solutions imply chemical mass conservation, i.e.

$$\int_\Omega c(t)\,\mathrm{d}x \equiv \text{const.}$$

2.2 Assumptions and main results

The general setting, the growth assumptions and the assumptions on the coefficient tensors which are mandatory for the existence theorems are summarized below.

(i) *Setting*

Space dimension	$n \in \mathbb{N}$,
Components in the alloy	$N \in \mathbb{N}$ with $N \geqslant 2$,
Regularization exponent	$p > n$,
Viscosity factors	$\alpha, \beta > 0$,
Domain	$\Omega \subseteq \mathbb{R}^n$ bounded Lipschitz domain,
Dirichlet boundary	$D \subseteq \partial\Omega$ with $\mathscr{H}^{n-1}(D) > 0$,
Time interval	$[0,T]$ with $T > 0$

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(ii) *Energy densities*

Elastic energy density $W^{\mathrm{el}} \in \mathscr{C}^1(\mathbb{R}^{n\times n} \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}_+)$ with

$$W^{\mathrm{el}}(e,c,z) = W^{\mathrm{el}}(e^t,c,z), \tag{A1}$$

$$W^{\mathrm{el}}(e,c,z) \leqslant C(|e|^2 + |c|^2 + 1), \tag{A2}$$

$$\eta|e_1 - e_2|^2 \leqslant (W^{\mathrm{el}}_{,e}(e_1,c,z) - W^{\mathrm{el}}_{,e}(e_2,c,z)) : (e_1 - e_2), \tag{A3}$$

$$|W^{\mathrm{el}}_{,e}(e_1 + e_2,c,z)| \leqslant C(W^{\mathrm{el}}(e_1,c,z) + |e_2| + 1), \tag{A4}$$

$$|W^{\mathrm{el}}_{,c}(e,c,z)| \leqslant C(|e|^2 + |c|^2 + 1), \tag{A5}$$

$$|W^{\mathrm{el}}_{,z}(e,c,z)| \leqslant C(|e|^2 + |c|^2 + 1). \tag{A6}$$

Chemical energy densities $W^{\mathrm{ch,pol}}, W^{\mathrm{ch,log}} \in \mathscr{C}^1(\mathbb{R}^N;\mathbb{R})$ with $W^{\mathrm{ch,pol}} \geqslant -C$,

$$|W^{\mathrm{ch,pol}}_{,c}(c)| \leqslant C(|c|^{2^\star/2} + 1), \tag{A7}$$

$$W^{\mathrm{ch,log}}(c) = \theta \sum_{k=1}^{N} c_k \log c_k + \frac{1}{2} c \cdot Ac,\ \ \theta > 0,\ A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}. \tag{A8}$$

(iii) *Tensors*

Mobility tensor $\mathbb{M} \in \mathbb{R}^{N\times N}$ symmetric and positive definite on $T\Sigma$ and

$$\sum_{l=1}^{N} \mathbb{M}_{kl} = 0 \text{ for all } k = 1,\dots,N,$$

Energy gradient tensor $\Gamma \in \mathscr{L}(\mathbb{R}^{N\times n};\mathbb{R}^{N\times n})$ symmetric and positive definite fourth order tensor.

Remark 2.4 Due to the effect of damage on the elastic response of the material, W^{el} is often modelled by the following ansatz:

$$W^{\mathrm{el}} = (\Phi(z) + \tilde{\eta})\, \hat{W}^{\mathrm{el}},$$

where $\Phi : [0,1] \to \mathbb{R}_+$ is a continuously differentiable and monotonically increasing function with $\Phi(0) = 0$ and $\tilde{\eta} > 0$ is a small value.

A typically form of the *elastically stored energy density* $\hat{W}^{\mathrm{el}}$ is as follows:

$$\hat{W}^{\mathrm{el}}(c,e) = \frac{1}{2}\big(e - e^*(c)\big) : \mathbb{C}(c)\big(e - e^*(c)\big). \tag{12}$$

Here $e^*(c)$ denotes the *eigenstrain*, which is usually linear in c, and $\mathbb{C}(c) \in \mathscr{L}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is a fourth order stiffness tensor, which is symmetric and positive definite. The elastic energy density is called *homogeneous* if the stiffness tensor does not depend on the concentration, i. e. $\mathbb{C}(c) = \mathbb{C}$.

Note that the inhomogeneous elastic energy (12) fits into our setting with the previous growth assumptions (A1)–(A6). In particular, we are not confined to homogeneous

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elasticity as in [27]. There the more restrictive growth condition $|W^{\mathrm{el}}_{,c}(e,c,z)| \leqslant C(|e|+|c|^2+1)$ is used instead of (A5).

The main results of this work are summarized in the following theorems.

Theorem 2.5 **(Existence theorem – polynomial case)** *Let the above assumptions be satisfied. Then for every*

$$
\begin{aligned}
&b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n)),\\
&c^0 \in H^1(\Omega;\mathbb{R}^N) \text{ with } c^0 \in \Sigma \text{ a.e. in } \Omega,\\
&z^0 \in H^1(\Omega) \text{ with } 0 \leqslant z^0 \leqslant 1 \text{ a.e. in } \Omega,
\end{aligned}
$$

there exists a weak solution q of the system (S_0) with $W^{\mathrm{ch}} = W^{\mathrm{ch,pol}}$ and the initial-boundary conditions in the sense of Definition 2.3.

Theorem 2.6 **(Existence theorem – logarithmic case)** *Let the above assumptions be satisfied and, in addition, let $D = \partial\Omega$ and $\Gamma = \gamma\,\mathrm{Id}$ with a constant $\gamma > 0$. Then for every*

$$
\begin{aligned}
&b \in W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^n)),\\
&c^0 \in H^1(\Omega;\mathbb{R}^N) \text{ with } c^0 \in \Sigma \text{ and } c^0_k > 0 \text{ a.e. in } \Omega \text{ for } k = 1,\dots,N,\\
&z^0 \in H^1(\Omega) \text{ with } 0 \leqslant z^0 \leqslant 1 \text{ a.e. in } \Omega,
\end{aligned}
$$

there exists a weak solution q of the system (S_0) with $W^{\mathrm{ch}} = W^{\mathrm{ch,log}}$ and the initial-boundary conditions in the sense of Definition 2.3. In addition, $c_k > 0$ a.e. in Ω_T for $k = 1,\dots,N$.

Remark 2.7 Note that for Theorem 2.6 the assumptions (A2), (A5) and (A6) can be replaced by

$$W^{\mathrm{el}}(e,c,z) \leqslant C(|e|^2+1), \tag{A2'}$$

$$|W^{\mathrm{el}}_{,c}(e,c,z)| \leqslant C(|e|^2+1), \tag{A5'}$$

$$|W^{\mathrm{el}}_{,z}(e,c,z)| \leqslant C(|e|^2+1), \tag{A6'}$$

for all $c \in \mathbb{R}^N$ with $0 \leqslant c_k \leqslant 1$ and $\sum_{k=1}^N c_k = 1$, all $e \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and all $z \in \mathbb{R}$ with $0 \leqslant z \leqslant 1$.

3 Existence of weak solutions of (S_ε)

The proof is based on [27]. Arguments similar to [27] are only sketched.

Since $\varepsilon > 0$ is fixed in this section, we omit the ε-dependence in the notation, e.g. here $\mathscr{E}$ always means $\mathscr{E}_\varepsilon$ and so on. Furthermore, in this section z^0 is assumed to be in $W^{1,p}(\Omega)$.

1. Step: constructing time-discrete solutions.
   Set u^0 to be a minimizer of $u \mapsto \mathscr{E}(u,c^0,z^0)$ defined on the space $W^{1,4}(\Omega)$ with the constraint $u|_D = b(0)|_D$ in the sense of traces.

Let the closed subspace $\mathscr{Q}_M^m$ of $H^1(\Omega;\mathbb{R}^n)\times H^1(\Omega;\mathbb{R}^N)\times W^{1,p}(\Omega)$ be defined by

$$\mathscr{Q}_M^m=\left\{\begin{array}{l|l} u\in H^1(\Omega;\mathbb{R}^n), & u|_D=b(m\tau)|_D,\\ c\in H^1(\Omega;\mathbb{R}^N), & \int_\Omega c-c^0\,\mathrm{d}x=0 \text{ for C-H systems},\\ z\in W^{1,p}(\Omega) & 0\leqslant z\leqslant z_M^{m-1}.\end{array}\right\}$$

Based on the initial triple (u^0,c^0,z^0), we construct (u_M^m,c_M^m,z_M^m) for $m=1,\dots,M$ recursively by minimizing the following functional $\mathbb{E}_M^m:\mathscr{Q}_M^m\to\mathbb{R}$:

$$\mathbb{E}_M^m(u,c,z):=\tilde{\mathscr{E}}(u,c,z)+\tau\tilde{\mathscr{R}}\left(\frac{z-z_M^{m-1}}{\tau}\right)+\frac{\tau}{2}\left\|\frac{c-c_M^{m-1}}{\tau}\right\|_X^2+\frac{\varepsilon\tau}{2}\left\|\frac{c-c_M^{m-1}}{\tau}\right\|_{L^2}^2, \quad (13)$$

where X denotes the space $\mathscr{D}$ (see (5)) with the scalar-product

$$(c_1\,|\,c_2)_X:=\int_\Omega \mathbb{M}\nabla\mathscr{S}^{-1}c_1\cdot\nabla\mathscr{S}^{-1}c_2\,\mathrm{d}x$$

for Cahn–Hilliard systems and $X=L^2(\Omega;\mathbb{R}^N)$ with the scalar-product

$$(c_1\,|\,c_2)_X:=\int_\Omega \mathbb{M}c_1\cdot c_2\,\mathrm{d}x$$

for Allen–Cahn systems.

