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Motion stability of a periodic system of bubbles in a liquid

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Wave-like motion in a periodic structure of bubbles that steadily moves through ideal incompressible liquid is considered. The wavelength is microscopically short. Some general local properties containing general information about two-phase flow are found. The dynamics of small-amplitude disturbances is studied in linear systems (called trains) and in spatial structures (such as a cubic lattice). The behaviour of one-dimensional waves in various structures is shown to differ widely: one-dimensional waves in the train do not magnify, whereas in the three-dimensional structure there may be stability and instability of one-dimensional waves. In the continuum limit the one-dimensional instability is demonstrated not to be related to the mean parameters of two-phase flow. The long-wave dynamics is shown to depend significantly on the relative velocity vector orientation in the lattice, but orientation is not included in the usual equations for the two-phase continuum. One result of this study is the relation between the short-wave-type instability of the periodic structure, on the one hand, and the instability of one-dimensional flow of inviscid bubbly liquid discovered by van Wijngaarden on the other. Long microscopic waves are analysed to determine the coefficients of one-dimensional equations for a two-phase continuum model. The velocity orientation at which the coefficients of the traditional one-dimensional model are obtained is found. Short waves in a stationary structure are studied by using the system of equations based on the equation of motion of a small sphere in a general potential flow. A refined equation for the force applied on a sphere in a non-uniform potential flow is derived.

1. Introduction

Consider the movement of multiple gas bubbles in an ideal liquid without body forces. Assume that the bubbles are spheres, the liquid flow is potential, and the volumetric concentration of bubbles is low. When bubbles are at nodes of a regular lattice and form a periodic structure, their configuration may be stationary due to symmetry and spatial periodicity of the system.

Stationary periodic structures have been used in many research papers in order to obtain information about macroscopic properties of bubbly liquids and to substantiate averaged equations. The added mass coefficient (Milne-Thomson 1949; Sedov 1976) and the macroscopic properties of the bubbly liquid, in connection with the averaged dynamic equations, have been investigated by using periodic structures (Cook & Harlow 1984; Wallis 1989, 1991, 1994*a*, *b*; Biesheuvel & Spoelstra 1989; Sangani, Zhang & Prosperetti 1991; Wallis, Cai & Luo 1992; Cai & Wallis 1992, 1993), the cell method (Zuber 1964; Nigmatulin 1979, 1991; Wallis 1992), the random state model (van Wijngaarden 1976*b*, 1993; Biesheuvel & van Wijngaarden 1984; Sangani & Didwania 1993*a*; Zhang & Prosperetti 1994), and the variational approach (Geurst 1985, 1988), etc.

Wallis (1989, 1991, 1992) has related the added mass coefficient theory to the classical theory of electric conductivity of a dispersion (Maxwell 1881; Rayleigh 1892). Sangani & Acrivos (1983) studied the conductivity of periodic arrays of spheres with a quite large concentration. Features of rectangular structures (or arrays) were studied to take into account the effect of microscopic properties on macroscopic equations (Wallis 1994*a*; Wallis, Luo & McDonald 1996). The importance of microstructure in the case of a random state of bubbly liquid was emphasized in Voinov & Petrov (1975).

There are various versions of the continuum equations describing the motion of a liquid with bubbles; these were derived by averaging methods in Garipov (1973), Nigmatulin (1979, 1991), Biesheuvel & van Wijngaarden (1984), Arnold, Drew & Lahey (1989), Wallis (1969, 1992). A bubble suspension may be described by analysing the interaction of pairs (van Wijngaarden 1976*a*; van Wijngaarden & Kapteyn 1990); a similar approach was first used by Batchelor (1972) to analyse suspension sedimentation in Stokes conditions. Unlike these and many similar works, the present study does not rely upon averaging. The work is based on an asymptotically accurate description of the dynamics of multiple bubbles in inviscid liquid within linear perturbation theory.

From Voinov & Petrov (1977) we know that a periodic structure of bubbles is not stable. This instability is similar to the instability of a system of dipoles in electrostatics and magnetostatics, a consequence of Earnshaw's theorem (Earnshaw 1842; Braunbek 1939). The instability of the random state of bubbly liquids was demonstrated in Sangani & Didwania (1993*b*) as the result of numerical analyses. The authors assumed that the gas bubbles do not coalesce during collisions but jump apart from one to the other – that is, behave as solids.

The objective of the present work is to investigate the dynamics of disturbances to bubble structures – that is, to investigate waves in these systems. Below, waves that are short compared with the lattice parameter are called short waves. Perturbations to the ordered bubble structure may be studied to obtain information about features of two-phase flows and the basic equations for a two-phase continuum. Of particular interest are one-dimensional movements in a bubble system because, in the limiting case of long waves, we can use them to model the phase interaction in one-dimensional two-phase flow. Knowledge of the perturbations to periodic spatial structures is essential in connection with the instability of one-dimensional bubbly flow in the continuum two-phase model, as noticed for the first time in van Wijngaarden (1976*b*).

Up to now, waves in lattices with bubbles have not been studied. However, there exist numerous studies on waves in bubbly liquids, based on averaged equations; see, for example, van Wijngaarden & Biesheuvel (1988), van Wijngaarden & Kapteyn (1990), and Lammers & Biesheuvel (1996). In the present study we will consider continuum equations for only the long-wave limit of the equations for short waves in a stationary bubble structure. Short waves will be studied on the basis of equations which describe bubble dynamics at the micro-level – the microscopic equations. Such equations for a system with multiple bubbles (Voinov & Petrov 1975) include that describing the motion of a small sphere subjected to the force from non-uniform potential flow (Voinov 1973; Voinov & Petrov 1973; Voinov, Voinov & Petrov 1973).

The reaction to a small body in non-uniform flow was studied for the first time by Zhukovsky in 1896 (refer to Zhukovsky 1949). Later, the problem was considered for various particular conditions (Taylor 1928; Khaskind 1956; Birkhoff 1960). Solutions in Voinov & Petrov (1973) and Voinov *et al.* (1973) describe reactions to a small body in unsteady non-uniform flow, where the body shape and volume can vary.

The present article deals only with vortex-free flow; readers should bear in mind that the potential flow model is limited and that there exist important results (Lighthill 1956; Auton 1987; Drew & Lahey 1987, 1990) about the force applied by vortical flow to the sphere.

The article is structured as follows. First (in § 2 and § 3), we discuss the equations of motion of an infinite system of bubbles. In § 4 we derive equations for perturbed motions in an infinite periodic system of bubbles. Then, waves in periodic structures are studied (§ 5–§ 9). The results are used to determine the coefficients of the equivalent continuum model. In § 10 the short-wave theory is generalized to the case of bubble structures in viscous fluid at large Reynolds numbers. In the Appendix the refined equation for the hydrodynamic force applied to a sphere by the non-uniform flow will be derived.

2. Equations for the dynamics of an infinite system of bubbles

2.1. *Microscopic velocity field*

Equations for bubbly liquid at the microscopic level are proposed in Voinov & Petrov (1975). These equations take into account a variation of bubble radius, but here, they are derived for a system of bubbles with constant radius. The derivation of the principal equations is dealt with in detail in order for the essence of the problem statement to be clear.

Assume that ideal incompressible liquid fills the entire space and is the medium through which many spheres of radius R move; the liquid motion is vortex-free, and the velocity has a potential

$$\boldsymbol{v} = \nabla \Phi, \quad \nabla^2 \Phi = 0. \tag{2.1}$$

On each sphere S_m the kinematic condition is met:

$$\boldsymbol{x} \in S_m, \quad \boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{u}^m \cdot \boldsymbol{n}. \tag{2.2}$$

Here $\boldsymbol{x}$ is the radius vector in a Cartesian coordinate system, $\boldsymbol{n}$ is the normal vector of the sphere, and $\boldsymbol{u}^m$ is the velocity of sphere m, where m, is either an integer (for a linear system when the spheres are all in a line) or a triplet of integers (in the case of a cubic lattice).

If the structure is limited in at least one direction, we require the liquid to have a constant velocity at infinity:

$$\boldsymbol{v} \to \boldsymbol{v}_\infty, \quad r \to \infty. \tag{2.3}$$

Here, r is the distance from the structure. In the case of a spatial system, (2.3) should be replaced by specifying a mean velocity $\boldsymbol{v}_*$ of the liquid between the spheres.

Let us assume that the minimum distance a between the centres of neighbouring spheres is much greater than the radius R,

$$a \gg R.$$

To describe the velocity field, we introduce the sphere S_∞ with a large radius r_0:

$$r_0 \gg a,$$

the sphere centre being at $\boldsymbol{x} = \boldsymbol{x}_0$.

Note first of all that the sphere S_∞ may intersect some of the bubbles. When $r_0 \gg a$, the velocity field near the sphere centre does not change if such bubbles are replaced by a liquid. For a point in the liquid within S_∞, Green's identity for harmonic functions is

$$\Phi = \Phi_0 + \sum_m \Phi_m, \quad 4\pi\Phi_m = \int_{S_m} \left(\frac{1}{r}\frac{\partial \Phi}{\partial n} - \Phi \frac{\partial}{\partial n}\frac{1}{r} \right) \mathrm{d}S, \tag{2.4a,b}$$

where the potential Φ_0 is a harmonic function inside S_∞, Φ_m is a harmonic function everywhere outside sphere $S_m(|\boldsymbol{x} - \boldsymbol{x}^m| \geqslant R)$, and r is the distance from a point on S_m. Summation in (2.4a) and relevant subsequent formulas is only carried out for the number m of spheres inside S_∞ ($|\boldsymbol{x}_0 - \boldsymbol{x}^m| < r_0$).

Consider flow around sphere n. The potential ϕ in (2.4a) may be represented as a sum:

$$\Phi = \Phi_n + \Phi_n', \quad \Phi_n' = \Phi_0 + \sum_{m \neq n} \Phi_m, \tag{2.5a,b}$$

where the potential Φ_n' is an analytic function in the vicinity of the centre of the nth sphere (including the interior of the sphere). The radius of this vicinity is close to the distance to the neighbouring bubble centre.

The potential Φ_n' (2.5b) gives the external velocity $\boldsymbol{v}'$:

$$\boldsymbol{v}'^n = \nabla \Phi_n' = \nabla \Phi_0 + \sum_{m \neq n} \nabla \Phi_m. \tag{2.6}$$

To meet condition (2.2) for a small sphere S_n, we use a Taylor series expansion for the external velocity vector at a centre $\boldsymbol{x}^n$:

$$\boldsymbol{v}'^n(\boldsymbol{x}) = \boldsymbol{v}'(\boldsymbol{x}^n) + \frac{\partial \boldsymbol{v}'}{\partial x_i}(\boldsymbol{x}^n)(x_i - x_i^n) + \cdots. \tag{2.7}$$

Here, the repeated subscript i is the index for summation from 1 to 3. The potential Φ_m is the sum of a dipole potential and the small contribution $\tilde{\Phi}_m$ of multipoles of higher orders (Voinov & Petrov 1975):

$$\Phi_m = \tfrac{1}{2} R^3 \boldsymbol{w}'^m \cdot \nabla \frac{1}{r_m} + \tilde{\Phi}_m, \quad r_m = |\boldsymbol{x} - \boldsymbol{x}^m|, \tag{2.8}$$

$$\boldsymbol{w}'^m = \boldsymbol{u}^m - \boldsymbol{v}'(\boldsymbol{x}^m). \tag{2.9}$$

Here, $\boldsymbol{w}'^m$ is the sphere velocity relative to the liquid. The summand $\tilde{\Phi}_m$ in (2.8) is small when R/a is small; flow non-uniformity may be neglected in the leading approximation.

