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MULTIFRACTAL ANALYSIS OF FUNCTIONS ON HEISENBERG AND CARNOT GROUPS

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Abstract In this article, we investigate the pointwise behaviors of functions on the Heisenberg group. We find wavelet characterizations for the global and local Hölder exponents. Then we prove some a priori upper bounds for the multifractal spectrum of all functions in a given Hölder, Sobolev, or Besov space. These upper bounds turn out to be optimal, since in all cases they are reached by typical functions in the corresponding functional spaces. We also explain how to adapt our proof to extend our results to Carnot groups.

Keywords: measure and integration; partial differential equations; functional analysis; harmonic analysis on Euclidean spaces

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1. Introduction

In this article, we draw the first results for a multifractal analysis for functions defined on the Heisenberg group $\mathbb{H}$. Multifractal analysis is now a widespread issue in analysis. Its objective is to provide a description of the variety of local behaviors of a given function or a given measure. The local behaviors are measured thanks to the pointwise Hölder exponent, and one aims at describing the distribution of the iso-Hölder sets i.e., the sets of points $x \in \mathbb{H}$ with same pointwise exponent. What makes the Heisenberg group interesting for our dimensional considerations is that its Hausdorff dimension is $\dim_H(\mathbb{H}) = 4$ while it is defined using only three topological coordinates. This is due to the special form of the metric in the 'vertical' direction. This induces surprising properties from the geometric measure theoretic standpoint which are currently being investigated: for instance, Besicovitch's covering theorem and Marstrand's projection theorem are not true; see [1, 2, 23, 26, 29]. In this paper, we pursue this investigation by studying the multifractal properties of functions defined on $\mathbb{H}$. We find an a priori upper bound for the Hausdorff dimensions of iso-Hölder sets for all functions in a given Hölder and Besov space, and we prove that these bounds are optimal, since they are reached for *generic* functions (in the sense of Baire's categories) in these function spaces. To do so, we develop methods based on wavelets on $\mathbb{H}$ [24]. In the last section, we explain how to adapt our proof to extend our results to Carnot groups.

Let us start by some basic facts on $\mathbb{H}$. The first Heisenberg group $\mathbb{H}$ consists [31, p. 530] of the set $\mathbb{R}^3$ equipped with a non-commutative group law

$$(p, q, r) * (p', q', r') = (p + p', q + q', r + r' + 2(qp' - pq'))$$

which is also denoted by the absence of multiplicative symbol when the context is clear. The inverse of $x = (p, q, r)$ is $x^{-1} = (-p, -q, -r)$. The Haar measure is $dx = dp \wedge dq \wedge dr$, and is also denoted by ℓ. It is a homogeneous group [31, p. 618] whose dilations are defined by

$$\lambda \circ (p, q, r) = (\lambda p, \lambda q, \lambda^2 r).$$

A left-invariant distance $\delta(x, y) = \left\|x^{-1} * y\right\|_{\mathbb{H}}$ is given by the homogeneous pseudo-norm:

$$\|x\|_{\mathbb{H}} = \left\{(p^2 + q^2)^2 + r^2\right\}^{1/4}. \tag{1}$$

Remark 1. One can modify the coefficients in formula (1) so that it becomes a norm on $\mathbb{H}$; see [12]. We nonetheless keep this formulation so that the generalization of our results to Carnot groups (see § 7) will require fewer modifications.

The Lie algebra $\mathfrak{h}$ is the vector space of left-invariant vector fields on $\mathbb{H}$. It is nilpotent of step 2 (see [31, p. 544]); i.e., $\mathfrak{h} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1$ is spanned by

$$X = \frac{\partial}{\partial p} + 2q\frac{\partial}{\partial r} \quad \text{and} \quad Y = \frac{\partial}{\partial q} - 2p\frac{\partial}{\partial r}, \tag{2}$$

and $\mathfrak{n}_2$ is spanned by $Z = \frac{\partial}{\partial r}$. All commutators vanish except $[X, Y] = -4Z$. Hence $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{n}_2$ and $[\mathfrak{n}_2, \mathfrak{h}] = 0$. The homogeneous structure of $\mathbb{H}$ induces dilations of $\mathfrak{h}$:

$$\lambda \circ (\alpha X + \beta Y + \gamma Z) = \lambda(\alpha X + \beta Y) + \lambda^2 \gamma Z,$$

which satisfy $\lambda \circ [U, V] = [\lambda \circ U, \lambda \circ V]$.

The positive self-adjoint hypoelliptic Laplace operator on $\mathbb{H}$ is (see [15])

$$\mathcal{L} = -(X^2 + Y^2). \tag{3}$$

Sobolev spaces of regularity index $s \geqslant 0$ can be defined by functional calculus:

$$H^s(\mathbb{H}) = \{u \in L^2(\mathbb{H}) : \mathcal{L}^{s/2} u \in L^2(\mathbb{H})\}. \tag{4}$$

Throughout the article, $Q = 4$ denotes the homogeneous dimension of $\mathbb{H}$. In order to state the results quickly, we postpone the classical definitions and notation (horizontal paths, Carnot balls, polynomials, Hausdorff dimension, Besov and Hölder spaces) to § 2.

Let us define the pointwise Hölder regularity of a function.

Definition 1. Let $f : \mathbb{H} \to \mathbb{R}$ be a function belonging to $L^\infty_{loc}(\mathbb{H})$. For $s > 0$ and $x_0 \in \mathbb{H}$, f is said to belong to $C^s(x_0)$ if there exist constants $C > 0$, $\eta > 0$, and a polynomial P with homogeneous degree $\deg_{\mathbb{H}}(P) < s$ such that

$$\forall x \in \mathbb{H}, \quad \|x\| < \eta \Longrightarrow |f(x_0 x) - P(x)| \leqslant C \|x\|^s_{\mathbb{H}}. \tag{5}$$

One says that $f \in C^s_{\log}(x_0)$ if, instead of (5), the following holds:

$$|f(x_0x) - P(x)| \leqslant C\, \|x\|^s_{\mathbb{H}} \cdot |\log \|x\|_{\mathbb{H}}|\,. \tag{6}$$

Observe that this definition is left invariant: $f \in C^s(x_0)$ if and only if $f_y \in C^s(y^{-1}x_0)$ with $f_y : x \mapsto f(yx)$.

The following quantities are crucial in multifractal analysis.

Definition 2. Let $f : \mathbb{H} \to \mathbb{R}$ be a function belonging to $L^\infty_{loc}(\mathbb{H})$, and let $x_0 \in \mathbb{H}$.

The pointwise regularity exponent of f at x_0 is

$$h_f(x_0) = \sup\{s > 0 : f \in C^s(x_0)\}, \tag{7}$$

with the convention that $h_f(x_0) = 0$ if $f \notin C^s(x_0)$ for any $s > 0$.

The multifractal spectrum of f is the mapping $d_f : [0, \infty] \to \{-\infty\} \cup [0, Q]$:

$$d_f(h) = \dim_H \left(E_f(h)\right) \quad \text{where } E_f(h) = \{x \in \mathbb{H} : h_f(x) = h\},$$

where $\dim_H$ stands for the Hausdorff dimension on $\mathbb{H}$. By convention, $\dim_H \emptyset = -\infty$.

The multifractal spectrum of f describes the geometrical distribution of the singularities of f over $\mathbb{H}$. The Hausdorff dimension is the right notion to use here, since (at least intuitively, but also for generic functions) the iso-Hölder sets $E_f(h)$ are dense over the support of f and the Minkowski dimension does not distinguish dense sets.

Wavelets are a key tool in our analysis. The construction of wavelets on stratified Lie groups has been achieved in [24]. A convenient observation is that $\mathcal{Z} = \mathbb{Z}^3$ is a subgroup of $\mathbb{H}$. For $j \in \mathbb{Z}$ and $k = (k_p, k_q, k_r) \in \mathcal{Z}$, one defines

$$x_{j,k} = 2^{-j} \circ k = (2^{-j}k_p, 2^{-j}k_q, 2^{-2j}k_r).$$

Note that $x_{j,k}^{-1} = x_{j,-k}$. The dyadic cubes are defined in the following way:

$$C_0 = \{(p, q, r) \in \mathbb{H} : 0 \leqslant p, q, r < 1\} \quad \text{and} \quad C_{j,k} = x_{j,k} * (2^{-j} \circ C_0).$$

The left multiplication by $x_{j,k}$ maps affine planes of $\mathbb{R}^3$ on affine planes, and thus the shape of $C_{j,k}$ is a regular parallelogram with vertices on $2^{-j} \circ \mathcal{Z}$. But one will observe that two different cubes $C_{j,k}$ and $C_{j,k'}$ are not in general Euclidean translates of each other. A neighborhood $\Lambda_{j,k}$ of $C_{j,k}$ is given by

$$\Lambda_{j,k} = \bigcup_{k' \in \Xi} C_{j,k*k'}, \tag{8}$$

where Ξ is the set of 35 multi-integers $k' = (k'_p, k'_q, k'_r)$ given by (see Figure 1)

k'_p	k'_q	k'_r
0	0	−1, 0, 1
1	0	−3, −2, −1, 0, 1
1	1	−1, 0, 1

k'_p	k'_q	k'_r
0	1	−1, 0, 1, 2, 3
−1	1	1, 2, 3
−1	0	−1, 0, 1, 2, 3

k'_p	k'_q	k'_r
−1	−1	−1, 0, 1
0	−1	−3, −2, −1, 0, 1
1	−1	−3, −2, −1.

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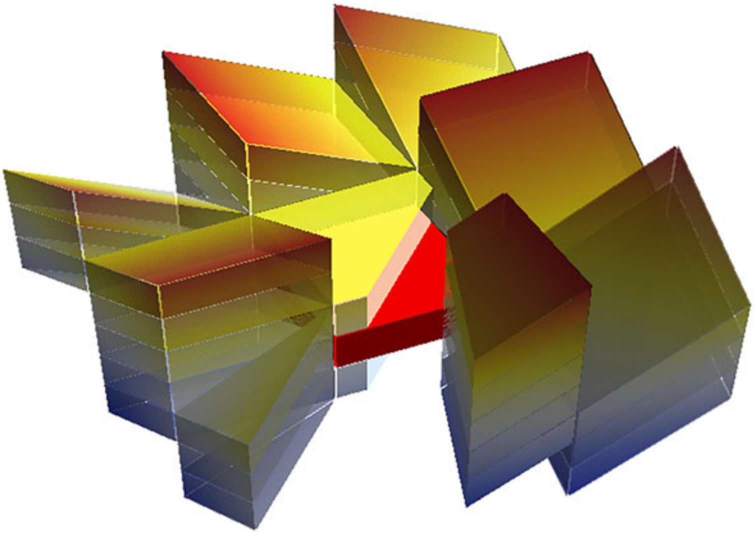

Figure 1. The cube C_0 admits 34 closest neighbors in $\mathbb{H}$ contrary to Euclidean cubes of $\mathbb{R}^3$ that admit only 26 neighbors.

Given $x \in \mathbb{H}$ and $j \in \mathbb{Z}$, there exists a unique $k \in \mathcal{Z}$ such that $x \in C_{j,k}$. For this choice of k, it is convenient to write

$$C_j(x) = C_{j,k} \quad \text{and} \quad \Lambda_j(x) = \Lambda_{j,k}. \tag{9}$$

The diameter of $C_{j,k}$ is $13^{1/4} \times 2^{-j} < 2^{1-j}$ (because the diameter of C_0 is $13^{1/4}$). In particular, if $\delta(x, y) < 2^{-j}$, then x, y belong simultaneously to at least one $\Lambda_{j,k}$.

Let us recall now the construction of wavelets on $\mathbb{H}$ by Lemarié [24]. For any integer $M > Q/2$, there exist $2^Q - 1 = 15$ functions $(\vartheta_\varepsilon)_{1\leqslant\varepsilon\leqslant 15}$ in $H^{4M}(\mathbb{H})$ such that the following hold.