Note that the last regularization term in (13) is not necessary for Allen–Cahn equations due to the term with the X-norm. To use a uniform approach, we consider this term in both systems. By direct methods of calculus of variations the triple

$$\left(u_M^m,c_M^m,z_M^m\right):=\underset{(u,c,z)\in\mathscr{Q}_M^m}{\arg\min}\ \mathbb{E}_M^m(u,c,z)$$

exists (cf. [27]). Furthermore, we set

$$w_M^m:=\begin{cases}-\mathscr{S}^{-1}\left(\frac{c_M^m-c_M^{m-1}}{\tau}\right)+\lambda_M^m & \text{for C–H systems},\\ -\mathscr{S}^{-1}\left(\frac{c_M^m-c_M^{m-1}}{\tau}\right) & \text{for A–C systems},\end{cases}$$

with the Lagrange multiplier λ_M^m (associated with the mass constraint for C–H systems) given by

$$\lambda_M^m:=\fint_\Omega W_{,c}^{\mathrm{ch,pol}}\left(c_M^m\right)+W_{,c}^{\mathrm{el}}\left(e(u_M^m),c_M^m,z_M^m\right)\mathrm{d}x.$$

We define the time incremental solutions as

$$q_M^m:=\left(u_M^m,c_M^m,w_M^m,z_M^m\right)$$

and introduce the piecewise constant interpolations q_M, q_M^-,t_M,t_M^- and the linear interpolation $\hat{q}_M$ as

$$t_M:=\min\{m\tau\,|\,m\in\mathbb{N}_0 \text{ and } m\tau\geqslant t\},$$

$$
\begin{aligned}
t_M^- &:= \min\{(m-1)\tau \,|\, m \in \mathbb{N}_0 \text{ and } m\tau \geqslant t\},\\
q_M(t) &:= q_M^m \text{ for } t \in \big((m-1)\tau, m\tau\big],\\
q_M^-(t) &:= q_M^m \text{ for } t \in \big[m\tau, (m+1)\tau\big),\\
\hat{q}_M(t) &:= \beta q_M^m + (1-\beta) q_M^{m-1} \text{ for } t \in \big[(m-1)\tau, m\tau\big) \text{ and } \beta = \frac{t}{\tau} - (m-1).
\end{aligned}
$$

Due to the minimization properties of (u_M^m, c_M^m, z_M^m), we establish the following variational formulas and energy estimate (cf. [27, Lemma 6.2]).

Lemma 3.1 (Euler–Lagrange equation, energy estimate) *The functions q_M, q_M^- and $\hat{q}_M$ satisfy the following properties for all $t \in (0,T)$:*

(i) *For all $\zeta \in H^1(\Omega;\mathbb{R}^N)$:*

$$\int_\Omega (\partial_t \hat{c}_M(t)) \cdot \zeta \,\mathrm{d}x = -\langle \mathscr{S} w_M(t), \zeta\rangle. \tag{14}$$

(ii) *For all $\zeta \in H^1(\Omega;\mathbb{R}^N)$:*

$$
\begin{aligned}
\int_\Omega w_M(t)\cdot\zeta\,\mathrm{d}x = &\int_\Omega \mathbb{P}\Gamma\nabla c_M(t) : \nabla\zeta + \mathbb{P}W^{\text{ch,pol}}_{,c}(c_M(t))\cdot\zeta\,\mathrm{d}x\\
&+\int_\Omega \mathbb{P}W^{\text{el}}_{,c}(e(u_M(t)), c_M(t), z_M(t))\cdot\zeta + \varepsilon\partial_t\hat{c}_M(t)\cdot\zeta\,\mathrm{d}x.
\end{aligned} \tag{15}
$$

(iii) *For all $\zeta \in W^{1,4}_D(\Omega;\mathbb{R}^n)$:*

$$\int_\Omega W^{\text{el}}_{,e}(e(u_M(t)), c_M(t), z_M(t)) : e(\zeta) + \varepsilon|\nabla u_M(t)|^2\nabla u_M(t) : \nabla\zeta\,\mathrm{d}x = 0. \tag{16}$$

(iv) *For all $\zeta \in W^{1,p}(\Omega)$ with $0 \leqslant \zeta + z_M(t) \leqslant z_M^-(t)$:*

$$
\begin{aligned}
&\int_\Omega (\varepsilon|\nabla z_M(t)|^{p-2} + 1)\nabla z_M(t)\cdot\nabla\zeta + W^{\text{el}}_{,z}(e(u_M(t)), c_M(t), z_M(t))\zeta\,\mathrm{d}x\\
&\quad+\int_\Omega(-\alpha + \beta(\partial_t\hat{z}_M(t)))\zeta\,\mathrm{d}x \geqslant 0.
\end{aligned} \tag{17}
$$

(v) *Energy estimate:*

$$
\begin{aligned}
&\mathscr{E}(u_M(t), c_M(t), z_M(t)) + \int_0^{t_M}\int_\Omega -\alpha\partial_t\hat{z}_M + \frac{\beta}{2}|\partial_t\hat{z}_M|^2 + \frac{\varepsilon}{2}|\partial_t\hat{c}_M|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^{t_M}\frac{1}{2}\langle\mathscr{S}w_M(s), w_M(s)\rangle\,\mathrm{d}s - \mathscr{E}(u^0, c^0, z^0)\\
&\leqslant \int_0^{t_M}\int_\Omega W^{\text{el}}_{,e}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\varepsilon\int_0^{t_M}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{18}
$$

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2. Step: identifying convergent subsequences.

The energy estimate (v) in Lemma 3.1, growth condition (A4) and the Gronwall estimation argument lead to *a priori* estimates for the energy $\mathscr{E}(u_M(t), c_M(t), z_M(t))$ and for $\|\partial_t \hat{z}_M\|_{L^2(\Omega_T)}$, $\|\partial_t \hat{c}_M\|_{L^2(\Omega_T)}$ and $\int_0^T \langle \mathscr{S} w_M(s), w_M(s)\rangle \,\mathrm{d}s$. By standard compactness arguments and a compactness theorem from Aubin and Lions (cf. [34]), we deduce the following weak convergence properties (cf. [27]).

Lemma 3.2 *There exist a subsequence* $\{M_k\}$ *and an element* $q = (u, c, w, z)$ *satisfying (i) from Definition 2.2 such that for a.e.* $t \in [0, T]$:

(i) $u_{M_k} \overset{\star}{\rightharpoonup} u$ *in* $L^\infty(0, T; W^{1,4}(\Omega))$,

(ii) $c_{M_k}, c^-_{M_k} \overset{\star}{\rightharpoonup} c$ *in* $L^\infty(0, T; H^1(\Omega; \mathbb{R}^N))$,
$c_{M_k}(t), c^-_{M_k}(t) \rightharpoonup c(t)$ *in* $H^1(\Omega; \mathbb{R}^N)$,
$c_{M_k}, c^-_{M_k} \to c$ *a.e. in* Ω_T,
$\hat{c}_{M_k} \rightharpoonup c$ *in* $H^1(0, T; L^2(\Omega; \mathbb{R}^N))$,

(iii) $z_{M_k}, z^-_{M_k} \overset{\star}{\rightharpoonup} z$ *in* $L^\infty(0, T; W^{1,p}(\Omega))$,
$z_{M_k}(t), z^-_{M_k}(t) \rightharpoonup z(t)$ *in* $W^{1,p}(\Omega)$,
$z_{M_k}, z^-_{M_k} \to z$ *a.e. in* Ω_T,
$\hat{z}_{M_k} \rightharpoonup z$ *in* $H^1(0, T; L^2(\Omega))$

and

(iv) $w_{M_k} \rightharpoonup w$ *in* $L^2(0, T; H^1(\Omega; \mathbb{R}^N))$ *for C–H systems,*
$w_{M_k} \rightharpoonup w$ *in* $L^2(\Omega_T; \mathbb{R}^N)$ *for A–C systems*

as $k \to \infty$.

Exploiting the Euler–Lagrange equations, we can even prove stronger convergence properties. To proceed, we recall an approximation lemma from [27].

Lemma 3.3 ([27, Lemma 5.2]) *Let* $q \geqslant 1$, $p > n$ *and* $f, \zeta \in L^q(0, T; W^{1,p}_+(\Omega))$ *with* $\{\zeta = 0\} \supseteq \{f = 0\}$. *Furthermore, let* $\{f_M\}_{M\in\mathbb{N}} \subseteq L^q(0, T; W^{1,p}_+(\Omega))$ *be a sequence with* $f_M(t) \rightharpoonup f(t)$ *in* $W^{1,p}(\Omega)$ *as* $M \to \infty$ *for a.e.* $t \in [0, T]$. *Then there exist a sequence* $\{\zeta_M\}_{M\in\mathbb{N}} \subseteq L^q(0, T; W^{1,p}_+(\Omega))$ *and constants* $v_{M,t} > 0$ *such that*

(i) $\zeta_M \to \zeta$ *in* $L^q(0, T; W^{1,p}(\Omega))$ *as* $M \to \infty$,

(ii) $\zeta_M \leqslant \zeta$ *a.e. in* Ω_T *for all* $M \in \mathbb{N}$,

(iii) $v_{M,t}\zeta_M(t) \leqslant f_M(t)$ *a.e. in* Ω *for a.e.* $t \in [0, T]$ *and for all* $M \in \mathbb{N}$.

If, in addition, $\zeta \leqslant f$ *a.e. in* Ω_T, *then condition (iii) can be refined to*

(iii)' $\zeta_M \leqslant f_M$ *a.e. in* Ω_T *for all* $M \in \mathbb{N}$.

We are now able to prove strong convergence results by using uniform convexity estimates.

Lemma 3.4 (Strong convergence of the time incremental solutions) *There exists a subsequence* $\{M_k\}$ *such that for a.e.* $t \in [0, T]$:

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(i) $u_{M_k}, u^-_{M_k} \to u$ *in* $L^4(0,T;W^{1,4}(\Omega;\mathbb{R}^n))$,
$u_{M_k}(t), u^-_{M_k}(t) \to u(t)$ *in* $W^{1,4}(\Omega;\mathbb{R}^n)$,
$u_{M_k}, u^-_{M_k} \to u$ *a.e. in* Ω_T,

(ii) $c_{M_k}, c^-_{M_k} \to c$ *in* $L^{2^\star}(0,T;H^1(\Omega;\mathbb{R}^N))$,
$c_{M_k}(t), c^-_{M_k}(t) \to c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$,
$c_{M_k}, c^-_{M_k} \to c$ *a.e. in* Ω_T,
$\hat{c}_{M_k} \rightharpoonup c$ *in* $H^1(0,T;L^2(\Omega;\mathbb{R}^N))$,

(iii) $z_{M_k}, z^-_{M_k} \to z$ *in* $L^p(0,T;W^{1,p}(\Omega))$,
$z_{M_k}(t), z^-_{M_k}(t) \to z(t)$ *in* $W^{1,p}(\Omega)$,
$z_{M_k}, z^-_{M_k} \to z$ *a.e. in* Ω_T,
$\hat{z}_{M_k} \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$

as $k \to \infty$.

