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Dynamics of two-sign point vortices in positive and negative temperature states

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Abstract. The characteristic features of a two-sign point vortex system in positive and negative temperature states are examined by massive numerical simulations using MDGRAPE-2. The temperature is determined by a density of states for a microcanonical ensemble consisting of randomly generated 10^7 states. Since the density of states has a single peak, the system has negative temperature states. The distributions of vortices in time-asymptotic equilibrium states in positive and negative temperature are obtained by time-development simulations. In positive temperature, both-sign vortices mix with each other and neutralize. In negative temperature, part of the vortices condense and form clumps exclusively consisting of the same-sign vortices, while the other part of the vortices distribute uniformly outside the clumps. It is found that the vortices inside the clumps gain energy and the vortices outside the clumps lose energy to keep the total energy constant. This suggests the common and essential role of the background vortices in the energy-conserving system that assists the formation of the clumps as well as the crystallization and generation of the symmetric configuration observed in the non-neutral plasma experiments.

1. Introduction

Equilibrium states for a two-dimensional Euler fluid have been widely discussed. Axisymmetric and non-axisymmetric solutions in a particular energy range were examined theoretically and numerically [1–4]. One of the remarkable features in a two-dimensional Euler fluid is the formation of large, isolated vortices. To tackle this problem, the concept of negative temperature for two-dimensional point-vortex systems confined in a finite configuration space was introduced by Onsager [5]. Such systems have a finite amount of phase space volume proportional to the number of available microcanonical states. In this case, the density of states $W(E)$ as a function of system energy E converges to zero in the limit of infinite energy, and $W(E)$ has a peak at a certain energy $E = E_c$. Thus, the inverse temperature $\beta = d \ln W(E)/dE$ is negative at $E > E_c$.

In this paper, we examine the characteristics of a system consisting of the same number of positive and negative point vortices in a two-dimensional circular boundary in positive and negative temperature states via numerical simulations

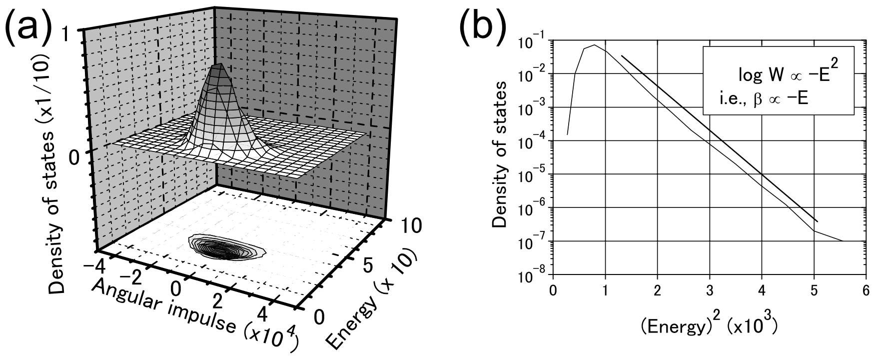

Figure 1. Profiles of the density of states are plotted against: (a) E and I; and (b) E^2 with $I = 0$.

using a special-purpose supercomputer for molecular dynamics simulations called MDGRAPE-2 [6, 7]. A numerically generated density of states exhibits that the system certainly has negative temperature states. The dynamics of such vortex systems are examined by time-development simulations starting from various initial distributions characterized by different values of β. When $\beta < 0$, mergers between patches of the same sign vortices occurs, while when $\beta > 0$, both-sign vortices mix and a neutralized distribution appears. Analyses by energy spectra elucidate an energy partition in the time-asymptotic equilibrium states in positive and negative temperature. It is found from the spectra in negative temperature that a part of the vortices obtain energy to form the clumps and the other part of the vortices lower energy to form the neutralized background distribution. We conclude that the background vortices enable the vortex condensation in energy-conserving systems.

2. Determination of β

We consider a system consisting of $N/2$ positive and $N/2$ negative point vortices bounded in a circular area of radius R with constant strength $\pm\Gamma_0$. The total number of vortices N is typically 6724. The constants of motion are Hamiltonian E and angular impulse I [6]. The boundary is represented by image vortices located at $\bar{\boldsymbol{r}}_i = R^2\boldsymbol{r}_i/|\boldsymbol{r}_i|^2$, where $\boldsymbol{r}_i$ is the position vector of the ith point vortex. To determine the temperature of the system, the density of states $W(E, I)$ is obtained from randomly generated 10^7 states for the microcanonical ensemble where E and I are kept constant. Similar work has been done widely [8, 9]. In our work, the maximum number of vortices is larger by one order of magnitude, and the number of states is larger by two orders of magnitude than the previous results. Thus, we can discuss the characteristics at negative β more precisely. The profile of $W(E, I)$ is shown in Fig. 1. There is a single peak at $(E_c, I_c) = (29.0, 0.0)$ in Fig. 1(a). When the system energy is larger than E_c in the $I = 0$ plane, the temperature is negative. From Fig. 1(b), an unusual dependence where β is proportional to $-E$ is found, while β is proportional to $1/E$ for ideal gases in positive temperature.

