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Existence of normalized solutions for nonlinear fractional Schrödinger equations with trapping potentials

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In this paper, we study the existence, nonexistence and mass concentration of L^2-normalized solutions for nonlinear fractional Schrödinger equations. Comparing with the Schrödinger equation, we encounter some new challenges due to the nonlocal nature of the fractional Laplacian. We first prove that the optimal embedding constant for the fractional Gagliardo–Nirenberg–Sobolev inequality can be expressed by exact form, which improves the results of [**17**, **18**]. By doing this, we then establish the existence and nonexistence of L^2-normalized solutions for this equation. Finally, under a certain type of trapping potentials, by using some delicate energy estimates we present a detailed analysis of the concentration behavior of L^2-normalized solutions in the mass critical case.

Keywords: Fractional Schrödinger equation; normalized solutions; mass concentration

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1. Introduction and main results

This paper is devoted to the existence, nonexistence and mass concentration of L^2-normalized solutions for a class of Schrödinger equations with the fractional Laplacian. More precisely, we are concerned with the following stationary (i.e., time-independent) fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = \mu u + af(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $0 < s < 1$, $N \geqslant 2$, $V : \mathbb{R}^N \to \mathbb{R}$ is an external potential function, $\mu \in \mathbb{R}$ and $a > 0$ are parameters, and f is a subcritical nonlinearity. The operator $(-\Delta)^s$ is the fractional Laplacian of order s, which, for a rapidly decreasing C^∞ function u, may be defined as

$$(-\Delta)^s u(x) = C_{N,s}\,\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy = C_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, \mathrm{d}y. \tag{1.2}$$

The symbol P. V. denotes the Cauchy principal value of the singular integral, and $C_{N,s}$ is a dimensional constant that depends on N and s, precisely given by $C_{N,s} = (\int_{\mathbb{R}^N} ((1 - \cos \zeta_1)/(|\zeta|^{N+2s}))\, \mathrm{d}\zeta)^{-1}$.

It is well known that equation (1.1) arises from looking for the standing wave type solutions $\Psi(x, t) = e^{-i\mu t} u(x)$ for the following time-dependent nonlinear fractional Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} = (-\triangle)^s \Psi + V(x)\Psi - a h(|\Psi|)\Psi, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where i is the imaginary unit and $\Psi : \mathbb{R}^N \times [0, \infty) \mapsto \mathbb{C}$. Obviously, Ψ solves (1.3) if and only if the standing wave $u(x)$ satisfies (1.1) with $f(u) = h(|u|)u$. Here $\Psi(x, t)$ represents the quantum mechanical probability amplitude for a given unit mass particle to have position x at time t (the corresponding probability density is $|\Psi|^2$), under a confinement due to the potential $V(x)$. Equation (1.3) is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes. A path integral over the Lévy flights paths and a fractional Schrödinger equation of fractional quantum mechanics are formulated by Laskin [**25**] from the idea of Feynman and Hibbs's path integrals. For more physical background of (1.3), we refer the reader to [**6**, **26**] and the references therein.

Note that $(-\Delta)^s$ on $\mathbb{R}^N$ with $0 < s < 1$ is a nonlocal operator. The nonlocal nature of the fractional Laplacian makes it difficult to study. To overcome this difficulty, Caffarelli and Silvestre [**2**] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. That is, for a function $u \in H^s(\mathbb{R}^N)$, one considers the extension $U : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ that satisfies

$$\begin{cases} -\text{div}(y^{1-2s} \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ U(x, 0) = u & \text{on } \mathbb{R}^N. \end{cases}$$

Then, it follows from [**2**] that

$$(-\triangle)^s u(x) = -C(N, s) \lim_{y \to 0^+} y^{1-2s} U_y(x, y),$$

where $C(N, s)$ is an appropriate constant depending on N and s. This extension method has been applied successfully to study equations involving the fractional Laplacian, and a series of significant results have been obtained.

Recently, the study on equations with the fractional Laplacian has been attracted much interest from many mathematicians. Coti Zelati and Nolasco [**7**] obtained the existence and regularity of positive stationary solutions for a class of nonlinear pseudo-relativistic Schrödinger equations involving the operator $(-\Delta + d^2)^{1/2}$ with $d > 0$. Cheng [**5**] proved the existence of ground state solutions for the following equation

$$(-\Delta)^s u + V(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $V(x)$ is an unbounded potential. In (1.4), when $V(x) \equiv 1$, Dipierro et al. [**13**] obtained the existence and symmetry results for solutions, and Felmer et al. [**15**] considered the same equation with a more general nonlinearity $f(x,u)$, they proved the existence, regularity and qualitative properties of ground state solutions. Secchi [**32**] obtained the existence of positive solutions for a more general fractional Schrödinger equation by the variational method. Chen and Zheng [**4**] studied the existence and concentration phenomenon for solutions of the following equation

$$(-\varepsilon^2\Delta)^s u + V(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.5}$$

under further constraints in the space dimension N and the values of s, by using the Lyapunov–Schmidt reduction method. Davillá, del Pino and Wei [**10**] generalized various existence results already known for (1.5) with $s = 1$ to the case of fractional Laplcians. For more results on this direction, see for example, [**1**, **3**, **14**, **34**–**37**].

In the aforementioned papers, the frequency μ is seen as a fixed parameter, and the critical point theory is used to look for solutions. However, nothing can be given a priori estimate on the L^2-norm of solutions. Motivated by the fact that physicists are often interested in 'L^2-normalized solutions', that is, solutions with normalized L^2-norm, in this paper, we study the existence, nonexistence and mass concentration of L^2-normalized solutions for equation (1.1). To this aim, we note that (1.1) is also the Euler–Lagrange equation of the following constrained minimization problem

$$e(a) := \inf_{u \in \mathcal{M}} E_a(u), \tag{1.6}$$

where the energy functional $E_a(u)$ is defined by

$$E_a(u) := \int_{\mathbb{R}^N} \left(|(-\Delta)^{s/2}u(x)|^2 + V(x)|u(x)|^2\right) \mathrm{d}x - 2a \int_{\mathbb{R}^N} F(u(x))\,\mathrm{d}x, \qquad u \in \mathcal{H}.$$

Here F is the primitive of f, and we define

$$\mathcal{H} := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^2\,\mathrm{d}x < \infty \right\} \tag{1.7}$$

with associated norm $\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^N} \left(|(-\Delta)^{s/2}u(x)|^2 + V(x)|u(x)|^2\right) \mathrm{d}x$, and

$$\mathcal{M} := \left\{ u \in \mathcal{H} : \int_{\mathbb{R}^N} |u(x)|^2\,\mathrm{d}x = 1 \right\}. \tag{1.8}$$

Obviously, if u_a is a minimizer of (1.6), it is the 'L^2-normalized solution' of (1.1) with a suitable Lagrange parameter μ_a associated with u_a. Consequently, in this paper, we are interested in minimizers of (1.6) under the unit mass constraint (1.8). Alternatively, one may want to impose the constraint $\int_{\mathbb{R}^N} |u(x)|^2\,\mathrm{d}x = \mathcal{K} > 0$, that is, L^2-spheres in $H^s(\mathbb{R}^N)$, but this latter case can easily be reduced to the previous one, by minimizing under the constraint (1.8) but simply replacing a by $\mathcal{K}a$. Therefore, we prefer to work with (1.8) instead.

In (1.1), if we set $s = 1$ and replace μ and a by λ and b, respectively, it reduces to the following nonlinear Schrödinger equation

$$-\Delta u + V(x)u = \lambda u + bf(u), \quad x \in \mathbb{R}^N. \tag{1.9}$$

There are many works focusing on equation (1.9). We just mention the earlier work by Floer and Weinstein [**16**], Oh [**29**], Rabinowitz [**31**], Wang [**38**], del Pino and Felmer [**11**], without any attempt to review the references here. When $f(u) = |u|^2u$ and $N = 2$, we can describe equation (1.9) by the following constrained minimization problem

$$\hat{e}_b(u) := \inf_{u\in\hat{\mathcal{M}}} \hat{E}_b(u), \tag{1.10}$$

where the energy functional $\hat{E}_b(u)$ is defined by

$$\hat{E}_b(u) := \int_{\mathbb{R}^2} \left(|\nabla u(x)|^2 + V(x)|u(x)|^2\right) \, dx - \frac{b}{2}\int_{\mathbb{R}^2} |u(x)|^4 \, dx, \quad u \in \hat{\mathcal{H}}.$$

Here we define

$$\hat{\mathcal{H}} := \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)|u(x)|^2 \, dx < \infty\right\},$$
$$\hat{\mathcal{M}} := \left\{u \in \hat{\mathcal{H}} : \int_{\mathbb{R}^2} |u(x)|^2 \, dx = 1\right\}.$$

We note that problem (1.10) is a mass critical problem. It is shown in [**9**, **30**] that attractive Bose–Einstein condensates can be described by the L^2-constraint minimizers of (1.10), where $b > 0$ represents the strengthen of the attractive interaction among the cold atoms. Assume that

$$(V_1) 0 \leqslant V(x) \in L^\infty_{\text{loc}}(\mathbb{R}^N), \lim_{|x|\to\infty} V(x) = \infty \quad \textit{and} \quad \inf_{x\in\mathbb{R}^N} V(x) = 0.$$

In case $N = 2$, Guo and Seiringer [**19**] proved the following existence and nonexistence of minimizers for (1.10): there exists a critical value $b^* > 0$ such that (1.10) has at least one minimizer if $0 \leqslant b < b^*$, and (1.10) has no minimizers if $b \geqslant b^*$. Furthermore, the critical value $b^* = \|\phi\|_2^2$. That is, the square of the L^2-norm of the unique (up to translations) positive solution of the famous nonlinear scalar field equation

$$-\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2, \quad u \in H^1(\mathbb{R}^2).$$

The limit behaviour of minimizers for (1.10) as $b \nearrow b^*$ is also explored in [**19**–**21**].

To our best knowledge, there is no result on the existence, nonexistence and mass concentration of L^2-constraint minimizers of (1.6). In this paper, we shall fill the gap of information. More precisely, the purpose of this paper is to obtain the existence and nonexistence of L^2-constraint minimizers of (1.6) by applying a constrained variational method. Moreover, under a certain type of trapping potentials, by using some delicate energy estimates, we present a detailed analysis of the concentration behaviour of L^2-constraint minimizers for the mass critical case of (1.6). Actually, we prove that all the mass concentrates at a global minimum point x_0 of the trapping potential $V(x)$.

Before we formulate the main results of this paper, let us first recall some facts in [**17**, **18**] that, up to translations, the fractional Schrödinger equation

$$(-\Delta)^s u + u = |u|^{p-2}u, \quad u \in H^s(\mathbb{R}^N), \tag{1.11}$$

where $0 < s < 1$, $N \geqslant 1$ and $2 < p < 2_s^*$ ($2_s^* := 2N/(N-2s)$ if $N > 2s$, and $2_s^* := \infty$ if $N \leqslant 2s$), has a unique radial positive ground state solution $\varphi(x)$ which can be taken to be radially symmetric about the origin. Moreover, the function $\varphi(x)$ is strictly decreasing in $|x|$, $\varphi \in H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, and it satisfies

$$\begin{aligned} \frac{c_1}{1+|x|^{N+2s}} \leqslant \varphi(x) \leqslant \frac{c_2}{1+|x|^{N+2s}} \quad &\text{for } x \in \mathbb{R}^N, \\ \left|\partial_{x_j}\varphi(x)\right| \leqslant \frac{c_3}{1+|x|^{N+2s}} \quad &\text{for } x \in \mathbb{R}^N \quad \text{and } j = 1, \cdots, N, \end{aligned} \tag{1.12}$$

with some constant $c_i > 0$, $i = 1, 2, 3$. Furthermore, every nonnegative optimizer $v \in H^s(\mathbb{R}^N)\backslash\{0\}$ for the fractional Gagliardo–Nirenberg–Sobolev inequality

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x)|^p \,\mathrm{d}x \leqslant C_{\text{opt}} \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(x)|^2 \,\mathrm{d}x \right)^{(N(p-2)/(4s))} \\ \times \left(\int_{\mathbb{R}^N} |u(x)|^2 \,\mathrm{d}x \right)^{p/2-(N(p-2)/(4s)} \end{aligned} \tag{1.13}$$

is of the form $v = \beta\varphi(\gamma(\cdot + y))$ with some $\beta > 0$, $\gamma > 0$ and $y \in \mathbb{R}^N$.

In order to establish the existence and nonexistence of minimizers for the minimization problem (1.6), we need to know the exact value of C_{opt} in (1.13). To this aim, we first address the fractional Gagliardo-Nirenberg-Sobolev inequality.

THEOREM 1.1. *For $N \geqslant 1$ and $2 < p < 2_s^*$, the fractional Gagliardo–Nirenberg–Sobolev inequality* (1.13) *is attained at a function $Q(x)$ with the following properties:*

(i) *$Q(x)$ is radial, positive, and strictly decreasing in $|x|$.*

(ii) *$Q(x)$ belongs to $H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and satisfies*

$$\begin{aligned} \frac{C_1}{1+|x|^{N+2s}} \leqslant Q(x) \leqslant \frac{C_2}{1+|x|^{N+2s}} \quad &\text{for } x \in \mathbb{R}^N, \\ |\partial_{x_j} Q(x)| \leqslant \frac{C_3}{1+|x|^{N+2s}} \quad &\text{for } x \in \mathbb{R}^N \text{ and } j = 1, \cdots, N, \end{aligned}$$

where C_i ($i = 1, 2, 3$) are positive constants.