- There exist $C_0, r_0 > 0$ such that for any multi-index α of length $|\alpha| < 4M - Q$:
$$\forall x \in \mathbb{H}, \quad |\nabla^\alpha_{\mathbb{H}} \vartheta_\varepsilon(x)| \leqslant C_0 \exp\left(-\|x\|_{\mathbb{H}}/r_0\right). \tag{10}$$
- Each function $\Psi_\varepsilon = \mathcal{L}^M \vartheta_\varepsilon$ has $2M$ vanishing moments; i.e., for every polynomial function P of homogeneous degree $\deg_{\mathbb{H}} P < 2M$,
$$\int_{\mathbb{H}} \Psi_\varepsilon(x) P(x)\, dx = 0. \tag{11}$$
Moreover, $|\Psi^\varepsilon(x)| \leqslant C_0 \exp\left(-\|x\|_{\mathbb{H}}/r_0\right)$ and $\Psi^\varepsilon \in H^\sigma(\mathbb{H})$ for $\sigma < 2M - Q$.
- The family of functions $(2^{jQ/2}\Psi^\varepsilon_{j,k})_{j\in Z, k\in\mathcal{Z}, 1\leqslant\varepsilon\leqslant 15}$, where
$$\Psi^\varepsilon_{j,k}(x) = \Psi_\varepsilon\left(2^j \circ (x_{j,k}^{-1} * x)\right),$$
forms a Hilbert basis of $L^2(\mathbb{H})$; i.e.,
$$f \overset{L^2}{=} \sum_{\varepsilon,j,k} d^\varepsilon_{j,k}(f)\Psi^\varepsilon_{j,k} \quad \text{with } d^\varepsilon_{j,k}(f) = 2^{jQ} \int_{\mathbb{H}} f(x)\Psi^\varepsilon_{j,k}(x)\, dx. \tag{12}$$

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The real numbers $d^{\varepsilon}_{j,k}(f)$ are called the *wavelet coefficients* of f. Note that we use an L^∞ normalization for the wavelet in (12) and that our choice implies that the family $(2^{-jQ/2}d^{\varepsilon}_{j,k}(f))$ belongs to ℓ^2 and thus tends to 0 when $j \to \pm\infty$ and $\|k\|_{\mathbb{H}} \to \infty$.

Remark 2. Instead of Lemarié's wavelet, one may use the wavelets built by Führ and Mayeli in [17] which can be constructed on any Carnot group. We will use these wavelets when explaining the generalization of our results to more general Carnot groups.

We can now state our main theorems. For non-integer regularity, Hölder classes can be totally described with wavelet coefficients, as in the Euclidean case.

Theorem 1. *For $s \in \mathbb{R}_+ \backslash \mathbb{N}$ and $[s] < 2M$, a function f belongs to $C^s(\mathbb{H})$ if and only if there exists a constant $C > 0$ such that*

$$\forall (\varepsilon, j, k) \in \{1, \dots, 2^Q - 1\} \times \mathbb{Z} \times \mathcal{Z}, \quad |d^{\varepsilon}_{j,k}(f)| \leqslant C2^{-js}. \tag{13}$$

Theorem 1 is essentially proved in [17, Theorems 5.4 and 6.1]. We give another proof here, using Lemarié's wavelets. The existence of the decomposition (12) allows us to obtain a straightforward equivalence between $C^s(\mathbb{H})$ and (13).

Up to a logarithmic factor, the pointwise regularity class $C^s(x_0)$ can also be described with wavelet coefficients.

Theorem 2. *Given $f \in L^2(\mathbb{H})$ and $x_0 \in \mathbb{H}$, the following properties hold.*

- *If $f \in C^s(x_0)$, then there is $R > 0$ such that, for any indices ε, j, k,*

$$\delta(x_{j,k}, x_0) < R \Longrightarrow |d^{\varepsilon}_{j,k}(f)| \leqslant C2^{-js}\left(1 + 2^j \delta(x_{j,k}, x_0)\right)^s. \tag{14}$$

- *Conversely, if f satisfies (14) and belongs to $C^\sigma(\mathbb{H})$ for an arbitrary small $\sigma > 0$, then f belongs to $C^s_{\log}(x_0)$.*

Remark 3. The important information contained in (14) does not just lie in the coefficients closest (at each dyadic scale) to x_0:

$$|d^{\varepsilon}_{j,k}(f)| \leqslant \begin{cases} C2^{-js} & \text{if } \delta(x_{j,k}, x_0) \lesssim 2^{-j}, \\ C\delta(x_{j,k}, x_0)^s & \text{if } 2^{-j} \lesssim \delta(x_{j,k}, x_0) < R. \end{cases}$$

Remark 4. In the Euclidean case, wavelet leaders [22] are more stable numerically. The wavelet leaders of a function $f \in L^2(\mathbb{H})$ is the sequence

$$D_j(f, x) = \sup\left\{|d^{\varepsilon}_{j',k'}(f)| : j' \geqslant j \text{ and } C_{j',k'} \subset \Lambda_j(x)\right\},$$

with $\Lambda_j(x)$ defined by (9). One checks easily that another statement equivalent to (14) is

$$f \in C^s(x_0) \Longrightarrow \forall j \geqslant 0, \quad D_j(f, x_0) \leqslant C2^{-js}.$$

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It is also obvious from the last two theorems that $f \in C^s(\mathbb{H})$ implies that $h_f(x) \geqslant s$ for every $x \in \mathbb{H}$. The optimality of this result is asserted by the following theorem. Recall that a property $\mathcal{P}$ is *generic* in a complete metric space E when it holds on a *residual* set i.e., a set with a complement of first Baire category. A set is of first Baire category if it is the union of countably many nowhere dense sets. As is often the case, it is enough to build a residual set which is a countable intersection of dense open sets in E.

Theorem 3. *There exists a dense open set (and hence a generic set) $\mathcal{R}$ of functions in $C^s(\mathbb{H})$ such that, for every $f \in \mathcal{R}$ and every $x \in \mathbb{H}$, $h_f(x) = s$.*

In particular, generic functions in $C^s(\mathbb{H})$ are monofractal; i.e., $E_f(h) = \emptyset$ if $h \neq s$.

One can also obtain a priori upper bounds for the multifractal spectrum of functions belonging to Besov and Sobolev spaces on $\mathbb{H}$ (see § 2.6 for precise definitions). These function spaces play a fundamental role in harmonic and functional analysis. There are two types of result one can naturally look for: general local regularity results that are valid for all functions in a given Besov space, and results that are only true for 'almost every' functions in this function space.

Theorem 4. *For $s > Q/p$, every $f \in B^s_{p,q}(\mathbb{H})$ satisfies*

$$\text{for all } h \geqslant s - Q/p, \quad d_f(h) \leqslant \min\left(Q,\, p\,(h - s + Q/p)\right) \tag{15}$$

and $d_f(h) = -\infty$ if $h < s - Q/p$.

The extremal points of Theorem 4 are illustrated in Figure 2. This theorem has many remarkable consequences. For instance, it illustrates the optimality of the Sobolev inclusion $B^s_{p,q}(\mathbb{H}) \hookrightarrow C^{s-Q/p}(\mathbb{H})$: the sets of points with the least possible pointwise Hölder exponent $s - Q/p$ has Hausdorff dimension at most 0. Similarly, as a consequence of the proof, the set of points whose pointwise Hölder exponent is at least s, and has full Haar measure in $\mathbb{H}$. The main difference with $C^s(\mathbb{H})$ is that functions in $B^s_{p,q}(\mathbb{H})$ may really be multifractal, meaning that many iso-Hölder sets $E_f(h)$ are non-empty with a non-trivial Hausdorff dimension. This is the case for generic functions in $B^s_{p,q}(\mathbb{H})$.

Theorem 5. *For $s > Q/p$, there is a residual set $\widetilde{\mathcal{R}} \subset B^s_{p,q}(\mathbb{H})$ such that, for all $f \in \widetilde{\mathcal{R}}$,*

$$\forall h \in [s - Q/p, s], \quad d_f(h) = p\,(h - s + Q/p)\,, \tag{16}$$

and $E_f(h) = \emptyset$ for all other exponents.

In particular, for generic functions $f \in B^s_{p,q}(\mathbb{H})$, almost every point with respect to the Haar measure has a pointwise Hölder exponent equal to s.

This paper is organized as follows. Section 2 contains the definitions and previous results that we use in what follows. In §§ 3 and 4, respectively, we deal with global and pointwise Hölder regularity (Theorems 1 and 2). In particular, the monofractality of generic functions (Theorem 3) in $C^s(\mathbb{H})$ is proved in § 4.3. The multifractal properties of functions in a Besov space are then investigated in §§ 5 and 6. Finally, we explain how to extend our results to general stratified nilpotent groups (i.e., Carnot groups) in § 7.

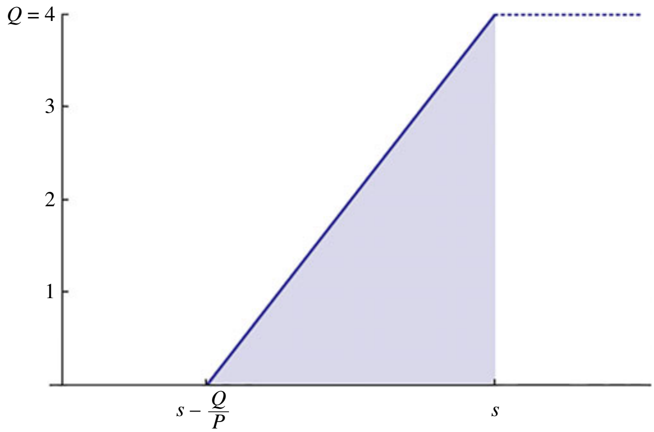

Figure 2. Upper bound for the multifractal spectrum of functions in $B^s_{p,q}(\mathbb{H})$.

Let us finish this introduction with a question. It would be very interesting to be able to represent the functions on $\mathbb{H}$, or at least the traces of such functions on affine subspaces of $\mathbb{R}^3$. Indeed, the natural anisotropy induced by the metric on $\mathbb{H}$ should create an anisotropic picture, and as of today, creating natural and simple models for anisotropic textures is a great challenge in image processing. Of course the starting point would be to understand how to draw a wavelet (be it a Lemarié wavelet, or another one!) on $\mathbb{H}$. We believe that this is a very promising research direction.

2. Definitions and classical results

2.1. Balls on $\mathbb{H}$

As the shape of balls is rather counterintuitive on $\mathbb{H}$, a few geometric statements will be useful in the following. The volume of the gage balls

$$\mathcal{B}(x, r) = \{y \in \mathbb{H} : \delta(x, y) < r\}$$

is denoted by $\ell(\mathcal{B}(x, r))$ and is equal to $\frac{\pi^2}{2} r^Q$ with $Q = 4$. In particular, the Haar measure ℓ has the doubling property: $l(B(x, 2r)) \leqslant Cl(B(x, r))$ for some universal constant C.

For any $x, x' \in \mathbb{H}$ and $r, r' > 0$, one has $x' * (r' \circ \mathcal{B}(x, r)) = \mathcal{B}(x' * (r' \circ x), rr')$ and $\mathcal{B}(x, r) = x * (r \circ \mathcal{B}(0, 1))$.

The triangular inequality holds in general with a constant depending on the metric.

Proposition 6 (Folland, Stein, Proposition 1.6). *There exists a constant $\gamma_1 > 0$ such that*

$$\forall x, y \in \mathbb{H}, \quad \|xy\|_{\mathbb{H}} \leqslant \gamma_1 \left(\|x\|_{\mathbb{H}} + \|y\|_{\mathbb{H}}\right). \tag{17}$$

In particular, the diameter of a gage ball $\mathcal{B}(x, r)$ does not exceed $2\gamma_1 r$. One will use this property later in the following form.