On the basis of (2.4a) the liquid velocity is

$$\boldsymbol{v} = \nabla \Phi_0 + \sum_m \nabla \Phi_m. \tag{2.10}$$

For a limited system of spheres the vector $\nabla \Phi_0$ is equal to $\boldsymbol{v}_\infty$, which is the flow velocity at infinity (see condition (2.3)). But for an infinite spatial system of spheres $\nabla \Phi_0$ is unknown. It can be determined by averaging the vectors $\boldsymbol{v} = \nabla \Phi$ and $\boldsymbol{v}' = \nabla \Phi'$.

2.2. *Relation of local velocities to averaged values*

Consider two types of averaging formulas from Voinov & Petrov (1975). The mean value $\langle f\rangle$ of a function $f(\boldsymbol{x})$ specified for a liquid volume is defined as

$$(1-c)\langle f\rangle = \frac{1}{V}\int_{V_f} f(\boldsymbol{x}')\,\mathrm{d}^3x'. \tag{2.11}$$

where the domain V_f is filled with the liquid, $V_f \in V$; c is the volumetric concentration of the spheres.

Assume that f is specified for bubble centres: $f = f_n$ at $\boldsymbol{x} = \boldsymbol{x}^n$. In this case the mean value $\bar{f}$ is defined as

$$c\bar{f} = \frac{1}{V}\sum_n f_n V_n, \quad |\boldsymbol{x}_0 - \boldsymbol{x}^n| < r_0, \tag{2.12}$$

where V_n is the volume of the nth bubble.

Now we can determine $\nabla\Phi_0$. First, we express $\nabla(\Phi'-\Phi_0)$ by utilizing (2.6) and carry out averaging on the basis of (2.12):

$$c\overline{\nabla(\Phi'-\Phi_0)} = \frac{1}{V}\sum_n\sum_{m\neq n}\int_{V_n}\nabla\Phi_m(\boldsymbol{x}')\,\mathrm{d}^3x'. \tag{2.13}$$

Here, we have used the fact that any function $\phi(\boldsymbol{x})$ that is harmonic within the sphere V_n satisfies the equation

$$\int_{V_n}\phi(\boldsymbol{x}')\,\mathrm{d}^3x' = \phi(\boldsymbol{x}^n)V_n. \tag{2.14}$$

Using (2.4*a*) to express $\nabla(\Phi-\Phi_0)$, and averaging on the basis of (2.11) results in

$$(1-c)\langle\nabla(\Phi-\Phi_0)\rangle = -\frac{1}{V}\sum_m\left(\int_{V\backslash V_m}\nabla\Phi_m(\boldsymbol{x}')\,\mathrm{d}^3x' - \sum_{n\neq m}\int_{V_n}\nabla\Phi_m(\boldsymbol{x}')\,\mathrm{d}^3x'\right). \tag{2.15}$$

Note that

$$\int_{V\backslash V_m}\nabla\Phi_m(\boldsymbol{x}')\,\mathrm{d}^3x' = 0, \tag{2.16}$$

because Φ_m is a result of superposition of a dipole and higher-order multipoles (at the centre of the sphere S_m).

From (2.13), (2.15), (2.16) it follows that

$$(1-c)\langle\nabla(\Phi-\Phi_0)\rangle + c\overline{\nabla(\Phi'-\Phi_0)} = 0. \tag{2.17}$$

So, by resorting to (2.11), (2.12), and (2.14), we derive

$$\nabla\Phi_0(\boldsymbol{x}_0) = (1-c)\langle\nabla\Phi\rangle + c\overline{\nabla\Phi'}. \tag{2.18}$$

Hereinafter, let $\boldsymbol{x}_0$ be replaced by $\boldsymbol{x}$. By substituting (2.18) into (2.10), we obtain

$$\boldsymbol{v}(\boldsymbol{x}) = (1-c)\langle\boldsymbol{v}\rangle + c\overline{\boldsymbol{v}'} + \sum_m\nabla\Phi_m(\boldsymbol{x}). \tag{2.19}$$

From this, the external velocity is

$$\boldsymbol{v}'^n(\boldsymbol{x}) = (1-c)\langle\boldsymbol{v}\rangle + c\overline{\boldsymbol{v}'} + \sum_{m\neq n}\nabla\Phi_m(\boldsymbol{x}). \tag{2.20}$$

Note that (2.19) is an exact equation in the limit $r_0/a \to \infty$. Hereinafter, we make use of a dipole-based representation of Φ_m upon assuming $\tilde{\Phi}_m = 0$ in (2.8).

From (2.8) and (2.20) one can derive the velocity of the non-uniform flow in which the nth bubble of the infinite system is placed:

$$v_i'(\boldsymbol{x}) = v_{*i} - c(v_{*i} - \overline{v_i'}) + \sum_{m\neq n} \tfrac{1}{2} R^3 w_j'^m \nabla_i \nabla_j \left(\frac{1}{r_m}\right), \tag{2.21}$$

$$r_m = |\boldsymbol{x} - \boldsymbol{x}^m|, \quad i,j = 1,2,3.$$

Hereafter, $\boldsymbol{v}_*$ is taken as the mean velocity of the liquid, $\boldsymbol{v}_* = \langle \boldsymbol{v} \rangle$; $\boldsymbol{w}'^m$ is the relative velocity (2.9); and m is the sphere identification number. The summation for m is for the interior of $S_\infty(|\boldsymbol{x} - \boldsymbol{x}^m| < r_0)$. The repeated subscripts are indices for summation from 1 to 3.

Relation (2.21) is the basis on which to write a set of simultaneous linear equations for $\boldsymbol{v}'(\boldsymbol{x}^n)$:

$$\left.\begin{aligned} v_i'(\boldsymbol{x}^n) = v_{*i} - c(v_{*i} - \overline{v_i'}) + \sum_{m\neq n} \tfrac{1}{2} R^3 w_j'^m \nabla_i \nabla_j \left(\frac{1}{r_m}\right), \quad \boldsymbol{x} = \boldsymbol{x}^n, \\ \boldsymbol{w}'^n = \boldsymbol{u}^n - \boldsymbol{v}'(\boldsymbol{x}^n). \end{aligned}\right\} \tag{2.22}$$

Here, $\boldsymbol{x}^n$ is the nth sphere centre.

From (2.8) and (2.19) the velocity field is

$$v_i(\boldsymbol{x}) = v_{*i} - c(v_{*i} - \overline{v_i'}) + \sum_{m} \tfrac{1}{2} R^3 w_j'^m \nabla_i \nabla_j \left(\frac{1}{r_m}\right). \tag{2.23}$$

Equations (2.21)–(2.23) are in agreement with similar relations obtained by Voinov & Petrov (1975): equation (2.23) is the same as their formula (2.5). When comparing with, Voinov & Petrov one should first take into account their formula (2.6), secondly their note after (2.6), and thirdly assume $R = \text{const}$.

Consider the system in which the points $\boldsymbol{x}^n$ form a regular lattice and, consequently, $w_i'^n$ does not depend on n. Then, due to symmetry,

$$\sum_{m\neq n} w_j'^m \nabla_i \nabla_j \left(\frac{1}{r_m}\right) = 0 \quad \text{at} \quad \boldsymbol{x} = \boldsymbol{x}^n. \tag{2.24}$$

From (2.22) and (2.24) we conclude that the mean velocity $\boldsymbol{v}_*$ of the liquid is equal to the external velocity at the sphere centre:

$$\boldsymbol{v}_* = \overline{\boldsymbol{v}'}. \tag{2.25}$$

Here, the averaging symbol may be omitted. The formula (2.25) is in agreement with a similar relation obtained by Voinov & Petrov (1975) in a different way for an isotropic mixture of a liquid with bubbles.

The accuracy of (2.25) is the same as that of (2.21). If the second term in the series (2.7) is taken into account, then we can obtain a more accurate formula than (2.21):

$$v_i'(\boldsymbol{x}) = v_{\infty i} + \sum_{m\neq n} \frac{R^3}{2} w_j'^m \nabla_i \nabla_j r_m^{-1} + \sum_{m\neq n} \frac{R^5}{3} \nabla_i (r_m^{-5} y_j^m y_k^m) \nabla_k v_j'(\boldsymbol{x}^m). \tag{2.26}$$

Here, $\nabla_k v_j'(\boldsymbol{x}^m)$ is used to designate the value of $\nabla_k v_j'$ at $\boldsymbol{x} = \boldsymbol{x}^m$ as provided by (2.21), $\boldsymbol{v}_\infty = (1-c)\boldsymbol{v}_* + \overline{\boldsymbol{v}'}$.

Relation (2.26) is useful in estimating the accuracy of (2.21). If the lattice is regular, then we use symmetry to obtain

$$\nabla_k v'_j(\boldsymbol{x}^m) = 0.$$

The second summand in (2.26) is zero. Therefore, the error of (2.21) or (2.22) for the regular lattice is $O(\varepsilon^{10})$, where $\varepsilon = R/a$. For disturbances to the regular lattice the error of (2.21) and (2.22) is $O(\varepsilon^8)$.

2.3. *Microscopic equations in bubble system dynamics*

The dynamic equations for multiple spheres can be written on the basis of the equation of motion for one sphere in the flow induced by the other spheres. Following Voinov & Petrov (1975) it suffices to consider (2.21) and (2.22) together with the equation of motion of a sphere, which in a non-uniform flow, is

$$\left(\tfrac{1}{2}\rho_1 + \rho_2\right)\frac{\mathrm{d}\boldsymbol{u}^n}{\mathrm{d}t} = \rho_1 \frac{3}{2}\frac{\mathrm{d}\boldsymbol{v}'^n}{\mathrm{d}t_v} + (\rho_2 - \rho_1)\,\boldsymbol{g}, \tag{2.27}$$

where the subscript v symbolizes the convective derivative, ρ_2 is the density of the dispersed matter (here, the basic case is a system with $\rho_2 = 0$), and $\boldsymbol{g}$ is the body force. The equation is written on the basis of the usual expression (A 1) for the force in the principal approximation, $O(R^3)$, when R is small. The Appendix provides a refined relation accurate to within $O(R^5)$.

Consider the mean velocity of the mixture volume

$$\boldsymbol{v}_\Sigma = (1-c)\boldsymbol{v}_* + c\bar{\boldsymbol{u}}, \tag{2.28}$$

where $\boldsymbol{v}_*$ and $\bar{\boldsymbol{u}}$ are mean phase velocities. We can specify a mean velocity $\boldsymbol{v}_\Sigma$ as constant in space and time.

We would like to understand small-amplitude wave disturbances to the bubble system uniform in the mean. Hereafter, let us assume that no body forces are acting – that is, $\boldsymbol{g} = 0$. The kinematic equations (2.21) and (2.22) include the mean velocity $\boldsymbol{v}_*$ and the mean volumetric concentration c. They correspond to a large amount of the mixture and are assumed constant in the sphere dynamics because the disturbance wavelength Λ is much shorter than the diameter of this domain, $\Lambda \ll r_0$.

Relations (2.27) with $\boldsymbol{v}'$ from (2.21) and (2.22) generate a set of ordinary differential equations which describe the motion of multiple spheres in a liquid and allow for hydrodynamic interaction of these spheres. The equations are valid if ε is quite small ($\varepsilon = R/a$). The error inherent in the kinematic equations is stated in § 2.2 above. The error in the right-hand side of (2.27) is $O(\varepsilon^9)$ in the general case, see (A 14), and $o(\varepsilon^9)$ if bubbles are in a regular lattice.

Our primary concern is to derive the disturbed motion equations in the principal approximation for low c values.