(iii) *$Q(x)$ is a solution of the fractional Schrödinger equation*

$$\begin{cases} \dfrac{N(p-2)}{4s}(-\Delta)^s u + \left(1 + \dfrac{p-2}{4}\left(2 - \dfrac{N}{s}\right)\right) u - |u|^{p-2}u = 0 \quad \textit{in } \mathbb{R}^N \\ u \in H^s(\mathbb{R}^N) \end{cases} \tag{1.14}$$

of minimal L^2 norm (the ground state). In addition,

$$C_{\text{opt}} = \frac{p}{2\|Q\|_2^{p-2}}. \tag{1.15}$$

Moreover, every nonnegative optimizer $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ for the fractional Gagliardo–Nirenberg–Sobolev inequality is of the form

$$v(x) = \frac{\|(-\Delta)^{s/2}v\|_2^{N/(2s)}}{\|v\|_2^{N/(2s)-1}\|Q\|_2} Q\left(\frac{\|(-\Delta)^{s/2}v\|_2^{1/s}}{\|v\|_2^{1/s}}(x+y)\right) \quad \textit{for some } y \in \mathbb{R}^N.$$

Remark 1.1. Note that, we find the exact value of C_{opt} in (1.15). Moreover, it follows from scaling arguments that $Q(x) = \widetilde{\beta}\varphi(\widetilde{\gamma}x)$, where φ is the unique radial positive ground state solution of (1.11), and

$$\widetilde{\beta} = \left(\frac{2N - p(N-2s)}{4s}\right)^{1/(p-2)}, \qquad \widetilde{\gamma} = \left(\frac{2N - p(N-2s)}{N(p-2)}\right)^{1/(2s)}.$$

Then, every nonnegative optimizer $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ for the fractional Gagliardo–Nirenberg–Sobolev inequality is of the form $v = \beta\varphi(\gamma(\cdot + y))$, where

$$\beta = \left(\frac{2N - p(N-2s)}{4s}\right)^{1/(p-2)} \frac{\|(-\Delta)^{s/2}v\|_2^{N/(2s)}}{\|v\|_2^{N/(2s)-1}\|Q\|_2^2},$$
$$\gamma = \left(\frac{2N - p(N-2s)}{N(p-2)}\right)^{1/(2s)} \frac{\|(-\Delta)^{\frac{s}{2}}v\|_2^{1/s}}{\|v\|_2^{1/s}}.$$

As a consequence, we conclude the exact values of C_{opt}, β and γ for the fractional Gagliardo–Nirenberg–Sobolev inequality and improve the results in [**17**, **18**].

The proof of Theorem 1.1 is evident from the following considerations: To compute C_{opt}, it suffices to minimize the corresponding 'Weinstein functional' (see [**39**]):

$$J(u) = \frac{\left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2 \, \mathrm{d}x\right)^{((N(p-2))/(4s))} \left(\int_{\mathbb{R}^N} |u(x)|^2 \, \mathrm{d}x\right)^{p/2-((N(p-2))/(4s))}}{\int_{\mathbb{R}^N} |u(x)|^p \, \mathrm{d}x}, \tag{1.16}$$

where $u \in H^s(\mathbb{R}^N)$ and $u \not\equiv 0$. In § 2, we will show that the minimum is attained at some function $\widetilde{v} \in H^s(\mathbb{R}^N)\backslash\{0\}$. By scaling, we can take $\|(-\Delta)^{s/2}\widetilde{v}\|_2 = 1$ and $\|\widetilde{v}\|_2 = 1$. Computing the Euler–Lagrange equation leads to (1.14) and (1.15). The main idea in the proof of Theorem 1.1 comes from [**39**], in which the author dealt

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with such a problem by working on the radially symmetric subspace $H^1_r(\mathbb{R}^N)$, which embeds compactly in $L^q(\mathbb{R}^N)$ for $2 < q < ((2N)/(N-2))$ if $N \geqslant 3$ and $q > 2$ if $N = 2$ [**40**]. In our theorem 1.1, since the embedding $H^s_r(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$, $q > 2$ is not compact, we cannot work on the radially symmetric subspace $H^s_r(\mathbb{R}^N)$ as in [**39**]. To overcome this difficulty, we use the concentration-compactness principle of P. L. Lions [**28**] (see also [**40**]).

Throughout this paper, we suppose that f satisfies the following assumptions:
(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $|f(t)| \leqslant c_1(|t| + |t|^{p-1})$ for some $c_1 > 0$ and $2 < p < 2 + 4s/N$.
(f_2) $f \in C(\mathbb{R}, \mathbb{R})$, $|f(t)| \leqslant c_2(|t| + |t|^{p-1})$ for some $c_2 > 0$ and $2 + 4s/N < p < 2^*_s$.
(f_3) there exist $\nu > 2 + 4s/N$ and $r_0 > 0$ such that

$$0 < \nu F(t) \leqslant t f(t) \quad \text{for all } |t| \geqslant r_0.$$

By applying directly the fractional Gagliardo–Nirenberg–Sobolev inequality (1.13) and scaling techniques, we shall establish the following existence and nonexistence of minimizers for the minimization problem (1.6).

THEOREM 1.2. *Suppose* $0 \leqslant V(x) \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ *satisfies* $\lim_{|x|\to\infty} V(x) = \infty$.

(i) *If* (f_1) *holds, then* $e(a)$ *has at least one minimizer and* $e(a) > -\infty$ *for each* $a > 0$.

(ii) *If* (f_2) *and* (f_3) *hold, then* $e(a)$ *has no minimizers and* $e(a) = -\infty$ *for each* $a > 0$.

REMARK 1.2.

(i) A typical example satisfying the condition (f_1) is the power function $f(t) = |t|^{p-2}t$ with $2 < p < 2 + 4s/N$. Another example is $f(t) = |t|^{p-2}t - |t|^{q-2}t$ with $2 < q < p < 2 + 4s/N$.

(ii) A typical example satisfying the conditions (f_2) and (f_3) is the power function $f(t) = |t|^{p-2}t$ with $2 + 4s/N < p < 2^*_s$. Another example is $f(t) = |t|^{p-2}t - |t|^{q-2}t$ with $2 + 4s/N < q < p < 2^*_s$.

(iii) For the case of power function $f(t) = |t|^{p-2}t$ with $2 < p < 2^*_s$, theorem 1.2 gives a complete classification of the existence and nonexistence of minimizers for (1.6), expect that $p = 2 + 4s/N$.

It is worth to point out that $p = 2 + 4s/N$ is the so-called mass critical exponent for (1.6). An interesting question now is whether the same existence or nonexistence results as theorem 1.2 occur to the mass critical case of (1.6), that is, $f(t) \sim |t|^{4s/N}t$. For this purpose, in order to keep the ideas and the results simple, in the sequel we only deal with the nonlinearity of the power-type function, that is, $f(t) = |t|^{4s/N}t$. In this paper, we shall derive the following existence and nonexistence of minimizers for (1.6) with $f(t) = |t|^{4s/N}t$.

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THEOREM 1.3. *Suppose $V(x)$ satisfies (V_1), and let $f(t) = |t|^{4s/N}t$. Then we have*

(i) *For all $a \in [0,\, a^*)$, $e(a)$ has at least one minimizer, and $e(a)$ has no minimizers if $a \geqslant a^*$, where $a^* = \|Q\|_2^{4s/N}$ and Q is the unique radial positive ground state solution of*

$$(-\Delta)^s u + \frac{2s}{N}u - |u|^{4s/N}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^s(\mathbb{R}^N). \tag{1.17}$$

Moreover, $e(a) > 0$ for $0 \leqslant a < a^*$, $\lim_{a \nearrow a^*} e(a) = e(a^*) = 0$ and $e(a) = -\infty$ for $a > a^*$.

(ii) *When $a \in [0,\, a^*)$ is suitably small, $e(a)$ has a unique nonnegative minimizer.*

REMARK 1.3.

(i) theorems 1.2 and 1.3 provides a complete classification of the existence and nonexistence of minimizers for (1.6) with $f(t) = |t|^{p-2}t$, where $2 < p < 2_s^*$. We note that, theorem 1.3 also implies that the trap shape does not affect the critical value a^*.

(ii) In order to prove theorem 1.3, the main difficulty is that we need to estimate the Gagliardo (semi) norm of some trial function, see the forthcoming estimate (3.19). For this purpose, stimulated by [**34**] we establish lemma 3.2 to circumvent this obstacle. Moreover, the function $Q(x)$ given by theorem 1.1 is polynomially decay at infinity, which is in contrast to the fact that the ground state exponentially decays at infinity in $s = 1$. So, we need to give more detailed analysis to establish the desired estimates of the trial function.

If u_a is a minimizer of (1.6) with $f(t) = |t|^{4s/N}t$, then we can assume that u_a is nonnegative, due to the fact that $E_a(u) \geqslant E_a(|u|)$ for any $u \in \mathcal{H}$. Consequently, without loss of generality, we can restrict the minimization to nonnegative functions. In view of theorem 1.3 (ii), we know that if $f(t) = |t|^{4s/N}t$, $e(a)$ defined in (1.6) has a unique nonnegative minimizer for any $a > 0$ small enough. So, we may define

$$a_* := \sup\left\{l > 0 : e(a) \text{ has a unique nonnegative minimizer for all } a \in [0,\, l)\right\}, \tag{1.18}$$

and $0 < a_* \leqslant a^*$. Note that any minimizer u_a of (1.6) with $f(t) = |t|^{4s/N}t$ satisfies the following fractional nonlinear Schrödinger equation:

$$(-\Delta)^s u + V(x)u = \mu_a u + a|u|^{4s/N}u, \quad x \in \mathbb{R}^N,$$

where $\mu_a \in \mathbb{R}$ is a suitable Lagrange multiplier associated with u_a. Now some natural questions arise: can we show that the Lagrange multiplier μ_a depends only on a and is independent of the choice of u_a? If so, can we determine the sign of μ_a? The following theorem shall give some answers for these questions.

THEOREM 1.4. *Suppose $V(x)$ satisfies (V_1), and let $f(t) = |t|^{4s/N}t$. Then we have*

(i) *For all $a \in [0,\, a_*)$ and for a.e. $a \in [a_*,\, a^*)$, μ_a depends only on a and is independent of the choice of u_a. Moreover, we have that $\mu_a > 0$ for any $a \in [0,\, a^*)$ small enough.*

(ii) *In addition, we assume that*
(V_2) *$V(x)$ is a weak differentiable function such that*

$$(\nabla V(x),\, x) \leqslant C_4 V(x) \quad \textit{for a.e. } x \in \mathbb{R}^N,$$

where $C_4 > 0$ is a constant and $(\cdot\,,\cdot)$ is the usual inner product in $\mathbb{R}^N$. Then $\mu_a < 0$ for any a sufficiently approaches a^*.

REMARK 1.4. The conclusion (i) of theorem 1.4 is similar to the case $s = 1$. We also prove a new result for μ_a in theorem 1.4 (ii).

Inspired by [**19**–**21**], we next focus on the concentration behaviour of nonnegative minimizers for the mass critical case of (1.6) as $a \nearrow a^*$. As for the mass critical case of equation (1.1), we can rewrite (1.1) as follows

$$(-\Delta)^s u + V(x)u = \mu u + a|u|^{4s/N}u, \qquad x \in \mathbb{R}^N. \tag{1.19}$$

Since $e(a^*) = 0$, it is easy to see that $\int_{\mathbb{R}^N} V(x)|u_a(x)|^2\,\mathrm{d}x \to 0 = \inf_{x\in\mathbb{R}^N} V(x)$ as $a \nearrow a^*$, hence this behaviour depends on the behaviour of V near its minima. The functions u_a can be expected to concentrate at the flattest minimum of V. If V has a unique minimum, $|u_a(x)|^2$ converges to a δ-function located at this minimum.

In what follows, we shall suppose that the trapping potential V has $n \geqslant 1$ isolated minima, and that in their vicinity, V behaves like a power of the distance from these points. More precisely, we shall assume that there exist $n \geqslant 1$ distinct points $x_i \in \mathbb{R}^N$ with $V(x_i) = 0$, while $V(x_i) > 0$ otherwise. Moreover, we assume that there exist $q_i \in (0,\, 2s)$ and $C > 0$ such that

$$V(x) = h(x)\prod_{i=1}^{n}|x - x_i|^{q_i} \quad \text{ with } C < h(x) < 1/C \quad \text{for all } x \in \mathbb{R}^N, \tag{1.20}$$

where $\lim_{x\to x_i} h(x)$ exists for all $1 \leqslant i \leqslant n$. Set $q = \max\{q_1, \cdots, q_n\}$ and let $\lambda_i \in (0,\, \infty]$ be given by

$$\lambda_i = \left(\frac{q}{N}\left(\frac{N}{2s}\right)^{-q/(2s)} \|Q\|_2^{((4s/N)-2)} \int_{\mathbb{R}^N} |x|^q |Q(x)|^2\,\mathrm{d}x \lim_{x\to x_i}\frac{V(x)}{|x - x_i|^q}\right)^{1/(q+2s)}. \tag{1.21}$$

Define $\lambda = \min\{\lambda_1, \cdots, \lambda_n\}$ and let

$$\mathcal{Z} := \{x_i :\ \lambda_i = \lambda\} \tag{1.22}$$

denote the locations of the flattest global minima of $V(x)$.

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Theorem 1.5. *Suppose $V(x)$ satisfies the above assumption, and let u_a be a non-negative minimizer of* (1.6) *with $f(t)=|t|^{4s/N}t$ for $a\nearrow a^*$. Given a sequence $\{a_k\}$ with $a_k\nearrow a^*$ as $k\to\infty$, then there exists a subsequence, still denoted by $\{a_k\}$, of $\{a_k\}$ and an $x_0\in\mathcal{Z}$ such that*

$$(a^*-a_k)^{\frac{N}{2(q+2s)}}u_{a_k}\left((a^*-a_k)^{((1)/(q+2s))}x+x_0\right)$$
$$\xrightarrow{k}\frac{\lambda_0^{N/2}}{\|Q\|_2}Q(\lambda_0 x)\quad \text{strongly in } H^s(\mathbb{R}^N), \tag{1.23}$$

where

$$\lambda_0=\left(\frac{N}{2s}\right)^{1/(2s)}\lambda. \tag{1.24}$$

Remark 1.5.