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Corollary 7. *There exists $C > 0$ such that, for any $\eta > 0$ and $x \in \mathcal{B}(0, \eta/C)$, one has*

$$\mathcal{B}(0, \eta/C) \subset \mathcal{B}(x, \eta).$$

2.2. Polynomial functions

A polynomial function P on $\mathbb{H}$ is a polynomial function of the coordinates (p, q, r); its homogeneous degree is defined by

$$\deg_{\mathbb{H}} P = \deg P(t, t, t^2),$$

where the right-hand side is computed in $\mathbb{R}[t]$. Given $x = (p, q, r)$ and $\alpha \in \{1, 2, 3\}^m$, one defines $x^\alpha = x(\alpha_1) \dots x(\alpha_m)$ with $x(1) = p$, $x(2) = q$, and $x(3) = r$. A polynomial function $\mathbb{H}$ of homogeneous degree at most N is thus a function of the form

$$P(x) = \sum_{|\alpha| \leqslant N} c_\alpha x^\alpha,$$

with $c_\alpha \in \mathbb{R}$ and $|\alpha| = \sum \omega(\alpha_i)$ with $\omega(1) = \omega(2) = 1$ and $\omega(3) = 2$.

2.3. The operator $\nabla_{\mathbb{H}}$

Let us denote by $\nabla_{\mathbb{H}} = (X, Y)$ the basis (equation (2)) of horizontal derivatives. Given a multi-index $\alpha \in \{1, 2, 3\}^m$, one will denote the horizontal derivative of order α by

$$\nabla_{\mathbb{H}}^\alpha f = V_{\alpha_1} \dots V_{\alpha_m} f \tag{18}$$

where $V_1 = X$, $V_2 = Y$ and $V_3 = Z$. As $Z = -\frac{1}{4}[X, Y]$, one may reduce $\nabla_{\mathbb{H}}^\alpha f$ to a linear combination of terms that contain exactly $|\alpha|$ powers of X and Y. One says that $\nabla_{\mathbb{H}}^\alpha f$ is a *horizontal derivative* of f of order $|\alpha|$.

2.4. Horizontal paths and Taylor formula

In this short section, we elaborate on Taylor expansions; see [16, 33] for details. We emphasize the notions needed later in this paper.

In this short section, we elaborate on Taylor expansions. Two points $x, y \in \mathbb{H}$ can always be joined by a subunitary horizontal path, i.e., a piecewise Lipschitz arc $\gamma : [0, L] \to \mathbb{H}$, such that, for almost every t, the tangent vector can be decomposed as

$$\gamma'(t) = \alpha(t) X(\gamma(t)) + \beta(t) Y(\gamma(t)),$$

with $\alpha^2(t) + \beta^2(t) \leqslant 1$. The so-called Carnot length $d_C(x, y) = \inf_\gamma (\int_0^L \alpha^2(t) + \beta^2(t)\, dt)^{1/2}$ is uniformly equivalent to $\delta(x, y)$. The shape of the d_C balls is illustrated by Figure 3. Integrating along such an arc provides the first-order Taylor formula:

$$f(y) = f(x) + \int_0^L \nabla_{\mathbb{H}} f(\gamma(t)) \gamma'(t)\, dt.$$

In turn, this identity provides a Lipschitz estimate.

Proposition 8 [16, Theorem 1.41]. *There exist $C > 0$ and $\gamma_2 > 0$ such that, for all $f \in C^1(\mathbb{H})$ and $x, y \in \mathbb{H}$,*

$$|f(y) - f(x)| \leqslant C \delta(x, y) \sup_{\|z\| \leqslant \gamma_2 \delta(x, y)} |\nabla_{\mathbb{H}} f(xz)|.$$

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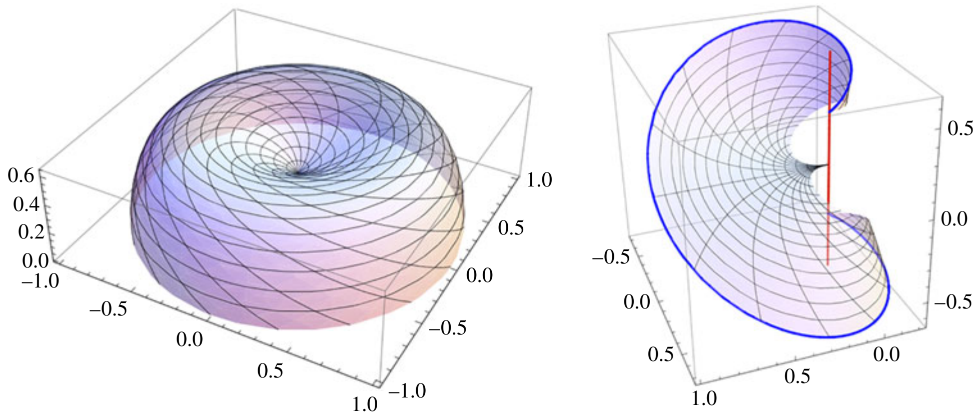

Figure 3. Left: upper half of the unit Carnot ball of $\mathbb{H}$. Right: one meridian arc of the unit ball and the unitary 'horizontal' geodesics joining the origin to each point of this meridian arc.

The left-invariant Taylor expansion of a function is given by the next definition.

Definition 3 [16]. The right Taylor polynomial of homogeneous degree k of a smooth function f at $x_0 \in \mathbb{H}$ is the unique polynomial P_{x_0} of homogeneous degree $\leqslant k$ such that

$$\forall \alpha \in \bigcup_{m\in\mathbb{N}} \{1,2,3\}^m, \quad |\alpha| \leqslant k \Longrightarrow \nabla^\alpha_{\mathbb{H}} f(x_0) = \nabla^\alpha_{\mathbb{H}} P_{x_0}(0).$$

To proceed with the subsequent calculations we will need to write down the Taylor expansion *explicitly*. It must be done carefully for various reasons. The most obvious one is that $XYf \neq YXf$ but $pq = qp$. The second problem induced by the anisotropy of the Heisenberg structure is that the traditional match between the index of the derivative and the index of the polynomial breaks down. For example, at the second order near the origin, $f(p,q,0)$ will be computed using only the first two powers of p and q but, contrary to the Euclidean setting, it will involve vertical derivatives at the origin, through $Zf(0)$.

With this in mind, a good way to write down the Taylor polynomial of order N is

$$P_{x_0}(y) = \sum_{k=0,\ldots,N} \sum_{|\alpha|=k} y^\alpha \left(\sum_{|\beta|=k} c_{\alpha,\beta} \nabla^\beta_{\mathbb{H}} f(x_0) \right) = \sum_{|\alpha|=|\beta|\leqslant N} c_{\alpha,\beta} \nabla^\beta_{\mathbb{H}} f(x_0) y^\alpha. \tag{19}$$

Beyond order 2, even though the polynomial P_{x_0} remains unique, the coefficients $c_{\alpha,\beta}$ in (19) are not, and thus a choice has to be done once and for all before starting a computation. For example, one possible writing of the polynomial of order 3 at the origin is

$$\begin{aligned} P_0(p,q,r) = {} & f(0) + pXf(0) + qYf(0) \\ & + \frac{1}{2}\left(p^2X^2f(0) + 2pqXYf(0) + q^2Y^2f(0)\right) + (2pq + r)\cdot Zf(0) \end{aligned}$$

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$$+\frac{1}{3!}\Big(p^3X^3f(0)+3p^2qX^2Yf(0)+3pq^2XY^2f(0)+q^3Y^3f(0)\Big)$$
$$+(2pq+r)\cdot(pXZf(0)+qYZf(0)).$$

The actual choice between the possible expressions is irrelevant. Since monomials are commutative and $Z=-\frac{1}{4}[X,Y]$, one can assume from now on that the formula is reduced to indices $\beta\in\bigcup_{m\in\mathbb{N}}\{1,2\}^m$. Further results and explicit Taylor formulas on homogenous groups can be found in [8].

As expected, the right Taylor polynomial approximates $f(x_0y)$ for y small enough.

Theorem 9 (Folland, Stein, Corollary 1.44). *If $f\in C^{k+1}(\mathbb{H})$, then the following estimate holds for some universal constant C_k:*

$$|f(x_0y)-P_{x_0}(y)|\leqslant C_k\,\|y\|_{\mathbb{H}}^{k+1}\sup_{|\alpha|=k+1}\left(\sup_{\|z\|\leqslant\gamma_2^{k+1}}|\nabla_{\mathbb{H}}^{\alpha}f(x_0z)|\right).$$

2.5. Hausdorff dimension on $\mathbb{H}$

The diameter of a set $E\subset\mathbb{H}$ will be denoted by

$$|E|=\sup\{\delta(x,y):x,y\in E\}.$$

Let us recall the definition of the Hausdorff measures and dimension. Let $s>0$ and $\eta>0$ be two positive real numbers. For any set $E\subset\mathbb{H}$, one defines

$$\mathcal{H}_\eta^s(E)=\inf_{\mathcal{R}}\sum_{B\in\mathcal{R}}|B|^s\in[0,+\infty],\tag{20}$$

where the infimum is taken over all possible coverings $\mathcal{R}$ of E by gage balls of radius less than η. Recall that a covering of E is a family $\mathcal{R}=\{B_i\}_{i\in I}$ of balls satisfying

$$E\subset\bigcup_{i\in I}B_i.$$

The mapping $\eta\mapsto\mathcal{H}_\eta^s(E)$ is decreasing with η, and hence one can define

$$\mathcal{H}^s(E)=\lim_{\eta\to0^+}\mathcal{H}_\eta^s(E)\in[0,+\infty].$$

From this definition, it is standard to see that $s\mapsto\mathcal{H}^s(E)$ is a decreasing function that jumps from infinity to zero at a unique real number called the Hausdorff dimension of E:

$$\dim_H E=\inf\{s:\mathcal{H}^s(E)=0\}=\sup\{s:\mathcal{H}^s(E)=+\infty\}.$$

2.6. Hölder, Sobolev, and Besov regularity

Definition 4. For $s=k+\sigma$ with $k\in\mathbb{N}$ and $\sigma\in]0,1[$, $C^s(\mathbb{H})$ is the set of functions such that, for any multi-index of length $|\alpha|\leqslant k$, the function $\nabla_{\mathbb{H}}^{\alpha}f$ is continuous, and

$$\sup_{|\alpha|=k}\frac{|\nabla_{\mathbb{H}}^{\alpha}f(x)-\nabla_{\mathbb{H}}^{\alpha}f(y)|}{\delta(x,y)^{\sigma}}<\infty.$$

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Hence, Hölder classes are defined as in the Euclidean case (see [10, 11]). Two equivalent Banach norms on $C^s(\mathbb{H})$, denoted by $\|f\|_{C^s(\mathbb{H})}$, are given by

$$\sup_{|\alpha|=[s]} \frac{|\nabla_{\mathbb{H}}^{\alpha} f(x) - \nabla_{\mathbb{H}}^{\alpha} f(y)|}{\delta(x,y)^{s-[s]}} \quad \text{and} \quad \sup_{\varepsilon,j,k} \left(2^{js} |d_{j,k}^{\varepsilon}(f)|\right).$$

Sobolev spaces, which we introduced before in (4), can also be described using the horizontal derivatives (18) (see [25] and references therein, and also [15] for more details on these functional spaces).