3. Dynamic equations for small-amplitude disturbances to a periodic bubble system

Assume that the spheres form an infinite periodic structure which moves unchanged through a liquid. Let us consider small disturbances to this motion by using the linear approximation. The undisturbed periodic structure has velocity $\boldsymbol{u}_0$, and the average liquid velocity $\boldsymbol{v}_*$ is specified. The primary challenge now is to describe the dynamics of disturbances to the sphere centre coordinates,

$$\delta x_i^n = x_i^n(t) - x_{0i}^n(t). \tag{3.1}$$

Here, undisturbed values x^n_{0i} are linear temporal functions. We suppose that disturbances to the nth bubble coordinates δx^n_i are small in comparison with the lattice spacing:

$$|\delta \boldsymbol{x}^n| \ll a.$$

Let the vector function $\boldsymbol{f}(\boldsymbol{x}, t)$ stand for the external velocity or acceleration. Use $\delta^n \boldsymbol{f}$ to denote a disturbance $\boldsymbol{f}$ at the sphere centre:

$$\delta^n \boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}^n, t) - \boldsymbol{f}_0(\boldsymbol{x}^n_0, t). \tag{3.2}$$

Hereafter, the subscript 0 denotes unperturbed values of functions. For the linear approximation we write

$$\delta^n \boldsymbol{f} = \delta^n_0 \boldsymbol{f} + \delta x^n_i \nabla_i \boldsymbol{f}, \tag{3.3}$$

where $\delta^n_0 \boldsymbol{f}$ is the perturbation at the centre of the nth sphere in the unperturbed structure,

$$\delta^n_0 \boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}^n_0, t) - \boldsymbol{f}_0(\boldsymbol{x}^n_0, t). \tag{3.4}$$

To write disturbed motion equations, we need to find the disturbance to the external flow acceleration present in the right-hand side of (2.27).

Consider the derivative $\partial/\partial t_{u0}$ in the coordinate system x_i fixed at the undisturbed structure and write the convective derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t_v} = \frac{\partial}{\partial t_{u0}} + (\boldsymbol{v}' - \boldsymbol{u}_0) \cdot \nabla. \tag{3.5}$$

Because of symmetry of the regular lattice the undisturbed values of $\nabla_i v'_j$ are zero at the nodes,

$$\nabla_i v'_{0j} = 0, \quad \boldsymbol{x} = \boldsymbol{x}^n; \quad i, j = 1, 2, 3.$$

Using (3.2)–(3.5) we find the small disturbance to the acceleration of the non-uniform (external) flow

$$\delta \frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t_v}(\boldsymbol{x}^n) = \delta_0 \frac{\partial \boldsymbol{v}'}{\partial t_{u0}} - (\boldsymbol{w}_0 \cdot \nabla)\, \delta_0 \boldsymbol{v}' - \delta x^n_i (\boldsymbol{w}_0 \cdot \nabla) \nabla_i \boldsymbol{v}'_0, \tag{3.6a}$$

$$\boldsymbol{w}_0 = \boldsymbol{u}^n_0 - \boldsymbol{v}'_0(\boldsymbol{x}^n_0) = \boldsymbol{u}_0 - \boldsymbol{v}_*. \tag{3.6b, c}$$

Here, the repeated index i is the summation variable; terms on the right-hand side of (3.6a) are for the undisturbed centre $\boldsymbol{x}^n_0$ of the sphere; (3.6c) is based on (2.25). The disturbance to the relative velocity is

$$\delta \boldsymbol{w}^n = \delta \boldsymbol{u}^n - \delta \boldsymbol{v}'(\boldsymbol{x}^n), \quad \boldsymbol{x}^n = \boldsymbol{x}^n_0 + \delta \boldsymbol{x}^n. \tag{3.7}$$

The disturbance to the external velocity $\boldsymbol{v}'(\boldsymbol{x}^n)$ is expressed via (2.22):

$$\delta \boldsymbol{v}'(\boldsymbol{x}^n) = \frac{R^3}{2} \nabla \sum_{m \neq n} [\nabla_k(-y^m_s r^{-3}_m) w_{0s}\, \delta y^{nm}_k - \delta w^n_s y^m_s r^{-3}_m], \tag{3.8}$$

$$\boldsymbol{x} = \boldsymbol{x}^n, \quad \boldsymbol{y}^m = \boldsymbol{x} - \boldsymbol{x}^m, \quad \delta \boldsymbol{y}^{nm} = \delta \boldsymbol{x}^n - \delta \boldsymbol{x}^m; \quad k, s = 1, 2, 3.$$

Here, indices n and m are either integers or triplets of integers corresponding to lattice nodes.

Now we need to estimate $\delta \boldsymbol{v}'(\boldsymbol{x}^n)$ defined in (3.8). When $\varepsilon \to 0$, the right-hand side of (3.8) is small:

$$|\delta \boldsymbol{v}'(\boldsymbol{x}^n)| = O(\varepsilon^3). \tag{3.9}$$

Since $\delta v'_j$ is small, relations (3.7) and (3.9) give

$$|\delta \boldsymbol{w}^n - \delta \boldsymbol{u}^n| = O(\varepsilon^3). \tag{3.10}$$

Hence, we can assume $\delta \boldsymbol{w}^m = \delta \boldsymbol{u}^m$ in the sum of (3.8). Using this approximation we find from (2.21) and (3.6)–(3.8) for the small disturbance to the external liquid acceleration

$$\delta \frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t_v}(\boldsymbol{x}^n) = \frac{R^3}{2}\nabla \sum_{m\neq n}\left(-f_j^m \frac{\mathrm{d}}{\mathrm{d}t} w_j^m + 2w_{0j}\,\delta u_k^m \nabla_k f_j^m + w_{0p} w_{0j} \delta y_k^{nm} \nabla_p \nabla_k f_j^m\right), \quad \boldsymbol{x} = \boldsymbol{x}_0^n, \tag{3.11}$$

where $\nabla = \partial/\partial \boldsymbol{x}$, $f^m = (\boldsymbol{x} - \boldsymbol{x}^m)/r_m^3$, $\delta \boldsymbol{u}^n = \mathrm{d}/\mathrm{d}t(\delta \boldsymbol{x}^n)$.

Upon substituting (3.11) into (2.27) we obtain the disturbed motion equations for the spheres:

$$K \frac{\mathrm{d}}{\mathrm{d}t}\delta u_i^n = \mu_i^n + \beta_i^n + \sigma_i^n, \tag{3.12}$$

$$\mu_i^n = -\frac{R^3}{2}\sum_{m\neq n}\left[\nabla_i \nabla_j \frac{-1}{r_m}\right]\frac{\mathrm{d}}{\mathrm{d}t}u_j^m, \tag{3.13}$$

$$\beta_i^n = R^3 w_{0j}\sum_{m\neq n}\left[\nabla_i \nabla_k \nabla_j \frac{-1}{r_m}\right]\delta u_k^m, \tag{3.14}$$

$$\sigma_i^n = \frac{R^3}{2} w_{0p} w_{0j}\sum_{m\neq n}\left[\nabla_i \nabla_p \nabla_k \nabla_j \frac{-1}{r_m}\right]\delta y_k^{nm}, \tag{3.15}$$

where $\boldsymbol{x} = \boldsymbol{x}_0^n$, $\quad r_m = |\boldsymbol{x} - \boldsymbol{x}^m|$, $\quad \delta \boldsymbol{y}^{nm} = \delta \boldsymbol{x}^n - \delta \boldsymbol{x}^m$, $\quad K = (2\rho_2 + \rho_1)/3\rho_1$. For bubbles $K = 1/3$. Hereafter, we replace $\boldsymbol{w}_0$ with $\boldsymbol{w}$ for brevity.

4. Added mass of a bubble in one-dimensional motion

Consider one-dimensional and spatially uniform motion of an infinite system of bubbles in a liquid:

$$\boldsymbol{v}_\Sigma = \boldsymbol{v}_\Sigma(t), \quad \boldsymbol{v}_* = \boldsymbol{v}_*(t), \quad \bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}(t). \tag{4.1}$$

Assume that the mixture microstructure has three orthogonal axes of symmetry. In particular, this assumption is valid when the bubbles are in a cubic lattice. Then (as shown in Voinov & Petrov 1975 and in § 2.2 above) the mean value of $\boldsymbol{v}$ throughout the volume is the same as the mean value of $\boldsymbol{v}'$ at centres. Also, it is obvious that the mean values of velocities $\boldsymbol{v}'$ and $\boldsymbol{u}$ are the same as their local values:

$$\boldsymbol{v}_* = \bar{\boldsymbol{v}}' = \boldsymbol{v}', \quad \bar{\boldsymbol{u}} = \boldsymbol{u}. \tag{4.2}$$

From (2.27):

$$\frac{1}{2}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{3}{2}\frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} - \boldsymbol{g}. \tag{4.3}$$

Also, from (2.28), (4.1), (4.2), and (4.3):

$$\frac{1}{2}\left(1 + \frac{3c}{1-c}\right)\left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}_\Sigma\right) = \frac{\mathrm{d}\boldsymbol{v}_\Sigma}{\mathrm{d}t} - \boldsymbol{g}. \tag{4.4}$$

The coefficient $1 + 3c/(1 - c)$ on the left-hand side of (4.4) was initially introduced Zuber (1964) and substantiated in many works (van Wijngaarden 1976*b*; Biesheuvel & Spoelstra 1989; Wallis 1989, 1992; Sangani *et al.* 1991). The accuracy of the Zuber coefficient is $O(c^{10/3})$ for a regular lattice.

Let us use the symbol Z to denote the coefficient by which the added mass of a bubble in the system of bubbles differs from the added mass of a single sphere in unbounded liquid. Then the added mass coefficient $m(c)$ (van Wijngaarden 1976*a*) of the bubble in a dispersion is

$$m(c) = \tfrac{1}{2}\rho_1 V Z, \tag{4.5}$$

where V is the sphere volume. The solution in Zuber (1964) corresponds to

$$Z = \frac{1+2c}{1-c} = 1 + 3c + O(c^2). \tag{4.6}$$

The added mass coefficient $m(c)$ obtained in van Wijngaarden (1976*a*) for the random state of a bubbly liquid corresponds to

$$Z = 1 + 2.78c, \tag{4.7}$$

and that from Biesheuvel & Spoelstra (1989) and Kok (1988*a*) corresponds to

$$Z = 1 + 3.32c. \tag{4.8}$$

Values of Z for waves in trains of bubbles and in spatial systems are discussed below.

5. Waves in an infinite bubble system

Now we consider disturbances in one- and three-dimensional systems. In both situations each bubble may be identified by its unique integer, m and n. However, for the spatial system the identification would be better based on triplets of integers. The undisturbed coordinates of sphere centres can be specified in terms of the spacing a,

$$x_{0j}^n = a n_j, \quad j = 1, 2, 3. \tag{5.1}$$

In the case of a cubic lattice the variable n stands for a triplet of integers

$$n_1, n_2, n_3 = 0, \pm 1, \pm 2, \dots . \tag{5.2}$$

Disturbances $\delta \boldsymbol{x}^n = \boldsymbol{x}^n - \boldsymbol{x}_0^n$ are small ($|\delta \boldsymbol{x}^n \ll a|$) and described by the system of linear equations (3.12)–(3.15).

The set of equations (3.12)–(3.15) is invariant in relation to translation along any lattice axis by the lattice spacing a. Therefore, the set possesses a wave-like solution

$$x_j^n - x_{0j}^n = \mathrm{d}_j \exp(\lambda t + \mathrm{i} n_1 \gamma_1 + \mathrm{i} n_2 \gamma_2 + \mathrm{i} n_3 \gamma_3), \tag{5.3}$$

where the real variables $\gamma_j \in (-\pi, \pi)$, d_j are initial ($t = 0$) wave amplitudes, and λ is the amplitude growth exponent. The wave parameters γ_j are expressed in terms of wavenumbers k_j:

$$\gamma_j = a k_j, \quad n_j \gamma_j = k_j x_{0j}^n. \tag{5.4}$$

The primary problem is to analyse the growth exponent λ for motion of one- and three-dimensional structures when wave parameters γ_j (or wavenumbers k_j) are specified.