(i) The proof of Theorem 1.5 follows from optimal energy estimates of $e(a)$. Motivated by [**19**], we are able to derive the following optimal energy estimate:

$$\lim_{a\nearrow a^*}\frac{e(a)}{(a^*-a)^{q/(q+2s)}}=\frac{\lambda^{2s}}{a^*}\left(\frac{N}{2s}+\frac{N}{q}\right).$$

(ii) Note that, the convergence in (1.23) also implies that

$$\int_{\mathbb{R}^N}|u_a(x)|^{2+4s/N}\,\mathrm{d}x\approx\frac{N+2s}{N}(a^*-a)^{-((2s)/(q+2s))}\frac{\lambda_0^{2s}}{a^*}\qquad\text{as } a\nearrow a^*$$

for a minimizer u_a.

(iii) Comparing with the case $s=1$ in [**19**], we need to make various modifications due to the non-locality of the fractional Laplacian operator. On the one hand, the ground state for (1.11) decays polynomially at infinity, which is in contrast to the fact that the ground state for $-\Delta$ decays exponentially at infinity. Consequently, we require the restrict condition for the order q_i of the polynomial potential function (1.20). We first have to suppose that $q<N+4s$. Indeed, if $q\geqslant N+4s$, the integral term of (1.21) makes no sense. In addition, in order to establish the optimal energy estimates of $e(a)$, we also need to assume that $q<2s$ (see the forthcoming estimate (5.3)). For this we assume that $0<q_i<2s$ for $i=1,\cdots,n$. On the contrary, since the non-locality of the fractional Laplacian operator, the value λ_0 defined by (1.24) is much more difficult to find for $0<s<1$ than for $s=1$.

Theorem 1.5 gives a detail description of the blow-up behaviour of minimizers as $a\nearrow a^*$ for $V(x)$ satisfying (1.20). As $a\nearrow a^*$, a minimizer u_a of (1.6) behaves like

$$u_a(x)\approx\frac{\lambda_0^{N/2}}{\|Q\|_2(a^*-a)^{((N)/(2(q+2s)))}}Q\left(\frac{\lambda_0(x-x_0)}{(a^*-a)^{((1)/(q+2s))}}\right),$$

with x_0 a minimum point of $V(x)$, and λ_0 defined in (1.24). Such a expression can, in general, hold only for a subsequence. However, if x_0 is unique, that is, $|\mathcal{Z}|=1$, it is not necessary to go to a subsequence, and the convergence (1.23) holds for any sequence. A simplest case of the polynomial potential function (1.20) is

$$V(x) = h(x)|x - x_0|^q \quad \text{with } C < h(x) < 1/C \quad \text{for all } x \in \mathbb{R}^N, \tag{1.25}$$

where $0 < q < N + 4s$, $C > 0$ is a constant and $\lim_{x \to x_0} h(x)$ exists, we have the following corollary.

COROLLARY 1.6. *Suppose $V(x)$ satisfies (1.25), and let u_a be a nonnegative minimizer of (1.6) with $f(t) = |t|^{4s/N} t$ for $a \nearrow a^*$. Then we have*

$$\lim_{a \nearrow a^*} (a^* - a)^{((N)/(2(q+2s)))} u_a((a^* - a)^{((1)/(q+2s))} x + x_0) = \frac{\lambda_0^{N/2}}{\|Q\|_2} Q(\lambda_0 x)$$

strongly in $H^s(\mathbb{R}^N)$, where λ_0 is given by (1.24).

REMARK 1.6. We note that, the existence range of the order q of the polynomial potential function (1.25) is sharp. Indeed, if $q \geqslant N + 4s$, then the integral term of (1.21) makes no sense. Therefore, the order q of (1.25) at most lies in $(0,\ N + 4s)$.

Furthermore, theorem 1.5 also indicates that symmetry breaking occurs in the minimizers when the potential $V(x)$ has a symmetry. For example, $V(x) = \prod_{i=1}^{n} |x - x_i|^q$ with $q \in (0,\ 2s)$ and the x_i arranged on the vertices of a regular polyhedron centred at the origin. It then follows from theorem 1.5 that all non-negative minimizers of (1.6) with $f(t) = |t|^{4s/N} t$ can concentrate at any vertex of this regular polyhedron. This further implies that there exists an $\bar{a}$ satisfying $0 < \bar{a} < a^*$ such that for any $a \in [\bar{a},\ a^*)$, $e(a)$ has (at least) n different nonnegative minimizers, each of which concentrates at a specific global minimum point x_i. However, $e(a)$ has a unique nonnegative minimizer u_a for all $a \in [0,\ a_*)$, where $a_* > 0$ is given by (1.18), and then by rotation u_a must be n-fold rotational symmetry with respect to the origin. Therefore, the above arguments yield immediately the following corollary.

COROLLARY 1.7. *Suppose $V(x) = \prod_{i=1}^{n} |x - x_i|^q$ with $q \in (0,\ 2s)$ and the x_i arranged on the vertices of a regular polyhedron centred at the origin. Let $f(t) = |t|^{4s/N} t$, then there exist two positive constant a_* and $\bar{a}$ satisfying $0 < a_* \leqslant \bar{a} < a^*$ such that*

(i) *If $a \in [0,\ a_*)$, $e(a)$ has a unique nonnegative minimizer which is n-fold rotational symmetry with respect to the origin.*

(ii) *If $a \in [\bar{a},\ a^*)$, $e(a)$ has (at least) n different nonnegative minimizers which is not n-fold rotational symmetry with respect to the origin.*

Finally, we note that symmetry breaking bifurcation of ground states for non-linear Schrödinger or Gross–Pitaevskii equations have been studied extensively in the literature, see that is, [**22**–**24**]. To the best of our knowledge, there seem few results concerning the symmetry breaking of L^2-normalized solutions for nonlinear fractional Schrödinger equations.

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The rest of this paper is organized as follows. In § 2, we present some preliminary results and give the proof of Theorem 1.1. Section 3 is devoted to the proofs of Theorems 1.2 and 1.3 on the existence and nonexistence of minimizers. In § 4, we present a detailed description of the Lagrange μ_a and give the proof of Theorem 1.4. In § 5, we shall first establish optimal energy estimates of nonnegative minimizers for the mass critical case of (1.6) as $a \nearrow a^*$, and we then make use of the blow-up analysis and energy methods to complete the proofs of Theorem 1.5 and corollary 1.6. Finally, theorem 1.2 (iv) is proved in the Appendix.

Throughout this paper, we shall make use of the following notations.

- $L^q(\mathbb{R}^N)$ with $1 \leqslant q \leqslant \infty$ denotes the usual Lebesgue space with standard norm $\|\cdot\|_q$.
- We denote by '$\to$' strong convergence and by '$\rightharpoonup$' weak convergence.
- The letters C, C_i, c and c_i will mean different positive constants that may vary from line to line but remain independent of the relevant quantities.
- For any $\rho > 0$ and $z \in \mathbb{R}^N$, $B_\rho(z)$ denotes the ball of radius ρ centred at z, and for simplicity of notations, we write $B_\rho := B_\rho(0)$.

2. Fractional Gagliardo–Nirenberg–Sobolev inequality

In this section, we give the proof of Theorem 1.1. First of all, we recall some useful facts of the fractional order Sobolev space.

For any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{N+2s/2}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$
$$= \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1+|\xi|^2)^s |\mathscr{F}(u)|^2 \, \mathrm{d}\xi < \infty \right\},$$

where $\mathscr{F}$ is the Fourier transform. The space $H^s(\mathbb{R}^N)$ is a Hilbert space endowed with the inner product and norm

$$(u,\, v)_{H^s} = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} uv \, \mathrm{d}x, \quad \|u\|_{H^s} = (u,u)_{H^s}^{1/2}.$$

Here the term

$$[u]_{H^s} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \right)^{1/2}$$

is the so-called Gagliardo (semi) norm. Notice that all the functional spaces L^2, H^s, and so on. are set in the whole of $\mathbb{R}^N$ unless explicitly mentioned. Let $\mathscr{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$. Indeed, the fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$, as stated in the following

Lemma 2.1 (See [12]). *Let $s \in (0,1)$ and let $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^N)$ be the fractional Laplacian operator defined by (1.2). Then, for any $u \in \mathscr{S}$,*

$$(-\Delta)^s u = \mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}(u)) \quad \text{for } \xi \in \mathbb{R}^N.$$

The following identity (propositions 3.4 and 3.6 of [12]) yields the relation between the fractional operator $(-\Delta)^s$ and the fractional Laplacian Sobolev space $H^s(\mathbb{R}^N)$,

$$[u]_{H^s}^2 = 2C_{N,s}^{-1}\int_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u|^2 \, \mathrm{d}\xi = 2C_{N,s}^{-1}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2 \, \mathrm{d}x.$$

As a consequence, the norms on $H^s(\mathbb{R}^N)$

$$\begin{aligned} &u \mapsto \|u\|_{H^s}, \\ &u \mapsto \left(\|u\|_2 + \|(-\Delta)^{s/2}u\|_2^2\right)^{1/2}, \\ &u \mapsto \left(\|u\|_2 + \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}(\xi)|^2 \, \mathrm{d}\xi\right)^{1/2}, \end{aligned}$$

are all equivalent.

For the reader's convenience, we review the main embedding results for this class of fractional Sobolev spaces.

Lemma 2.2 (See [4, 12]). *Let $s > 0$, then the following imbeddings are continuous:*

(i) $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, $2 \leqslant r \leqslant ((2N)/(N-2s))$, *if* $N > 2s$,

(ii) $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, $2 \leqslant r < \infty$, *if* $N = 2s$,

(iii) $H^s(\mathbb{R}^N) \hookrightarrow C_b^j(\mathbb{R}^N)$, *if $N < 2(s-j)$ for some nonnegative integer j, where*

$$C_b^j(\mathbb{R}^N) = \left\{u \in C^j(\mathbb{R}^N) : D^K u \text{ is bounded on } \mathbb{R}^N \text{ for } |K| \leqslant j\right\}.$$

Moreover, $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is locally compact whenever $q \in [1,\ 2_s^*)$.

One major tool in the proof of Theorem 1.1 is the following concentration compactness principle, originally proved by P. L. Lions [28].

Lemma 2.3 (See [15, 32]). *Assume that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and it satisfies*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_\rho(y)} |u_n(x)|^2 \, \mathrm{d}x = 0,$$

where $\rho > 0$. Then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_s^*$.

Applying lemma 2.3 and the idea of [39, theorem B], we now prove theorem 1.1.

Proof of Theorem 1.1. First of all, we define

$$\alpha := \inf_{u\in H^s(\mathbb{R}^N)\setminus\{0\}} J(u),$$

where $J(u)$ is given by (1.16). Note that if we set $u^{\lambda,\mu} = \mu u(\lambda x)$, then

$$\begin{aligned} J(u^{\lambda,\mu}) &= J(u), \\ \|u^{\lambda,\mu}\|_2^2 &= \lambda^{-N}\mu^2\|u\|_2^2, \\ \|(-\Delta)^{s/2}u^{\lambda,\mu}\|_2^2 &= \lambda^{2s-N}\mu^2\|(-\Delta)^{s/2}u\|_2^2. \end{aligned}$$

Let $\{u_n\} \subset H^s(\mathbb{R}^N)$, with $u_n \not\equiv 0$, be a minimizing sequence, that is, $0 < \alpha = \lim_{n\to\infty} J(u_n) < \infty$. Since $J(|u|) \leqslant J(u)$, without loss of generality, we can assume that $u_n \geqslant (\not\equiv)0$.

Choosing $\lambda_n = \|u_n\|_2^s/\|(-\Delta)^{s/2}u_n\|_2^s$ and $\mu_n = \|u_n\|_2^{Ns/2-1}/\|(-\Delta)^{s/2}u_n\|_2^{Ns/2}$, we obtain a sequence $v_n(x) = u_n^{\lambda_n,\mu_n}(x)$ such that

$$\|v_n\|_2^2 = 1,\ \|(-\Delta)^{s/2}v_n\|_2^2 = 1 \text{ and } \lim_{n\to\infty} J(v_n) = \alpha. \tag{2.1}$$

By (2.1), we deduce that $\lim_{n\to\infty}\|v_n\|_p^p = 1/\alpha$. Then lemma 2.3 implies that there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and $\rho, \delta > 0$ such that

$$\int_{B_\rho(y_n)} |v_n|^2 \,\mathrm{d}x \geqslant \delta > 0. \tag{2.2}$$

Set $\widetilde{v}_n(\cdot) := v_n(\cdot + y_n)$. Hence $\|\widetilde{v}_n\|_2^2 = 1$, $\|(-\Delta)^{s/2}\widetilde{v}_n\|_2^2 = 1$ and $\lim_{n\to\infty} J(\widetilde{v}_n) = \alpha$. Since $\{\widetilde{v}_n\}$ is a bounded sequence in $H^s(\mathbb{R}^N)$, we may assume, going if necessary to a subsequence,

$$\begin{cases} \widetilde{v}_n \rightharpoonup \widetilde{v} & \text{weakly in } H^s(\mathbb{R}^N), \\ \widetilde{v}_n \to \widetilde{v} & \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^N), \\ \widetilde{v}_n \to \widetilde{v} & \text{a.e. on } \mathbb{R}^N. \end{cases}$$

Furthermore, it follows from (2.2) that $\widetilde{v} \not\equiv 0$. Denote $\widetilde{w}_n := \widetilde{v}_n - \widetilde{v}$, we have

$$\|\widetilde{v}\|_2^2 + \lim_{n\to\infty}\|\widetilde{w}_n\|_2^2 = 1 \quad \text{and} \quad \|(-\Delta)^{s/2}\widetilde{v}\|_2^2 + \lim_{n\to\infty}\|(-\Delta)^{s/2}\widetilde{w}_n\|_2^2 = 1.$$