Proposition 10. *For $k \in \mathbb{N}$, a function f belongs to $H^k(\mathbb{H})$ if and only if, for any multi-index α of length $|\alpha| \leqslant k$, $\nabla_{\mathbb{H}}^{\alpha} f \in L^2(\mathbb{H})$.*

For $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in]0,1[$, one has $f \in H^s(\mathbb{H})$ if and only if $f \in H^k(\mathbb{H})$, and for $|\alpha| = k$ (with $Q = 4$, the homogeneous dimension of $\mathbb{H}$)

$$\iint_{\mathbb{H}\times\mathbb{H}} \frac{|\nabla_{\mathbb{H}}^{\alpha} f(x) - \nabla_{\mathbb{H}}^{\alpha} f(y)|^2}{\delta(x,y)^{Q+2\sigma}}\, dx\, dy < \infty. \tag{21}$$

One has the continuous inclusion $H^s(\mathbb{H}) \subset C^{s-Q/2}(\mathbb{H})$, which holds if $s > Q/2$ and $s - Q/2 \notin \mathbb{N}$. Namely, for any multi-index α such that $s = Q/2 + |\alpha| + \sigma$ and $\sigma \in]0,1[$,

$$|\nabla_{\mathbb{H}}^{\alpha} f(x) - \nabla_{\mathbb{H}}^{\alpha} f(y)| \leqslant C_s \|f\|_{H^s(\mathbb{H})}\, \delta(x,y)^{s-Q/2}.$$

As in the Euclidean case, this inclusion fails if $s - Q/2 \in \mathbb{N}$, because one can then find $f \in H^s(\mathbb{H})$ and $|\alpha| = s - Q/2$ such that $\nabla_{\mathbb{H}}^{\alpha} f$ is not a continuous function (note that $Q = 4$ is even). However, the corresponding inclusion in BMO holds (see [11]).

On $\mathbb{R}^n$, Besov spaces can be defined in various ways, and the equivalence between all those definitions is part of folklore. For nilpotent Lie groups, the situation is less straightforward, and all the equivalences should be checked carefully. For multifractal analysis, the most convenient definition of Besov spaces is the following.

Definition 5. The Besov space $B_{p,q}^s(\mathbb{H})$ of [28] consists of functions f on $\mathbb{H}$ such that

$$a_j = \left\| 2^{j(s-Q/p)} d_{j,k}^{\varepsilon}(f) \right\|_{\ell^p(k)} \in \ell^q(j). \tag{22}$$

Other definitions of Besov spaces involve Littlewood–Paley theory (see [30] for a historical background) and continuous wavelet decomposition (see [17] for instance), but all definitions coincide. The interested reader can have a look at [18, § 5]. Depending on the applications, other definitions have proved useful. Trace theory [13, 34] is better understood with the geometric norm (21), and real interpolation norms with operator theory [7, 28], while complex interpolation norms relate to microlocal analysis and Weyl calculus on $\mathbb{H}$ [9, 10, 25].

3. Global Hölder regularity with wavelet coefficients: Theorem 1

We first prove in § 3.1 that the wavelet coefficients of every function $f \in C^s(\mathbb{H})$ enjoy the decay property (13). The converse property is shown in § 3.2.

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3.1. Upper bound for the wavelet coefficients

Assume that $f \in C^s(\mathbb{H})$. Let $s = [s] + \sigma$ with $[s] \in \mathbb{N}$ and $0 < \sigma < 1$. The change of variable $y = 2^j \circ (x_{j,k}^{-1} x)$ yields

$$d_{j,k}^{\varepsilon} = \int_{\mathbb{H}} f(x_{j,k}(2^{-j} \circ y))\Psi_{\varepsilon}(y)\,dy.$$

When $[s] = 0$, one infers from the vanishing moment of Ψ_ε (i.e., $\int_{\mathbb{H}} \Psi_\varepsilon(x)\,dx = 0$) that

$$\begin{aligned} |d_{j,k}^{\varepsilon}| &= \left| \int_{\mathbb{H}} (f(x_{j,k}(2^{-j} \circ y)) - f(x_{j,k}))\Psi_{\varepsilon}(y)\,dy \right| \\ &\leqslant \int_{\mathbb{H}} |f(x_{j,k}(2^{-j} \circ y)) - f(x_{j,k})||\Psi_{\varepsilon}(y)|\,dy \\ &\leqslant 2^{-js}\, \|f\|_{C^s(\mathbb{H})} \int_{\mathbb{H}} \|y\|_{\mathbb{H}}^s\, |\Psi_\varepsilon(y)|\,dy = C2^{-js}. \end{aligned}$$

When $[s] \geqslant 1$, one uses $\Psi^{\varepsilon} = \mathcal{L}^M \vartheta_\varepsilon$, and proceeds with $[s]$ integrations by part against the function $g_{j,k}(y) = f(x_{j,k}(2^{-j} \circ y))$.

Observe that the homogeneity of the horizontal derivatives yields that, for every α,

$$\nabla_{\mathbb{H}}^{\alpha} g_{j,k}(y) = 2^{-j|\alpha|} \times [\nabla_{\mathbb{H}}^{\alpha} f]((x_{j,k} * (2^{-j} \circ y)). \tag{23}$$

When $[s] = 2m$ is even ($m \geqslant 1$), one writes $d_{j,k}^{\varepsilon} = \int_{\mathbb{H}} (\mathcal{L}^m g_{j,k})(x) \widetilde{\vartheta}_\varepsilon(x)\,dx$, with $\widetilde{\vartheta}_\varepsilon = \mathcal{L}^{M-m}\vartheta_\varepsilon$. The term $\mathcal{L}^m g_{j,k} = (-X^2 - Y^2)^m (g_{j,k})$ can be developed, and one gets

$$\mathcal{L}^m g_{j,k} = \sum_{|\alpha| = [s]} l_\alpha \nabla_{\mathbb{H}}^{\alpha}(g_{j,k}),$$

for some coefficients l_α independent of the problem. Recalling (10), $\widetilde{\vartheta}_\varepsilon$ is a well localized function. One can use the vanishing moments of ϑ_ε (and thus of $\widetilde{\vartheta}_\varepsilon$) to get

$$\begin{aligned} d_{j,k}^{\varepsilon} &= \sum_{|\alpha|=[s]} l_\alpha \int_{\mathbb{H}} \left(\nabla_{\mathbb{H}}^{\alpha} g_{j,k}(y) - \nabla_{\mathbb{H}}^{\alpha} g_{j,k}(0)\right) \widetilde{\vartheta}_\varepsilon(y)\,dy, \\ &= 2^{-j[s]} \sum_{|\alpha|=[s]} l_\alpha \int_{\mathbb{H}} \left(\nabla_{\mathbb{H}}^{\alpha} f((x_{j,k} * (2^{-j} \circ y)) - \nabla_{\mathbb{H}}^{\alpha} f(x_{j,k})\right) \widetilde{\vartheta}_\varepsilon(y)\,dy. \end{aligned}$$

The assumption that $f \in C^s(\mathbb{H})$ implies that $\nabla_{\mathbb{H}}^{\alpha} f \in C^{\sigma}(\mathbb{H})$, thus ultimately providing

$$|d_{j,k}^{\varepsilon}| \leqslant 2^{-j([s]+\sigma)}\, \|f\|_{C^s(\mathbb{H})} \sum_{|\alpha|=[s]} |l_\alpha| \int_{\mathbb{H}} \|y\|_{\mathbb{H}}^{\sigma}\, |\widetilde{\vartheta}_\varepsilon(y)|\,dy = C2^{-js}.$$

When $[s] = 2m - 1$ is odd ($m \geqslant 1$), one has

$$d_{j,k}^{\varepsilon} = \int_{\mathbb{H}} (X\mathcal{L}^{m-1} g_{j,k})(X\widetilde{\vartheta}_\varepsilon) + \int_{\mathbb{H}} (Y\mathcal{L}^{m-1} g_{j,k})(Y\widetilde{\vartheta}_\varepsilon).$$

Again, $X\widetilde{\vartheta}_\varepsilon$ and $Y\widetilde{\vartheta}_\varepsilon$ are well localized functions, with at least one vanishing moment. Using the same arguments as above,

$$d_{j,k}^{\varepsilon} = \sum_{|\alpha|=[s]} l'_\alpha \int_{\mathbb{H}} \left(\nabla_{\mathbb{H}}^{\alpha} g_{j,k}(y) - \nabla_{\mathbb{H}}^{\alpha} g_{j,k}(0)\right) \widetilde{\Psi}_{\varepsilon,\alpha}(y)\,dy,$$

with $\widetilde{\Psi}_{\varepsilon,\alpha} = X\widetilde{\vartheta}_\varepsilon$ or $Y\widetilde{\vartheta}_\varepsilon$ (depending on whether the first slot in α codes for X or Y) and some other coefficients l'_α. The rest of the proof is the same, giving finally $|d_{j,k}^{\varepsilon}| \leqslant C2^{-js}$.

3.2. Hölder estimate derived from wavelet coefficients

Let us now focus on the converse assertion in Theorem 1, and assume that (13) holds. The normal convergence of the series (12) up to the $[s]^{\text{th}}$ derivatives ensures that, for any multi-index α such that $|\alpha| \leqslant [s]$, the function $\nabla^{\alpha}_{\mathbb{H}} f$ is continuous, and that the following identity holds:

$$\nabla^{\alpha}_{\mathbb{H}} f(x) = \sum_{\varepsilon, j,k} 2^{j|\alpha|} d^{\varepsilon}_{j,k}(f) \cdot (\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}) \left(2^j \circ (x^{-1}_{j,k} x)\right).$$

Let us estimate the $[s]$th derivatives. As before, $s = [s] + \sigma$, and for $|\alpha| = [s]$ one gets

$$|\nabla^{\alpha}_{\mathbb{H}} f(x) - \nabla^{\alpha}_{\mathbb{H}} f(y)| \leqslant \sum_{\varepsilon, j,k} 2^{-j\sigma} \left|(\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}) \left(2^j \circ (x^{-1}_{j,k} x)\right) - (\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}) \left(2^j \circ (x^{-1}_{j,k} y)\right)\right|.$$

Let $j_0 \in \mathbb{Z}$ such that $2^{-j_0-1} \leqslant \delta(x, y) < 2^{-j_0}$. There exists $\widetilde{k} = (k_1, k_2, k_3) \in \mathcal{Z}$ such that x and y both belong to the dyadic neighborhood of $x_{j_0,\widetilde{k}}$, namely $x, y \in \Lambda_{j_0,\widetilde{k}}$, where $\Lambda_{j_0,\widetilde{k}}$ has been defined by (8) and the remarks that follow.

For $j \leqslant j_0$, one uses that $\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}$ is Lipschitz:

$$\left|(\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}) \left(2^j \circ (x^{-1}_{j,k} x)\right) - (\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}) \left(2^j \circ (x^{-1}_{j,k} y)\right)\right| \leqslant C 2^j \delta(x, y) \times \xi_{j,k},$$

where $\xi_{j,k}$ satisfies the following estimate (Proposition 8):

$$\begin{aligned} \xi_{j,k} &= \sup_{\|z\|_{\mathbb{H}} \leqslant \gamma_2 2^{j-j_0}} |\nabla_{\mathbb{H}} \nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}((2^j \circ (x^{-1}_{j,k} x_{j_0,\widetilde{k}})) * z)| \\ &\leqslant \sup_{z \in k^{-1} * (2^{j-j_0} \circ \mathcal{B}(\widetilde{k}, \gamma_2))} (C_0 \exp(-\|z\|_{\mathbb{H}}/r_0)) \leqslant C'_0 \exp\left(-\delta(k, 2^{j-j_0} \circ \widetilde{k})/r'_0\right). \end{aligned}$$

For $j > j_0$, one uses simply the boundedness of $\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}$:

$$\begin{aligned} &\left|(\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}) \left(2^j \circ (x^{-1}_{j,k} x)\right) - (\nabla^{\alpha}_{\mathbb{H}} \Psi_{\varepsilon}) \left(2^j \circ (x^{-1}_{j,k} y)\right)\right| \\ &\qquad \leqslant C_0 \left\{\exp\left(-\delta(k, 2^j \circ x)/r_0\right) + \exp\left(-\delta(k, 2^j \circ y)/r_0\right)\right\}. \end{aligned}$$

Each right-hand side is obviously summable in the variable $k \in \mathcal{Z}$ because there exists a constant C such that

$$\forall z \in \mathbb{H}, \quad \sum_{k \in \mathcal{Z}} \exp(-\delta(k, z)/r_0) \leqslant C.$$

Combining the previous estimates and the fact that the ε variable belongs to a finite set $\{1, \ldots, 2^Q - 1\}$, one gets the following upper bound for $|\nabla^{\alpha}_{\mathbb{H}} f(x) - \nabla^{\alpha}_{\mathbb{H}} f(y)|$:

$$\sum_{j \leqslant j_0} 2^{j(1-\sigma)} \delta(x, y) + \sum_{j > j_0} 2^{-j\sigma} \leqslant C(2^{j_0(1-\sigma)} \delta(x, y) + 2^{-j_0\sigma}) \leqslant C' \delta(x, y)^{\sigma};$$

i.e., $f \in C^s(\mathbb{H})$.