6. Waves in bubble trains

Consider disturbances to an infinite bubble train that moves in an unbounded liquid which is at rest at infinity. Assume that the train is oriented along the x_1-axis:

$$n_1 = 0, \pm 1, \pm 2, \dots ; \quad n_2 = 0, \quad n_3 = 0. \tag{6.1}$$

(i) If the bubble train is moving along its own axis x_1 with a velocity w, then it can feature low-amplitude waves of two types.

(a) In waves of the first type the displacements are along the train (i.e. one-dimensional movement):

$$\delta x_1^n = d\mathrm{e}^{\lambda t + \mathrm{i}n\gamma}, \quad \delta x_2^n = \delta x_3^n = 0. \tag{6.2}$$

Here i is the imaginary unit, and $\gamma \in (-\pi, \pi)$.

To derive the equation for λ, we substitute (6.2) into (3.13)–(3.15) and take into account (6.1) and the equalities

$$\sum_{m \neq n} r_{mn}^{-3} \mathrm{e}^{\lambda t + \mathrm{i}m\gamma} = \frac{2}{a^3} \mathrm{e}^{\lambda t + \mathrm{i}n\gamma} a_3, \quad r_{mn} = |x_1^m - x_1^n|, \quad a_3 = \sum_{l=1}^{\infty} \frac{\cos l\gamma}{l^3}, \tag{6.3}$$

$$\sum_{m \neq n} \frac{x_1^n - x_1^m}{r_{mn}^5} \mathrm{e}^{\lambda t + \mathrm{i}m\gamma} = -2 \frac{\mathrm{i}}{a^4} \mathrm{e}^{\lambda t + \mathrm{i}n\gamma} a_4, \quad a_4 = \sum_{l=1}^{\infty} \frac{\sin l\gamma}{l^4}, \tag{6.4}$$

$$\sum_{m \neq n} \frac{1}{r_{mn}^5} \left\{ \mathrm{e}^{\lambda t + \mathrm{i}n\gamma} - \mathrm{e}^{\lambda t + \mathrm{i}m\gamma} \right\} = \frac{2}{a^5} \mathrm{e}^{\lambda t + \mathrm{i}n\gamma} a_5, \quad a_5 = \sum_{l=1}^{\infty} \frac{1 - \cos l\gamma}{l^5}. \tag{6.5}$$

The sum a_3 governs the sign of the correction, $Z - 1$, to the added mass coefficient of a single sphere:

$$Z = 1 - 6\varepsilon^3 a_3(\gamma), \tag{6.6}$$

where

$$\varepsilon = R/a \tag{6.7}$$

In the continuum limit (at $\gamma \to 0$) the added mass is less than that for a sphere in unbounded liquid ($Z < 1$). In the case of quite short waves (at $|\gamma| > 1.45035$) the added mass is greater than that for a single sphere ($Z > 1$).

From (3.12)–(3.15) we obtain the characteristic equation

$$(K - 2\varepsilon^3 a_3)\lambda^2 + 12\mathrm{i}\varepsilon^3 (w/a) a_4 \lambda + 24\varepsilon^3 (w^2/a^2) a_5 = 0, \tag{6.8}$$

which has two imaginary roots. In the principal approximation with a small ε we have

$$\lambda_{11} = \pm \mathrm{i} \frac{w}{a} \left(\varepsilon^{3/2} \sqrt{24 a_5 K^{-1}} + O(\varepsilon^3) \right). \tag{6.9}$$

Hence, the train in a longitudinal flow is stable to longitudinal perturbations. One-dimensional motion of bubbles in a train looks like that of spheres connected by springs.† By analogy, it can be assumed that one-dimensional disturbed motion in a spatial system behaves in the same way. Below, in § 9, this hypothesis is proven. Qualitatively, the effect may be explained by using a system with two spheres which move one following the other in unbounded liquid. The mutual repulsion force of the two spheres is known to monotonically increase as their separation distance

† The possibility of this effect in a train of bubbles was suggested by R. I. Nigmatulin.

decreases (Basset 1961; Lamb 1932; Voinov 1969). The dynamics of an infinite train is mainly governed by the interaction of neighbouring bubbles, which is similar to the interaction of two spheres in unbounded liquid.

(b) Disturbances of the second type are displacements of spheres normal to the axis of the train:

$$\delta x_1^n = \delta x_3^n = 0; \quad \delta x_2^n = d_2 \mathrm{e}^{\lambda t + \mathrm{i} n\gamma}, \quad n = 0, \pm 1, \pm 2, \ldots . \tag{6.10}$$

The summands $\boldsymbol{\mu}$, $\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$ in (3.13)–(3.15) need to be written which corresponds to (6.10), to give, from (3.12),

$$(K + \varepsilon^3 a_3)\lambda^2 - 6\mathrm{i}\varepsilon^3 (w/a) a_4 \lambda - 12\varepsilon^3 (w^2/a^2) a_5 = 0. \tag{6.11}$$

Relation (6.11) gives

$$\lambda_{12} = \frac{w}{a}\left(\pm\varepsilon^{3/2}\sqrt{12 a_5 K^{-1}} + O(\varepsilon^3)\right). \tag{6.12}$$

This indicates that the train in a longitudinal flow is unstable with respect to transverse disturbances.

(ii) Consider the motion of a train normal to its axis x_1 and along the x_2-axis:

$$w_2 = w, \quad w_1 = w_3 = 0.$$

Two situations might occur.

(a) The train may feature the longitudinal perturbations

$$\delta x_1^n = d_1 \mathrm{e}^{\lambda t + \mathrm{i} n\gamma}, \quad \delta x_2^n = \delta x_3^n = 0. \tag{6.13}$$

Then from (3.12)–(3.15):

$$(K - 2\varepsilon^3 a_3)\lambda^2 - 12\varepsilon^3 (w^2/a^2) a_5 = 0, \tag{6.14}$$

and we obtain two values of the exponent:

$$\lambda_{21} = \pm\varepsilon^{3/2}\sqrt{12 a_5 K^{-1}}(1 + O(\varepsilon^3))\frac{w}{a}. \tag{6.15}$$

It is interesting to note that in its principal approximation (6.15) is the same as (6.12): $\lambda_{12} = \lambda_{21}$. Hence, disturbances with identical values of γ in these two cases grow at the same rate. It is clear that in both cases the instability is caused by the Bernoulli effect.

We now determine the disturbances with the fastest growth. The maximum of the real part of λ corresponds to the maximum of $a_5(\gamma)$. In accordance with (6.10) the function $a_5(\gamma)$ reaches its maximum values at $\gamma = \pm\pi$. This corresponds to the pairing mode for which the wavelength is $2a$. Note that the pairing mode is of importance in the development of the instability of a single infinite row of point vortices in an ideal liquid which were studied by von Kármán (1911, 1956); refer to Lamb (1932) and Aref (1995).

(b) Transverse perturbations (i.e. displacements along the x_2-axis) to the train moving normal to its x_1-axis (i.e. along the x_2-axis) do not grow. The growth exponent is purely imaginary:

$$\lambda_{22} = \pm\mathrm{i}\frac{w}{a}(\varepsilon^{3/2}\sqrt{24 a_5 K^{-1}} + O(\varepsilon^3)). \tag{6.16}$$

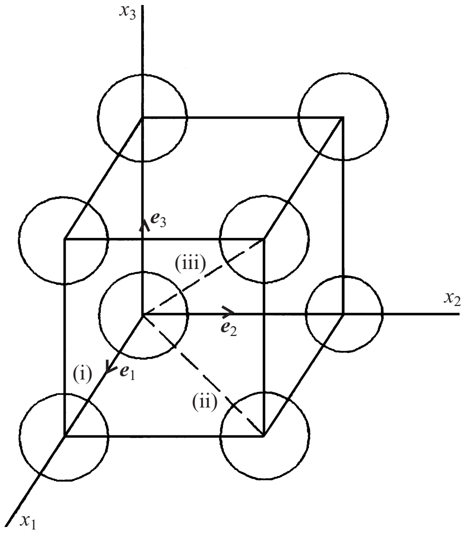

FIGURE 1. Directions of one-dimensional disturbed motion of bubbles in a cubic lattice.

7. Waves in a cubic lattice of bubbles and coefficients of the dynamic equation for a continuous medium

We next analyse linear waves in a cubic lattice of bubbles. A typical element of the system is sketched in figure 1.

Assume that the unperturbed velocity vector $\boldsymbol{w}$ is oriented along one of the lattice symmetry axes. In this case a one-dimensional solution to the equations of the motion is a possibility.

In order for the wave-like disturbances (5.3) to correspond to one-dimensional motion of the bubble system, the vectors of the perturbation to the coordinates of each bubble (i.e. the real part of $\delta \boldsymbol{x}^n$) should be collinear to the velocity vector $\boldsymbol{w}$. The motion is symmetric, and without restricting the general character of the model we can assume the initial amplitudes d_j to be real numbers. Then the one-dimensional motion condition is formulated as follows: vectors $\boldsymbol{d}$, $\boldsymbol{\gamma}$, and $\boldsymbol{w}$ are collinear.

$$d_j = b w_j, \quad \gamma_j = h w_j, \tag{7.1}$$

where b and h are real numbers and $j = 1, 2, 3$.

The problem is to establish the dependence of the exponent λ on wave parameters γ_j. Therefore we should first analyse the summands $\boldsymbol{\mu}$, $\boldsymbol{\beta}$, and $\boldsymbol{\sigma}$ in (3.12). The summand $\boldsymbol{\mu}$ makes it possible to find the correction to the added mass of the sphere involved in wave motion.

7.1. *Added mass of a sphere*

The added mass of a sphere in a wave motion differs from that in an unbounded liquid. By writing the mth sphere's acceleration as

$$\frac{\mathrm{d}}{\mathrm{d}t} u_j^m = \lambda^2 d_j \exp(\lambda t + \mathrm{i} m_h \gamma_h), \tag{7.2}$$

we find from (3.13) and (5.3) that

$$\mu_i^n = -c\lambda^2 d_i \exp(\mathrm{i} n_j \gamma_j + \lambda t)\alpha_0, \tag{7.3a}$$

$$\alpha_0 = \frac{3}{8\pi}\sum_{y\neq 0}\left[1 - 3\frac{(\boldsymbol{y}\cdot\boldsymbol{e}_0)^2}{r^2}\right]\frac{\cos(\gamma\boldsymbol{y}\cdot\boldsymbol{e}_0)}{r^3}, \tag{7.3b}$$

$$e_{0j} = \gamma_j/\gamma, \quad \gamma = |\boldsymbol{\gamma}|; \quad y_j = 0, \pm 1, \pm 2, \ldots; \quad r = |\boldsymbol{y}|$$

(e_0 is the unit wave vector.) Hence, the corrected multiplier to the added mass coefficient in accordance with (3.12), (4.5) and (7.3) is

$$Z = 1 + 3c\alpha_0. \tag{7.4}$$

Obviously, for (7.4) to conform to the Zuber coefficient (4.6) in the continuum model we must have $\alpha_0 = 1$.