3. Characterization of the system by β

Time asymptotic equilibrium distributions of the point vortices are obtained by time-development simulations. The time evolution of each point vortex is traced by

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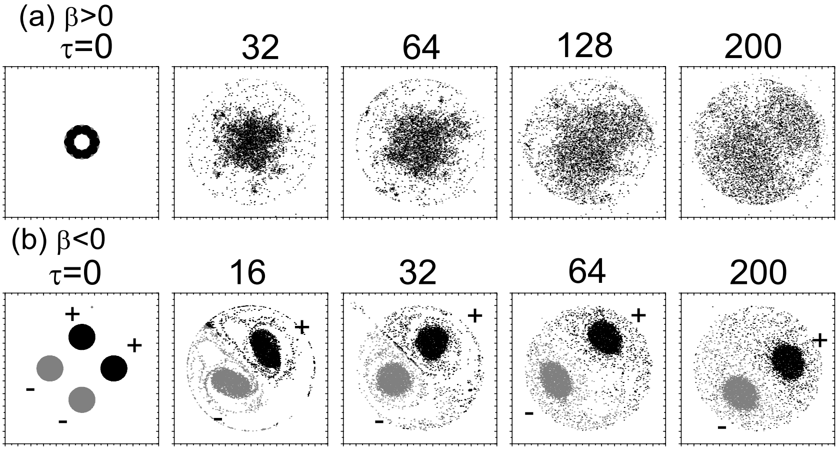

Figure 2. Time evolutions of the vortices in (a) positive and (b) negative temperature states. The values of the energy and angular impulse are: (a) $E = 24.1$, $I = 0$; and (b) $E = 2.69 \times 10^4$, $I = 0$.

the following Biot–Savart integral;

$$\frac{d\boldsymbol{r}_i}{dt} = -\frac{1}{2\pi}\sum_{j\neq i}^{N}\Gamma_j \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^2} + \frac{1}{2\pi}\sum_{j}^{N}\Gamma_j \frac{\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j}{|\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j|^2}, \tag{3.1}$$

where Γ_i is the circulation of the ith point vortex. Two results corresponding to positive and negative β are shown in Fig. 2. The system energy is a constant of motion. Thus, initial energy determined by the initial configuration of the vortices gives the temperature. The positive β case is shown in Fig. 2(a). In the initial doughnut-like profile, positive and negative small clumps are arranged, in turn, at regular intervals. Both-sign vortices mix and broaden during the time evolution. At the final state ($\tau = 200$), the distributions are neutralized. On the other hand, in the negative β case shown in Fig. 2(b), positive vortices condense with the other positive vortices and form a clump as do the negative vortices. By assuming the two clumps as two point vortices located at $(r_0, 0)$ and $(-r_0, 0)$, the energy of a two-vortex system confined in the circular boundary of radius R is evaluated. The energy is maximized when $r_0 = (\sqrt{5} - 2)^{1/2}R$. The two clumps are likely to be arranged so that the energy belonging to the two clumps are maximized under the constraint of the constant total energy. Thus, vortex condensation is enabled due to the existence of the background vortices (outside the clumps) whose energy is absorbed by the vortices inside the clumps. This suggests a common and essential role of the background vortices as is seen in the symmetrization (crystallization) observed in non-neutral plasma experiments.

A characteristic feature in the negative β state also appears in the energy spectrum shown in Fig. 3 defined by a histogram of point vortices as a function of energy of each point vortex. When $\beta > 0$, the profile at a time-asymptotic equilibrium state has a single peak at the average energy of the point vortex and is symmetric about the peak. When $\beta < 0$, there are two peaks in the high- and the low-energy regions. The probability of realizing a state is proportional to the Boltzmann factor $\exp(-\beta E)$. Thus, the peak at the high-energy region indicates $\beta < 0$ and corresponds to the vortices inside the clumps. It is also found that the positive

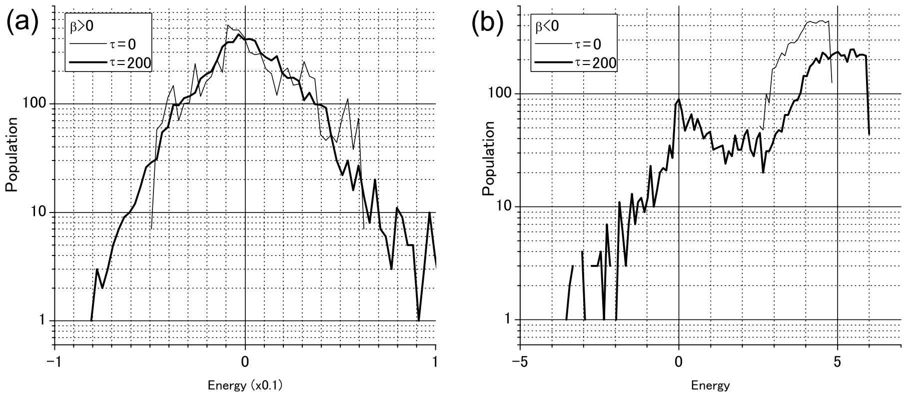

Figure 3. Energy spectra in (a) positive and (b) negative temperature states.

slope at the high-energy region is approximately proportional to the system energy squared. In Fig. 3(b), the profile around the peak at the low-energy region resembles the profile for $\beta > 0$. Thus, the peak in the low-energy region corresponds to the background distribution. It is clearly understood that a part of the vortices gain energy and the others lose the energy to keep the total energy constant.

4. Discussion

We have numerically examined the dynamics of two-sign point vortices confined in the circular boundary using MDGRAPE-2. By using the large-scale numerical sampling for the microcanonical ensemble, the density of states and the temperature have been determined. The time-asymptotic equilibrium distribution of the vortices with given energy and angular impulse shows a clear correlation with the temperature. In negative temperature, the same-sign vortices tend to condense into clumps by gaining the energy from other vortices remaining in the background. Energy share between the vortices in the clumps and background is quantitatively evaluated by the energy spectra. The correspondence between the vortex dynamics and the temperature supports the long-standing interpretation in terms of the negative temperature.

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