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4. Pointwise Hölder regularity

The wavelet decay property 14 of functions f belonging to $C^s(x_0)$ is obtained in § 4.1. The second part of Theorem 2 is more delicate and is explained in § 4.2. Finally, § 4.3 contains the proof of Theorem 3, which gives the existence of a generic set of functions in C^s with a pointwise Hölder exponent everywhere equal to s. This proves in turn the optimality of Theorem 1.

4.1. Upper bound for the wavelet coefficients

Assume first that $f \in C^s(x_0)$. Let P be the unique polynomial of degree $\deg_{\mathbb{H}}(P) < s$ and $\eta > 0$ such that (5) holds on a small neighborhood of the origin $\mathcal{B}(0,\eta)$. Using the vanishing moments of Ψ^ε, one has

$$d_{j,k}^\varepsilon(f) = 2^{jQ} \int_{\mathbb{H}} (f(x_0 x) - P(x)) \Psi_{j,k}^\varepsilon(x_0 x)\, dx,$$

and thus

$$\begin{aligned} |d_{j,k}^\varepsilon(f)| \leqslant\ & C 2^{jQ} \int_{\mathcal{B}(0,\eta)} \|x\|_{\mathbb{H}}^s \left|\Psi_{j,k}^\varepsilon(x_0 x)\right| dx + 2^{jQ} \int_{\mathbb{H}\setminus\mathcal{B}(x_0,\eta)} |f(x)| \left|\Psi_{j,k}^\varepsilon(x)\right| dx \\ & + 2^{jQ} \int_{\mathbb{H}\setminus\mathcal{B}(x_0,\eta)} |P(x_0^{-1}x)| \left|\Psi_{j,k}^\varepsilon(x)\right| dx. \end{aligned}$$

We denote the three integrals above by, respectively, I_1, I_2, and I_3. In the following, we assume (as in the statement of Theorem 2) that

$$\delta(x_{j,k}, x_0) < R \tag{24}$$

for some constant $R \leqslant 1$ that we will adjust on the way. Observe that R will not depend on j or k.

The change of variable $y = 2^j \circ (x_{j,k}^{-1} x_0 x)$, Hölder's inequality $\|ab\|_{\mathbb{H}}^s \leqslant C_s(\|a\|_{\mathbb{H}}^s + \|b\|_{\mathbb{H}}^s)$, and the exponential decay (10) of the mother wavelet yield

$$\begin{aligned} I_1 &= C \int_{\mathcal{B}(2^j(x_{j,k}^{-1}x_0), 2^j\eta)} \|x_0^{-1} x_{j,k} (2^{-j}\circ x)\|_{\mathbb{H}}^s\, |\Psi_\varepsilon(x)|\, dx \\ &\leqslant C \int_{\mathbb{H}} C_s \left(\delta(x_0, x_{j,k})^s + \|2^{-j}\circ x\|_{\mathbb{H}}^s\right) \times C_0 \exp\left(-\|x\|_{\mathbb{H}}/r_0\right) dx \\ &\leqslant C \left(\delta(x_0, x_{j,k})^s + 2^{-js}\right) \end{aligned}$$

for some other constant C, independent of x_0, j, and k. The decay property (10) of the wavelet was used to obtain the second line.

For the second integral, one uses the Cauchy–Schwarz inequality:

$$I_2 = \int_{\mathbb{H}\setminus\mathcal{B}(x_0,\eta)} |f(x)| \times 2^{jQ} \left|\Psi_{j,k}^\varepsilon(x)\right| dx \leqslant \|f\|_{L^2(\mathbb{H})} \left(\int_{\mathbb{H}\setminus\mathcal{B}(x_0,\eta)} 2^{2jQ} \left|\Psi_{j,k}^\varepsilon(x)\right|^2 dx\right)^{1/2}.$$

The weight of the tail of the wavelet depends on how $\delta(x_{j,k}, x_0)$ compares to η. Let us assume that $R = \eta/C$ in (24), with the constant C given by Corollary 7; thus

$$\mathcal{B}(0, 2^j\eta/C) \subset \mathcal{B}(2^j \circ (x_{j,k}^{-1}x_0), 2^j\eta).$$

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Observe that $\|y\|_{\mathbb{H}} \geqslant 2^j \eta/C$ implies that $1 \leqslant 2^{-j}\|y\|_{\mathbb{H}} C/\eta$ and $\left\|x_0^{-1}x_{j,k}\right\|_{\mathbb{H}} \leqslant R = \eta/C \leqslant 2^{-j}\|y\|_{\mathbb{H}}$. Then, the usual change of variable $y = 2^j \circ (x_{j,k}^{-1}x)$ yields

$$\begin{aligned}
\int_{\mathbb{H}\setminus\mathcal{B}(x_0,\eta)} 2^{2jQ}\left|\Psi^{\varepsilon}_{j,k}(x)\right|^2 dx &= 2^{jQ}\int_{\mathbb{H}\setminus\mathcal{B}(2^j\circ(x_{j,k}^{-1}x_0),2^j\eta)} |\Psi_{\varepsilon}(y)|^2\,dy\\
&\leqslant 2^{jQ}\int_{\{y\in\mathbb{H}:\|y\|_{\mathbb{H}}\geqslant 2^j\eta/C\}}\left(\frac{C\,\|y\|}{2^j\eta}\right)^{Q+2s}|\Psi_{\varepsilon}(y)|^2\,dy\\
&\leqslant C2^{-2js}.
\end{aligned}$$

The last inequality uses the decay property (10) of Ψ^{ε}. Finally, one gets $I_2 \leqslant C2^{-js}$.

For I_3, the idea is similar, except that one has to compensate for $P \notin L^2(\mathbb{H})$ by adding an extra weight. For example, one chooses an integer $N > s$ such that $(1+\|x\|_{\mathbb{H}})^{-N}P(x)$ is bounded. Then, as previously, one has

$$\begin{aligned}
I_3 &\leqslant 2^{jQ}\left\|(1+\|\cdot\|_{\mathbb{H}})^{-N}P\right\|_{L^\infty(\mathbb{H})} \times \int_{\mathbb{H}\setminus\mathcal{B}(x_0,\eta)}(1+\|x_0^{-1}x\|_{\mathbb{H}})^N\left|\Psi^{\varepsilon}_{j,k}(x)\right| dx\\
&\leqslant C\int_{\|y\|_{\mathbb{H}}\geqslant 2^j\eta/C}\left(1+\|x_0^{-1}x_{j,k}\|+2^{-j}\|y\|_{\mathbb{H}}\right)^N\left|\Psi^{\varepsilon}(y)\right| dy.
\end{aligned}$$

Hence, using the same arguments as those developed for I_2, the estimates boil down to

$$I_3 \leqslant C\int_{\|y\|_{\mathbb{H}}\geqslant 2^j\eta/C}\left(2^{-j}\|y\|_{\mathbb{H}}\right)^N\left|\Psi^{\varepsilon}(y)\right| dy \leqslant C2^{-jN}\int_{\mathbb{H}}\|y\|_{\mathbb{H}}^N\exp\left(-\frac{\|y\|_{\mathbb{H}}}{r_0}\right) dy.$$

Finally, one gets $I_3 \leqslant C2^{-jN} \leqslant C2^{-js}$.

Putting together the estimates for I_1, I_2 and I_3, one gets

$$|d^{\varepsilon}_{j,k}(f)| \leqslant C\left(2^{-js}+\delta(x_{j,k},x_0)^s\right)$$

when $\delta(x_{j,k},x_0) < \eta/C = R$, which is equivalent to (14). This concludes the proof of the first half of Theorem 2. □

4.2. Pointwise Hölder estimate derived from wavelet coefficients

Let us move to the second part of Theorem 2 and prove the converse property. One assumes that $f \in C^{\sigma}(\mathbb{H})$ for some $\sigma > 0$, and that (14) holds for all triplets (ε, j, k) such that $\delta(x_{j,k}, x_0) \leqslant R$ for some $R > 0$. Let us fix x such that $\delta(x, x_0) \leqslant R$, and let j_0 and j_1 be the unique integers such that

$$2^{-j_0-1} \leqslant \delta(x,x_0) < 2^{-j_0} \quad\text{and}\quad j_1 = \left[\frac{s}{\sigma}\cdot j_0\right]. \tag{25}$$

We aim at proving (6) for x close enough to x_0; i.e., for j_0 large enough.

The wavelet decomposition of f is $f = \sum_{j\in\mathbb{Z}} f_j(x)$, where, for every $j \in \mathbb{Z}$,

$$f_j(x) = \sum_{\varepsilon\in\{1,\ldots,15\}}\sum_{k\in\mathcal{Z}} d^{\varepsilon}_{j,k}(f)\Psi^{\varepsilon}_{j,k}(x).$$

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For subsequent use, let us notice immediately that the low-frequency term

$$f^\flat(x) = \sum_{j=-\infty}^{0} f_j(x)$$

is as regular as the wavelet itself. In particular, at least $C^{[s]+2}(\mathbb{H})$.

Assumption (14) reads

$$|d^\varepsilon_{j,k}(f)| \leqslant C(2^{-js} + \|x_{j,k}^{-1}x_0\|_{\mathbb{H}}^s) \leqslant C(2^{-js} + \|x_{j,k}^{-1}x\|_{\mathbb{H}}^s + \|x^{-1}x_0\|_{\mathbb{H}}^s). \tag{26}$$

For every $n \in \{0, \dots, [s]+1\}$ and any multi-index α with $|\alpha| = n$, one has

$$\nabla^\alpha_{\mathbb{H}} f_j(x) = \sum_\varepsilon \sum_{k\in\mathcal{Z}} d^\varepsilon_{j,k}(f)\cdot(\nabla^\alpha_{\mathbb{H}}\Psi^\varepsilon_{j,k})(x). \tag{27}$$

As $\Psi^\varepsilon_{j,k}(x) = \Psi_\varepsilon(2^j \circ (x_{j,k}^{-1}x))$, and using (10), a computation similar to (23) gives

$$|\nabla^\alpha_{\mathbb{H}}\Psi^\varepsilon_{j,k}(x)| \leqslant C2^{j|\alpha|}\exp\left(-\|2^j\circ(x_{j,k}^{-1}x)\|_{\mathbb{H}}/r_0\right) \leqslant \frac{C2^{j|\alpha|}}{(1+\|2^j\circ(x_{j,k}^{-1}x)\|_{\mathbb{H}})^{Q+1+s}}. \tag{28}$$

Next, let us notice that, for a constant that does not depends on $j \in \mathbb{Z}$ or $x \in \mathbb{H}$,

$$\forall\gamma > 0, \quad \exists C > 0, \quad \sum_{k\in\mathcal{Z}} \frac{\|x_{j,k}^{-1}x\|_{\mathbb{H}}^\gamma}{(1+\|2^j\circ(x_{j,k}^{-1}x)\|_{\mathbb{H}})^{Q+1+\gamma}} \leqslant C2^{-j\gamma}. \tag{29}$$

Combining (28), (26) and (29) provides, in a neighborhood of x_0,

$$|\nabla^\alpha_{\mathbb{H}} f_j(x)| \leqslant C2^{j|\alpha|}\sum_{k\in\mathcal{Z}} \frac{(2^{-js} + \|x_{j,k}^{-1}x\|_{\mathbb{H}}^s + \|x^{-1}x_0\|_{\mathbb{H}}^s)}{(1+\|2^j\circ(x_{j,k}^{-1}x)\|_{\mathbb{H}})^{Q+1+s}} \leqslant C2^{j|\alpha|}\left(2^{-js} + \|x^{-1}x_0\|_{\mathbb{H}}^s\right). \tag{30}$$

In particular, $|\nabla^\alpha f_j(x_0)| \leqslant C2^{j(|\alpha|-s)}$, and the series (used subsequently) $\sum_{j=0}^\infty \nabla^\alpha_{\mathbb{H}} f_j(x_0)$ converges absolutely for every α such that $|\alpha| \leqslant [s]$.