Proof We omit the index k in the proof.

(i) We refer to [27, Lemma 5.9].

(ii) The weak convergence properties for c_M, c^-_M and $\hat{c}_{M_k}$ follow from Lemma 3.2. It remains to show strong convergence of ∇c_M to ∇c in $L^2(\Omega_T;\mathbb{R}^N)$.

By the compact embedding $H^1(\Omega;\mathbb{R}^N) \hookrightarrow L^{2^\star/2+1}(\Omega;\mathbb{R}^N)$ and Lemma 3.2, we get $\|c_M(t)-c(t)\|_{L^{2^\star/2+1}(\Omega;\mathbb{R}^N)} \to 0$ as $M\to\infty$ for a.e. $t\in[0,T]$. The boundedness property $\operatorname{ess\,sup}_{t\in[0,T]}\|c_M(t)-c(t)\|_{L^{2^\star/2+1}(\Omega;\mathbb{R}^N)} < C$ for all $M\in\mathbb{N}$ and Lebesgue's convergence theorem yield $c_M \to c$ as $M\to\infty$ in $L^{2^\star/2+1}(\Omega_T;\mathbb{R}^N)$. Testing (15) with $\zeta = c_M(t)$ and $\zeta = c(t)$ gives after integration from $t=0$ to $t=T$:

$$\int_{\Omega_T} \mathbb{P}\Gamma\nabla c_M : \nabla c_M \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} w_M\cdot c_M - \mathbb{P}W^{\mathrm{ch,pol}}_{,c}(c_M)\cdot c_M \,\mathrm{d}x\mathrm{d}t - \int_{\Omega_T} \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_M),c_M,z_M)\cdot c_M + \varepsilon\partial_t\hat{c}_M\cdot c_M \,\mathrm{d}x\mathrm{d}t,$$

$$\int_{\Omega_T} \mathbb{P}\Gamma\nabla c_M : \nabla c \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} w_M\cdot c - \mathbb{P}W^{\mathrm{ch,pol}}_{,c}(c_M)\cdot c \,\mathrm{d}x\mathrm{d}t - \int_{\Omega_T} \mathbb{P}W^{\mathrm{el}}_{,c}(e(u_M),c_M,z_M)\cdot c + \varepsilon\partial_t\hat{c}_M\cdot c \,\mathrm{d}x\mathrm{d}t.$$

Passing to $M\to\infty$ and comparing the right sides of the equations show

$$\int_{\Omega_T} \mathbb{P}\Gamma\nabla c_M : \nabla c_M \,\mathrm{d}x\mathrm{d}t \to \int_{\Omega_T} \mathbb{P}\Gamma\nabla c : \nabla c \,\mathrm{d}x\mathrm{d}t.$$

By using the properties $\mathbb{P}\nabla c_M = \nabla c_M$ and $\mathbb{P}\nabla c = \nabla c$, we eventually obtain

$$\int_{\Omega_T} \Gamma\nabla c_M : \nabla c_M \,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \Gamma\nabla c : \nabla c \,\mathrm{d}x\,\mathrm{d}t.$$

We end up with

$$\int_{\Omega_T} \Gamma(\nabla c_M - \nabla c) : (\nabla c_M - \nabla c)\,\mathrm{d}x\,\mathrm{d}t \to 0.$$

Therefore, $\nabla c_M \to \nabla c$ in $L^2(\Omega_T;\mathbb{R}^N)$ since Γ is positive definite.

(iii) Applying Lemma 3.3 with $f = z$ and $f_M = z^-_M$ and $\zeta = z$ gives an approximation

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sequence $\{\zeta_M\} \subseteq L^p(0,T;W^{1,p}_+(\Omega))$ with the properties:

$$\zeta_M \to z \text{ in } L^p(0,T;W^{1,p}(\Omega)), \tag{19a}$$
$$0 \leqslant \zeta_M \leqslant z_M^- \text{ for all } M \in \mathbb{N}. \tag{19b}$$

The estimate

$$C_{\mathrm{uc}}|\nabla z_M - \nabla z|^p \leqslant (|\nabla z_M|^{p-2}\nabla z_M - |\nabla z|^{p-2}\nabla z)\cdot\nabla(z_M - z)$$

where $C_{\mathrm{uc}} > 0$ is a constant, and equation (17) tested with $\zeta = \zeta_M(t) - z_M(t)$ (possibly due to (19b)) yields:

$$\begin{aligned}
&C_{\mathrm{uc}}\int_{\Omega_T}\varepsilon|\nabla z_M - \nabla z|^p\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T}|\nabla z_M - \nabla z|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad\leqslant \int_{\Omega_T}\big((\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M - (\varepsilon|\nabla z|^{p-2}+1)\nabla z\big)\cdot\nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t\\
&\quad\leqslant \int_{\Omega_T}(\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M\cdot\nabla(z_M - \zeta_M)\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{\Omega_T}(\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M\cdot\nabla(\zeta_M - z) - (\varepsilon|\nabla z|^{p-2}+1)\nabla z\cdot\nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t\\
&\quad\leqslant \int_{\Omega_T}(W^{\mathrm{el}}_{,z}(e(u_M),c_M,z_M) - \alpha + \beta\partial_t\hat{z}_M)(\zeta_M - z_M)\,\mathrm{d}x\mathrm{d}t\\
&\qquad+\int_{\Omega_T}(\varepsilon|\nabla z_M|^{p-2}+1)\nabla z_M\cdot\nabla(\zeta_M - z) - (\varepsilon|\nabla z|^{p-2}+1)\nabla z\cdot\nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t\\
&\quad\leqslant \underbrace{\|W^{\mathrm{el}}_{,z}(e(u_M),c_M,z_M) - \alpha + \beta\partial_t\hat{z}_M\|_{L^2(\Omega_T)}}_{\text{bounded}}\|\zeta_M - z_M\|_{L^2(\Omega_T)}\\
&\qquad+\underbrace{(\varepsilon\|\nabla z_M\|^{p-1}_{L^p(\Omega_T)} + \|\nabla z_M\|_{L^{p/(p-1)}(\Omega_T)})}_{\text{bounded}}\|\nabla\zeta_M - \nabla z\|_{L^p(\Omega_T)}\\
&\qquad-\int_{\Omega_T}(\varepsilon|\nabla z|^{p-2}+1)\nabla z\cdot\nabla(z_M - z)\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

Due to (19a) and $z_M \overset{\star}{\rightharpoonup} z$ in $L^\infty(0,T;W^{1,p}(\Omega))$ as well as $z_M \to z$ in $L^2(\Omega_T)$, each term on the right-hand side converges to 0 as $M \to \infty$. □

3. Step: establishing a precise energy inequality.

In this step we establish an asymptotic energy inequality, which is sharper than the energy inequality in (18). Note that compared with (18), the factor 1/2 in front of $\langle\mathscr{S}w_M(s), w_M(s)\rangle$ is missing. To simplify notation, we omit the index k in the following.

Lemma 3.5 *For every $t \in [0,T]$:*

$$\begin{aligned}
&\mathscr{E}(u_M(t),c_M(t),z_M(t)) + \int_0^{t_M}\int_\Omega -\alpha\partial_t\hat{z}_M + \beta|\partial_t\hat{z}_M|^2 + \varepsilon|\partial_t\hat{c}_M|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^{t_M}\langle\mathscr{S}w_M(s),w_M(s)\rangle\,\mathrm{d}s - \mathscr{E}(u^0,c^0,z^0)
\end{aligned}$$

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$$\leqslant \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \varepsilon \int_0^{t_M}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2 \nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\,\mathrm{d}s + \kappa_M$$

with $\kappa_M \to 0$ *as* $M \to \infty$.

Proof Applying the estimate $\mathbb{E}_M^m(q_M^m) \leqslant \mathbb{E}_M^m(u_M^{m-1} + b_M^m - b_M^{m-1}, c_M^m, z_M^m)$ for $m = 1$ to $\frac{t_M}{\tau}$ yields (cf. [27, Lemma 6.10]):

$$\begin{aligned}
&\mathscr{E}(u_M(t), c_M(t), z_M(t)) - \mathscr{E}(u^0, c^0, z^0)\\
&\leqslant \varepsilon \int_0^{t_M}\int_\Omega |\nabla(u_M^- + b(s) - b_M^-)|^2 \nabla(u_M^- + b(s) - b_M^-) : \nabla\partial_t b(s)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-), c_M^-, z_M^-) : e(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \underbrace{\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,c}(e(u_M^- + b_M - b_M^-), \hat{c}_M, z_M^-)\cdot\partial_t\hat{c}_M\,\mathrm{d}x\,\mathrm{d}s}_{(\star)_1}\\
&\quad + \underbrace{\int_0^{t_M}\int_\Omega \Gamma\nabla\hat{c}_M : \nabla\partial_t\hat{c}_M + W^{\mathrm{ch,pol}}_{,c}(\hat{c}_M)\cdot\partial_t\hat{c}_M\,\mathrm{d}x\,\mathrm{d}s}_{(\star)_2}\\
&\quad + \underbrace{\int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,z}(e(u_M^- + b_M - b_M^-), c_M, \hat{z}_M)\partial_t\hat{z}_M\,\mathrm{d}x\,\mathrm{d}s}_{(\star\star)_1}\\
&\quad + \underbrace{\int_0^{t_M}\int_\Omega \varepsilon|\nabla\hat{z}_M|^{p-2}\nabla\hat{z}_M\cdot\nabla\partial_t\hat{z}_M + \nabla\hat{z}_M\cdot\nabla\partial_t\hat{z}_M\,\mathrm{d}x\,\mathrm{d}s}_{(\star\star)_2}.
\end{aligned}\tag{20}$$