This, together with Brezis–Lieb Lemma and Young inequality, implies that

$$\begin{aligned}
\frac{1}{\alpha} &= \lim_{n\to\infty} \|\widetilde{v}_n\|_p^p = \lim_{n\to\infty} \|\widetilde{w}_n\|_p^p + \|\widetilde{v}\|_p^p \\
&\leqslant \frac{1}{\alpha}\left(\lim_{n\to\infty} \|(-\Delta)^{s/2}\widetilde{w}_n\|_2^{((N(p-2))/(2s))}\|\widetilde{w}_n\|_2^{p-\frac{N(p-2)}{2s}}\right. \\
&\qquad \left. + \|(-\Delta)^{s/2}\widetilde{v}\|_2^{((N(p-2))/(2s))}\|\widetilde{v}\|_2^{p-((N(p-2))/(2s))}\right) \\
&\leqslant \frac{\theta}{\alpha}\left(\lim_{n\to\infty} \|(-\Delta)^{s/2}\widetilde{w}_n\|_2^p + \|(-\Delta)^{s/2}\widetilde{v}\|_2^p\right) + \frac{1-\theta}{\alpha}\left(\lim_{n\to\infty} \|\widetilde{w}_n\|_2^p + \|\widetilde{v}\|_2^p\right) \\
&\leqslant \frac{\theta}{\alpha}\left(\left(\|(-\Delta)^{s/2}\widetilde{v}\|_2^2\right)^{p/2} + \left(1 - \|(-\Delta)^{s/2}\widetilde{v}\|_2^2\right)^{p/2}\right) \\
&\quad + \frac{1-\theta}{\alpha}\left((\|\widetilde{v}\|_2^2)^{p/2} + (1 - \|\widetilde{v}\|_2^2)^{p/2}\right) \\
&\leqslant \frac{\theta}{\alpha} + \frac{1-\theta}{\alpha} = \frac{1}{\alpha},
\end{aligned}$$

where $\theta = N(p-2)/(2sp)$. Since $\widetilde{v} \not\equiv 0$, we obtain $\|(-\Delta)^{s/2}\widetilde{v}\|_2 = 1$, $\|\widetilde{v}\|_2 = 1$, and $\alpha = J(\widetilde{v})$. Hence, the minimizer $\widetilde{v} \in H^s(\mathbb{R}^N)$ satisfies the Euler–Lagrange equation:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J(\widetilde{v} + \varepsilon\psi) = 0 \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^N).$$

Taking into account that $\|\widetilde{v}\|_2 = 1$ and $\|(-\Delta)^{s/2}\widetilde{v}\|_2 = 1$, we have

$$\frac{N(p-2)}{4s}(-\Delta)^s\widetilde{v} + \left(1 + \frac{p-2}{4}\left(2 - \frac{N}{s}\right)\right)\widetilde{v} - \frac{\alpha p}{2}|\widetilde{v}|^{p-2}\widetilde{v} = 0 \quad \text{in } \mathbb{R}^N.$$

Let $Q(x) = (\alpha p/2)^{1/(p-2)}\widetilde{v}(x)$, then $Q(x)$ satisfies (1.14), and (1.15) holds.

Note that $Q \in H^s(\mathbb{R}^N)$ is also a nonnegative minimizer for $J(u)$. By [**17**, **18**], we know that $Q(x) = \beta\varphi(\gamma(x + y_0))$ with some $\beta > 0$, $\gamma > 0$ and $y_0 \in \mathbb{R}$. Here φ is the unique radial positive ground state solution of (1.11). Without loss of generality, we may assume that $y_0 = 0$.

It then follows from the properties of φ that the conclusions (i) and (ii) hold. Moreover, it is easy to verify that every nonnegative optimizer $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ for the fractional Gagliardo–Nirenberg–Sobolev inequality is of the form

$$v(x) = \frac{\|(-\Delta)^{s/2}v\|_2^{N/(2s)}}{\|v\|_2^{N/(2s)-1}\|Q\|_2} Q\left(\frac{\|(-\Delta)^{\frac{s}{2}}v\|_2^{1/s}}{\|v\|_2^{1/s}}(x+y)\right) \quad \text{for some } y \in \mathbb{R}^N.$$

This completes the proof of Theorem 1.1. □

3. Existence and nonexistence of minimizers

In this section, we study the existence and nonexistence of minimizers for (1.6) and give the proofs of Theorems 1.2 and 1.3. For this we need the following compactness result, see for example [**5**].

LEMMA 3.1. *Suppose* $0 \leqslant V(x) \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ *satisfies* $\lim_{|x|\to\infty} V(x) = \infty$. *Then for all* $2 \leqslant r < 2^*_s$, *the embedding* $\mathcal{H} \hookrightarrow L^r(\mathbb{R}^N)$ *is compact, where* $\mathcal{H}$ *is given in* (1.7).

By applying directly the fractional Gagliardo–Nirenberg–Sobolev inequality (1.13) and recalling techniques, we are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. (i) From (f_1) we see that

$$|F(t)| \leqslant \frac{c_1}{2}|t|^2 + \frac{c_1}{p}|t|^p, \qquad \forall t \in \mathbb{R}. \tag{3.1}$$

For any $a > 0$ and $u \in \mathcal{M}$, by (3.1) and (1.13), we have

$$\begin{aligned} E_a(u) &= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x - 2a\int_{\mathbb{R}^N} F(u)\,\mathrm{d}x \\ &\geqslant \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x - c_1 a - \frac{2c_1 a}{p}\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x \\ &\geqslant \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x - c_1 a - \frac{c_1 a}{\|Q\|_2^{p-2}} \\ &\quad \times \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x\right)^{((N(p-2))/(4s))}. \end{aligned} \tag{3.2}$$

Since $2 < p < 2 + 4s/N$, we see that

$$0 < \frac{N(p-2)}{4s} < 1.$$

Thus $e(a)$ is bounded from below on $\mathcal{M}$; that is, $e(a)$ is well defined. Let $\{u_n\} \subset \mathcal{H}$ be a minimizing sequence satisfying $\int_{\mathbb{R}^N} |u_n(x)|^2 \,\mathrm{d}x = 1$ and $\lim_{n\to\infty} E_a(u_n) = e(a)$. Because of (3.2), we infer that both $\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_n|^2 \,\mathrm{d}x$ and $\int_{\mathbb{R}^N} V(x)|u_n|^2 \,\mathrm{d}x$ are uniformly bounded in n. By the compactness of lemma 3.1, we can extract a subsequence such that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } \mathcal{H}, \qquad u_n \to u_0 \quad \text{strongly in } L^q(\mathbb{R}^N) \quad \text{with } 2 \leqslant q < 2^*_s,$$

for some $u_0 \in \mathcal{H}$. Then we conclude that $\int_{\mathbb{R}^N} |u_0(x)|^2 \mathrm{d}x = 1$ and $E_a(u_0) = e(a)$, by weak lower semicontinuity. This implies the existence of minimizers for all $a > 0$.

(ii) By (f_3) and (f_4) , we see that there exist constants $C_1, C_2 > 0$ such that

$$F(t) \geqslant C_1|t|^\nu - C_2|t|^2 \qquad \text{for all } t \in \mathbb{R}. \tag{3.3}$$

For any $a > 0$, choose $\hat{u} \in C_0^\infty(\mathbb{R}^N) \cap \mathcal{M}$ and set $\hat{u}_\tau(x) := \tau^{N/2}\hat{u}(\tau x)$ for $\tau \geqslant 1$, then $\hat{u}_\tau \in \mathcal{M}$ and

$$\begin{aligned} E_a(\hat{u}_\tau) &= \tau^{2s}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\hat{u}|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x/\tau)|\hat{u}|^2\,\mathrm{d}x - 2a\int_{\mathbb{R}^N} F(\hat{u}_\tau)\,\mathrm{d}x \\ &\leqslant \tau^{2s}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\hat{u}|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x/\tau)|\hat{u}|^2\,\mathrm{d}x - 2aC_1\tau^{((N(\nu-2))/(2))} \\ &\times \int_{\mathbb{R}^N} |\hat{u}|^\nu\,\mathrm{d}x + 2aC_2 \to -\infty \quad \text{as } \tau \to \infty, \end{aligned}$$

since $\nu > 2 + 4s/N$. It then follows that

$$e(a) \leqslant \lim_{\tau\to\infty} E_a(u_\tau) = -\infty.$$

This implies the nonexistence of minimizers for all $a > 0$. This completes the proof.

□

Let $\eta : \mathbb{R}^N \to \mathbb{R}$ be a smooth function such that $\eta(x) = 1$ for $|x| \leqslant \tau_1$, $\eta(x) = 0$ for $|x| \geqslant 2$, $0 \leqslant \eta \leqslant 1$ and $|\nabla\eta| \leqslant 2$. Then, we define

$$Q_\tau(x) = \eta(x/\tau)Q(x), \quad x \in \mathbb{R}^N, \tag{3.4}$$

for any $\tau > 0$, where $Q(x)$ is given in theorem 1.1. In order to prove theorem 1.3, we will estimate the Gagliardo (semi) norm of Q_τ. In the setting of the fractional Laplacian, this estimate is more delicate than in the case of the Laplacian, due to the nonlocal nature of the operator $-(\Delta)^s$. For this purpose, inspired by the proof of [**34**, proposition 21], we establish the following lemma.

Lemma 3.2. *Let $s \in (0, 1)$ and $N \geqslant 2$. Then the following estimate holds true:*

$$\int_{\mathbb{R}^{2N}} \frac{|Q_\tau(x) - Q_\tau(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^{2N}} \frac{|Q(x) - Q(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + O(\tau^{-4s})$$

as $\tau \to \infty$, where Q_τ is given in (3.4).

Proof. We first show that the following assertions hold true:

(a) for any $x \in \mathbb{R}^N$ and $y \in B_\tau^c$, with $|x - y| \leqslant \tau/2$,

$$|Q_\tau(x) - Q_\tau(y)| \leqslant C\tau^{-N-2s}|x-y|; \tag{3.5}$$

(b) for any $x, y \in B_\tau^c$,

$$|Q_\tau(x) - Q_\tau(y)| \leqslant C\tau^{-N-2s}\min\{1,\ |x-y|\} \tag{3.6}$$

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for any $\tau \geqslant 2$ and for some constant $C > 0$. In fact, let us start by proving assertion (a). For this let $x \in \mathbb{R}^N$ and $y \in B_\tau^c$ with $|x-y| \leqslant \tau/2$, and let ξ be any point on the segment joining x and y. Then we have

$$\xi = tx + (1-t)y \quad \text{for some } t \in [0,1],$$

so that

$$|\xi| = |y + t(x-y)| \geqslant |y| - t|x-y| \geqslant \tau - t(\tau/2) \geqslant \tau/2.$$

This and theorem 1.1 imply that $|\nabla Q_\tau(\xi)| \leqslant C\tau^{-N-2s}$, and so, by a first-order Taylor expansion,

$$|Q_\tau(x) - Q_\tau(y)| \leqslant C\tau^{-N-2s}|x-y|,$$

which proves (3.5). Now, we show (b). For this let $x, y \in B_\tau^c$. If $|x-y| \leqslant 1$, then (b) follows from (a) since $\tau \geqslant 2$, so we may suppose $|x-y| > 1$. Then, from theorem 1.1, we deduce that

$$|Q_\tau(x) - Q_\tau(y)| \leqslant |Q(x)| + |Q(y)| \leqslant C\tau^{-N-2s},$$

and this completes the proof of (3.6).

Now we introduce the notation

$$\mathbb{D} := \{(x,y) \in \mathbb{R}^{2N} : \, x \in B_\tau, \; y \in B_\tau^c \quad \text{and} \quad |x-y| > \tau/2\}$$

and

$$\mathbb{E} := \{(x,y) \in \mathbb{R}^{2N} : \, x \in B_\tau, \; y \in B_\tau^c \quad \text{and} \quad |x-y| \leqslant \tau/2\}.$$

By (3.4), we have

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \frac{|Q_\tau(x) - Q_\tau(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y &= \int_{B_\tau \times B_\tau} \frac{|Q(x) - Q(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \\ &+ 2\int_{\mathbb{D}} \frac{|Q_\tau(x) - Q_\tau(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y + 2\int_{\mathbb{E}} \frac{|Q_\tau(x) - Q_\tau(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \\ &+ \int_{B_\tau^c \times B_\tau^c} \frac{|Q_\tau(x) - Q_\tau(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.7}$$

By (3.4) and (3.6), we obtain

$$\begin{aligned} &\int_{B_\tau^c \times B_\tau^c} \frac{|Q_\tau(x) - Q_\tau(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y \\ &\quad \leqslant C\tau^{-2N-4s} \int_{B_{2\tau} \times \mathbb{R}^N} \frac{\min\{1, \, |x-y|^2\}}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y = O(\tau^{-N-4s}), \end{aligned} \tag{3.8}$$

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while, by (3.5),

$$\begin{aligned}\int_{\mathbb{E}} \frac{|Q_\tau(x)-Q_\tau(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y &\leqslant C\tau^{-2N-4s}\int_{\substack{x\in B_\tau,\ y\in B_\tau^c\\ |x-y|\leqslant \tau/2}} \frac{|x-y|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\ &\leqslant C\tau^{-2N-4s}\int_{|x|\leqslant\tau}\mathrm{d}x\int_{|\xi|\leqslant\tau/2}\frac{1}{|\xi|^{N+2s-2}}\,\mathrm{d}\xi = O(\tau^{-N-6s+2}), \end{aligned} \tag{3.9}$$

as $\tau\to\infty$.