Let us now introduce the (right) Taylor polynomial P_j of f_j at x_0. According to (19), it can be written as

$$P_j(y) = \sum_{|\alpha|=|\beta|\leqslant[s]} c_{\alpha,\beta}\nabla^\beta_{\mathbb{H}} f_j(x_0) y^\alpha. \tag{31}$$

The coefficients $c_{\alpha,\beta}$ are chosen once and for all for the rest of this computation. Let $P^\flat(y)$ stand for the (right) Taylor polynomial of the low-frequency part $f^\flat$ at x_0.

The polynomial P that we are going to use to prove (6) is defined by

$$P(y) = P^\flat(y) + \sum_{j=0}^{\infty} P_j(y).$$

The previous estimates ensure that P is indeed well defined and of degree at most $[s]$. One gets the following decomposition:

$$|f(x) - P(x_0^{-1}x)| \leqslant |f^\flat(x) - P^\flat(x_0^{-1}x)| + \sum_{j=0}^{j_0}\left|f_j(x) - P_j(x_0^{-1}x)\right| + R_1(x) + R_2(x) \tag{32}$$

with two remainders:

$$R_1(x) = \sum_{j=j_0}^{\infty} |f_j(x)| \quad \text{and} \quad R_2(x) = \sum_{j=j_0}^{\infty} \left| P_j(x_0^{-1}x) \right|.$$

The low frequency is instantaneously dealt with by Theorem 9:

$$|f^\flat(x) - P^\flat(x_0^{-1}x)| \leqslant C \left\| x_0^{-1}x \right\|_{\mathbb{H}}^{[s]+1}. \tag{33}$$

Let us now focus on the three other terms. One uses the Taylor development of the wavelet at x_0 and the unicity of the Taylor expansion to recover the polynomial P_j. Let us thus write

$$\Psi^\varepsilon_{j,k}(x) = \sum_{|\alpha|=|\beta|\leqslant[s]} c_{\alpha,\beta} \nabla^\beta_{\mathbb{H}} \Psi^\varepsilon_{j,k}(x_0)(x_0^{-1}x)^\alpha + R^\varepsilon_{j,k}(x).$$

Theorem 9 ensures that, for some constant $r_1 > 0$ and x in the neighborhood of x_0,

$$|R^\varepsilon_{j,k}(x)| \leqslant C \left\| x_0^{-1}x \right\|_{\mathbb{H}}^{[s]+1} \sup_{|\alpha|=[s]+1, \|z\|\leqslant r_1} \left| \nabla^\alpha_{\mathbb{H}} \Psi^\varepsilon_{j,k}(x_0 z) \right|. \tag{34}$$

Substitution in the definition of f_j gives

$$f_j(x) = \sum_{\varepsilon,k} d^\varepsilon_{j,k}(f) \left(\sum_{|\alpha|=|\beta|\leqslant[s]} c_{\alpha,\beta} \nabla^\beta_{\mathbb{H}} \Psi^\varepsilon_{j,k}(x_0)(x_0^{-1}x)^\alpha + R^\varepsilon_{j,k}(x) \right).$$

In the first double sum, by combining (27) and (31), one recognizes $P_j(x_0^{-1}x)$, and thus

$$f_j(x) - P_j(x_0^{-1}x) = \sum_{\varepsilon,k} d^\varepsilon_{j,k}(f) R^\varepsilon_{j,k}(x).$$

Combining (26), (34), and definition (25) of j_0 gives

$$\begin{aligned} \left| f_j(x) - P_j(x_0^{-1}x) \right| &\leqslant \sum_{\varepsilon,k} |d^\varepsilon_{j,k}(f)| |R^\varepsilon_{j,k}(x)| \\ &\leqslant C 2^{-j_0([s]+1)} \sum_{\varepsilon,k} (2^{-js} + 2^{-j_0 s} + \|x_{j,k}^{-1}x\|^s_{\mathbb{H}}) \sup_{\substack{|\alpha|=[s]+1 \\ \|z\|\leqslant r_1}} \left| \nabla^\alpha_{\mathbb{H}} \Psi^\varepsilon_{j,k}(x_0 z) \right|. \end{aligned}$$

To deal with the summation in k, one uses (28) and (29): for all $\varepsilon = 1, \ldots, 2^Q - 1$,

$$\sum_k \sup_{\|z\|\leqslant r_1} \left| \nabla^\alpha_{\mathbb{H}} \Psi^\varepsilon_{j,k}(x_0 z) \right| \leqslant C \sum_k \frac{2^{j|\alpha|}}{(1 + \|2^j \circ (x_{j,k}^{-1}x_0)\|_{\mathbb{H}})^{Q+1+s}} \leqslant C 2^{j|\alpha|},$$

and similarly

$$\sum_k \|x_{j,k}^{-1}x\|^s_{\mathbb{H}} \sup_{\|z\|\leqslant r_1} \left| \nabla^\alpha_{\mathbb{H}} \Psi^\varepsilon_{j,k}(x_0 z) \right| \leqslant C \sum_k \frac{2^{j|\alpha|} \|x_{j,k}^{-1}x_0\|^s_{\mathbb{H}}}{(1 + \|2^j \circ (x_{j,k}^{-1}x_0)\|_{\mathbb{H}})^{Q+1+s}} \leqslant C 2^{j(|\alpha|-s)}.$$

The summation in ε plays no role. Putting it all together, one gets

$$\left| f_j(x) - P_j(x_0^{-1}x) \right| \leqslant C 2^{-(j_0-j)([s]+1)} \left(2^{-js} + 2^{-j_0 s} \right).$$

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Finally, the sum over $j \in \{0, \dots, j_0\}$ boils down to

$$\sum_{j=0}^{j_0} \left| f_j(x) - P_j(x_0^{-1}x) \right| \leqslant C 2^{-j_0([s]+1)} \sum_{j=0}^{j_0} 2^{j([s]+1-s)} + 2^{-j_0([s]+1+s)} \sum_{j=0}^{j_0} 2^{j([s]+1)}$$
$$\leqslant C 2^{-j_0 s} \leqslant C \left\| x_0^{-1} x \right\|_{\mathbb{H}}^s. \tag{35}$$

The term R_1 contains the high-frequency components of the Littlewood–Paley decomposition of f, and is responsible for the logarithmic correction in (6). By (25) and (30),

$$\forall j \geqslant j_0, \quad |f_j(x)| \leqslant C(2^{-js} + \|x^{-1}x_0\|_{\mathbb{H}}^s) \leqslant C(2^{-js} + 2^{-j_0 s}) \leqslant C\|x^{-1}x_0\|_{\mathbb{H}}^s.$$

Let us split this remainder depending on whether $j_0 \leqslant j < j_1$ or $j \geqslant j_1$:

$$R_1(x) \leqslant \sum_{j=j_0}^{j_1} |f_j(x)| + \sum_{j=j_1}^{\infty} |f_j(x)|.$$

Our choice (25) for j_1 and j_0 gives $j_1 \sim s j_0/\sigma \sim s/\sigma \cdot |\log \|x^{-1}x_0\|_{\mathbb{H}}^s|$. Hence

$$\sum_{j=j_0}^{j_1} |f_j(x)| \leqslant j_1 \cdot C\|x^{-1}x_0\|_{\mathbb{H}}^s \leqslant C\|x^{-1}x_0\|_{\mathbb{H}}^s \cdot |\log \|x^{-1}x_0\|_{\mathbb{H}}^s|. \tag{36}$$

When $j \geqslant j_1$, one uses $f \in C^\sigma(\mathbb{H})$ instead and Theorem 1, which gives $|d_{j,k}^\varepsilon(f)| \leqslant C 2^{-j\sigma}$. Combining with (10), one deduces that

$$|f_j(x)| \leqslant \sum_{\varepsilon} \sum_{k \in \mathcal{Z}} C 2^{-j\sigma} e^{-\delta(x_{j,k},x)/r_0} \leqslant C 2^{-j\sigma}.$$

Using (25) one last time yields to the expected conclusion:

$$\sum_{j=j_1}^{\infty} |f_j(x)| \leqslant C \sum_{j=j_1}^{\infty} 2^{-j\sigma} \leqslant C 2^{-j_1\sigma} \leqslant C \left\| x_0^{-1} x \right\|_{\mathbb{H}}^s. \tag{37}$$

Let us move to R_2, which contains the Taylor expansions of the high-frequency components of the Littlewood–Paley decomposition of f. Intuitively, it is small because of the natural spectral separation between polynomials and highly oscillatory functions.

Using (25) and (30), each term of the sum boils down to

$$\left| P_j(x_0^{-1}x) \right| \leqslant \sum_{|\alpha|=|\beta| \leqslant [s]} |c_{\alpha,\beta}| |\nabla_{\mathbb{H}}^\beta f_j(x_0)| \| (x_0^{-1}x)^\alpha \|_{\mathbb{H}} \leqslant C \sum_{n=0}^{[s]} 2^{j(n-s)-j_0 n},$$

and thus

$$\sum_{j=j_0}^{\infty} \left| P_j(x_0^{-1}x) \right| \leqslant C \sum_{n=0}^{[s]} 2^{-j_0 n} \left(\sum_{j \geqslant j_0} 2^{-j(s-n)} \right) \leqslant C 2^{-j_0 s} \leqslant C \left\| x_0^{-1} x \right\|_{\mathbb{H}}^s. \tag{38}$$

Substituting (33), (35), (36), (37), and (38) back in the original question (32) proves that (6) holds in a neighborhood of x_0, and concludes the proof of Theorem 2. □

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4.3. Generic monofractality of functions in $C^s(\mathbb{H})$

The proof of Theorem 3 is classical in the Euclidean context [21] and can be adapted quickly to ours.

Let us recall that, for any $f \in C^s(\mathbb{H})$, Theorem 1 gives a constant $C > 0$ such that

$$f = \sum_{\varepsilon,j,k} d^{\varepsilon}_{j,k}(f)\Psi^{\varepsilon}_{j,k} \quad \text{with } |d^{\varepsilon}_{j,k}(f)| \leqslant C2^{-js}, \tag{39}$$

and $\|f\|_{C^s} = \inf\{C > 0 : (39) \text{ is satisfied for all } \varepsilon, j, k\}$ is a Banach norm on $C^s(\mathbb{H})$. For each integer N, let us define

$$\begin{aligned} E_N &= \left\{ f \in C^s(\mathbb{H}) : \forall(\varepsilon, j, k), 2^{js+N} d^{\varepsilon}_{j,k}(f) \in \mathbb{Z}^* \right\} \\ F_N &= \left\{ g \in C^s(\mathbb{H}) : \exists f \in E_N, \|f - g\|_{C^s(\mathbb{H})} < 2^{-N-2} \right\}. \end{aligned} \tag{40}$$

Lemma 11. *For every $N \geqslant 1$, all functions in F_N are monofractal of exponent s:*

$$\forall g \in F_N, \quad \forall x \in \mathbb{H}, \quad h_g(x) = s.$$

Proof. This simply follows from the fact that, given $f \in E_N$, all the wavelet coefficients of f satisfy

$$2^{-N-js} \leqslant |d^{\varepsilon}_{j,k}(f)| \leqslant \|f\|_{C^s} 2^{-js}.$$

Thus, for any function $g \in F_N$ and its associated $f \in E_N$,

$$2^{-N-js} - 2^{-N-2-js} \leqslant |d^{\varepsilon}_{j,k}(g)| \leqslant \|f\|_{C^s} 2^{-js} + 2^{-N-2-js};$$

i.e.,

$$2^{-N-1-js} \leqslant |d^{\varepsilon}_{j,k}(g)| \leqslant \left(\|f\|_{C^s} + 2^{-N-2}\right) 2^{-js}.$$