The elementary inequalities

$$(|\nabla\hat{z}_M|^{p-2}\nabla\hat{z}_M - |\nabla z_M|^{p-2}\nabla z_M)\cdot\nabla\partial_t\hat{z}_M \leqslant 0 \quad\text{and}\quad (\nabla\hat{z}_M - \nabla z_M)\cdot\nabla\partial_t\hat{z}_M \leqslant 0$$

and (17) tested with $\zeta := -\partial_t\hat{z}_M(t)\tau$ lead to the following estimate:

$$\begin{aligned}
&(\star\star)_1 + (\star\star)_2\\
&\leqslant -\int_0^{t_M}\int_\Omega -\alpha\partial_t\hat{z}_M + \beta|\partial_t\hat{z}_M|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \underbrace{\int_0^{t_M}\int_\Omega (W^{\mathrm{el}}_{,z}(e(u_M^- + b_M - b_M^-), c_M, \hat{z}_M) - W^{\mathrm{el}}_{,z}(e(u_M), c_M, z_M))\partial_t\hat{z}_M\,\mathrm{d}x\,\mathrm{d}s}_{=:\kappa_M^3}.
\end{aligned}$$

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Furthermore,

$$(\star)_1 \leqslant \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,c}(e(u_M), c_M, z_M)\cdot \partial_t \hat{c}_M \,\mathrm{d}x\,\mathrm{d}s \\ + \underbrace{\int_0^{t_M}\int_\Omega (W^{\mathrm{el}}_{,c}(e(u_M^- + b_M - b_M^-), \hat{c}_M, z_M^-) - W^{\mathrm{el}}_{,c}(e(u_M), c_M, z_M))\cdot \partial_t \hat{c}_M \,\mathrm{d}x\,\mathrm{d}s.}_{=:\kappa_M^1}$$

Using the elementary estimate $\Gamma(\nabla\hat{c}_M - \nabla c_M) : \nabla\partial_t\hat{c}_M \leqslant 0$ gives

$$(\star)_2 \leqslant \int_0^{t_M}\int_\Omega \Gamma\nabla c_M : \nabla\partial_t\hat{c}_M + W^{\mathrm{ch,pol}}_{,c}(c_M)\cdot\partial_t\hat{c}_M\,\mathrm{d}x\,\mathrm{d}s \\ + \underbrace{\int_0^{t_M}\int_\Omega (W^{\mathrm{ch,pol}}_{,c}(\hat{c}_M) - W^{\mathrm{ch,pol}}_{,c}(c_M))\cdot\partial_t\hat{c}_M\,\mathrm{d}x\,\mathrm{d}s.}_{=:\kappa_M^2}$$

Hence, applying equation (15) with $\zeta = \partial_t\hat{c}_M(t)$ and (14) with $\zeta = w_M(t)$ by noticing $\mathbb{P}\partial_t\hat{c}_M(t) = \partial_t\hat{c}_M(t)$ shows

$$(\star)_1 + (\star)_2 \leqslant -\int_0^{t_M}\langle \mathscr{S}w_M(s), w_M(s)\rangle\,\mathrm{d}s - \int_0^{t_M}\int_\Omega \varepsilon|\partial_t\hat{c}_M|^2\,\mathrm{d}x\,\mathrm{d}s + \kappa_M^1 + \kappa_M^2.$$

Lebesgue's generalized convergence theorem, growth conditions (A5)–(A7) and Lemma 3.4 show $\kappa_M := \kappa_M^1 + \kappa_M^2 + \kappa_M^3 \to 0$ as $M\to\infty$. We would like to emphasize that we need the boundedness of ∇u_M in $L^4(\Omega_T;\mathbb{R}^{n\times n})$ and the boundedness of $\partial_t\hat{c}_M$ and $\partial_t\hat{z}_M$ in $L^2(\Omega_T)$ with respect to M. □

4. Step: passing to $M\to\infty$.

Using Lemmas 3.2 and 3.4 and (14), (15) and (16) we establish (ii), (iii) and (iv) of Definition 2.2. Moreover, Lemma 3.5 implies

$$\mathscr{E}(u_M(t), c_M(t), z_M(t)) + \int_{\Omega_t} -\alpha\partial_t\hat{z}_M + \beta|\partial_t\hat{z}_M|^2 + \varepsilon|\partial_t\hat{c}_M|^2\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t \langle\mathscr{S}w_M(s), w_M(s)\rangle\,\mathrm{d}s - \mathscr{E}(u^0, c^0, z^0) \\ \leqslant \int_0^{t_M}\int_\Omega W^{\mathrm{el}}_{,e}(e(u_M^- + b - b_M^-), c_M^-, z_M) : e(\partial_t b)\,\mathrm{d}x\,\mathrm{d}s \\ + \varepsilon\int_0^{t_M}\int_\Omega |\nabla u_M^- + \nabla b - \nabla b_M^-|^2\nabla(u_M^- + b - b_M^-) : \nabla\partial_t b\,\mathrm{d}x\,\mathrm{d}s + \kappa_M.$$

The energy estimate (vi) from Definition 2.2 follows from above by using the known convergence properties and weak semi-continuity arguments.

It remains to show (v) of Definition 2.2. To proceed, we cite the following lemma from [27] which provides a tool to drop a restriction on the space of test-functions for a variational inequality of a specific form.

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Lemma 3.6 ([27, Lemma 5.3]) *Let $f \in L^p(\Omega;\mathbb{R}^n)$, $g \in L^p(\Omega)$ and $z \in W^{1,p}(\Omega)$ with $z \geqslant 0$, $f \cdot \nabla z \geqslant 0$ and $\{f = 0\} \supseteq \{z = 0\}$ a.e. Furthermore, we assume that*

$$\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x \geqslant 0 \quad \textit{for all } \zeta \in W^{1,p}_-(\Omega) \textit{ with } \{\zeta = 0\} \supseteq \{z = 0\}.$$

Then

$$\int_\Omega f \cdot \nabla\zeta + g\zeta \,\mathrm{d}x \geqslant \int_{\{z=0\}} [g]^+\zeta \,\mathrm{d}x \quad \textit{for all } \zeta \in W^{1,p}_-(\Omega).$$

We are now able to prove the remaining property.

Lemma 3.7 *We have*

$$\int_\Omega (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t) \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x$$
$$\geqslant -\langle r(t), \zeta\rangle, \tag{21}$$

for all $\zeta \in W^{1,p}_-(\Omega)$ and for a.e. $t \in [0, T]$, where $r(t) \in L^1(\Omega) \subseteq (W^{1,p}(\Omega))^\star$ is given by

$$r(t) := -\chi_{\{z(t)=0\}}[W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+. \tag{22}$$

Proof First of all, we take any test-function $\zeta \in L^p(0, T; W^{1,p}_-(\Omega))$ with $\{\zeta = 0\} \supseteq \{z = 0\}$. Lemma 3.3 gives a sequence $\{\zeta_M\} \subseteq L^p(0, T; W^{1,p}_-(\Omega))$ with $\zeta_M \to \zeta$ in $L^p(0, T; W^{1,p}(\Omega))$ and $0 \geqslant v\zeta_M(t) \geqslant -z_M(t)$, where v depends on M and t. Therefore, (17) holds for $\zeta = \zeta_M(t)$. Integration from 0 to T and passing to $M \to \infty$ gives

$$\int_{\Omega_T} (\varepsilon|\nabla z|^{p-2} + 1)\nabla z \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u), c, z) - \alpha + \beta(\partial_t z))\zeta \,\mathrm{d}x\mathrm{d}t \geqslant 0.$$

In other words,

$$\int_\Omega (\varepsilon|\nabla z(t)|^{p-2} + 1)\nabla z(t) \cdot \nabla\zeta + W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))\zeta \,\mathrm{d}x$$
$$+ \int_\Omega (-\alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x \geqslant 0$$

holds for every $\zeta \in W^{1,p}_-(\Omega)$ with $\{\zeta = 0\} \supseteq \{z(t) = 0\}$ and a.e. $t \in [0, T]$. To finish the proof, we need to extend the variational inequality to the whole space $W^{1,p}_-(\Omega)$.

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Setting $f = (\varepsilon|\nabla z(t)|^{p-2}+1)\nabla z(t)$ and $g = W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t))$, Lemma 3.6 shows for every $\zeta \in W^{1,p}_{-}(\Omega)$

$$\begin{aligned}
&\int_\Omega (\varepsilon|\nabla z(t)|^{p-2}+1)\nabla z(t)\cdot\nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t)))\zeta\,\mathrm{d}x\\
&\quad \geqslant \int_{\{z(t)=0\}} [W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t)) - \alpha + \beta(\partial_t z(t))]^+\zeta\,\mathrm{d}x\\
&\quad \geqslant \int_{\{z(t)=0\}} [W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+\zeta\,\mathrm{d}x.
\end{aligned}$$

Now variational inequality (21) follows by setting

$$r(t) := -\chi_{\{z(t)=0\}}[W^{\mathrm{el}}_{,z}(e(u(t)), c(t), z(t))]^+. \qquad \square$$

Remark 3.8 Lemma 3.7 gives more information than (v) from Definition 2.2. It provides a special choice for $r(t)$ given by (22).

4 Existence of weak solutions of (S_0) – polynomial case

In this section we show that an appropriate subsequence of regularized solutions q_ε for $\varepsilon \in (0,1]$ of Definition 2.2 converges in 'some sense' to q, which satisfies the limit equations given in Definition 2.3. Besides this the initial damage profile z^0 in this section is in $H^1(\Omega)$. We approximate $z^0 \in H^1(\Omega)$ by a sequence $\{z^0_\varepsilon\}$ in $W^{1,p}(\Omega)$ such that $z^0_\varepsilon \to z^0$ in $H^1(\Omega)$ as $\varepsilon \searrow 0$.

Using the energy inequality and Gronwall's inequality, we establish again the following energy estimate.