Now, in (3.7) it remains to estimate the integral on $\mathbb{D}$, that is,

$$\int_{\mathbb{D}} \frac{|Q_\tau(x)-Q_\tau(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y. \tag{3.10}$$

For this, recalling that $Q_\tau(x)=Q(x)$ for any $x\in B_\tau$ thanks to (3.4), we note that for any $(x,y)\in\mathbb{D}$,

$$\begin{aligned}|Q_\tau(x)-Q_\tau(y)|^2 &= |(Q(x)-Q(y))+(Q(y)-Q_\tau(y))|^2\\ &\leqslant |Q(x)-Q(y)|^2+|Q(y)-Q_\tau(y)|^2\\ &\quad+2|Q(x)-Q(y)||Q(y)-Q_\tau(y)|\end{aligned}$$

so that

$$\begin{aligned}\int_{\mathbb{D}} \frac{|Q_\tau(x)-Q_\tau(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y &\leqslant \int_{\mathbb{D}} \frac{|Q(x)-Q(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{D}} \frac{|Q(y)-Q_\tau(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\ &\quad+2\int_{\mathbb{D}} \frac{|Q(x)-Q(y)||Q(y)-Q_\tau(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.11}$$

Hence, in order to estimate (3.10), we bound the last two terms in the right-hand side of (3.11). By theorem 1.1, we yield

$$\begin{aligned}\int_{\mathbb{D}} \frac{|Q(y)-Q_\tau(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y &\leqslant 4\int_{\mathbb{D}} \frac{|Q(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\ &\leqslant C\tau^{-2N-4s}\int_{\substack{x\in B_\tau,\ y\in B_\tau^c\\ |x-y|>\tau/2}} \frac{1}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\ &\leqslant C\tau^{-2N-4s}\int_{|x|\leqslant\tau}\mathrm{d}x\int_{|\xi|>\tau/2}\frac{1}{|\xi|^{N+2s}}\,\mathrm{d}\xi = O(\tau^{-N-6s}), \end{aligned} \tag{3.12}$$

as $\tau\to\infty$. In order to estimate the last term in the right-hand side of (3.11), first of all we note that, once more by theorem 1.1,

$$|Q(x)Q(y)|\leqslant C\tau^{-N-2s} \quad \text{for any } (x,y)\in\mathbb{D}.$$

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As a consequence, by (3.4) we infer that

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|Q(x)||Q(y)-Q_\tau(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&\quad\leqslant 2\int_{\mathbb{D}} \frac{|Q(x)||Q(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \leqslant C\tau^{-N-2s}\int_{\mathbb{D}} \frac{1}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&\quad\leqslant C\tau^{-N-2s}\int_{|x|\leqslant\tau}\mathrm{d}x\int_{|\xi|>\tau/2}\frac{1}{|\xi|^{N+2s}}\,\mathrm{d}\xi = O(\tau^{-4s}), \qquad (3.13)
\end{aligned}$$

as $\tau\to\infty$. On the contrary, again by (3.4) and theorem 1.1,

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|Q(y)||Q(y)-Q_\tau(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&\quad\leqslant 2\int_{\mathbb{D}} \frac{|Q(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \leqslant C\tau^{-2N-4s}\int_{\mathbb{D}} \frac{1}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&\quad\leqslant C\tau^{-2N-4s}\int_{|x|\leqslant\tau}\mathrm{d}x\int_{|\xi|>\tau/2}\frac{1}{|\xi|^{N+2s}}\,\mathrm{d}\xi = O(\tau^{-N-6s}), \qquad (3.14)
\end{aligned}$$

as $\tau\to\infty$. Putting together (3.13) with (3.14), we infer that

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|Q(x)-Q(y)||Q(y)-Q_\tau(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&\leqslant \int_{\mathbb{D}} \frac{|Q(x)||Q(y)-Q_\tau(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{D}} \frac{|Q(y)||Q(y)-Q_\tau(y)|}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&= O(\tau^{-4s}), \qquad (3.15)
\end{aligned}$$

as $\tau\to\infty$.

Finally, by (3.7)–(3.9), (3.11), (3.12) and (3.15), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{2N}} \frac{|Q_\tau(x)-Q_\tau(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y &= \int_{B_\tau\times B_\tau} \frac{|Q(x)-Q(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\int_{\mathbb{D}} \frac{|Q(x)-Q(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + O(\tau^{-4s}) \\
&\quad \leqslant \int_{\mathbb{R}^{2N}} \frac{|Q(x)-Q(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + O(\tau^{-4s}),
\end{aligned}$$

as $\tau\to\infty$. This completes the proof of Lemma 3.2. □

REMARK 3.1. Note that, it is interesting to observe that the energy interaction outside $B_\tau\times B_\tau$ is not negligible. Indeed, while the contributions in (3.8), (3.9), (3.12) and (3.15) are all $O(\tau^{-4s})$, the integral in $\mathbb{D}$ provides a relevant part that needs to be taken into account in the full energy.

Now we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3.

(i) Motivated by the proof of [**19**, theorem 1], we divide our proof into three parts. We first show that (1.6) admits at least one minimizer for all $0 \leqslant a < \|Q\|_2^{4s/N}$. If $u \in \mathcal{H}$ and $\int_{\mathbb{R}^N} |u(x)|^2 \, \mathrm{d}x = 1$, we observe from (1.13) and the nonnegativity of $V(x)$ that

$$\begin{aligned} E_a(u) &= \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, \mathrm{d}x \\ &\quad - \frac{Na}{N+2s} \int_{\mathbb{R}^N} |u(x)|^{2+4s/N} \, \mathrm{d}x \\ &\geqslant \left(1 - \frac{a}{\|Q\|_2^{4s/N}}\right) \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(x)|^2 \, \mathrm{d}x, \end{aligned} \tag{3.16}$$

which implies that $E_a(u)$ is bounded from below. Let $\{u_n\} \subset \mathcal{H}$ be a sequence satisfying $\|u_n\|_2 = 1$ and $\lim_{n\to\infty} E_a(u_n) = e(a)$. In view of (3.16), both $\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 \, \mathrm{d}x$ and $\int_{\mathbb{R}^N} V(x)|u_n|^2 \, \mathrm{d}x$ are bounded uniformly with respect to n. According to lemma 3.1, we may assume, passing if necessary to a subsequence,

$$u_n \rightharpoonup u \quad \text{weak in } \mathcal{H}, \qquad u_n \to u \quad \text{strongly in } L^q(\mathbb{R}^N) \text{ with } 2 \leqslant q < 2_s^*,$$

for some $u \in \mathcal{H}$. We then deduce that $\int_{\mathbb{R}^N} |u(x)|^2 \, \mathrm{d}x = 1$ and $E_a(u) = e(a)$, by weak lower semicontinuity. This implies the existence of minimizers for all $0 \leqslant a < \|Q\|_2^{4s/N}$.

We next prove that there is no minimizer for (1.6) as soon as $a \geqslant \|Q\|_2^{4s/N}$. Choose a nonnegative cut-off function $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $\eta(x) = 1$ for $|x| \leqslant 1$, $\eta(x) = 0$ for $|x| \geqslant 2$, $0 \leqslant \eta \leqslant 1$ and $|\nabla \eta| \leqslant 2$. For $x_0 \in \mathbb{R}^N$, $\tau > 0$ and $R > 0$, let

$$u(x) = A_{R,\tau} \frac{\tau^{N/2}}{\|Q\|_2} \eta\left(\frac{x - x_0}{R}\right) Q(\tau(x - x_0)), \tag{3.17}$$

where $A_{R,\tau} > 0$ is chosen so that $\int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x = 1$. By scaling, $A_{R,\tau}$ depends only on the product $R\tau$ and $\lim_{R\tau\to\infty} A_{R,\tau} = 1$. In fact, according to the polynomial decay of Q, we have

$$\frac{1}{A_{R,\tau}^2} = \frac{1}{\|Q\|_2^2} \int_{\mathbb{R}^N} \eta^2 \left(\frac{x}{R\tau}\right) Q^2(x) \, \mathrm{d}x = 1 + O((R\tau)^{-N-4s}) \quad \text{as } R\tau \to \infty. \tag{3.18}$$

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For example, in the following, we could set $R = 1$. Using (1.13), lemma 3.2 and the polynomial decay of Q, we also obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2\,\mathrm{d}x - \frac{Na}{N+2s}\int_{\mathbb{R}^N} |u(x)|^{2+4s/N}\,\mathrm{d}x \\ \leqslant \frac{\tau^{2s}}{\|Q\|_2^2}\bigg[\int_{\mathbb{R}^N} |(-\Delta)^{s/2}Q|^2\,\mathrm{d}x \\ - \frac{Na}{(N+2s)\|Q\|_2^{4s/N}}\int_{\mathbb{R}^N} |Q|^{2+((4s)/(N))}\,\mathrm{d}x + O(\tau^{-4s})\bigg] \quad \text{as } \tau \to \infty.\end{aligned} \tag{3.19}$$

Since $\|(-\Delta)^{s/2}Q\|_2^2 = ((N)/(N+2s))\|Q\|_{2+((4s)/(N))}^{2+((4s)/(N))}$, we further have

$$(3.19) = \frac{N\tau^{2s}}{(N+2s)\|Q\|_2^2}\left[\left(1 - \frac{a}{\|Q\|_2^{4s/N}}\right)\int_{\mathbb{R}^N} |Q|^{2+((4s)/(N))}\,\mathrm{d}x + O(\tau^{-4s})\right] \quad \text{as } \tau \to \infty. \tag{3.20}$$

On the contrary, since the function $x \mapsto V(x)\eta((x-x_0)/R)$ is bounded and has compact support, it follows from [**27**] that

$$\lim_{\tau\to\infty}\int_{\mathbb{R}^N} V(x)|u(x)|^2\,\mathrm{d}x = V(x_0) \quad \text{for a.e. } x_0 \in \mathbb{R}^N. \tag{3.21}$$

For $a > \|Q\|_2^{4s/N}$, it follows from (3.20) and (3.21) that

$$e(a) \leqslant \lim_{\tau\to\infty} E_a(u) = -\infty.$$

This implies that for any $a > \|Q\|_2^{4s/N}$, $e(a)$ is unbounded from below, and nonexistence of minimizers is, therefore, proved.

We finally deal with the case $a = \|Q\|_2^{4s/N}$. Combining (3.20) and (3.21), we infer that $e(a) \leqslant V(x_0)$. This holds for almost every x_0; taking the infimum over x_0 yields $e(a) \leqslant 0$. We further use (3.16) to derive that $e(a) = 0$. Suppose now that there exists a minimizer u at $a = \|Q\|_2^{4s/N}$. As pointed out in the Introduction, we can assume u to be nonnegative. We would then have

$$\int_{\mathbb{R}^N} V(x)|u(x)|^2\,\mathrm{d}x = \inf_{x\in\mathbb{R}^N} V(x) = 0$$

and

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2\,\mathrm{d}x = \frac{Na}{N+2s}\int_{\mathbb{R}^N} |u(x)|^{2+((4s)/(N))}\,\mathrm{d}x.$$

This is a contradiction since for the first inequality u would have the compact support, while for the second one it has to be equal to the translation and scaling of Q.

Moreover, by (3.16) we see that $e(a) > 0$ for $0 \leqslant a < a^* = \|Q\|_2^{4s/N}$. We have already shown that $e(a^*) = 0$ and $e(a) = -\infty$ for $a > a^*$, hence it remains

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to show that $\lim_{a\nearrow a^*} e(a) = 0$. This follows easily from (3.20) and (3.21), by first taking $a \nearrow a^*$, followed by $\tau \to \infty$. This implies that $\limsup_{a\nearrow a^*} e(a) \leqslant V(x_0)$ which, after taking the infimum over x_0, yields the result.

(ii) The proof is similar to that of [**20**, theorem 1.1], and we write out the details in the Appendix for the reader's convenience. This completes the proof of Theorem 1.3.

□

4. Some properties of the Lagrange multiplier μ_a

In this section, we always assume that $f(t) = |t|^{4s/N}t$. For any $a \in [0,\, a^*)$, let

$$\Lambda_a := \{u_a :\ u_a \text{ is a minimizer of } e(a) \text{ in } (1.6)\}\,. \tag{4.1}$$

If $u_a \in \Lambda_a$, as illustrated in the Introduction, we may assume that $u_a \geqslant 0$ and u_a satisfies (1.19), where μ_a is a Lagrange multiplier associated with u_a. The aim of this section is to present a detailed description of the Lagrange μ_a. For this, we first study some properties of $e(a)$, upon which we give the proof Theorem 1.4. Inspired by [**21**], in the following, we give our result on the smoothness of $e(a)$ with respect to a.

LEMMA 4.1. *Suppose $V(x)$ satisfies (V_1). Then for $a \in (0,\, a^*)$, the left and right derivative of $e(a)$ always exist in $[0,\, a^*)$ and satisfy*

$$e'_-(a) = -\frac{\alpha_a}{2} \quad \textit{and} \quad e'_+(a) = -\frac{\gamma_a}{2}$$

where

$$\begin{aligned} \alpha_a &:= \inf\left\{\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\,\mathrm{d}x :\ u_a \in \Lambda_a\right\},\\ \gamma_a &:= \sup\left\{\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\,\mathrm{d}x :\ u_a \in \Lambda_a\right\}, \end{aligned} \tag{4.2}$$

and Λ_a is given by (4.1).