In particular, $g \in C^s(x)$ for any $x \in \mathbb{H}$, and there is no $x_0 \in \mathbb{H}$ and $s' > s$ such that $g \in C^{s'}(x_0)$. Indeed, (14) with $s' > s$ is not compatible when j tends to infinity with the left-hand side of the above inequality. □

Lemma 12. *The set $\mathcal{R} = \bigcup_{N \geqslant 1} F_N$ is a dense open set in $C^s(\mathbb{H})$ containing only monofractal functions with exponent s.*

Proof. The preceding lemma ensures that $\mathcal{R}$ is composed of monofractal functions. According to (40), F_N is an open set, and thus so is $\mathcal{R}$. Let us check the density. Given $f \in C^s(\mathbb{H})$ and $\eta > 0$, let us choose $N \in \mathbb{N}$ so that $2^{-N} < \eta$. Let us define the 'non-zero integer part' function

$$E^*(x) = \begin{cases} 1 & \text{if } 0 \leqslant x < 2, \\ [x] & \text{else.} \end{cases}$$

Obviously $E^* : \mathbb{R} \to \mathbb{Z}^*$ and $|x - E^*(x)| \leqslant 1$. Let us finally define a function $g \in F_N$ by its wavelet coefficients:

$$d^{\varepsilon}_{j,k}(g) = 2^{-js-N} E^*\left(2^{js+N} d^{\varepsilon}_{j,k}(f)\right).$$

By construction,

$$2^{js}\left|d^{\varepsilon}_{j,k}(f)-d^{\varepsilon}_{j,k}(g)\right| = 2^{-N}|2^{js+N}d^{\varepsilon}_{j,k}(f)-E^{*}\left(2^{js+N}d^{\varepsilon}_{j,k}(f)\right)| \leqslant 2^{-N} < \eta;$$

thus $\|f-g\|_{C^s} < \eta$. This proves the density of $\mathcal{R}$ in $C^s(\mathbb{H})$. □

5. Upper bound for the multifractal spectrum in a Besov space

The classical Sobolev embedding $B^s_{p,q}(\mathbb{H}) \hookrightarrow C^{s-Q/p}(\mathbb{H})$ can be retrieved easily using wavelets. Indeed, definition (22) reads

$$\left\|2^{j(s-Q/p)}d^{\varepsilon}_{j,k}(f)\right\|_{\ell^p(k)} \in \ell^q(j),$$

and implies the existence of a constant $C_0 > 0$ such that, for every triplet (ε, j, k),

$$|d^{\varepsilon}_{j,k}(f)| \leqslant C_0 2^{-j(s-Q/p)}. \tag{41}$$

Thus (13) holds, and Theorem 1 ensures that $f \in C^{s-Q/p}(\mathbb{H})$. In particular, (14) is satisfied around any point $x_0 \in \mathbb{H}$, and thus, by Theorem 2, one has

$$\forall x \in \mathbb{H}, \quad h_f(x) \geqslant s - Q/p.$$

It is worth mentioning that the index q of the Besov space $B^s_{p,q}(\mathbb{H})$ does not play any role in the Sobolev embedding, and neither does it in the multifractal analysis of f.

Let us now establish Theorem 4, i.e., that, for any $h \geqslant s - Q/p$, the iso-Hölder set of regularity h is of Hausdorff dimension

$$d_f(h) \leqslant \min\left(Q,\, p\left(h - s + Q/p\right)\right).$$

The inequality is obvious as soon as $h \geqslant s$, since the upper bound reduces to Q, which is the Hausdorff dimension of $\mathbb{H}$ itself. Thus one can now assume that

$$s - Q/p \leqslant h < s, \tag{42}$$

and, in particular, that $1 \leqslant p < +\infty$.

By Theorem 2, heuristically, the wavelet coefficients that might give rise to an exponent $h_f(x_0) \leqslant h$ for some $x_0 \in \mathbb{H}$ satisfy $|d^{\varepsilon}_{j,k}| \geqslant 2^{-jh}$. Hence we focus on

$$N_f(j,h) = \left\{k \in \mathcal{Z} : \exists \varepsilon \in \{1, 2, \dots, 2^Q - 1\}, |d^{\varepsilon}_{j,k}(f)| \geqslant C_0 2^{-jh}\right\}.$$

We focus on the set $N_f(j,h)$ obtained by taking the constant C_0 to be the one from the Sobolev embedding (41). This choice is made for technical reasons that will be clear later on (see equation (45)).

Lemma 13. *There exists $C > 0$ such that, for every $j \geqslant 1$ and for every $h' \in (s - Q/p, s]$,*

$$\#N_f(j,h') \leqslant C 2^{jp(h'-s+Q/p)}.$$

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Proof. Obviously, from (22), $B^s_{p,q}(\mathbb{H}) \subset B^s_{p,\infty}(\mathbb{H})$, and thus there is a constant $C > 0$ such that, for any $j \in \mathbb{Z}$, one has $\sum_{k\in\mathcal{Z}} 2^{j(ps-Q)}|d^\varepsilon_{j,k}(f)|^p \leqslant C$. Hence,

$$C2^{-j(ps-Q)} \geqslant \sum_\varepsilon \sum_{k\in\mathcal{Z}} |d^\varepsilon_{j,k}(f)|^p \geqslant \sum_{k\in N_f(j,h')} \sum_\varepsilon \left|d^\varepsilon_{j,k}(f)\right|^p \geqslant C_0^p \times (\# N_f(j,h')) \times 2^{-jph'},$$

which yields the result. Observe that this result also holds when $h' \geqslant s$, but it is useless for our purpose. □

Lemma 14. *The set*

$$E^{\leqslant}_f(h) = \{x \in \mathbb{H} : h_f(x) \leqslant h\}$$

has the following property:

$$\dim_H E^{\leqslant}_f(h) \leqslant p\,(h - s + Q/p)\,. \tag{43}$$

Estimate (43) is stronger than (15) since $E_f(h) \subset E^{\leqslant}_f(h) = \bigcup_{h'\leqslant h} E_f(h')$. In particular, $\dim_H E_f(h) \leqslant \dim_H E^{\leqslant}_f(h)$, and Theorem 4 follows immediately.

Proof. Definition (7) of h_f as a supremum implies that

$$E^{\leqslant}_f(h) = \left\{x \in \mathbb{H} : \forall h' > h,\ f \notin C^{h'}(x)\right\}.$$

Together with Theorem 2, this observation provides the following:

$$E^{\leqslant}_f(h) = \left\{x \in \mathbb{H} : \forall h' > h,\ \sup_{\varepsilon,j,k}\left[2^{-j} + \delta(x, x_{j,k})\right]^{-h'} |d^\varepsilon_{j,k}(f)| = +\infty\right\}. \tag{44}$$

Note that, as soon as $f \in L^2(\mathbb{H})$, one has $|d^\varepsilon_{j,k}| \leqslant C2^{jQ/2}$, and thus the only real constraint contained in (44) concerns the regime $j \to +\infty$ and $h' \in [h, \alpha h]$ for an arbitrary $\alpha > 1$.

We fix $\alpha > 1$, and let T be a large integer. As said above, the only interesting wavelet coefficients are those satisfying $|d^\varepsilon_{j,k}| \geqslant C_0 2^{-jh'}$ (it is not enough to consider only those greater than $C_0 2^{-jh}$). One splits $[s - Q/p, \alpha h]$ into intervals of length

$$\eta = (\alpha h - s + Q/p)/T,$$

namely the intervals $I_m = [h_{m-1}, h_m]$ for $m \in \{1, \dots, T\}$ and $h_m = s - Q/p + m\eta$. One chooses T large enough such that $h_0 - \eta = s - Q/p - \eta > 0$. The next idea is that, if $x \in \mathbb{H}$ is far from the dyadic set $(x_{j,k})_{j\in\mathbb{Z},k\in\mathcal{Z}}$ and simultaneously the wavelet coefficient $|d^\varepsilon_{j,k}|$ is too small, then it cannot contribute to (44). More precisely, if

$$|d^\varepsilon_{j,k}(f)| \leqslant C_0 2^{-jh\alpha} \quad \text{or} \quad \exists m \in \{0, \dots, n_1 - 1\}, \quad \begin{cases} |d^\varepsilon_{j,k}(f)| \leqslant C_0 2^{-jh_m}, \\ \delta(x, x_{j,k}) \geqslant 2^{-j(h_m-\eta)/\alpha h}, \end{cases} \tag{45}$$

with the same constant $C_0 > 0$ as in the Sobolev embedding (41), then

$$\left[2^{-j} + \delta(x, x_{j,k})\right]^{-h'} |d^\varepsilon_{j,k}(f)| \leqslant \begin{cases} C_0 2^{-j(\alpha h - h')} & \text{in the first case,} \\ C_0 2^{j(h_m-\eta)(h'/\alpha h)} 2^{-jh_m} & \text{in the others.} \end{cases}$$

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In the range $1 < h'/h \leqslant \alpha$, and for large j, one infers in both cases that

$$\left[2^{-j} + \delta(x, x_{j,k})\right]^{-h'} |d^{\varepsilon}_{j,k}(f)| \leqslant C_0.$$

Therefore, if (45) holds for *any* family $(\varepsilon_n, j_n, k_n)$ with $j_n \to +\infty$, then $x \notin E^{\leqslant}_f(h)$.

By contraposition, for any $x \in E^{\leqslant}_f(h)$, there exists a family $(\varepsilon_n, j_n, k_n)$ with $j_n \to +\infty$, contradicting (45). By the Sobolev embedding (41), each wavelet coefficient is bounded above by $C_0 2^{-jh_0}$. Hence, for each n, there exists necessarily $m \in \{1, \ldots, T\}$ such that

$$C_0 2^{-j_n h_m} < |d^{\varepsilon_n}_{j_n,k_n}(f)| \leqslant C_0 2^{-j_n h_{m-1}} \quad \text{and} \quad x \in \mathcal{B}(x_{j_n,k_n}, 2^{-j_n(h_{m-1}-\eta)/\alpha h}).$$

The previous statement can be expressed more easily in term of lim-sup sets:

$$E^{\leqslant}_f(h) \subset \bigcup_{m=1}^{T} S_{m,\eta}, \quad \text{with } S_{m,\eta} = \bigcap_{J \geqslant 1} \bigcup_{j \geqslant J, k \in N_f(j,h_m)} \mathcal{B}(x_{j,k}, 2^{-j(h_m-2\eta)/\alpha h}). \tag{46}$$

Let us now establish an upper bound for the Hausdorff dimension of each set $S_{m,\eta}$. Given $\xi > 0$, one chooses an integer J_ξ so large that $2\gamma_1 \times 2^{-J_\xi(s-Q/p-\eta)/\alpha h} \leqslant \xi$ (with the constant γ_1 from (17)). A covering of $S_{m,\eta}$ by balls of diameter less than ξ is thus provided by

$$S_{m,\eta} \subset \bigcup_{j \geqslant J_\xi, k \in N_f(j,h_m)} \mathcal{B}(x_{j,k}, 2^{-j(h_m-2\eta)/\alpha h}).$$

For any $d \geqslant 0$, the $\mathcal{H}^d_\xi$-pre-measure of $S_{m,\eta}$ can then be estimated easily:

$$\mathcal{H}^d_\xi(S_{m,\eta}) \leqslant \sum_{j \geqslant J_\xi} \sum_{k \in N_f(j,h_m)} \left(C 2^{-j(h_m-2\eta)/\alpha h}\right)^d.$$

Using Lemma 13 to estimate $\#N_f(j, h_m) \leqslant C 2^{jp(h_m-s+Q/p)}$ gives

$$\mathcal{H}^d_\xi(S_{m,\eta}) \leqslant C \sum_{j \geqslant J_\xi} 2^{j[p(h_m-s+Q/p)-d(h_m-2\eta)/\alpha h]}.$$