Lemma 4.1 *We have*

$$\begin{aligned}
&\mathscr{E}_\varepsilon(u_\varepsilon(t), c_\varepsilon(t), z_\varepsilon(t)) + \int_0^t\int_\Omega -\alpha\partial_t z_\varepsilon + \beta|\partial_t z_\varepsilon|^2 + \varepsilon|\partial_t c_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle \mathscr{S}w_\varepsilon(s), w_\varepsilon(s)\rangle\,\mathrm{d}s\\
&\quad \leqslant C(\mathscr{E}_\varepsilon(u^0_\varepsilon, c^0, z^0_\varepsilon)+1)
\end{aligned}$$

for a.e. $t \in [0,T]$ *and every* $\varepsilon \in (0,1]$.

Since $\mathscr{E}_\varepsilon(u^0_\varepsilon, c^0, z^0_\varepsilon) \leqslant \mathscr{E}_\varepsilon(u^0_1, c^0, z^0_\varepsilon) \leqslant \mathscr{E}_1(u^0_1, c^0, z^0_\varepsilon)$, the left-hand side is also uniformly bounded with respect to a.e. $t \in [0,T]$ and every $\varepsilon \in (0,1]$. By using standard compactness theorems and uniform convexity properties of W^{el} (see (A3)), we obtain the following convergence properties (cf. [27]).

Lemma 4.2 (**Convergence properties of** q_ε) *There exists a subsequence* $\{\varepsilon_k\}$ *with* $\varepsilon_k \searrow 0$ *as* $k \to \infty$ *and an element* $q = (u, c, w, z)$ *satisfying (i) of Definition 2.3 such that for a.e.* $t \in [0,T]$

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(i) $u_{\varepsilon_k} \to u$ *in* $L^2(0,T;H^1(\Omega;\mathbb{R}^n))$,
$\sqrt[3]{\varepsilon_k}\nabla u_{\varepsilon_k} \to 0$ *in* $L^\infty(0,T;L^4(\Omega;\mathbb{R}^{n\times n}))$,
$u_{\varepsilon_k}(t) \to u(t)$ *in* $H^1(\Omega;\mathbb{R}^n)$,
$u_{\varepsilon_k} \to u$ *a.e. in* Ω_T,
$u^0_{\varepsilon_k} \to u^0$ *in* $H^1(\Omega;\mathbb{R}^n)$,
$\sqrt[3]{\varepsilon_k}\nabla u^0_{\varepsilon_k} \to 0$ *in* $L^4(\Omega;\mathbb{R}^{n\times n})$,

(ii) $c_{\varepsilon_k} \overset{\star}{\rightharpoonup} c$ *in* $L^\infty(0,T;H^1(\Omega;\mathbb{R}^N))$,
$\varepsilon_k\partial_t c_{\varepsilon_k} \to 0$ *in* $L^2(\Omega_T;\mathbb{R}^N)$,
$c_{\varepsilon_k}(t) \rightharpoonup c(t)$ *in* $H^1(\Omega;\mathbb{R}^N)$,
$c_{\varepsilon_k} \to c$ *a.e. in* Ω_T,

(iii) $z_{\varepsilon_k} \overset{\star}{\rightharpoonup} z$ *in* $L^\infty(0,T;H^1(\Omega))$,
$\sqrt[p-1]{\varepsilon_k}\nabla z_{\varepsilon_k} \to 0$ *in* $L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))$,
$z_{\varepsilon_k}(t) \rightharpoonup z(t)$ *in* $H^1(\Omega)$,
$z_{\varepsilon_k} \to z$ *a.e. in* Ω_T,
$z_{\varepsilon_k} \rightharpoonup z$ *in* $H^1(0,T;L^2(\Omega))$

as $k \to \infty$. *We, in addition, obtain for Cahn–Hilliard systems*

$$w_{\varepsilon_k} \rightharpoonup w \text{ in } L^2(0,T;H^1(\Omega;\mathbb{R}^N))$$

and for Allen–Cahn systems

$$\begin{aligned} &w_{\varepsilon_k} \rightharpoonup w \text{ in } L^2(\Omega_T;\mathbb{R}^N),\\ &c_{\varepsilon_k} \rightharpoonup c \text{ in } H^1(0,T;L^2(\Omega;\mathbb{R}^N)) \end{aligned}$$

as $k \to \infty$.

As before, we will omit index k in subscripts below.

Remark 4.3 We would like to mention that the arguments in [27, Lemma 6.14] cannot be adapted to prove strong convergence properties of ∇c_ε and ∇z_ε due to more generous growth condition (A5) as well as the use of Lemma 3.3 where the compact embedding $W^{1,p}(\Omega) \hookrightarrow \mathscr{C}^{0,\alpha}(\overline{\Omega})$ for $p > n$ with $\alpha > 0$ and $\alpha < 1 - \frac{n}{p}$ is exploited.

We are now able to establish existence of weak solutions of (S_0) in the polynomial case.

Proof of Theorem 2.5 Whenever we refer in the following to (7)–(11) the functions u, c, w, z and r are substituted by $u_\varepsilon, c_\varepsilon, w_\varepsilon, z_\varepsilon$ and r_ε. Moreover, Lemma 4.2 is used without mention in the following.

(i) Let $\zeta \in L^2(0,T;H^1(\Omega;\mathbb{R}^N))$ with $\partial_t\zeta \in L^2(\Omega_T;\mathbb{R}^N)$ and $\zeta(T) = 0$. Integration from $t = 0$ to $t = T$ of (7) and integration by parts yield

$$\int_{\Omega_T}(c_\varepsilon - c^0)\cdot\partial_t\zeta \,\mathrm{d}x\,\mathrm{d}s = \int_0^T \langle \mathscr{S}w_\varepsilon, \zeta\rangle \,\mathrm{d}s.$$

Passing to $\varepsilon \searrow 0$ shows (ii) of Definition 2.3.

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(ii) Let $\zeta \in L^2(0,T;H^1(\Omega;\mathbb{R}^N)) \cap L^\infty(\Omega_T;\mathbb{R}^N)$. Integration from $t=0$ to $t=T$ of (8) and passing to $\varepsilon \searrow 0$ yield

$$\int_{\Omega_T} w \cdot \zeta \,\mathrm{d}x\,\mathrm{d}s = \int_{\Omega_T} \mathbb{P}\Gamma\nabla c : \nabla\zeta + (\mathbb{P}W^{\mathrm{ch,pol}}_{,c}(c) + \mathbb{P}W^{\mathrm{el}}_{,c}(e(u),c,z)) \cdot \zeta \,\mathrm{d}x\,\mathrm{d}s.$$

Note that

$$\left| \int_{\Omega_T} \varepsilon\partial_t c_\varepsilon \cdot \zeta \,\mathrm{d}x\,\mathrm{d}s \right| \leqslant \varepsilon \|\partial_t c_\varepsilon\|_{L^2(\Omega_T;\mathbb{R}^N)} \|\zeta\|_{L^2(\Omega_T;\mathbb{R}^N)} \to 0$$

as $\varepsilon \searrow 0$. This shows (iii) of Definition 2.3 with $W^{\mathrm{ch}}_{,c} = W^{\mathrm{ch,pol}}_{,c}$.

(iii) Let $\zeta \in W^{1,4}_D(\Omega;\mathbb{R}^n)$ be arbitrary. Passing to $\varepsilon \searrow 0$ in (9) yields for a.e. $t \in [0,T]$

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u(t)),c(t),z(t)) : e(\zeta)\,\mathrm{d}x = 0, \tag{23}$$

by noticing

$$\left| \int_\Omega \varepsilon |\nabla u_\varepsilon(t)|^2 \nabla u_\varepsilon(t) : \nabla\zeta \,\mathrm{d}x \right| \leqslant \varepsilon \|\nabla u_\varepsilon(t)\|^3_{L^4(\Omega)} \|\zeta\|_{L^4(\Omega)} \to 0.$$

A density argument shows that (23) also holds for all $\zeta \in H^1_D(\Omega;\mathbb{R}^n)$. Therefore, (iv) of Definition 2.3 is shown.

(iv) The characteristic functions $\chi_{\{z_\varepsilon=0\}}$ are bounded in $L^\infty(\Omega_T)$ with respect to $\varepsilon \in (0,1]$. We select a subsequence such that $\chi_{\{z_{\varepsilon_k}=0\}} \overset{\star}{\rightharpoonup} \chi$ in $L^\infty(\Omega_T)$ as $k \to \infty$. In the following, we will omit index k in the notation. Integrating (10) from $t=0$ to $t=T$ and passing to $\varepsilon \searrow 0$ show

$$\int_{\Omega_T} \nabla z \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u),c,z) - \alpha + \beta(\partial_t z))\zeta \,\mathrm{d}x \geqslant \int_{\Omega_T} \chi[W^{\mathrm{el}}_{,z}(e(u),c,z)]^+ \zeta \,\mathrm{d}x\,\mathrm{d}s \tag{24}$$

for all $\zeta \in L^p(0,T;W^{1,p}_-(\Omega)) \cap L^\infty(\Omega_T)$. We also used the fact that

$$\left| \int_{\Omega_T} \varepsilon |\nabla z_\varepsilon|^{p-2} \nabla z_\varepsilon \cdot \nabla\zeta \,\mathrm{d}x\,\mathrm{d}s \right| \leqslant \varepsilon \|\nabla z_\varepsilon\|^{p-1}_{L^p(\Omega_T)} \|\nabla\zeta\|_{L^p(\Omega_T)} \to 0.$$

It follows that

$$\begin{aligned} &\int_\Omega \nabla z(t) \cdot \nabla\zeta + (W^{\mathrm{el}}_{,z}(e(u(t)),c(t),z(t)) - \alpha + \beta(\partial_t z(t)))\zeta \,\mathrm{d}x \\ &\quad \geqslant \int_\Omega \chi(t)[W^{\mathrm{el}}_{,z}(e(u(t)),c(t),z(t))]^+ \zeta \,\mathrm{d}x \end{aligned}$$

for all $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ and a.e. $t \in [0,T]$. Set $r := -\chi[W^{\mathrm{el}}_{,z}(e(u),c,z)]^+$. For every $\xi \in L^\infty([0,T])$ with $\xi \geqslant 0$ a.e. on $[0,T]$ and every $\zeta \in H^1_+(\Omega) \cap L^\infty(\Omega)$ we also have

$$0 \geqslant \int_0^T \left(\int_\Omega r_\varepsilon(t)(\zeta - z_\varepsilon(t))\,\mathrm{d}x \right) \xi(t)\,\mathrm{d}t = \int_{\Omega_T} r_\varepsilon(\zeta - z_\varepsilon)\xi \,\mathrm{d}x\mathrm{d}t$$

$$\rightarrow \int_{\Omega_T} r(\zeta - z)\xi \,\mathrm{d}x\mathrm{d}t = \int_0^T \left(\int_{\Omega} r(t)(\zeta - z(t))\,\mathrm{d}x \right) \xi(t)\,\mathrm{d}t.$$

This shows $\int_\Omega r(t)(\zeta - z(t))\,\mathrm{d}x \leqslant 0$ for a.e. $t \in [0, T]$. Hence, we obtain inequalities (v) of Definition 2.3.