Proof. Since $V(x)$ satisfies (V_1), from the definition of $e(a)$ one can infer that $e(a)$ is decreasing in $a \in [0,\, a^*)$ and satisfies

$$0 \leqslant \inf_{x\in\mathbb{R}^N} V(x) \leqslant e(a) \leqslant e(0) = \mu_1 \quad \text{ for all } a \in [0,\, a^*),$$

where (1.13) is used in the above inequality and μ_1 is the first eigenvalue of $(-\Delta)^s + V(x)$ in $\mathcal{H}$. Moreover, by (1.13), we have

$$\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\,\mathrm{d}x \leqslant \frac{(N+2s)e(a)}{N(a^*-a)} \leqslant \frac{(N+2s)\mu_1}{N(a^*-a)} \quad \text{ for } a \in [0,\, a^*). \tag{4.3}$$

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For any $a_1, a_2 \in [0,\ a^*)$, we obtain

$$e(a_1) \geqslant e(a_2) + \frac{N(a_2 - a_1)}{N + 2s} \int_{\mathbb{R}^N} |u_{a_1}|^{2+((4s)/(N))}\,\mathrm{d}x, \qquad \forall u_{a_1} \in \Lambda_{a_1}, \tag{4.4}$$

$$e(a_2) \geqslant e(a_1) + \frac{N(a_1 - a_2)}{N + 2s} \int_{\mathbb{R}^N} |u_{a_2}|^{2+((4s)/(N))}\,\mathrm{d}x, \qquad \forall u_{a_2} \in \Lambda_{a_2}, \tag{4.5}$$

and hence $\lim_{a_2 \to a_1} e(a_2) = e(a_1)$. This implies that

$$e(a) \in C([0,\ a^*),\ \mathbb{R}^+). \tag{4.6}$$

Furthermore, it follows from (4.4) and (4.5) that

$$\begin{aligned} \frac{N(a_2 - a_1)}{N + 2s} \int_{\mathbb{R}^N} |u_{a_1}|^{2+((4s)/(N))}\,\mathrm{d}x &\leqslant e(a_1) - e(a_2) \\ &\leqslant \frac{N(a_2 - a_1)}{N + 2s} \int_{\mathbb{R}^N} |u_{a_2}|^{2+((4s)/(N))}\,\mathrm{d}x. \end{aligned} \tag{4.7}$$

Set $0 < a_1 < a_2 < a^*$, it then follows from (4.7) that

$$\begin{aligned} -\frac{N}{N + 2s} \int_{\mathbb{R}^N} |u_{a_2}|^p\,\mathrm{d}x &\leqslant \frac{e(a_2) - e(a_1)}{a_2 - a_1} \\ &\leqslant -\frac{N}{N + 2s} \int_{\mathbb{R}^N} |u_{a_1}|^p\,\mathrm{d}x, \qquad \forall u_{a_i} \in \Lambda_{a_i},\ i = 1, 2. \end{aligned} \tag{4.8}$$

This implies that

$$-\frac{N}{N + 2s} \inf_{u_{a_2} \in \Lambda_{a_2}} \int_{\mathbb{R}^N} |u_{a_2}|^p\,\mathrm{d}x \leqslant \frac{e(a_2) - e(a_1)}{a_2 - a_1} \leqslant -\frac{N}{N + 2s}\gamma_{a_1}. \tag{4.9}$$

By (4.3) and lemma 3.1, there exists $\bar{u} \in \mathcal{H}$ such that for all $a_2 \searrow a_1$,

$$u_{a_2} \rightharpoonup \bar{u} \quad \text{weakly in } \mathcal{H}, \quad u_{a_2} \to \bar{u} \quad \text{strongly in } L^q(\mathbb{R}^N) \quad \text{with } 2 \leqslant q < 2_s^*.$$

It then follows from (4.6) that

$$e(a_1) = \lim_{a_2 \searrow a_1} e(a_2) = \lim_{a_2 \searrow a_1} E_{a_2}(u_{a_2}) \geqslant E_{a_1}(\bar{u}) \geqslant e(a_1),$$

which yields that for all $a_2 \searrow a_1$,

$$u_{a_2} \to \bar{u} \in \mathcal{H} \quad \text{and} \quad \bar{u} \in \Lambda_{a_1}.$$

We thus deduce from (4.9) that

$$\begin{aligned} -\frac{N}{N + 2s} \int_{\mathbb{R}^N} |\bar{u}|^{2+((4s)/(N))}\,\mathrm{d}x &\leqslant \liminf_{a_2 \searrow a_1} \frac{e(a_1) - e(a_2)}{a_2 - a_1} \leqslant \limsup_{a_2 \searrow a_1} \frac{e(a_1) - e(a_2)}{a_2 - a_1} \\ &\leqslant -\frac{N}{N + 2s}\gamma_{a_1}. \end{aligned} \tag{4.10}$$

On the contrary, from (4.2), we see that

$$\int_{\mathbb{R}^N} |\bar{u}|^{2+((4s)/(N))}\,\mathrm{d}x \leqslant \gamma_{a_1}.$$

Therefore, all inequalities in (4.10) are indeed identities, from which we obtain

$$e'_+(a_1) = -\frac{N}{N+2s}\gamma_{a_1}.$$

Similarly, if $a_2 < a_1 < a^*$, letting $a_2 \to a_1^-$ and repeating the above arguments, one can infer that

$$e'_-(a_1) = -\frac{N}{N+2s}\alpha_{a_1}.$$

This completes the proof of Lemma 4.1. □

Remark 4.1. Lemma 4.1 implies that if $e(a)$ has a unique nonnegative minimizer, then $e(a) \in C^1([0,\ a^*),\ \mathbb{R}^+)$. However, this is true generally for $a \in [0,\ a_*)$, where $a_* > 0$ is given by (1.18), in view of the possible multiplicity of nonnegative minimizers as $a \nearrow a^*$, see corollary 1.7.

Now we are ready to give the proof of Theorem 1.4.

Proof of Theorem 1.4.

(i) By (4.7), we have

$$\begin{aligned} |e(a_1) - e(a_2)| &\leqslant \frac{N}{N+2s}|a_2 - a_1| \max \\ &\quad \times \left\{ \int_{\mathbb{R}^N} |u_{a_1}|^{2+((4s)/(N))}\,\mathrm{d}x,\ \int_{\mathbb{R}^N} |u_{a_2}|^{2+((4s)/(N))}\,\mathrm{d}x \right\} \\ &\leqslant \frac{N}{N+2s}\max\{\gamma_{a_1},\ \gamma_{a_2}\}|a_2 - a_1| \quad \text{for all } a_1, a_2 \in [0,\ a^*), \end{aligned}$$

where γ_{a_i} $(i = 1, 2)$ are given by (4.2). This implies that $e(a)$ is locally Lipschitz continuous in $[0,\ a^*)$. It then follows from Rademacher's theorem that $e(a)$ is differential for a.e. $a \in [0,\ a^*)$. Moreover, from lemma 4.1 and remark 4.1, we see that

$$\begin{aligned} &e'(a) \text{ exists for all } a \in [0,\ a_*) \text{ and } a.e.\ a \in [a_*,\ a^*), \\ &\text{and } e'(a) = -\frac{N}{N+2s}\int_{\mathbb{R}^N} |u|^{2+4s/N}\,\mathrm{d}x,\ \forall u \in \Lambda_a, \end{aligned} \tag{4.11}$$

and hence all minimizers of $e(a)$ have the same $L^{2+((4s)/(N))}(\mathbb{R}^N)$-norm. Taking each nonnegative function $u_a \in \Lambda_a$, where $a \in [0,\ a^*)$ such that $e'(a)$ satisfies (4.11), then u_a satisfies (1.19) for a suitable Lagrange multiplier

$\mu_a \in \mathbb{R}$ associated with u_a. We can easily conclude from (1.19) and (4.11) that

$$\mu_a = e(a) - \frac{2sa}{N+2s}\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x = e(a) + \frac{2sa}{N}e'(a), \tag{4.12}$$

which implies that μ_a depends only on a and is independent of the choice of u_a. Therefore, for any given $a \in [0,\, a_*)$ and a.e. $a \in [a_*,\, a^*)$, all minimizer of $e(a)$ satisfy equation (1.19) with the same Lagrange multiple μ_a. Moreover, it follows from (4.3) and (4.12) that

$$\mu_a \to \mu_1 \quad \text{as } a \searrow 0,$$

which implies that $\mu_a > 0$ for any $a \in [0,\, a^*)$ small enough.

(ii) Now, we suppose that the assumptions of (V_1) and (V_2) are satisfied, it remains to show that there exists $a_0 \in [0,\, a^*)$ such that $\mu_a < 0$ for $a \in [a_0,\, a^*)$. Since u_a satisfies equation (1.19),

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_a|^2\, \mathrm{d}x + \int_{\mathbb{R}^N} V(x)u_a^2\, \mathrm{d}x = \mu_a + a\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x. \tag{4.13}$$

Moreover, u_a satisfies the following Pohozaev identity [**33**]:

$$\begin{aligned}\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_a|^2\, \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} (\nabla V(x),\, x)u_a^2\, \mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N} V(x)u_a^2\, \mathrm{d}x \\ = \frac{N\mu_a}{2} + \frac{N^2 a}{2N+4s}\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x,\end{aligned}$$

which, together with (4.13) and (V_2), implies that

$$\begin{aligned}\mu_a &= \frac{1}{2s}\int_{\mathbb{R}^N} (\nabla V(x),\, x)u_a^2\, \mathrm{d}x + \int_{\mathbb{R}^N} V(x)u_a^2\, \mathrm{d}x \\ &\quad - \frac{2as}{N+2s}\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x \\ &\leqslant C\int_{\mathbb{R}^N} V(x)u_a^2\, \mathrm{d}x - \frac{2as}{N+2s}\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x,\end{aligned} \tag{4.14}$$

where C is a positive constant. Since $e(a^*) = 0$, it is easy to see that

$$\int_{\mathbb{R}^N} V(x)|u_a(x)|^2\, \mathrm{d}x \to 0 \quad \text{as } a \nearrow a^*. \tag{4.15}$$

Now, we claim that

$$\liminf_{a\nearrow a^*}\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x \geqslant \sigma > 0. \tag{4.16}$$

Assume by contradiction, there exists a subsequence $\{a_k\}$ with $a_k \nearrow a^*$ as $k \to \infty$, such that

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} |u_{a_k}|^{2+((4s)/(N))}\, \mathrm{d}x = 0, \tag{4.17}$$

which together with $e(a^*) = 0$ and (4.15), implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} |(-\triangle)^{s/2} u_{a_k}|^2 \, dx = 0. \tag{4.18}$$

It then follows from (4.15) and (4.18) that

$$u_{a_k} \xrightarrow{k} 0 \quad \text{strongly in } \mathcal{H},$$

which contradicts with $\|u_{a_k}\|_2^2 = 1$. Hence, (4.16) holds. Combining (4.14), (4.15) and (4.16) gives

$$\limsup_{a\nearrow a^*} \mu_a \leqslant -\frac{2sa^*}{N+2s}\sigma < 0.$$

Consequently, there exists $a_0 \in [0,\ a^*)$ such that $\mu_a < 0$ for $a \in [a_0,\ a^*)$. This completes the proof of Theorem 1.4.

□

5. Mass concentration

In this section, we analyse the limit behaviour of nonnegative minimizers for (1.6) with $f(t) = |t|^{4/N} t$ as $a \nearrow a^*$, in the case that the trapping potential $V(x)$ satisfies (1.20). In this case, note from theorem 1.3 that there exist minimizers for (1.6), if and only if $0 \leqslant a < a^*$. In the following, we shall derive refined estimates on $e(a)$.

Lemma 5.1. *Assume that $V(x)$ satisfies (1.20), then there exists two positive constants $C_1 < C_2$, independent of a, such that*

$$C_1(a^* - a)^{((q)/(q+2s))} \leqslant e(a) \leqslant C_2(a^* - a)^{((q)/(q+2s))} \quad \textit{for } 0 \leqslant a \leqslant a^*, \tag{5.1}$$

where $q = \max\{q_1, \cdots, q_n\} > 0$.

Proof. Since $e(a)$ is decreasing and bounded uniformly for $0 \leqslant a \leqslant a^*$, it suffices to consider the case when a is close to a^*. Let us start with the lower bound. From (1.13), we deduce that for any $\gamma > 0$ and $u \in \mathcal{M}$,

$$\begin{aligned} E_a(u) &\geqslant \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx + \frac{N(a^*-a)}{N+2s} \int_{\mathbb{R}^N} |u(x)|^{2+((4s)/(N))} \, dx \\ &= \gamma + \int_{\mathbb{R}^N} \left[(V(x) - \gamma)|u(x)|^2 + \frac{N(a^*-a)}{N+2s} |u(x)|^{2+((4s)/(N))} \right] dx \\ &\geqslant \gamma - \frac{2s}{(N+2s)(a^*-a)^{((N)/(2s))}} \int_{\mathbb{R}^N} [\gamma - V(x)]_+^{1+((N)/(2s))} \, dx, \end{aligned} \tag{5.2}$$

where $[\,\cdot\,]_+ = \max\{0,\ \cdot\}$ denotes the positive part. For small enough γ, the set

$$\{x \in \mathbb{R}^N : \ V(x) < \gamma\}$$

is contained in the disjoint union of n ball of radius at most $M\gamma^{1/q}$, centred at the minima x_i, for a suitable constant $M > 0$. Moreover, $V(x) \geqslant (|x - x_i|/M)^q$ on

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these balls. Thus

$$\int_{\mathbb{R}^N} [\gamma - V(x)]_+^{1+((N)/(2s))} \,\mathrm{d}x \leqslant n \int_{\mathbb{R}^N} [\gamma - (|x|/M)^q]_+^{1+((N)/(2s))} \,\mathrm{d}x$$
$$\leqslant C\gamma^{1+((N)/(2s))+((N)/(q))},$$

where $C > 0$ is a suitable constant. The lower bound in (5.1), therefore, follows from (5.2) by taking γ to be equal to $([(N+2s)q/(4Cs(q+1)]^{2s/N}(a^* - a))^{q/(q+2s)}$.