7.2. *Contribution to* (3.12) *from coordinate disturbances*

Consider now the sum $\boldsymbol{\sigma}$ which defines the contribution to (3.12) from bubble coordinate disturbances. To compute the sum $\boldsymbol{\sigma}$ on the basis of (3.15), take into account that

$$\delta y_k^{nm} = d_k \exp(\mathrm{i} n_j\gamma_j + \lambda t)[1 - \exp(\mathrm{i}(m_j - n_j)\gamma_j)]. \tag{7.5}$$

Due to flow symmetry the summand $\boldsymbol{\sigma}$ in (3.15) is proportional to the initial amplitude: $\sigma_j^n = Bd_j$. The coefficient B will be found by using (7.1) and after computing the scalar product $\sigma_j^n d_j$,

$$\sigma_i^n = cw^2a^{-2}d_i\exp(\mathrm{i}n_j\gamma_j + \lambda t)\alpha_1, \quad c = \tfrac{4}{3}\pi R^3a^{-3}, \tag{7.6a}$$

$$\alpha_1 = \frac{3}{8\pi}\sum_{y\neq 0}\left[-9 + 90\frac{(\boldsymbol{y}\cdot\boldsymbol{e}_0)^2}{r^2} - 105\frac{(\boldsymbol{y}\cdot\boldsymbol{e}_0)^4}{r^4}\right]\frac{1-\cos(\gamma\boldsymbol{y}\cdot\boldsymbol{e}_0)}{r^5}, \tag{7.6b}$$

$$e_{0j} = \gamma_j/\gamma; \quad y_j = 0, \pm 1, \pm 2, \ldots; \quad r = |\boldsymbol{y}|.$$

Summation is over lattice nodes. In connection with the analysis of long-wave motions we introduce the coefficient

$$A_1 = \alpha_1/\gamma^2 \tag{7.7}$$

and re-write (7.6) to obtain

$$\sigma_i^n = c(\gamma^2/a^2)w^2d_i\exp(\mathrm{i}n_j\gamma_j + \lambda t)A_1. \tag{7.8}$$

7.3. *Contribution to* (3.12) *from velocity disturbances*

Lastly, the summand $\boldsymbol{\beta}$ in (3.12) represents the effect of the sphere velocities on the force. Determine β_i^n in accordance with (3.14) for one-dimensional motion. By calculating the scalar product $\beta_i^n d_i$ we obtain

$$\beta_i^n = \mathrm{i}cwa^{-1}\lambda d_i\exp(\mathrm{i}n_j\gamma_j + \lambda t)\alpha_2, \quad \boldsymbol{w}\cdot\boldsymbol{e}_0 > 0, \tag{7.9}$$

$$\alpha_2 = \frac{3}{4\pi}\sum_{y\neq 0}\left[9 - 15\frac{(\boldsymbol{y}\cdot\boldsymbol{e}_0)^2}{r^2}\right](\boldsymbol{y}\cdot\boldsymbol{e}_0)\frac{\sin(\gamma\boldsymbol{y}\cdot\boldsymbol{e}_0)}{r^5}, \tag{7.10}$$

$$y_j = 0, \pm 1, \pm 2, \ldots; \quad r = |\boldsymbol{y}|.$$

Note that in the case of $\boldsymbol{w}\cdot\boldsymbol{e}_0 < 0$ the sign in (7.9) is opposite. However, then the exponent λ is replaced with its complex conjugate value; so it suffices to assume that $\boldsymbol{w}\cdot\boldsymbol{e}_0 > 0$.

To analyse long waves, we need to also write $\boldsymbol{\beta}$ in terms of the coefficient

$$A_2 = \alpha_2/\gamma, \tag{7.11}$$

$$\beta_i^n = \mathrm{i}c(\gamma/a)w\lambda d_i \exp(\mathrm{i}n_j\gamma_j + \lambda t)A_2, \quad \boldsymbol{w}\cdot\boldsymbol{e}_0 > 0. \tag{7.12}$$

7.4. *Growth exponent for one-dimensional waves*

From (3.12)–(3.15), (7.5), (7.8), and (7.12) two versions of the equation for the exponent λ appear:

$$(K + c\alpha_0)\lambda^2 = c(\gamma/a)^2 w^2 A_1 + \mathrm{i}c(\gamma/a)wA_2\lambda, \tag{7.13a}$$

$$(K + c\alpha_0)\lambda^2 = ca^{-2}w^2\alpha_1 + \mathrm{i}ca^{-1}w\alpha_2\lambda. \tag{7.13b}$$

In the principal approximation for small c the growth exponent is

$$\lambda = \pm(w/a)\sqrt{c\alpha_1/K} = \pm(\gamma/a)w\sqrt{cA_1/K}. \tag{7.14}$$

For bubbles $K = 1/3$. If $\alpha_1 > 0$ (and, correspondingly, $A_1 > 0$) then the system is unstable to one-dimensional disturbances, whereas it is stable if $\alpha_1 < 0$ (or $A_1 < 0$).

7.5. *The continuum limit*

The above coefficients enable us to write linearized dynamic equations (3.12)–(3.15) for one-dimensional waves as follows:

$$(\tfrac{1}{2}\rho_1(1 + 3\alpha_0 c) + \rho_2)\frac{\mathrm{d}\boldsymbol{u}^n}{\mathrm{d}t} = \tfrac{3}{2}c\rho_1(A_1 w^2 k^2\,\delta\boldsymbol{x}^n + A_2 w\mathrm{i}k\,\delta\boldsymbol{u}^n), \tag{7.15}$$

where $k = \gamma/a$.

In the limiting case of long one-dimensional waves ($\gamma \to 0$) a change in concentration c can be related to the displacement $\delta\boldsymbol{x}$:

$$\delta c = -\mathrm{i}kc_0\,\delta x, \quad \mathrm{i}k(\ldots) = \frac{\partial}{\partial x}(\ldots), \quad \delta x = \delta\boldsymbol{x}\cdot\boldsymbol{e}_0,$$

where $\boldsymbol{e}_0$ is the unit wave vector. With this, equation (7.15) corresponds to

$$(\tfrac{1}{2}\rho_1(1 + 3\alpha_0 c) + \rho_2)\frac{\mathrm{d}u}{\mathrm{d}t} = \tfrac{3}{2}A_1\rho_1 w^2\frac{\partial c}{\partial x} + \tfrac{3}{2}A_2\rho_1 wc\frac{\partial u}{\partial x}. \tag{7.16}$$

This is the equation for the continuum description of the two-phase system.

7.6. *Symmetry relations for wave dynamics coefficients*

Our intention is to demonstrate that equation (7.13) holds in certain situations with two-dimensional disturbed motion. One-dimensional motion is symmetric, so it does not depend on transverse disturbances to the bubble coordinates. The latter are not related to longitudinal disturbances and have a separate equation system.

Further, we will also deal with transverse waves for which

$$\boldsymbol{d}\cdot\boldsymbol{k} = 0, \quad \boldsymbol{k} = h\boldsymbol{w}, \tag{7.17}$$

where h is a real number. This disturbance is similar to the type-(i)(b) wave in a train (see §6). Let us assume that each of the vectors $\boldsymbol{w}$ and $\boldsymbol{d}$ is collinear to one of the lattice vectors $\boldsymbol{e}_j$. By introducing $\boldsymbol{e}_*$ as the unit vector directed along $\boldsymbol{d}$, we have

$$\boldsymbol{d} = \pm|\boldsymbol{d}|\boldsymbol{e}_*, \quad \boldsymbol{e}_0\cdot\boldsymbol{e}_* = 0. \tag{7.18}$$

Then we re-write (3.15) in the following form:

$$\sigma_{*1}^n = cw^2a^{-2}d_i \exp(\mathrm{i}n_j\gamma_j + \lambda t)\alpha_{1*}, \tag{7.19a}$$

$$\alpha_{i*} = \frac{3}{8\pi}\sum_{y\neq 0}\left[-3+15\frac{(\boldsymbol{y}\cdot\boldsymbol{e}_0)^2+(\boldsymbol{y}\cdot\boldsymbol{e}_*)^2}{r^2}-105\frac{(\boldsymbol{y}\cdot\boldsymbol{e}_0)^2(\boldsymbol{y}\cdot\boldsymbol{e}_*)^2}{r^4}\right]\frac{1-\cos(\gamma\boldsymbol{y}\cdot\boldsymbol{e}_0)}{r^5}. \tag{7.19b}$$

Assume that $\boldsymbol{e}_0 = \boldsymbol{e}_1$, and write the expression (7.19*b*) for α_{1*} for two cases:

$$\boldsymbol{e}_* = \boldsymbol{e}_2 \quad \text{and} \quad \boldsymbol{e}_* = \boldsymbol{e}_3. \tag{7.20}$$

Sum the expressions for α_{1*} and the expression (7.6*b*) for α_1; then transform the sum on the basis of the identity

$$y_1^2 + y_2^2 + y_3^2 = r^2.$$

The transverse wave coefficient (7.19*a*) may now be expressed through (7.6*b*):

$$\alpha_{1*} = -\tfrac{1}{2}\alpha_1. \tag{7.21}$$

The coefficient A_{1*} is based on (7.7):

$$A_{1*} = \alpha_{1*}/\gamma^2 = -\tfrac{1}{2}A_1. \tag{7.22}$$

Note that the difference in signs of coefficients α_1 and α_{1*} points to the fact that one of them is positive and, in accordance with (7.14), corresponds to instability.

By analogy with (7.21), we proceed from (3.13), (7.3*b*) and (3.14), (7.10) to find

$$\alpha_{0*} = -\tfrac{1}{2}\alpha_0, \quad \alpha_{2*} = -\tfrac{1}{2}\alpha_2. \tag{7.23a, b}$$

These symmetry relations between coefficients of dynamics of longitudinal and transverse disturbances make it possible to deal with α_0, α_1, and α_2, without involving other entities.

In the continuum limit the expression (7.23*a*) suggests that the transverse wave added mass coefficient differs greatly from the Zuber value (4.6):

$$Z = 1 - 1.5c. \tag{7.24}$$

Consider a longitudinal wave with the wave vector $\boldsymbol{k}$ orthogonal to the velocity $\boldsymbol{w}$:

$$\boldsymbol{d} = b\boldsymbol{k}, \quad \boldsymbol{w}\cdot\boldsymbol{k} = 0. \tag{7.25}$$

Here b is a real number. Let each of the vectors $\boldsymbol{w}$ and $\boldsymbol{k}$ be collinear to one of the lattice vectors. The wave is similar to the type-(ii)(a) wave in a train (see § 6). The combination (7.25) resembles (7.17) in that $\boldsymbol{d}\cdot\boldsymbol{w} = 0$. Therefore, the dynamic coefficients for the wave will be obtained by replacing $\boldsymbol{e}_0$ with $\boldsymbol{e}_*$ in sine or cosine functions in formulas for α_{0*}, α_{1*}, and α_{2*}. In this case the relation (7.23*a*) for α_{0*} becomes the formula for α_0, the formula (7.21) for α_{1*} remains unchanged, and the formula (7.23*b*) for α_{2*} gives us zero. So the coefficients for the (7.25) wave are

$$\tilde{\alpha}_0 = \alpha_0, \quad \tilde{\alpha}_1 = \alpha_{1*} = -\tfrac{1}{2}\alpha_1, \quad \tilde{\alpha}_2 = 0. \tag{7.26}$$

The symmetry relations (7.26) are similar to those which relate dynamic coefficients for type-(ii)(a) waves and other waves in trains (see § 6). With this, stability is defined by the same coefficient α_{1*} as for the transverse wave. However, unlike the latter, the added mass coefficient in the continuum model is described by the Zuber equation, owing to the equality $\tilde{\alpha}_0 = \alpha_0$.

Now address the problem of one-dimensional motion stability while considering one-dimensional perturbations only. This is of interest in connection with the one-dimensional theory of two-phase flows.

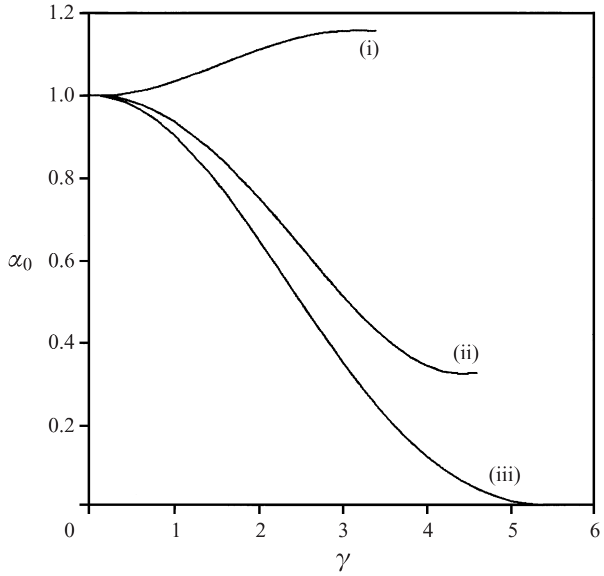

FIGURE 2. The effect of the wave parameter γ on the coefficient α_0, governing added mass of a bubble in the system. Curves are shown for three orientations of the relative velocity vector $\boldsymbol{w}$, (i), (ii) and (iii), and α_0 is given by equation (7.3*b*)

8. Wave dynamics in a bubble lattice: the role of velocity orientation

The coefficients in (7.13) which defines λ depend on $\boldsymbol{e}_j \cdot \boldsymbol{e}_0$ where $\boldsymbol{e}_j$ are lattice vectors and $\boldsymbol{e}_0$ is the wave unit vector collinear to the undisturbed velocity $\boldsymbol{w}$.