(v) Weakly semi-continuity arguments lead to

$$\liminf_{\varepsilon \searrow 0} \left(\mathscr{E}_\varepsilon(u_\varepsilon(t), c_\varepsilon(t), z_\varepsilon(t)) + \int_{\Omega_t} \alpha|\partial_t z_\varepsilon| + \beta|\partial_t z_\varepsilon|^2 + \varepsilon|\partial_t c_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle \mathscr{S} w_\varepsilon, w_\varepsilon\rangle \,\mathrm{d}s \right)$$
$$\geqslant \mathscr{E}(u(t), c(t), z(t)) + \int_{\Omega_t} \alpha|\partial_t z| + \beta|\partial_t z|^2 + \int_0^t \langle \mathscr{S} w, w\rangle \,\mathrm{d}s.$$

Testing (9) with $\zeta = u_\varepsilon^0 - b(0)$ and (iv) of Definition 2.3 with $\zeta = u^0 - b(0)$ yield

$$\begin{aligned}
\varepsilon \int_\Omega |\nabla u_\varepsilon^0|^4 \,\mathrm{d}x = \varepsilon &\int_\Omega |\nabla u_\varepsilon^0|^2 \nabla u_\varepsilon^0 : \nabla b(0)\,\mathrm{d}x \\
&- \int_\Omega W^{\mathrm{el}}_{,e}(e(u_\varepsilon^0), c^0, z_\varepsilon^0) : e(u_\varepsilon^0 - b(0))\,\mathrm{d}x \\
&\rightarrow - \int_\Omega W^{\mathrm{el}}_{,e}(e(u^0), c^0, z^0) : e(u^0 - b(0))\,\mathrm{d}x = 0
\end{aligned}$$

as $\varepsilon \searrow 0$.

Therefore, we can pass to the limit $\varepsilon \searrow 0$ in (11) and obtain (vi) from Definition 2.3. □

5 Higher integrability of the strain tensor

To prove existence results for chemical-free energies of logarithmic type, a higher integrability result for the strain tensor based on [20, 22] will be established. We adapt the higher integrability result for solutions of the elliptic equation of the form

$$\left\{ \begin{array}{ll} \mathrm{div}(W^{\mathrm{el}}_{,e}(e(u), c)) = 0 & \text{on } \Omega_T, \\ W^{\mathrm{el}}_{,e}(e(u), c) \cdot \overrightarrow{v} = \sigma^\star \cdot \overrightarrow{v} & \text{on } (\partial\Omega)_T \end{array} \right\}$$

to our setting with the non-constant Dirichlet boundary data b and the additional damage variable z in (S_0). In the following, we will use the assumption $D = \partial\Omega$.

The proof of the higher integrability result is based on the following special cases of Sobolev–Poincaré inequalities and on a reverse Hölder inequality.

Theorem 5.1 (Sobolev–Poincaré-type inequalities) *Let $1 \leqslant p < n$. There exists a constant $C > 0$ such that*

(i) *for all rectangles $Q \subseteq \mathbb{R}^n$ and all $u \in W^{1,p}(Q)$:*

$$\left(\fint_Q \left| u - \fint_Q u \right|^{p^\star} \right)^{\frac{1}{p^\star}} \leqslant C \left(\fint_Q |\nabla u|^p \right)^{\frac{1}{p}} (\mathrm{diam} Q),$$

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(ii) *for all rectangles* $Q = \prod_{i=1}^{n}(a_i, b_i) \subseteq \mathbb{R}^n$ *and all* $u \in W^{1,p}(Q)$ *with* $u = 0$ *on* $\{(x_1, \ldots, x_{n-1}, a_n) \,|\, a_i \leqslant x_i \leqslant b_i,\ i = 1, \ldots, n-1\} \subseteq \partial Q$ *(in the sense of traces)*:

$$\left(\fint_Q |u|^{p^\star} \right)^{\frac{1}{p^\star}} \leqslant C \left(\fint_Q |\nabla u|^p \right)^{\frac{1}{p}} (\operatorname{diam} Q).$$

Theorem 5.1 can be obtained by considering the corresponding inequalities on the unit cube $(0,1)^n$ (for instance, the case $1 < p < n$ was proven by Sobolev [35], while Nirenberg [33] gave a proof to $p = 1$) and then using a scaling argument.

Theorem 5.2 (Reverse Hölder inequality, see [24]) *Let* $Q \subseteq \mathbb{R}^n$ *be a cube,* $g \in L^q_{\mathrm{loc}}(Q)$ *for some* $q > 1$ *and* $g \geqslant 0$. *Suppose that there exist a constant* $b > 0$ *and a function* $f \in L^r_{\mathrm{loc}}(Q)$ *with* $r > q$ *and* $f \geqslant 0$ *such that*

$$\fint_{Q_R(x_0)} g^q \,\mathrm{d}x \leqslant b \left(\fint_{Q_{2R}(x_0)} g \,\mathrm{d}x \right)^q + \fint_{Q_{2R}(x_0)} f^q \,\mathrm{d}x$$

for each $x_0 \in Q$ *and all* $R > 0$ *with* $2R < \operatorname{dist}(x_0, \partial Q)$. *Then* $g \in L^s_{\mathrm{loc}}(Q)$ *for* $s \in [q, q+\varepsilon)$ *with some* $\varepsilon > 0$ *and*

$$\left(\fint_{Q_R(x_0)} g^s \,\mathrm{d}x \right)^{\frac{1}{s}} \leqslant c \left(\left(\fint_{Q_{2R}(x_0)} g^q \,\mathrm{d}x \right)^{\frac{1}{q}} + \left(\fint_{Q_{2R}(x_0)} f^s \,\mathrm{d}x \right)^{\frac{1}{s}} \right)$$

for all $x_0 \in Q$ *and* $R > 0$ *such that* $Q_{2R}(x_0) \subseteq Q$. *The positive constants* $c, \varepsilon > 0$ *depend on* b, q, n *and* r.

Theorem 5.3 (Higher integrability) *Let* $b \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, $z \in L^\infty(\Omega)$ *with* $0 \leqslant z \leqslant 1$ *a.e. in* Ω *and* $c \in L^\mu(\Omega; \mathbb{R}^N)$ *for some* $\mu > 4$. *Then there exists some* $p \in (2, \mu/2]$ *such that for all* $u \in H^1(\Omega; \mathbb{R}^n)$ *which satisfy* $u|_D = b|_D$ *and*

$$\int_\Omega W^{\mathrm{el}}_{,e}(e(u), c, z) : e(\zeta) \,\mathrm{d}x = 0 \text{ for all } \zeta \in H^1_D(\Omega; \mathbb{R}^n), \tag{25}$$

we obtain $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ *and*

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \leqslant C(\|\nabla u\|_{L^2(\Omega;\mathbb{R}^{n\times n})} + \|c\|^2_{L^{2p}(\Omega;\mathbb{R}^N)} + 1). \tag{26}$$

The positive constants p *and* C *are independent of* u, c, z.

Proof The proof is based on [20, Lemma 4.4 and Theorem 4.3] and uses a covering argument. However, due to the non-constant boundary condition, we need to apply a more general Sobolev–Poincaré inequality (see Theorem 5.1 (ii)) than given in [20].

(i) Higher integrability at the boundary.

Let $x_0 \in \partial\Omega$. Then there exist an $R_0 > 0$ and a bi-Lipschitz function $\tau : Q \to \mathbb{R}^n$

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with the open cube $Q := Q_{R_0}(0)$ such that $x_0 \in \tau(Q)$ and

$$\tau(Q^+) \subseteq \Omega,$$
$$\tau(Q^-) \subseteq \mathbb{R}^n \setminus \overline{\Omega},$$

where $Q^+ := \{x \in Q \,|\, x_n > 0\}$ and $Q^- := \{x \in Q \,|\, x_n < 0\}$. Define the transformed functions $\tilde{u}, \tilde{b} \in H^1(Q^+;\mathbb{R}^n)$, $\tilde{c} \in H^1(Q^+)$ and $\tilde{z} \in L^\infty(Q^+)$ as

$$(\tilde{u}, \tilde{b}, \tilde{c}, \tilde{z})(x) := (u, b, c, z)(\tau(x)).$$

To proceed, let $y_0 \in Q$ and $R < \frac{1}{2}\mathrm{dist}(y_0, \partial Q)$ and define for each $R' > 0$ the sets

$$Q^{\pm}_{R'}(y_0) := \{x \in Q_{R'}(y_0) \,|\, x_n \gtrless 0\}.$$

We distinguish the following three cases:

Case 1. We first consider the case $Q^+_R(y_0) \neq \emptyset$ and $Q^-_{\frac{3}{2}R}(y_0) \neq \emptyset$.