Next, we shall prove the upper bound in (5.1). Similarly to the proof of Theorem 1.2, we use a trial function of the form (3.17). Choose $x_0 \in \mathcal{Z}$, where $\mathcal{Z}$ is defined in (1.22) and fix $R > 0$ small enough so that

$$V(x) \leqslant C|x - x_0|^q \quad \text{for } |x - x_0| \leqslant 2R,$$

in which case, we infer that

$$\int_{\mathbb{R}^N} V(x)|u(x)|^2 \,\mathrm{d}x \leqslant C\tau^{-q} A_{R,\tau}^2 \int_{\mathbb{R}^N} |x|^q |Q(x)|^2 \,\mathrm{d}x,$$

where the constant $A_{R,\tau} > 0$ satisfies (3.18). From the estimates (3.18)–(3.20), we thus conclude that for large τ,

$$e(a) \leqslant \frac{N\tau^{2s}(a^* - a)}{(N+2s)\|Q\|_2^4} \int_{\mathbb{R}^N} |Q(x)|^{2+((4s)/(N))} \,\mathrm{d}x$$
$$+ C\tau^{-q} \int_{\mathbb{R}^N} |x|^q |Q(x)|^2 \,\mathrm{d}x + O(\tau^{-2s}). \tag{5.3}$$

By taking $\tau = (a^* - a)^{-((1)/(q+2s))}$, we obtain the desired upper bound of $e(a)$. This completes the proof of the lemma. □

Let u_a be a nonnegative minimizer of (1.6). The following estimates on the $L^{2+((4s)/(N))}(\mathbb{R}^N)$ norm of u_a is a simple consequence of lemma 5.1.

LEMMA 5.2. *Assume that u_a is a minimizer of (1.6) with $V(x)$ satisfying (1.20), then there exists a positive constant K, independent of a, such that*

$$0 < K(a^* - a)^{-((2s)/(q+2s))} \leqslant \int_{\mathbb{R}^N} |u_a(x)|^{2+((4s)/(N))} \,\mathrm{d}x$$
$$\leqslant \frac{1}{K}(a^* - a)^{-((2s)/(q+2s))} \quad \textit{for } 0 \leqslant a < a^*. \tag{5.4}$$

Proof. By (3.16) and (1.13), we have

$$e(a) \geqslant \frac{N(a^* - a)}{N+2s} \int_{\mathbb{R}^N} |u_a(x)|^{2+((4s)/(N))} \,\mathrm{d}x.$$

Then, the upper bounded in (5.4) follows immediately from lemma 5.1.

To show the lower bound in (5.4), we pick a $0 < b < a$ and employ that

$$e(b) \leqslant E_b(u_a) = e(a) + \frac{a-b}{2}\int_{\mathbb{R}^N} |u_a(x)|^{2+((4s))/(N))}\,\mathrm{d}x.$$

By applying lemma 5.1, the above inequality implies that there exist two positive constant $C_1 < C_2$ such that for any $0 < b < a < a^*$,

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N} |u_a(x)|^{2+((4s)/(N))}\,\mathrm{d}x &\geqslant \frac{e(b)-e(a)}{a-b}\\ &\geqslant \frac{C_1(a^*-b)^{q/(q+2s)} - C_2(a^*-a)^{q/(q+2s)}}{a-b}.\end{aligned} \tag{5.5}$$

With $b = a - \gamma(a^* - a)$, we can write the right side of (5.5) as

$$(a^*-a)^{-2s/(q+2s)}\frac{C_1(1+\gamma)^{q/(q+2s)} - C_2}{\gamma}.$$

The last fraction is positive for γ large enough. For a close to a^*, this then gives the desired lower bound. For smaller a, we can simply use the fact that $\int_{\mathbb{R}^N} |u_a(x)|^4\,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} |u_0(x)|^4\,\mathrm{d}x$ for any $0 \leqslant a \leqslant a^*$, which follows from the bounds $e(a) \leqslant E_a(u_0)$ and $e(0) \leqslant E_0(u_a)$. This completes the proof of the lemma. □

In the sequel, we will complete the proofs of Theorem 1.5 and corollary 1.6.

Proof of Theorem 1.5. Let u_a be a nonnegative minimizer of (1.6), and we now define

$$\varepsilon := (a^*-a)^{((1)/(q+2s))} > 0. \tag{5.6}$$

From the fractional Gagliardo–Nirenberg–Sobolev inequality (1.13), we deduce that

$$e(a) \geqslant \left(1 - \frac{a}{a^*}\right)\int_{\mathbb{R}^N} |(-\Delta)^s u_a|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u_a|^2\,\mathrm{d}x,$$

and it thus follows from lemma 5.1 that

$$\int_{\mathbb{R}^N} |(-\Delta)^s u_a|^2\mathrm{d}\,x \leqslant C\varepsilon^{-2s} \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)|u_a|^2\,\mathrm{d}x \leqslant C\varepsilon^q. \tag{5.7}$$

For $1 \leqslant i \leqslant n$, we define the $L^2(\mathbb{R}^N)$-normalized functions

$$w_a^i(x) := \varepsilon^{N/2} u_a(\varepsilon x + x_i). \tag{5.8}$$

It follows from (5.7) and lemma 5.2 that

$$0 < K \leqslant \int_{\mathbb{R}^N} |w_a^i|^{2+((4s)/(N))}\,\mathrm{d}x \leqslant \frac{1}{K}, \quad \int_{\mathbb{R}^N} |(-\Delta)^s w_a^i|^2\,\mathrm{d}x \leqslant C \tag{5.9}$$

and also

$$\int_{\mathbb{R}^N} V(\varepsilon x + x_i)|w_a^i|^2\,\mathrm{d}x \leqslant C\varepsilon^q. \tag{5.10}$$

In particular, the functions w_a^i are bounded uniformly in $H^s(\mathbb{R}^N)$.

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For any $\gamma > 0$, by (5.7), we obtain

$$\int_{\{V(x)\geqslant\gamma\varepsilon^q\}} |u_a|^2 \,\mathrm{d}x \leqslant \frac{1}{\gamma\varepsilon^q}\int_{\mathbb{R}^N} V(x)|u_a|^2 \,\mathrm{d}x \leqslant \frac{C}{\gamma}.$$

For ε small enough, that is, for a sufficiently close to a^*, the set $\{x \in \mathbb{R}^N : V(x) \leqslant \gamma\varepsilon^q\}$ is contained in n disjoint balls with radius at most $C\gamma^{1/q}\varepsilon$, for some $C > 0$, centred at the points x_i. We thus conclude from the above inequality that

$$\begin{aligned}\frac{C}{\gamma} &\geqslant \int_{\{V(x)\geqslant\gamma\varepsilon^q\}} |u_a|^2 \,\mathrm{d}x = 1 - \int_{\{V(x)\leqslant\gamma\varepsilon^q\}} |u_a|^2 \,\mathrm{d}x \\ &\geqslant 1 - \sum_{i=1}^{n}\int_{\{|x-x_i|\leqslant C\gamma^{1/q}\varepsilon\}} |u_a(x)|^2 \,\mathrm{d}x = 1 - \sum_{i=1}^{n}\int_{\{|x|\leqslant C\gamma^{1/q}\}} |w_a^i(x)|^2 \,\mathrm{d}x,\end{aligned}$$

which implies that

$$1 \geqslant \sum_{i=1}^{n}\int_{\{|x|\leqslant C\gamma^{1/q}\}} |w_a^i(x)|^2 \,\mathrm{d}x \geqslant \frac{C}{\gamma}. \tag{5.11}$$

Using the fact that the functions w_a^i are uniformly bounded in $H^s(\mathbb{R}^N)$, up to a subsequence $\{a_k\}$, satisfying $a_k \nearrow a^*$ as $k \to \infty$, we can deduce that for suitable functions $w_0^i \in H^s(\mathbb{R}^N)$,

$$w_a^i \rightharpoonup w_0^i \quad \text{weakly in } H^s(\mathbb{R}^N),\ 1 \leqslant i \leqslant n, \tag{5.12}$$

$$w_a^i \to w_0^i \quad \text{strongly in } L^r(\{|x| \leqslant C\gamma^{1/q}\}) \quad \text{for any } 2 \leqslant r < 2_s^*. \tag{5.13}$$

Therefore, from (5.11), we deduce that

$$1 \geqslant \sum_{i=1}^{n}\int_{\{|x|\leqslant C\gamma^{1/q}\}} |w_0^i(x)|^2 \,\mathrm{d}x \geqslant \frac{C}{\gamma}.$$

Since this bound holds for any $\gamma > 0$, we finally yield that

$$\sum_{i=1}^{n} \|w_0^i\|_2^2 = 1. \tag{5.14}$$

Since u_a is a nonnegative minimizer of (1.6), it satisfies the following Euler–Lagrange equation

$$(-\Delta)^s u_a(x) + V(x)u_a(x) = \mu_a u_a(x) + a u_a^{1+4s/N}(x) \quad \text{in } \mathbb{R}^N, \tag{5.15}$$

where $\mu_a \in \mathbb{R}$ is a suitable Lagrange multiplier and satisfies

$$\mu_a = e(a) - \frac{2as}{N+2s}\int_{\mathbb{R}^N} |u_a|^{2+((4s/(N))} \,\mathrm{d}x$$

The functions w_a^i in (5.8) are thus nonnegative solutions of

$$(-\Delta)^s w_a^i(x) + \varepsilon^{2s}V(\varepsilon x + x_i)w_a^i(x) = \varepsilon^{2s}\mu_a w_a^i(x) + a w_a^i(x)^{1+4s/N} \quad \text{in } \mathbb{R}^N, \tag{5.16}$$

It follows from lemma 5.2 that $\varepsilon^{2s}\mu_a$ is uniformly bounded as $a \nearrow a^*$, and strictly negative for a close to a^*. Passing if necessary to a subsequence of $\{a_k\}$, still denoted

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by $\{a_k\}$, we can assume that $\lim_{a\nearrow a^*}\varepsilon^{2s}\mu_a = -\beta^{2s} < 0$ for some $\beta > 0$. By passing to the weak limit (5.12), we see that the nonnegative functions w_0^i satisfy

$$(-\Delta)^s w_0^i(x) = -\beta^{2s} w_0^i(x) + a^* w_0^i(x)^{1+4s/N} \quad \text{in } \mathbb{R}^N. \tag{5.17}$$

Using the maximum principle, either $w_0^i = 0$ identically, or otherwise $w_0^i > 0$ for all $x \in \mathbb{R}^N$. In the latter case, a simple rescaling together with the uniqueness of positive ground state solutions of (1.17) up to translations allows us to deduce that

$$w_0^i(x) = (N\beta^{2s}/(2sa^*))^{N/(4s)} Q((N\beta^{2s}/(2s))^{1/2s}(x-y_i)), \tag{5.18}$$

for some $y_i \in \mathbb{R}^N$. Therefore, either $w_0^i = 0$ or $\|w_0^i\|_2^2 = 1$. Because of (5.14), we see that exactly one w_0^i is of form (5.18), which all the others are zero.

Let $1 \leqslant j \leqslant n$ be such that $\|w_0^j\|_2^2 = 1$. From the norm preservation, we infer that w_a^j converges to w_0^j strongly in $L^2(\mathbb{R}^N)$ and, in fact, strongly in $L^r(\mathbb{R}^N)$ for $2 \leqslant r < 2_s^*$ because of $H^s(\mathbb{R}^N)$ boundedness. Moreover, since w_a^j and w_0^j satisfy (5.16) and (5.17), respectively, a simple analysis shows that w_a^j converges to w_0^j strongly in $H^s(\mathbb{R}^N)$. By going to a subsequence, if necessary, we can also assume that the convergence holds pointwise almost everywhere.

To complete the proof of Theorem 1.5, we calculate from (5.8) that

$$\begin{aligned} e(a) = E_a(u_a) =& \frac{1}{\varepsilon^{2s}}\left[\int_{\mathbb{R}^N} |(-\Delta)^{s/2} w_a^j(x)|^2 \,\mathrm{d}x - \frac{Na^*}{N+2s}\int_{\mathbb{R}^N} |w_a^j(x)|^{2+((4s)/(N))}\,\mathrm{d}x\right] \\ &+ \frac{N\varepsilon^q}{N+2s}\int_{\mathbb{R}^N} |w_a^j(x)|^{2+((4s)/(N))}\,\mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x + x_i)|w_a^j(x)|^2\,\mathrm{d}x. \end{aligned}$$

The term in square brackets is nonnegative and can be ignored for a lower bound of $e(a)$. The $L^{2+((4s)/(N))}(\mathbb{R}^N)$ norm of w_a^j converges to the one of w_0^j, and from Fatou's lemma it follows that

$$\liminf_{\varepsilon\to 0} \varepsilon^{-q}\int_{\mathbb{R}^N} V(\varepsilon x + x_i)|w_a^j(x)|^2\,\mathrm{d}x \geqslant \kappa_j \int_{\mathbb{R}^N} |x|^q |w_0^j(x)|^2\,\mathrm{d}x$$

where $\kappa_j = \lim_{x\to x_j} V(x)|x-x_j|^{-q} \in (0,\infty]$. Moreover, since $Q(x)$ ia a radially symmetric decreasing function and decays polynomially as $|x| \to \infty$, we then conclude that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^q |w_0^j(x)|^2\,\mathrm{d}x &= \frac{1}{(N/(2s))^{q/2s}\beta^q \|Q\|_2^2}\int_{\mathbb{R}^N} |x+y_j|^q |Q(x)|^2\,\mathrm{d}x \\ &\geqslant \frac{1}{(N/(2s))^{q/2s}\beta^q \|Q\|_2^2}\int_{\mathbb{R}^N} |x|^q |Q(x)|^2\,\mathrm{d}x \end{aligned} \tag{5.19}$$

Moreover, the last equality of (5.19) holds if and only if $y_j = 0$. We thus derive from (5.18) and (5.19) that

$$\begin{aligned} \liminf_{a\nearrow a^*} \frac{e(a)}{(a^*-a)^{q/(q+2s)}} &\geqslant \frac{N}{N+2s}\|w_0^j\|_{2+((4s)/(N))}^{2+((4s)/(N))} + \kappa_j \int_{\mathbb{R}^N} |x|^q |w_0^j(x)|^2\,\mathrm{d}x \\ &\geqslant \frac{N}{a^*}\left(\frac{1}{2s}\beta^{2s} + \lambda_j^{2s+q}\frac{1}{q\beta^q}\right) \end{aligned} \tag{5.20}$$

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where λ_j is defined in (1.21), and we have used that $\|Q\|_2^2 = ((N)/(N+2s))\|Q\|_{2+((4s)/(N))}^{2+((4s)/(N))}$. Take the infimum of (5.20), which is achieved for $\beta = \lambda_j$. We then obtain that

$$\liminf_{a \nearrow a^*} \frac{e(a)}{(a^* - a)^{q/(q+2s)}} \geqslant \frac{N(q+2s)}{2sqa^*} \lambda^{2s}, \tag{5.21}$$

where $\lambda = \min_j \lambda_j$ is as before.