One-dimensional movement occurs when

(i) $\boldsymbol{e}_0$ is collinear to one of lattice vectors,
(ii) $\boldsymbol{e}_0$ is direct along the cube face diagonal, and
(iii) $\boldsymbol{e}_0$ is directed along the main cube diagonal.

Consider these situations separately. For each of them the scalar product $\boldsymbol{y}\cdot\boldsymbol{e}_0$ included in (7.6*b*) etc. may be written more specifically.

(i) The sphere velocity is directed along the lattice vector $\boldsymbol{e}_1$ (figure 1). The total number of these axes is only three. For these cases we have in (7.6*b*):

$$\boldsymbol{y} \cdot \boldsymbol{e}_0 = y_1. \tag{8.1}$$

(ii) A wave along a short cube face diagonal. In this case there is a possible six lines (figure 1). If we choose the diagonal in the plane $\boldsymbol{y}_3 = 0$, then

$$\boldsymbol{y} \cdot \boldsymbol{e}_0 = (y_1 + y_2)/\sqrt{2}. \tag{8.2}$$

(iii) A wave runs along one of the four main internal diagonals (figure 1). Then

$$\boldsymbol{y} \cdot \boldsymbol{e}_0 = (y_1 + y_2 + y_3)/\sqrt{3}. \tag{8.3}$$

Relations (7.3*b*) and (8.1)–(8.3) were used to compute the coefficient α_0 which defines the correction multiplier Z to the added mass coefficient of a bubble in a system of bubbles.

$$Z = 1 + 3c\alpha_0. \tag{8.4}$$

The results are depicted in figure 2.

The continuum limit ($\gamma \to 0$) provides the single values, $\alpha_0 = 1$, corresponding to

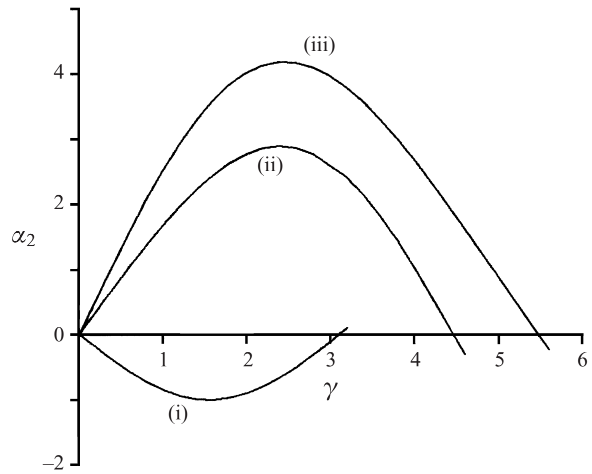

FIGURE 3. Coefficient α_2 of equation (7.13*b*) vs. wave parameter γ for three orientations of the relative velocity vector $\boldsymbol{w}$.

the Zuber solution (4.6). From the plot it is clear that the solution for the short-wave range may differ substantially from the Zuber solution. For type (i) waves directed along the lattice vector the difference between α_0 and 1.0 does not exceed 16%. But the difference may be large if the waves along diagonals of the elementary cube are considered. For example, for type (iii) waves (running along the main diagonal) the coefficient α_0 is zero at the minimum point. It is interesting to note that $\alpha_0(\gamma)$ values are non-negative for all three types of waves, and the added mass coefficient is greater than that for a single sphere. This constitutes the qualitative difference with waves in trains (see §6) for which the correction to the added mass may have any sign. The extreme values of α_0 in figure 2 correspond to the pairing mode considered below.

Also, relation (7.10) was utilized to evaluate the coefficient α_2 – see figure 3. Curves (i) to (iii) correspond to types (i) to (iii) above.

Results of computing coefficients α_{1*} and α_1 using (7.19*b*), (7.6*b*), and (7.10) taking into account (8.1)–(8.3) are shown in figure 4. Curve (i) is for the transverse wave coefficient α_{1*} in the case of movement along a cube edge. Curve (ii) represents the function $\alpha_1(\gamma)$ for a one-dimensional wave when the velocity is along a face diagonal. Curve (iii) is $\alpha_1(\gamma)$ for a one-dimensional wave when the velocity is along the main diagonal. In all situations the coefficients α_1 (or α_{1*}) are positive, which means instability. In case (i) the instability is with respect to transverse waves, whereas the system is stable with respect to one-dimensional perturbations.

Of particular interest are the maximum values of the $\alpha_1(\gamma)$ coefficient; these correspond to the fastest growing one-dimensional disturbances in the principal approximation (at low concentration c). From figure 4 it is clear that the extremes of $\alpha_1(\gamma)$ occur at the following values of γ:

$$\text{(i)}\ \gamma = \pi, \quad \text{(ii)}\ \gamma = \pi\sqrt{2}, \quad \text{(iii)}\ \gamma = \pi\sqrt{3}. \tag{8.5}$$

The extremes correspond to those wave-like disturbances of the lattice for which the disturbance velocities of every pair of neighbouring bubbles are symmetric (this is the pairing mode). This limiting phenomenon in the collective dynamics of bubbles has its ‘counterpart’ in the dynamics of vortical systems – the predominance of the

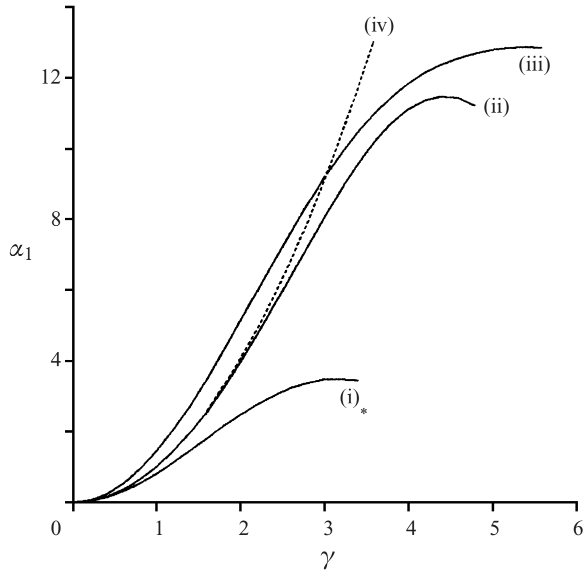

FIGURE 4. The effect of the wave parameter γ on the coefficient α_1, governing stability of the system of spheres. Curves are shown for three orientations of the relative velocity vector $\boldsymbol{w}$: (ii) and (iii) are for α_1 given by equation (7.6*b*); (i)$_*$ α_* given by equation (7.21). Curve (iv) is described in §9.

pairing mode in the perturbed single row of vortices (von Kármán 1911, 1956; Lamb 1932; Aref 1995).

When the wave is type (i) and the velocity $\boldsymbol{w}$ is along a lattice axis the pairing mode does not develop and oscillations take place along one of the lattice axes. However, in this case the maximum growth rate is attained by the transverse wave. For type (ii) or (iii) waves (along one of diagonals of the elementary cube (figure 1)) the maximum growth rate corresponds to symmetric motion of bubbles within pairs.

Thus, the system of bubbles is stable to one-dimensional perturbations for movement along three lines in the cubic lattice and unstable to one-dimensional perturbations along ten lines. This means that the one-dimensional instability dominates one-dimensional stability amongst the 13 possible lines of one-dimensional motion. That property of spatial motion is a notable contrast to the stability (described in §6) of one-dimensional motion in a train.

It should also be emphasized that, unlike trains, the question of one-dimensional stability of spatial structures cannot be given a certain answer. What is important is the derivation of three models of one-dimensional motion for the same cubic lattice. The dynamics is shown to depend significantly on the orientation of the velocity $\boldsymbol{w}$.

In order to analyse long waves in a lattice, relations (7.6*b*), (7.8), (7.19*b*), and (7.12) were employed to calculate coefficients $A_{1*}(A_{1*} = \alpha_{1*}/\gamma^2)$, A_1, and A_2 which are present in (7.13)–(7.16). Curves (ii) and (iii) in figure 5 provide values of A_1 for cases (ii) and (iii). Case (i) corresponds to curve (i)$_*$ (for A_{1*} values) and curve (i) (for A_1 values).

The limiting case of extremely long waves was also studied. The ultimate values as $\gamma \to 0$ are as follows:

$$\text{(i)}\ A_1 = -1.72, \quad A_2 = -1.02, \quad A_{1*} = 0.86; \tag{8.6a}$$

$$\text{(ii)}\ A_1 = 0.986, \quad A_2 = 1.756; \tag{8.6b}$$

$$\text{(iii)}\ A_1 = 1.493, \quad A_2 = 2.685. \tag{8.6c}$$

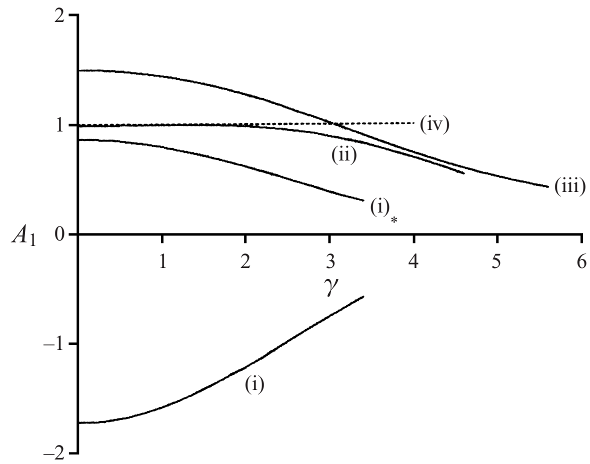

FIGURE 5. Coefficient A_1 of continuum equation (7.16) vs. wave parameter γ for various orientations of the relative velocity vector w, described in the text; curve (iv) is described in § 9.

These values lead us to the conclusion that unusual properties of the one-dimensional short waves take place in the continuum limit. To describe wave behaviour in a lattice, three models could be used. Since there is no unambiguous solution to the stability problem, the one-dimensional stability of the cubic lattice does not correlate with the usual mean values for two-phase media. The key part here is played by the motion orientation factor, not accounted for in conventional equations of two-phase system dynamics.

9. Dynamics of waves in lattices: comparison with one-dimensional theory for two-phase flows

Consider the continuum model (Zuber 1964; Wallis 1969; van Wijngaarden 1976*b*; Nigmatulin 1979, 1991; Arnold *et al.* 1989) of one-dimensional two-phase flow. Taken together with the continuity equations, the equations of motion for the dispersed phase with no dissipation at low concentration c would be

$$\rho_2 \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{3}{2}\rho_1 \frac{\mathrm{d}v}{\mathrm{d}t} - \frac{1}{2}\rho_1 \frac{\mathrm{d}u}{\mathrm{d}t},$$

$$\frac{\partial c}{\partial t} + \nabla(cu) = 0, \qquad -\frac{\partial c}{\partial t} + \nabla\{(1-c)v\} = 0.$$

If disturbances are small and can be described as

$$v = v_0 + v_1 \mathrm{e}^{\mathrm{i}kx+\lambda t}, \quad u = u_0 + u_1 \mathrm{e}^{\mathrm{i}kx+\lambda t}, \quad c = c_0 + c_1 \mathrm{e}^{\mathrm{i}kx+\lambda t},$$

then we have

$$\left(\tfrac{1}{2}\rho_1(1+3c)+\rho_2\right)\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{3}{2}\rho_1 w^2 \frac{\partial c}{\partial x} + 3\rho_1 w c_0 \frac{\partial u}{\partial x}. \tag{9.1}$$

This equation is analogous to (7.16). From (9.1) it follows that

$$(K + c_0)\lambda^2 = c_0 k^2 w^2 + 2\mathrm{i}kc_0 w\lambda,$$

and we have the positive growth exponent

$$\lambda \approx |wk|\sqrt{c/K}.$$

This corresponds to instability of two-phase flow. Such an instability is known in one-dimensional theories of two-phase flows. A similar effect was noticed by van Wijngaarden (1976*b*).