The bi-Lipschitz continuity of τ ensures

$$\mathrm{dist}(\tau(\partial Q^+_{2R}(y_0)) \cap \Omega, \tau(\partial Q^+_R(y_0)) \cap \Omega) > RC_1,$$

where $C_1 > 0$ is independent of R and y_0. Let $\xi \in \mathscr{C}^\infty_0(\Omega)$ be a cutoff function with the following properties:

(a) $\xi = 0$ in $\Omega \setminus \tau(Q_{2R}(y_0))$,

(b) $0 \leqslant \xi \leqslant 1$ in Ω,

(c) $\xi \equiv 1$ in $\tau(Q_R(y_0)) \cap \Omega$,

(d) $|\nabla\xi| \leqslant \frac{2}{C_1} R^{-1}$.

Testing (25) with $\zeta = \xi^2(u - b)$, using the computation

$$e(\zeta) = \xi^2 e(u) - \xi^2 e(b) + \xi((u - b)(\nabla\xi)^t + \nabla\xi(u - b)^t),$$

and (A1), we obtain

$$\begin{aligned} &\int_\Omega \xi^2 W^{\mathrm{el}}_{,e}(e(u), c, z) : e(u)\,\mathrm{d}x \\ &\quad = \int_\Omega \xi^2 W^{\mathrm{el}}_{,e}(e(u), c, z) : e(b)\,\mathrm{d}x - 2\int_\Omega \xi W^{\mathrm{el}}_{,e}(e(u), c, z) : ((u - b)(\nabla\xi)^t)\,\mathrm{d}x. \end{aligned} \tag{27}$$

By (A3), (A4) and (A2) we also have the estimates

$$\begin{aligned} &\eta|e(u)|^2 \leqslant W^{\mathrm{el}}_{,e}(e(u), c, z) : e(u) + C(|c|^2 + 1)|e(u)|, \\ &|W^{\mathrm{el}}_{,e}(e(u), c, z) : ((u - b)(\nabla\xi)^t| \leqslant \frac{C}{R}(|e(u)| + |c|^2 + 1)|u - b|, \\ &|W^{\mathrm{el}}_{,e}(e(u), c, z) : e(b)| \leqslant (|e(u)| + |c|^2 + 1)|e(b)|. \end{aligned}$$

Therefore, (27) can be estimated by

$$\begin{aligned} \eta\int_\Omega \xi^2|e(u)|^2\,\mathrm{d}x \leqslant C\int_\Omega \xi^2(|c|^2 + 1)|e(u)|\,\mathrm{d}x + \frac{C}{R}\int_\Omega \xi(|e(u)| + |c|^2 + 1)|u - b|\,\mathrm{d}x \\ + C\int_\Omega \xi^2(|e(u)| + |c|^2 + 1)|e(b)|\,\mathrm{d}x. \end{aligned}$$

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Young's inequality yields

$$c_1 \int_\Omega \xi^2 |e(u)|^2 \, dx \leqslant C \int_\Omega \xi^2 (|c|^4 + 1) \, dx + \frac{C}{R^2} \int_\Omega |u - b|^2 \, dx. \tag{28}$$

We choose $\mu = \fint_{Q^+_{2R}(y_0)} \tilde{u} \, dx$. The calculation $e(\xi(u-\mu)) = \xi e(u) + \frac{1}{2}((u-\mu)(\nabla\xi)^t + \nabla\xi(u-\mu)^t)$ leads to

$$\int_\Omega |e(\xi(u-\mu))|^2 \, dx \leqslant 2 \left(\int_\Omega \xi^2 |e(u)|^2 \, dx + \int_\Omega |u-\mu|^2 |\nabla\xi|^2 \, dx \right). \tag{29}$$

Combining (28) and (29), applying Korn's inequality for H^1-functions with zero boundary values and using (a) and (b) gives

$$\int_\Omega |\nabla(\xi(u-\mu))|^2 \, dx \leqslant C \int_{\tau(Q^+_{2R}(y_0))} (|c|^4 + 1) \, dx + \frac{C}{R^2} \int_{\tau(Q^+_{2R}(y_0))} |u - b|^2 \, dx + \frac{C}{R^2} \int_{\tau(Q^+_{2R}(y_0))} |u - \mu|^2 \, dx.$$

Because of $\nabla(\xi(u-\mu)) = \xi\nabla u + (u-\mu)(\nabla\xi)^t$ we derive by (a) and (c) the following type of Caccioppoli-inequality:

$$\int_{\tau(Q^+_R(y_0))} |\nabla u|^2 \, dx \leqslant C \int_{\tau(Q^+_{2R}(y_0))} (|c|^4 + 1) \, dx + \frac{C}{R^2} \int_{\tau(Q^+_{2R}(y_0))} |u - b|^2 \, dx + \frac{C}{R^2} \int_{\tau(Q^+_{2R}(y_0))} |u - \mu|^2 \, dx.$$

Integral transformation by τ implies

$$\int_{Q^+_R(y_0)} |\nabla \tilde{u}|^2 \, dx \leqslant C \int_{Q^+_{2R}(y_0)} (|\tilde{c}|^4 + 1) \, dx + \frac{C}{R^2} \int_{Q^+_{2R}(y_0)} |\tilde{u} - \tilde{b}|^2 \, dx + \frac{C}{R^2} \int_{Q^+_{2R}(y_0)} |\tilde{u} - \mu|^2 \, dx.$$

Conditions $Q^-_{\frac{3}{2}R}(y_0) \neq \emptyset$ and $D = \partial\Omega$ imply that $\tilde{u} - \tilde{b}$ vanishes on $\partial\big(Q^+_{2R}(y_0)\big) \cap \mathbb{R}^{n-1} \times \{0\}$. Therefore, we obtain by applying both variants of the Poincaré–Sobolev inequality in Theorem 5.1 for $p = 2n/(n+2)$:

$$\begin{aligned}\int_{Q^+_R(y_0)} |\nabla \tilde{u}|^2 \, dx \leqslant{}& C \int_{Q^+_{2R}(y_0)} (|\tilde{c}|^4 + 1) \, dx + \frac{C}{R^2} \mathscr{L}^n(Q^+_{2R}(y_0))^{-\frac{2}{n}} \operatorname{diam}(Q^+_{2R}(y_0))^2 \\ &\times \left[\left(\int_{Q^+_{2R}(y_0)} |\nabla\tilde{u} - \nabla\tilde{b}|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{n}} + \left(\int_{Q^+_{2R}(y_0)} |\nabla\tilde{u}|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{n}} \right]. \end{aligned} \tag{30}$$

Note that if $n = 1$, we cannot apply Theorem 5.1 because of $p = 2n/(n+2) < 1$. In this case, we can work with inequalities in Theorem 5.1 where p is substituted by 1 and $p^\star$ is substituted by 2. However, we will only treat the more delicate case $n \geqslant 2$ in the following.

The estimates $\operatorname{diam}(Q_{2R}^+(y_0)) \leqslant CR$ and $\mathscr{L}^n(Q_{2R}^+(y_0)) \geqslant R^n$ (because of $Q_R^+(y_0) \neq \emptyset$) show

$$\mathscr{L}^n(Q_{2R}^+(y_0))^{-\frac{2}{n}} \operatorname{diam}(Q_{2R}^+(y_0))^2 \leqslant C. \tag{31}$$

Now dividing (30) by $\mathscr{L}^n(Q_R(y_0))$ and using (31) and

$$\frac{1}{R^2}\frac{1}{\mathscr{L}^n(Q_{2R}(y_0))} \leqslant C\left(\frac{1}{\mathscr{L}^n(Q_{2R}(y_0))}\right)^{\frac{n+2}{n}}$$

gives

$$\begin{aligned}\frac{1}{\mathscr{L}^n(Q_R(y_0))}\int_{Q_R^+(y_0)} |\nabla\tilde{u}|^2\,\mathrm{d}x \leqslant{}& \frac{C}{\mathscr{L}^n(Q_{2R}(y_0))}\int_{Q_{2R}^+(y_0)} (|\tilde{c}|^4+1)\,\mathrm{d}x\\ &+C\left(\frac{1}{\mathscr{L}^n(Q_{2R}(y_0))}\int_{Q_{2R}^+(y_0)} |\nabla\tilde{u}|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}}\\ &+C\left(\frac{1}{\mathscr{L}^n(Q_{2R}(y_0))}\int_{Q_{2R}^+(y_0)} |\nabla\tilde{b}|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}}.\end{aligned}$$

Observe that

$$\left(\frac{1}{\mathscr{L}^n(Q_{2R}(y_0))}\int_{Q_{2R}^+(y_0)} |\nabla\tilde{b}|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}} \leqslant \|\nabla b\|^2_{L^\infty(\Omega)}.$$

Define the following functions on Q:

$$g(x) := \begin{cases} |\nabla\tilde{u}(x)|^{\frac{2n}{n+2}} & \text{for } x \in Q^+,\\ 0 & \text{for } x \in Q\setminus Q^+\end{cases}$$

and

$$f(x) := \begin{cases} C(|\tilde{c}|^4 + \|\nabla b\|^2_{L^\infty(\Omega)}+1)^{\frac{n}{n+2}} & \text{for } x \in Q^+,\\ 0 & \text{for } x \in Q\setminus Q^+.\end{cases}$$

We eventually get

$$⨍_{Q_R(y_0)} g^{\frac{n+2}{n}}\,\mathrm{d}x \leqslant ⨍_{Q_{2R}(y_0)} f^{\frac{n+2}{n}}\,\mathrm{d}x + C\left(⨍_{Q_{2R}(y_0)} g\,\mathrm{d}x\right)^{\frac{n+2}{n}}. \tag{32}$$

Case 2. Assume $Q_R^+(y_0) \neq \emptyset$ and $Q^-_{\frac{3}{2}R}(y_0) = \emptyset$.

The bi-Lipschitz continuity of τ implies

$$\operatorname{dist}(\tau(\partial Q_{\frac{3}{2}R}(y_0)), \tau(\partial Q_R(y_0))) > RC_1,$$

where $C_1 > 0$ is independent of R and y_0. Therefore, we can choose a cutoff function $\xi \in \mathscr{C}_0^\infty(\Omega)$, which satisfies

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(a) $\xi = 0$ in $\Omega \setminus \tau(Q_{\frac{3}{2}R}(x_0))$, (c) $\xi \equiv 1$ in $\tau(Q_R(x_0))$,
(b) $0 \leqslant \xi \leqslant 1$ in Ω, (d) $|\nabla \xi| \leqslant \frac{2}{C_1} R^{-1}$.