The limit in (5.21) actually exists, and is equal to the right side. To see this, we can simply take

$$u(x) = \frac{(N/2s)^{N/4s} \beta^{N/2}}{\varepsilon^{N/2} \|Q\|_2} Q \left((N/2s)^{1/2s} \beta \frac{x - x_j}{\varepsilon} \right)$$

as a trial function for E_a, and minimizes over $1 \leqslant j \leqslant n$ and $\beta > 0$. The result is that

$$\lim_{a \nearrow a^*} \frac{e(a)}{(a^* - a)^{q/(q+2s)}} = \frac{N(q+2s)}{2sqa^*} \lambda^{2s}. \tag{5.22}$$

From the equality (5.22), we can draw several conclusions. First, the j defined above is such that $\lambda_j = \lambda$, that is, $x_j \in \mathcal{Z}$. Second, β is unique (i.e., independent of the choice of the subsequence) and equal to the expression minimizing (5.20), that is, $\beta = \lambda$. Finally, $y_j = 0$, since the inequality (5.19) is strict for $y_j \neq 0$. We, therefore, have

$$w_a^j(x) = \varepsilon^{N/2} u_a(\varepsilon x + x_j) \to \frac{\lambda^{N/2} \left(N/(2s)\right)^{N/(4s)}}{\|Q\|_2} Q \left(\lambda \left(N/(2s)\right)^{1/2s} x \right) \quad \text{as } a \nearrow a^*,$$

strongly in $H^s(\mathbb{R}^N)$, where $x_j \in \mathcal{Z}$. This completes the proof of Theorem 1.5. □

Proof of Corollary 1.6. The proof is almost the same to the proof of Theorem 1.5, and we only need to modify the proof of the upper estimate of (5.1).

For this purpose, we take a new trial function

$$u(x) = \frac{\tau^{N/2}}{\|Q\|_2} Q(\tau(x - x_0)) \quad \text{for any } \tau > 0.$$

Then $\int_{\mathbb{R}^N} |u(x)|^2 \, \mathrm{d}x = 1$, it follows from (1.13) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(x)|^2 \, \mathrm{d}x - \frac{Na}{N+2s} \int_{\mathbb{R}^N} |u(x)|^{2+((4s)/(N))} \, \mathrm{d}x \\
&= \frac{\tau^{2s}}{\|Q\|_2^2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} Q|^2 \, \mathrm{d}x - \frac{Na}{(N+2s)\|Q\|_2^{4s/N}} \int_{\mathbb{R}^N} |Q|^{2+((4s)/(N))} \, \mathrm{d}x \right) \\
&= \frac{N\tau^{2s}}{(N+2s)\|Q\|_2^2} \left(1 - \frac{a}{\|Q\|_2^{4s/N}} \right) \int_{\mathbb{R}^N} |Q|^{2+((4s)/(N))} \, \mathrm{d}x
\end{aligned} \tag{5.23}$$

Moreover, by the polynomial decay of $Q(x)$, we have

$$\int_{\mathbb{R}^N} h(x)|x-x_0|^q|u(x)|^2\,\mathrm{d}x \leqslant \frac{\tau^{-q}}{C\|Q\|_2^2}\int_{\mathbb{R}^N}|x|^q|Q(x)|^2\,\mathrm{d}x \leqslant \frac{C_1}{\tau^q}. \tag{5.24}$$

It then follows from (5.23) and (5.24) that

$$e(a) \leqslant C_2(a^*-a)\tau^{2s} + \frac{C_1}{\tau^q}.$$

By taking $\tau = (a^*-a)^{-((1)/(q+2s))}$, the above inequality implies the desired upper estimate of (5.1). This completes the proof of Corollary 1.6. □

Appendix A Proof of Theorem 1.3 (ii)

In this appendix, we always assume that $V(x)$ satisfies condition (V_1). Define

$$\mu_1 = \inf\left\{\int_{\mathbb{R}^N}(|(-\Delta)^{s/2}u|^2 + V(x)u^2)\,\mathrm{d}x:\ u\in\mathcal{H} \quad \text{and} \int_{\mathbb{R}^N}u^2\,\mathrm{d}x = 1\right\}. \tag{A.1}$$

Usually, μ_1 is called the first eigenvalue of $(-\Delta)^s + V(x)$ in $\mathcal{H}$. Furthermore, from lemma 3.1, we can easily know that μ_1 is simple and can be attained by positive function $\phi_1 \in \mathcal{H}$. Here, $\phi_1 > 0$ is called the first eigenfunction of $(-\Delta)^s + V(x)$ in $\mathcal{H}$. We now define

$$\mu_2 = \inf\left\{\int_{\mathbb{R}^N}(|(-\Delta)^{s/2}u|^2 + V(x)u^2)\,\mathrm{d}x:\ u\in Z \quad \text{and} \int_{\mathbb{R}^N}u^2\,\mathrm{d}x = 1\right\}. \tag{A.2}$$

where

$$Z = \mathrm{span}\{\phi_1\}^\perp = \left\{u:\ u\in\mathcal{H},\ \int_{\mathbb{R}^N}u\phi_1\,\mathrm{d}x = 0\right\}.$$

It is known that $\mu_2 > \mu_1$ and

$$\mathcal{H} = \mathrm{span}\{\phi_1\}\oplus Z. \tag{A.3}$$

Then, we have the following lemma, its proof is somewhat standard, and we omit here.

LEMMA A.1. *Suppose $V(x)$ satisfies (V_1). Then we have*

(i) $\ker((-\triangle)^s + V(x) - \mu_1) = \mathrm{span}\{\phi_1\}$;

(ii) $\phi_1 \notin ((-\triangle)^s + V(x) - \mu_1)Z$;

(iii) $\mathrm{Im}((-\triangle)^s + V(x) - \mu_1) = ((-\triangle)^s + V(x) - \mu_1)Z$ *is closed in* $\mathcal{H}^*$;

(iv) $\mathrm{codim}\ \mathrm{Im}((-\triangle)^s + V(x) - \mu_1) = 1$.

where $\mathcal{H}^*$ denotes the dual space of $\mathcal{H}$.

Inspired by [8, theorem 3.2], we have the following lemma. For the sake of completeness, we give a short proof here.

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Lemma A.2. *Define the following C^1 functional $F : \mathcal{H} \times \mathbb{R}^2 \to H^*$*

$$F(u, \mu, a) = ((-\triangle)^s + V(x) - \mu)u - au^{1+((4s)/(N))}. \tag{A.4}$$

Then, there exist $\delta > 0$ and a unique function $(u(a),\ \mu(a)) \in C^1(B_\delta(0);\ B_\delta(\mu_1,\ \phi_1))$ such that

$$\begin{cases} \mu(0) = \mu_1, \quad u(0) = \phi_1; \\ F(u(a),\ \mu(a),\ a) = 0; \\ \|u(a)\|_2^2 = 1. \end{cases} \tag{A.5}$$

Proof. Let $g \in Z \times \mathbb{R}^3 \to \mathcal{H}^*$ be defined by

$$g(z, \tau, t, a) := F((1 + t)\phi_1 + z,\ \mu_1 + \tau,\ a).$$

Then $g \in C^1(Z \times \mathbb{R}^3,\ \mathcal{H}^*)$ and

$$\begin{cases} g(0, 0, 0, 0) = F(\phi_1,\ \mu_1,\ 0) = 0, \\ g_t(0, 0, 0, 0) = F_u(\phi_1,\ \mu_1,\ 0)\phi_1 = ((-\Delta)^s + V(x) - \mu_1)\phi_1 = 0. \end{cases} \tag{A.6}$$

Moreover, for any $(\hat{z},\ \hat{\tau}) \in Z \times \mathbb{R}$, we obtain

$$\begin{aligned} g_{(z,\tau)}(0, 0, 0, 0)(\hat{z},\ \hat{\tau}) &= F_u(\phi_1,\ \mu_1,\ 0)\hat{z} + F_\mu(\phi_1,\ \mu_1,\ 0)\hat{\tau} \\ &= ((-\Delta)^s + V(x) - \mu_1)\hat{z} - \hat{\tau}\phi_1. \end{aligned} \tag{A.7}$$

Then, by lemma A.1 and (A.7), $g_{(z,\tau)}(0, 0, 0, 0) : Z \times \mathbb{R} \to \mathcal{H}^*$ is an isomorphism. Hence, it follows from the implicit function theorem that there exist $\delta_1 > 0$ and a unique function $(z(t, a),\ \tau(t, a)) \in C^1(B_{\delta_1}(0, 0);\ B_{\delta_1}(0, 0))$ such that

$$\begin{cases} g(z(t, a), \tau(t, a), t, a) = F((1 + t)\phi_1 + z(t, a),\ \mu_1 + \tau(t, a),\ a) = 0, \\ z(0, 0) = 0,\ \tau(0, 0) = 0, \\ z_t(0, 0) = -g^{-1}_{z,\tau}(0, 0, 0, 0) \cdot g_t(0, 0, 0, 0) = 0. \end{cases} \tag{A.8}$$

Now, set

$$u(t, a) = (1 + t)\phi_1 + z(t, a), \qquad (t, a) \in B_{\delta_1}(0, 0),$$

and let us define

$$f(t, a) = \|u(t, a)\|_2^2 = (1 + t)^2 + \int_{\mathbb{R}^N} z^2(t, a)\,\mathrm{d}x, \qquad (t, a) \in B_{\delta_1}(0, 0).$$

It follows from (A.8) that

$$f(0, 0) = 1, \quad f_t(0, 0) = 2 + 2\int_{\mathbb{R}^N} z_t(0, 0)z(0, 0)\,\mathrm{d}x = 2.$$

Then, by applying implicit function theorem again, there exist $0 < \delta < \delta_1$ and a unique function $t = t(a) \in C^1(B_\delta(0);\ B_\delta(0))$ such that

$$f(t(a),\ a) = \|u(t(a),\ a)\|_2^2 = f(0, 0) = 1, \qquad a \in B_\delta(0).$$

This, together with (A.8), implies that for $a \in B_\delta(0)$, there exists a unique function

$$(u(a) := u(t(a), a), \;\; \mu_a := \mu_1 + \tau(t(a), a)) \in C^1(B_\delta(0);\; B_\delta(\phi_1, \mu_1))$$

such that (A.5) holds, and the proof is, therefore, complete. □

We are now ready to show the uniqueness of nonnegative minimizers of (1.6) with $f(t) = |t|^{4s/N}t$.

Proof of Theorem 1.3 (ii). Let u_a be a nonnegative minimizer of $e(a)$ with $a \in [0,\, a^*)$. We can easily see that

$$e(0) = \mu_1 \quad \text{and} \quad e(a) \leqslant e(0) = \mu_1, \tag{A.9}$$

where μ_1 is the first eigenvalue of $(-\Delta)^s + V(x)$. Since u_a is a minimizer of (1.6), it satisfies the following Euler–Lagrange equation

$$(-\Delta)^s u_a(x) + V(x)u_a(x) - \mu_a u_a(x) - a u_a^{1+4s/N}(x) = 0 \quad \text{in } \mathbb{R}^N,$$

where $\mu_a \in \mathbb{R}$ is a suitable Lagrange multiplier associated with u_a, that is,

$$F(u_a,\, \mu_a,\, a) = 0 \quad \text{where } F(\cdot) \text{ is defined by (A.4).} \tag{A.10}$$

since

$$\mu_a = e(a) - \frac{2sa}{N+2s}\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x,$$

it then follows from (A.9), (4.3) and (4.6) that there exists $a_1 > 0$ small such that

$$|\mu_a - \mu_1| \leqslant |e(a) - \mu_1| + \frac{2sa}{N+2s}\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x \leqslant \delta \quad \text{for } 0 \leqslant a < a_1, \tag{A.11}$$

where $\delta > 0$ is as in lemma A.2. On the contrary, since

$$E_0(u_a) = e(a) + \frac{Na}{N+2s}\int_{\mathbb{R}^N} |u_a|^{2+((4s)/(N))}\, \mathrm{d}x \to e(0) = \mu_1 \quad \text{as } a \searrow 0,$$

i.e. $\{u_a\}$ is a nonnegative minimizing sequence of $e(0) = \mu_1$ as $a \searrow 0$. Note that μ_1 is a simple eigenvalue, we can easily yield from lemma 3.1 that

$$u_a \to \phi_1 \quad \text{in } \mathcal{H} \quad \text{for all } a \searrow 0.$$

This implies that there exists $a_2 > 0$ such that

$$\|u_a - \phi_1\|_{\mathcal{H}} < \delta \quad \text{for } 0 \leqslant a < a_2. \tag{A.12}$$

Then using (A.10)–(A.12) and lemma A.2, we obtain that

$$\mu_a = \mu(a), \quad u_a = u(a) \quad \text{for } 0 \leqslant a < \min\{a_1,\, a_2\},$$

that is, $e(a)$ has a unique nonnegative minimizer $u(a)$ if $a > 0$ is small. □

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