It would be instructive to compare, on one hand, the instability of one-dimensional flow and, on the other hand, the above theory of linear waves in a periodic bubble structure. In accordance with (9.1) the coefficients present in (7.16) are

$$A_1 = 1, \quad A_2 = 2. \tag{9.2}$$

The coefficient $\alpha_1 = \gamma^2$ is depicted with the dashed line (iv) in figure 4, and the coefficient $A_1 = 1$ with the straight line in figure 5. Values in (9.2) compare well with (8.6*b*) for the case of movement along a cube face diagonal (figure 1). The pattern is an example flow for which an accurate computation within the potential flow framework gives almost the same wave growth exponent as the one-dimensional theory of two-phase flows. Note that case (i) in § 8 is the other example of one-dimensional motion (along the lattice vector) with no amplitude growth.

In the set of bubbles in a cubic lattice the total number of directions of one-dimensional motion in the lattice in which motion is unstable (with respect to one-dimensional disturbances) exceeds the total number of directions for which motion is stable; this is a principal feature of the two-phase medium model with relative movement of the phases, which is described within the potential flow framework.

10. Bubble lattice stability in a viscous liquid

Above, we have considered bubbles in an inviscid liquid. Now the short-wave theory will be generalized to cover a system of bubbles in a viscous liquid. The mathematics developed above applies to the disturbed motion of a system of spherical bubbles at large Reynolds numbers. In the case of low Reynolds numbers we would employ a different approach (Voinov 1997), but at high Reynolds numbers the velocity field can be close to potential.

From Levich (1962) and Moore (1963); see also Batchelor (1967) it is known that viscous flow around a spherical bubble at large Reynolds numbers is close to potential. This fact is the basis of the first contribution to the dynamics of a system with multiple bubbles in a low-viscosity liquid (Golovin 1966) and of a number of subsequent works (Voinov & Petrov 1975; van Wijngaarden & Kapteyn 1990; Sangani & Didwania 1993*b*). Work on bubble dynamics at high Reynolds numbers is reviewed in Voinov & Petrov (1976). Viscous drag during the movement of two bubbles at high Reynolds numbers has been studied in Golovin (1966), Voinov (1971), Kok (1988*b*) and van Wijngaarden & Kapteyn (1990).

This explanation that follows is based on equations in Voinov & Petrov (1975). In the preceding text a version of the equations for inviscid liquids has been used. To allow for viscosity, (2.27) should be replaced by (3.9) from Voinov & Petrov (1975):

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = 3\frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} - 2\boldsymbol{g} - \frac{2}{\tau_0}(\boldsymbol{u} - \boldsymbol{v}'), \quad \tau_0 = \frac{R^2}{9\nu}, \tag{10.1}$$

where $\boldsymbol{g}$ is the body force and ν is the kinematic viscosity. The τ_0 parameter in (10.1) has the dimension of time and corresponds to the fact that tangential stress over a sphere is zero, and the drag of a bubble is described by the formula from Levich

(1962). Equation (10.1) corresponds to the following equation of disturbed motion of the nth bubble in the system:

$$\frac{\mathrm{d}\,\delta \boldsymbol{u}^n}{\mathrm{d}t} = 3\delta \frac{\mathrm{d}\boldsymbol{v}'^n}{\mathrm{d}t} - \frac{2}{\tau_0}(\delta \boldsymbol{u}^n - \delta \boldsymbol{v}'^n), \tag{10.2}$$

where the external velocity disturbance $\delta \boldsymbol{v}'^n$ is governed by (3.8). As stated above, we can assume that $\delta \boldsymbol{w} = \delta \boldsymbol{u}$ in the sum in (3.8). Relations (3.11)–(3.15) and (10.2) may be used to write

$$\frac{\mathrm{d}\,\delta \boldsymbol{u}^n}{\mathrm{d}t} = 3(\boldsymbol{\mu}^n + \boldsymbol{\beta}^n + \boldsymbol{\sigma}^n) - \frac{2}{\tau_0}(\delta \boldsymbol{u}^n - \delta \boldsymbol{v}'^n), \quad \frac{\mathrm{d}\delta \boldsymbol{x}^n}{\mathrm{d}t} = \delta \boldsymbol{u}^n. \tag{10.3}$$

Substitute the wave-like solution (5.3) into the disturbed motion equations (10.3) and (3.13)–(3.15). By using (3.8) we then obtain

$$\lambda\, \delta v_j'^n(\boldsymbol{x}^n) = \mu_j^n + \tfrac{1}{2}\beta_j^n, \tag{10.4}$$

where μ_j^n and β_j^n are from (3.13) and (3.14).

Consider a system with one-dimensional waves. Substitute μ_j^n and β_j^n from (7.3) and (7.9) into (10.4) to describe $\delta \boldsymbol{v}'^n$ in terms of $\delta \boldsymbol{u}^n$ and $\delta \boldsymbol{x}^n$. Then, the disturbance to the relative velocity of the nth bubble is

$$\delta \boldsymbol{u}^n - \delta \boldsymbol{v}'^n(\boldsymbol{x}^n) = (1 + \alpha_0 c)\, \delta \boldsymbol{u}^n - \frac{\mathrm{i}w}{2a} c\alpha_2\, \delta \boldsymbol{x}^n, \tag{10.5}$$

where w is undisturbed relative velocity. It is valid not only for one-dimensional waves but also for transverse waves in the case stated in § 7.6. In this situation the coefficients α_{0*} and α_{2*} are from (7.23a,b). The formula (10.5) is also suitable for describing longitudinal waves such as in (7.25). Note that here the disturbed motion is not one-dimensional. The $1 + \alpha_0 c$ coefficient in the first term in (10.5) is similar to the coefficient $Z = 1 + 3\alpha_0 c$ governing the added mass of a bubble in a system. In the continuum limit we have $\alpha_0 = 1$ for one-dimensional waves, and the added mass is described by Zuber's solution (4.6). Hence, in the continuum limit, the first summand on the right-hand side of (10.5) for one-dimensional waves is similar to Zuber's solution.

The second summand in the right-hand side of (10.5) is due to the deformation of the lattice by the wave.

We now analyse now the growth exponent λ in (5.3) for one-dimensional waves. From (10.3) and (10.5):

$$\left(\tfrac{1}{3} + c\alpha_0\right) \tilde{\lambda}^2 = \left[-\frac{12}{\varepsilon Re}(1 + c\alpha_0) + \mathrm{i}c\alpha_2\right] \tilde{\lambda} + c\alpha_1 + 6\mathrm{i}c\frac{\alpha_2}{\varepsilon Re}. \tag{10.6}$$

Here the non-dimensional exponent $\tilde{\lambda}$ and the Reynolds number Re are

$$\lambda = \frac{w}{a}\tilde{\lambda}, \quad Re = \frac{2wR}{\nu}. \tag{10.7a,b}$$

Equation (10.6) has the solution

$$\tilde{\lambda} = \frac{3}{2Z}\left\{-b + \mathrm{i}c\alpha_2 \pm \sqrt{(b - \mathrm{i}c\alpha_2)^2 + \tfrac{4}{3}cZ\left(\alpha_1 + \frac{6\mathrm{i}\alpha_2}{\varepsilon Re}\right)}\right\}, \tag{10.8a}$$

$$b = \frac{12}{\varepsilon Re}(1 + c\alpha_0), \quad Z = 1 + 3\alpha_0 c. \tag{10.8b,c}$$

It contains general information on the effect of viscosity on the dynamics of one-dimensional disturbances. Dependence of $\tilde{\lambda}$ on Reynolds number, (10.8*a*), was analysed to establish the following theorem: if a system with low viscosity ($Re \to \infty$) has an exponent with positive real part, then a system with a finite Reynolds number also has an exponent with positive real part.

From this theorem it follows that all symmetric motions in a system of bubbles in inviscid liquid which are not stable under one-dimensional disturbances will also be unstable thereunder in a viscous liquid. The conclusion about the dominance of one-dimensional instability over one-dimensional stability remains in force. Lastly, the major feature is that the conclusion about the crucial dependence of one-dimensional stability on orientation of the relative velocity of phases is general and valid for any viscosity.

An interesting subproblem is to determine the values of parameters which correspond to the inviscid model (at large Reynolds numbers). From § 7 and § 8 it follows that α_1 is proportional to γ^2, with their multiplier being about 1.0; therefore, in (10.8*a*) the viscosity contribution can be neglected if

$$b \ll \gamma c^{1/2} \quad \text{or} \quad c^{5/6}\gamma Re \gg 20. \tag{10.9}$$

Clearly, if the wavelength is quite large (in the limit $\gamma \to 0$) then (10.9) may fail at any Reynolds number. Now we need to estimate the role of viscosity in typical conditions. In bubbly liquids the usual Re value is 100. So, if concentration is a few percent, then (10.9) is valid if $\gamma \gg 1$. However, $\gamma \sim 1$ (or less), so viscosity is important.

Viscosity influences the system behaviour greatly if $b \sim \gamma c^{1/2}$. The growth exponent may be obtained for $b \gg \gamma c^{1/2}$, when the role of viscosity is significant:

$$\tilde{\lambda} \approx cb^{-1}\alpha_1 + \tfrac{1}{2}\mathrm{i}c\alpha_2. \tag{10.10}$$

If $\alpha_1 > 0$, then this equation reflects the retardation of instability development by viscosity.

11. Summary and conclusions

(i) The short waves in a periodic structure of bubbles that moves steadily through an inviscid liquid have been theoretically found and studied. The volumetric concentration of bubbles is assumed to be low. The behaviour of short waves represents information on the dynamic behaviour of the two-phase system. The continuum limit for short-wave solutions provides models useful for understanding the continuum description of two-phase media.

(ii) One-dimensional waves in a periodic system of bubbles have been studied. The behaviour of one-dimensional disturbed motion depends on the structure type, which could be linear or spatial (a train or a three-dimensional structure). In a train the one-dimensional wave amplitude does not grow, although the train is not stable. This important property of trains is shown not to apply to the general case of one-dimensional disturbed motion in a spatial structure.

(iii) For a cubic lattice three models describing one-dimensional disturbed motions characterized by various dynamic coefficients have been proposed. Wave behaviour crucially depends on the type of model. Only in the continuum limit do all three models provide identical values of the added mass coefficient. Values of the stability-governing coefficient differ substantially. Therefore, knowledge of the mean characteristics of a bubbly liquid does not allow one to conclude whether the relative motion of phases is stable to longitudinal disturbances directed along the relative velocity direction.

Conditions under which the added mass coefficient for a bubble in a wave is notably different from Zuber's value have been found. This occurs for short one-dimensional waves, with the difference depending on the wave type (see figure 2). Also, the difference is characteristic of transverse waves and seen in the continuum limit.

(iv) The orientation of the relative velocity is the principal governing factor in wave dynamics for periodical systems. The orientation defines the type of model required to describe one-dimensional disturbed motion. The role of the orientation is clearly seen in the dynamics of one-dimensional disturbances, which are possible when the relative velocity is along one of the symmetry axes.