Testing (25) with $\xi = \zeta^2(u-\mu)$ and $\mu := \fint_{Q_{\frac{3}{2}R}(x_0)} \tilde{u}\,\mathrm{d}x$ yields as in the previous case

$$\int_{\tau(Q_R(x_0))} |\nabla u|^2\,\mathrm{d}x \leqslant C \int_{\tau(Q_{\frac{3}{2}R}(x_0))} (|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_{\tau(Q_{\frac{3}{2}R}(x_0))} |u-\mu|^2\,\mathrm{d}x.$$

Consequently,

$$\fint_{Q_R(x_0)} |\nabla \tilde{u}|^2\,\mathrm{d}x \leqslant C\fint_{Q_{\frac{3}{2}R}(x_0)} (|\tilde{c}|^4+1)\,\mathrm{d}x + C\left(\fint_{Q_{\frac{3}{2}R}(x_0)} |\nabla \tilde{u}|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}}.$$

Therefore, inequality (32) is also satisfied in this case.

Case 3. Assume $Q_R^+(y_0) = \emptyset$.

In this case, inequality (32) trivially holds.

In all three cases, the reverse Hölder inequality (see Theorem 5.2) shows $g \in L^s_{\mathrm{loc}}(Q)$ for all $s \in \left[\frac{n+2}{n}, \frac{n+2}{n}+\varepsilon\right)$ and some $\varepsilon > 0$ depending on R_0 and n.

(ii) Higher integrability in the interior.

This case follows with much less effort and is only sketched here.

Let $x_0 \in \Omega$ be arbitrary and $R > 0$ such that $Q_{2R}(x_0) \subseteq \Omega$. We take a cutoff function $\xi \in \mathscr{C}_0^\infty(\Omega)$ with

(a) $\xi = 0$ in $\Omega \setminus Q_{2R}(x_0)$, (c) $\xi \equiv 1$ in $Q_R(x_0)$,
(b) $0 \leqslant \xi \leqslant 1$ in Ω, (d) $|\nabla \xi| \leqslant \frac{2}{R}$.

Testing (25) with $\xi = \zeta^2(u-\mu)$ and $\mu = \fint_{Q_{2R}(x_0)} u\,\mathrm{d}x$ yields with the same computation as in case (i):

$$\int_{Q_R(x_0)} |\nabla u|^2\,\mathrm{d}x \leqslant C \int_{Q_{2R}(x_0)} (|c|^4+1)\,\mathrm{d}x + \frac{C}{R^2}\int_{Q_{2R}(x_0)} |u-\mu|^2\,\mathrm{d}x.$$

The Poincaré–Sobolev inequality implies

$$\fint_{Q_R(x_0)} |\nabla u|^2\,\mathrm{d}x \leqslant C\fint_{Q_{2R}(x_0)} (|c|^4+1)\,\mathrm{d}x + C\left(\fint_{Q_{2R}(x_0)} |\nabla u|^{\frac{2n}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{n}}.$$

Applying Theorem 5.2 with $g = |\nabla u|^{\frac{2n}{n+2}}$, $q = \frac{n+2}{n}$ and $f = C(|c|^4+1)^{\frac{n}{n+2}}$ finishes the proof. □

6 Existence of weak solutions of (S_0) – logarithmic case

The challenge here is to establish the integral equation (iii) in Definition 2.3 because the derivative of the logarithmic-free chemical energy (A8) becomes singular if one of the c_k's approaches 0. We only sketch the proof in this section since all essential ideas can

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be found in [20, 22]. We use the regularization method suggested in [16] and also used in [20, 22].

The energy gradient tensor is assumed to be of the form $\Gamma = \gamma\,\mathrm{Id}$ with a constant $\gamma > 0$. Define a $\mathscr{C}^2(\mathbb{R}^N)$ regularization with the regularization parameter $\delta > 0$ as

$$W^{\mathrm{ch},\delta}(c) := \theta \sum_{k=1}^{N} \phi^\delta(c^k) + \frac{1}{2} c \cdot Ac,$$

with

$$\phi^\delta(x) := \begin{cases} x\log(x) & \text{for } d \geqslant \delta, \\ x\log(\delta) - \frac{\delta}{2} + \frac{x^2}{2\delta} & \text{for } x < \delta. \end{cases}$$

Elliott and Luckhaus [16] showed that the regularization $W^{\mathrm{ch},\delta}$ is uniformly bounded from below.

Lemma 6.1 (cf. [16]) *There exist constants $\delta_0 > 0$ and $C > 0$ such that*

$$W^{\mathrm{ch},\delta}(c) \geqslant -C \qquad \textit{for all } c \in \Sigma,\ \delta \in (0, \delta_0).$$

Let q_δ denote a weak solution in the sense of Definition 2.3 with the free chemical energy $W^{\mathrm{ch}} = W^{\mathrm{ch},\delta}$. By applying Lemma 6.1 and using Gronwall's inequality in the energy inequality (vi) of Definition 2.3, we can show *a priori* estimates analogous as in Section 4 except the *a priori* estimate of w_δ.

In the Allen–Cahn case, we have $\partial_t c_\delta = -\mathbb{M} w_\delta$ and consequently the boundedness of c_δ in $L^2(\Omega;\mathbb{R}^N)$ and $w_\delta \in T\Sigma$ pointwise lead to boundedness of w_δ in $L^2(\Omega;\mathbb{R}^N)$.

In the case of Cahn–Hilliard systems, we can use the following lemma.

Lemma 6.2 ([20, Lemma 4.3]) *There exists a constant $C > 0$ such that for all $\delta \in (0, \delta_0)$*

$$\int_0^T \left(\fint_\Omega \mathbb{P} W^{\mathrm{ch},\delta}_{,c}(c_\delta(t))\,\mathrm{d}x \right)^2 \mathrm{d}t < C.$$

The proof of this lemma is similar to [20, Lemma 4.3], since all arguments can be adapted to our case. Therefore, we will omit the proof.

This lemma and the integral equation

$$\int_\Omega w_\delta(t)\,\mathrm{d}x = \int_\Omega \mathbb{P} W^{\mathrm{ch},\delta}_{,c}(c_\delta(t)) + \mathbb{P} W^{\mathrm{el}}_{,c}(e(u_\delta(t)), c_\delta(t), z_\delta(t))\,\mathrm{d}x$$

together with the already known boundedness properties show

$$\int_0^T \left(\fint_\Omega w_\delta(t)\,\mathrm{d}x \right)^2 \mathrm{d}t < C$$

for constant $C > 0$. Therefore, w_δ is bounded in $L^2(0, T; H^1(\Omega))$ by Poincaré's inequality.

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In conclusion, we can extract a subsequence $\{q_{\delta_k}\}$ such that we have the same convergence properties as in Lemma 4.2. As before, we will omit subscript k.

Proof of Theorem 2.6 The remaining crucial step is to show that the limit c satisfies $c_k > 0$ a.e. on Ω_T for all $k = 1, \ldots, N$ and $W^{\mathrm{ch},\delta}_{,c}(c_\delta) \to W^{\mathrm{ch,log}}_{,c}(c)$ in $L^1(\Omega_T)$ as $\varepsilon \searrow 0$.

To this end, we need an additional boundedness property.

Lemma 6.3 *There exists constants $q > 1$ and $C > 0$ such that for all $\delta \in (0, \delta_0)$ and all $k = 1, \ldots, N$*

$$\|(\phi^\delta)'(c^k_\delta)\|_{L^q(\Omega_T)} < C.$$

We omit the proof of this lemma, since by utilizing Theorem 5.3 the arguments are analogous to [20, Lemma 4.5].

Note that

$$\lim_{\delta \searrow 0} (\phi^\delta)'(c^k_\delta) = \begin{cases} \log(c^k) + 1 & \text{if } \lim_{\delta \searrow 0} c^k_\delta = c^k > 0, \\ \infty & \text{otherwise} \end{cases}$$

holds pointwise a.e. on Ω_T and for all $k = 1, \ldots, N$. Together with Lemma 6.3, we obtain

$$c^k > 0 \text{ a.e. on } \Omega_T$$

and

$$(\phi^\delta)'(c^k_\delta) \to \log(c^k) + 1 \text{ a.e. on } \Omega_T.$$

This and Lemma 6.3 further show

$$(\phi^\delta)'(c^k_\delta) \to \log(c^k) + 1 \text{ in } L^1(\Omega_T)$$

by Vitali's convergence theorem. Finally, we can pass to $\delta \searrow 0$ in equation

$$\int_{\Omega_T} w_\delta \cdot \zeta \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} \gamma \nabla c_\delta : \nabla \zeta + \mathbb{P} W^{\mathrm{ch},\delta}_{,c}(c_\delta) \cdot \zeta + \mathbb{P} W^{\mathrm{el}}_{,c}(e(u_\delta), c_\delta, z_\delta) \cdot \zeta \,\mathrm{d}x\mathrm{d}t$$

and obtain (iii) from Definition 2.3.

The remaining properties can be easily established as in Section 4. Hence, Theorem 2.6 is proven. □

7 Conclusion

Materials, which enable the functionality of technical products, change the micro-structure over time. Phase separation and coarsening phenomena take place and the complete failure of electronic devices often results from micro-cracks in solder joints.

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In this work, we have investigated mathematical models describing both phenomena, phase separation and damage processes, in a unifying approach. The main aim has been to prove existence of weak solutions for elastic Cahn–Hilliard and Allen–Cahn systems coupled with damage phenomena under mild assumptions where the free energy contains

- a chemical potential of polynomial or logarithmic type,
- an inhomogeneous elastic energy, e.g. $W^{\text{el}}(e, c, z) = \frac{1}{2}(z + \varepsilon)\mathbb{C}(c)(e - e^{\star}(c)) : (e - e^{\star}(c))$,
- a quadratic gradient term of the damage variable.

To this end, several approximation results have been established, as well as different variational techniques, regularization methods and higher integrability results for the strain have been applied.

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