The motion of a cubic lattice of bubbles along a lattice vector is stable with respect to one-dimensional wave-like disturbances. Oppositely, the motion along a cube face diagonal or along a main diagonal is unstable to those disturbances. Consequently, the one-dimensional motion is stable to one-dimensional disturbances for only three lines in the lattice, whereas no stability is seen for ten lines.

(v) The equations of the continuum limit that correspond to the case of one-dimensional waves have been analysed. For motion along one of the cube face diagonals the coefficients of the continuum equation are shown to be almost the same as those for the previously known version of the one-dimensional two-phase flow equations. A model example of instability of steady-state flow is found, which is similar to the instability revealed by van Wijngaarden (1976*b*).

(vi) The influence of a small liquid viscosity on disturbance dynamics in a stationary bubble system was studied on the basis of the dynamic equations in Voinov & Petrov (1975). It has been found that instability of movement along symmetry axes (with respect to one-dimensional disturbances) of the bubble system dominates stability. When allowing for the liquid viscosity, the wave amplitude growth exponent depends significantly on orientation of the phase relative velocity, as in the case of an ideal liquid.

(vii) For the short-wave theory under consideration a quite high accuracy may be attained by using the usual equation describing the force that acts on a small sphere in a non-uniform flow. A refined formula (taking into account a higher order with respect to the sphere radius) is derived in the Appendix. According to the new formula, the bubble acceleration can differ substantially from that in a uniformly accelerated flow; in the latter case the acceleration is thrice that of the liquid.

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Appendix. Refined equation for the force on a small sphere in a non-uniform flow

Consider a small sphere in a non-uniform flow with no vortices present. A body placed in a flow may be assumed to be small if its radius R is small relative to the distance l_0 to an external boundary of the flow: $l_0 \gg R$. Previous equations for computing the force $\boldsymbol{F}$ applied to the body by a non-uniform potential flow rely upon representing the flow in terms of both the velocity at the body centroid and derivatives of the velocity; for example see Voinov (1973), Voinov & Petrov (1973), and van Wijngaarden (1976*a*). Re-write the usual simplified equation for the force applied to a sphere with a constant volume ($V = \text{const}$),

$$\boldsymbol{F} = \rho_1 V \left(\frac{3}{2} \frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} - \frac{1}{2} \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} - \boldsymbol{g} \right). \tag{A 1}$$

Our main interest is to find a more general equation than (A 1) for hydrodynamic force.

It is known that the motion of a system of bodies in ideal liquid with no vortices can be described through the Lagrange equations (Thomson & Tait 1867; Lamb 1932; Birkhoff 1960). For a small body in a non-uniform potential flow we know the exact Lagrange function (Voinov & Petrov 1973; Voinov *et al.* 1973) that defines the hydrodynamic reaction on the body:

$$L = \frac{\rho}{2} \int_{\Omega} |\boldsymbol{v} - \boldsymbol{v}'|^2 \, \mathrm{d}\tau - \int_{V} p' \, \mathrm{d}\tau. \tag{A 2}$$

Here, $\boldsymbol{v}'$ is the liquid velocity in the specified non-uniform velocity field if the small body is absent; $\boldsymbol{v}$ is the liquid velocity disturbed by the body; p' is the pressure in the non-uniform field $\boldsymbol{v}'$; $\mathrm{d}\tau$ is the volume element; Ω is the domain outside the body; and V is the domain occupied by the body. At some distance from the body the velocity field is not disturbed:

$$\boldsymbol{v} - \boldsymbol{v}' \to 0, \quad r/R \to \infty.$$

The hydrodynamic force F_i acting on the sphere is described by the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = -F_i, \tag{A 3}$$

where q_i is position of the sphere centre, and dot symbolizes differentiation with respect to time.

The well-known equation (A 1) is a simple consequence of (A 2) and (A 3) in the main approximation with respect to small R values (Voinov & Petrov 1973; Voinov *et al.* 1973).

For a sphere the present author has derived independently the force equation, proceeding from the expression for pressure over the sphere (Voinov 1973). However to derive the refined relation, we shall not consider this alternative approach because using (A 2) ensures significant saving in transformations.

Because the sphere radius R is small, the potential of velocity $\boldsymbol{v}'$ in a vicinity of the sphere may be presented as a Taylor's series (Voinov 1973):

$$\Phi' = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \Phi'}{\partial x_i \partial x_j \ldots \partial x_k} y_i y_j \ldots y_k, \quad y_i = x_i - q_i, \quad \boldsymbol{v}' = \nabla \Phi'. \tag{A 4}$$

Here, the derivatives are taken at the point $\boldsymbol{x} = \boldsymbol{q}$.

If we use only three terms in the expansion (with a linear relation between the velocity and the coordinates) then the flow is described by five independent components from the nine in $\nabla \boldsymbol{v}'$. This local representation corresponds to the principal force approximation accurate to $O(R^3)$. In the higher approximation we must describe Φ' with the use of four expansion terms. Compared with the basic problem, this adds a further seven independent components from the twenty seven in $\nabla\nabla\boldsymbol{v}'$.

For the potential Φ in the presence of the sphere conditions (2.2) can be neglected and

$$\Phi - \Phi' \to 0 \quad \text{at} \quad r/R \to \infty.$$

This problem for the Laplace equation is solved by equation (1.3) in Voinov (1973); in our problem with $R =$ const. becomes

$$\Phi = -\frac{R^3}{2r^3} y_i u_i + \sum_{n=0}^{\infty} \frac{1}{n!}\left(1 + \frac{nR^{2n+1}}{(n+1)r^{2n+1}}\right) \frac{\partial^n \Phi'}{\partial x_i \partial x_j \ldots \partial x_k} y_i y_j \ldots y_k, \tag{A 5}$$

where $u_i = q_i^*$ and $r = |\boldsymbol{y}|$. Correspondingly, the velocity is

$$v_l = -u_l \frac{R^3}{2r^3} + \frac{3R^3}{2r^4} u_i y_i y_l + \sum_{n=1}^{\infty} \frac{n}{n!}\left\{\left(1 + \frac{nR^{2n+1}}{(n+1)r^{2n+3}}\right)\delta_{il} - \frac{2n+1}{n+1}\frac{R^{2n+1}}{r^{2n+3}} x_i x_l\right\} \frac{\partial^n \Phi'}{\partial x_i \partial x_j \ldots \partial x_k} y_j \ldots y_k. \tag{A 6}$$

Here, δ_{il} is Kronecker's delta function. For the sphere (at $r = R$) relations (A 5) and (A 6) yield

$$\Phi - \Phi' = -\tfrac{1}{2} w_i y_i + \frac{1}{3}\frac{\partial v_i'}{\partial x_j} y_i y_j + \Phi_3 \Big|_{r=R} + O(y^4), \quad w_i = u_i - v_i', \tag{A 7}$$

$$v_l - v_l' = -w_i\left(\tfrac{1}{2}\delta_{il} - \frac{3}{2}\frac{y_i y_l}{R^2}\right) + \left(\tfrac{2}{3}\delta_{il} - \frac{5}{3}\frac{y_i y_l}{R^2}\right)\frac{\partial v_i'}{\partial x_j} y_j + (\nabla_l \Phi_3)\Big|_{r=R} + O(y^3), \tag{A 8}$$

where $\Phi_3 = O(y^3)$ at $r = R$; the potential Φ_3 is a linear form of $\nabla_i \nabla_j v_k'$. The first integral in (A 2) is equal to a surface integral:

$$\tfrac{1}{2}\rho \int_\Omega |\boldsymbol{v} - \boldsymbol{v}'|^2 \mathrm{d}\tau = \tfrac{1}{2}\rho \int_S (\Phi - \Phi')(v_j - v_j') n_j \, \mathrm{d}S. \tag{A 9}$$

From (A 7)–(A 9) we have, for small R values,

$$\tfrac{1}{2}\rho \int_\Omega |\boldsymbol{v} - \boldsymbol{v}'|^2 \, \mathrm{d}\tau = \tfrac{1}{4}\rho V |\boldsymbol{u} - \boldsymbol{v}'|^2 + \tfrac{1}{15}\rho V R^2 \nabla_j v_k' \nabla_j v_k' + O(R^7). \tag{A 10}$$

There is no contribution from Φ_3:

$$T_3 = \Lambda_{ijk\ell}(u_i - v_i')\nabla_j \nabla_k v_l', \quad \Lambda_{ijkl} = V R^2 C(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where the value of C is of no importance. The sum T_3 is zero because $\nabla_j \nabla_j v_l' = \nabla_j \nabla_k v_k' = 0$.

We expand $p'(\boldsymbol{x})$ as a Taylor series at the point $\boldsymbol{x} = \boldsymbol{q}$ and substitute the series in the second integral of (A 2). Upon evaluating the integral and using the Cauchy–Lagrange integral we obtain

$$\int_V p' \mathrm{d}\tau = p' V - \tfrac{1}{10}\rho V R^2 \nabla_j v_k' \nabla_j v_k' + O(R^7). \tag{A 11}$$

Substitution of (A 9) and (A 11) into (A 2) leads us to

$$L = \tfrac{1}{4}\rho V |\boldsymbol{u} - \boldsymbol{v}'|^2 - p' V + \tfrac{1}{6}\rho V R^2 \nabla_j v_k' \nabla_j v_k'. \tag{A 12}$$

In accordance with (A 3) and (A 12) the force applied to the sphere becomes

$$\boldsymbol{F} = \rho_1 V \left(\frac{3}{2}\frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} + \tfrac{1}{3} R^2 \Delta \frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} - \frac{1}{2}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} - \boldsymbol{g}\right). \tag{A 13}$$

Here, ρ_1 is the liquid density; $\boldsymbol{u}$ is the sphere velocity; and V is the sphere volume

(V = const). This formula differs from the usual (A 1) by the summand with the Laplacian of liquid acceleration. The formula (A 13) includes some new information about the force in non-uniform flow. From Birkhoff (1960) it is known that gas bubble acceleration in a uniform flow is thrice the liquid acceleration. According to (A 13), the bubble acceleration in a non-uniform flow may differ significantly from that value. It may occur, for example, at a critical point ($\boldsymbol{v}' = 0$) of a non-uniform flow.

The force expression (A 13) does not contain a contribution (behaving as $O(R^6)$) from an external flow disturbance at a large distance from the small constant-radius sphere. The contribution cannot be introduced while using the concept of a body in non-uniform flow. Therefore, (A 13) cannot be refined. It is the expression of the highest possible accuracy for systems with constant-radius spheres.

If the sphere radius is variable, then it is easy to complement (A 13) with the relevant summand. However, we shall not do this because, for $R \neq$ const, the contribution of far boundaries is $O(R^5)$, the order of behaviour of the second summand in (A 13). This contribution cannot be taken into account in the general form; therefore, in a system with a variable sphere the maximum accuracy is ensured by the usual general formula (A 1) complemented with the term $(\rho_1/2)(\boldsymbol{v}' - \boldsymbol{u})\mathrm{d}V/\mathrm{d}t$. In Voinov (1973) the radius varies, so (A 13) was not discussed.

If we deal with a system in which the neighbouring bubbles are quite far from one another (so that $R/a = \varepsilon \ll 1$) then the second summand in (A 13) is a small correction. Indeed, we may use (2.21) and (2.22) to estimate the order of the second summand in brackets in (A 13):

$$\tfrac{1}{3}R^2 \left| \Delta \frac{d\boldsymbol{v}'}{dt} \right| = \tfrac{1}{3}R^2 \left| \Delta(\boldsymbol{v}' \cdot \nabla)\boldsymbol{v}' \right| = O(\varepsilon^9). \qquad \text{(A 14)}$$

The first summand in brackets in (A 13) is $O(\epsilon^3)$. Therefore, when utilizing (2.21) and (2.22) together with (A 13), the term $\Delta(\mathrm{d}\boldsymbol{v}'/\mathrm{d}t)$ is a small correction, ε^9, to the first summand.

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