This series converges when

$$d > \frac{\alpha p h}{h_m - 2\eta}(h_m - s + Q/p), \tag{47}$$

and in that case

$$\mathcal{H}^d_\xi(S_{m,\eta}) \leqslant C 2^{-J_\xi[d(h_m-2\eta)/\alpha h - p(h_m-s+Q/p)]}.$$

As ξ tends to zero, the d-Hausdorff measure of $S_{m,\eta}$ is 0, which in turn implies that $\dim_H(S_{m,\eta}) \leqslant d$. Finally, optimizing for any d that satisfies (47) provides

$$\dim_H(S_{m,\eta}) \leqslant \frac{\alpha p h}{h_m - 2\eta}(h_m - s + Q/p) = \alpha p h \left(1 - \frac{s - Q/p - 2\eta}{h_m - 2\eta}\right).$$

Looking back at (46), one deduces that

$$\dim_H E^{\leqslant}_f(h) \leqslant \max_{m=1,\ldots,T} \alpha p h \left(1 - \frac{s - Q/p - 2\eta}{h_m - 2\eta}\right) \leqslant \alpha p h \left(1 - \frac{s - Q/p - 2\eta}{\alpha h - 2\eta}\right).$$

The limit $\eta \to 0$ provides

$$\dim_H E_f^{\leqslant}(h) \leqslant \alpha ph \left(1 - \frac{s - Q/p}{\alpha h}\right) = p\,(\alpha h - s + Q/p)\,.$$

Finally, letting $\alpha \to 1$ gives (43) and Theorem 4. □

6. Almost all functions in $B^s_{p,q}(\mathbb{H})$ are multifractal

To prove Theorem 5, one will explicitly construct a G_δ set of functions in $B^s_{p,q}(\mathbb{H})$ that satisfies (16). The proof is adapted from the one of [21]; modifications are due to the metric on $\mathbb{H}$. We first construct a subset $\mathcal{R}_0$ of $B^s_{p,q}(\mathbb{H})$ whose restriction to $[0,1]^3$ is generic in $B^s_{p,q}([0,1]^3)$ and satisfies (16). Next, we define

$$\forall k \in \mathcal{Z}, \quad \mathcal{R}_k = \{f(k^{-1}x) : f \in \mathcal{R}_0\}.$$

Finally, the intersection $\mathcal{R} = \bigcap_{k\in\mathcal{Z}} \mathcal{R}_k$ is generic in $B^s_{p,q}(\mathbb{H})$ because it is a countable intersection of G_δ sets and thus still a G_δ set. By construction, it will still satisfy (16).

The actual proof of Theorem 5 is contained in § 6.3. To build up for it, one needs to recall a few classical definitions and results on dyadic approximation in § 6.1. One then constructs a single function that satisfies (16) in § 6.2, which is the starting point for growing the set $\mathcal{R}_0$ in § 6.3.

6.1. Dyadic approximation in $\mathbb{H}$

For any $j \in \mathbb{N}$, one considers the subset of indices

$$\mathscr{L}_0(j) = \left\{k \in \mathcal{Z} : x_{j,k} = 2^{-j} \circ k \in [0,1)^3\right\}. \tag{48}$$

For later use, let us observe immediately that

$$\#(\mathscr{L}_0(j)) = 2^{Qj}. \tag{49}$$

Definition 6. A dyadic point $x_{J,K}$ is called irreducible if $K = (K_p, K_q, K_r)$ and at least one of the three fractions $2^{-J}K_p$, $2^{-J}K_q$, or $2^{-2J}K_r$ is irreducible. A point $x_{J,K}$ is called the *irreducible version* of $x_{j,k}$ if $x_{J,K}$ is irreducible and $x_{j,k} = x_{J,K}$.

One can check immediately that for a given couple (j,k) the corresponding irreducible couple (J,K) is unique. Note that one may have $(j,k) = (J,K)$ but that one always has $J \leqslant j$. Conversely, given an irreducible $x_{J,K}$ and $j \geqslant J$, there exists a unique $k \in \mathcal{Z}$ such that $x_{j,k} = x_{J,K}$, namely $k = 2^{j-J} \circ K$.

Given an integer $J \in \mathbb{N}$, the number of irreducible elements $x_{J,K} \in [0,1)^3$ is

$$\#\left\{K \in \mathscr{L}_0(J) : x_{J,K} \text{ irreducible}\right\} = (2^Q - 1) \times 2^{Q(J-1)}. \tag{50}$$

Indeed, $K = (K_1, K_2, K_3)$ provides a non-irreducible $x_{J,K} \in [0,1)^3$ if and only if $0 \leqslant K_1, K_2 < 2^J$, $0 \leqslant K_3 < 2^{2J}$ and

$$K_1 \equiv 0\ [\text{mod } 2] \quad \text{and} \quad K_2 \equiv 0\ [\text{mod } 2] \quad \text{and} \quad K_3 \equiv 0\ [\text{mod } 4].$$

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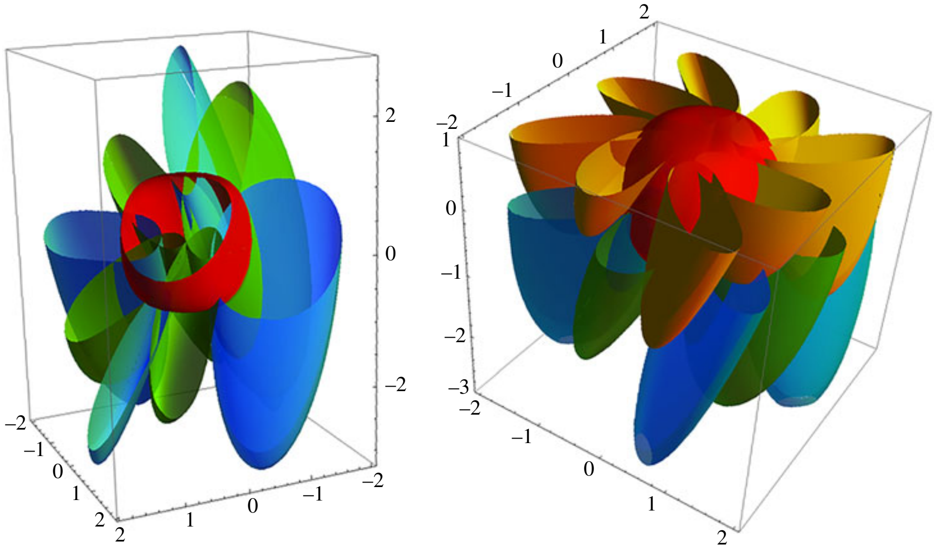

Figure 4. Left: section of the unit ball centered at the origin (red) and the balls of radius 1 centered at $(k_1, k_2, 0)$ with $k_1^2 + k_2^2 = 1$ (green) or $k_1^2 + k_2^2 = 2$ (blue). Right: another section of the same balls (same colors) and the balls centered in $(k_1, k_2, 2)$ (orange) (available in color online).

The complementary set in $\mathscr{L}_0(J)$ is thus of cardinal $2^{J-1} \times 2^{J-1} \times 2^{2J-2} = 2^{Q(J-1)}$, and (50) follows from (49).

Recall that $\mathcal{B}(x, r) = \{y \in \mathbb{H} : \|x^{-1}y\|_{\mathbb{H}} < r\}$ denotes the open gage ball of radius r. For fixed $j \in \mathbb{Z}$, the dyadic elements $\{x_{j,k} : k \in \mathcal{Z}\}$ are well distributed in $\mathbb{H}$, in the sense that the open balls $\{\mathcal{B}(x_{j,k}, 2^{-j}) : k \in \mathcal{Z}\}$ do not intersect too much. One can check easily the following lemma (see Figure 4).

Lemma 15. *For a given $j \in \mathbb{N}$ and $k \in \mathcal{Z}$, the only parameters $k' \in \mathcal{Z}$ such that $\mathcal{B}(x_{j,kk'}, 2^{-j}) \cap \mathcal{B}(x_{j,k}, 2^{-j}) \neq \emptyset$ are the 43 cubes defined by $k' = (k'_1, k'_2, k'_3)$, with*

$$\begin{cases} k'_1 = k'_2 = 0, \\ |k'_3| \leqslant 1, \end{cases} \qquad \text{or} \qquad \begin{cases} 1 \leqslant {k'_1}^2 + {k'_2}^2 \leqslant 2, \\ |k'_3| \leqslant 2. \end{cases}$$

Proof. By scale invariance of the pseudo-norm $\|\cdot\|_{\mathbb{H}}$, it is sufficient to investigate $j = 0$ and $k = (0, 0, 0)$. Then a counting argument applies. □

Observe that, if $r_0^2 = \frac{\sqrt{3}}{2} < 1$, then the cylinder $\Gamma_0 = \{(p, q, r) \in \mathbb{H} : p^2 + q^2 < r_0^2\}$ is included in $\bigcup_{k_3 \in \mathbb{Z}} \mathcal{B}((0, 0, k_3), 1)$. Since, for any $k = (k_1, k_2, k_3) \in \mathcal{Z}$, the left translation of Γ_0 is another vertical cylinder, one can choose a constant $C > 0$ such that, for each $j \in \mathbb{N}$, the family of balls

$$\{\mathcal{B}(x_{j,k}, C2^{-j}) : k \in \mathcal{Z}\}$$

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covers the whole space $\mathbb{H}$, and that, for any strictly increasing sequence $(j_m)_{m\geqslant 1} \in \mathbb{N}^{\mathbb{N}^*}$,

$$[0,1)^3 = \limsup_{m\to+\infty} \bigcup_{k\in\mathcal{L}_0(j_m)} \mathcal{B}(x_{j_m,k}, C2^{-j_m}). \tag{51}$$

Since each point $x \in [0,1)^3$ belongs to an infinite number of balls $\mathcal{B}(x_{j_m,k}, C2^{-j_m})$, one can wonder about the exact proximity of x to the dyadic elements of $\mathbb{H}$.

Definition 7. Let $\mathcal{J} = (j_m)_{m\geqslant 1}$ be a strictly increasing sequence of integers. For $\xi > 0$, an element $x \in \mathbb{H}$ is said to be ξ-approximable with respect to $\mathcal{J}$ when the inequality

$$\delta(x, x_{j_m,k}) \leqslant C2^{-j_m\xi} \tag{52}$$

holds true for an infinite number of couples (m,k) and the same constant C that appears in (51). For a given $\xi > 0$, one defines also

$$S_\xi(\mathcal{J}) = \{x \in [0,1)^3 : x \text{ is } \xi\text{-approximable with respect to } \mathcal{J}\}.$$

The dyadic approximation rate of x with respect to $\mathcal{J}$ is the real number

$$\xi_x(\mathcal{J}) = \sup\{\xi > 0 : x \text{ is } \xi\text{-approximable with respect to } \mathcal{J}\}.$$

Finally, the iso-approximable set of rate ξ is

$$\widetilde{S}_\xi(\mathcal{J}) = \{x \in [0,1)^3 : \xi_x(\mathcal{J}) = \xi\}.$$

When $\mathcal{J} = \mathbb{N}$, one simply writes S_ξ, $\widetilde{S}_\xi$, and ξ_x, and in that case it is sufficient to restrict oneself in (52) to irreducible dyadic elements $x_{j,k}$. Let us observe that, because of (51),

$$\forall x \in \mathbb{H}, \quad \xi_x(\mathcal{J}) \geqslant 1.$$

The size of the sets $S_\xi(\mathcal{J})$ and $\widetilde{S}_\xi(\mathcal{J})$ in terms of Hausdorff dimension and measures can be described thanks to the so-called *mass transference principle* by Beresnevich, Dickinson, and Velani [6].

Proposition 16. *For every strictly increasing sequence of integers $\mathcal{J} = (j_m)_{m\geqslant 1}$, and every $\xi \geqslant 1$, one has*

$$\dim_H \widetilde{S}_\xi(\mathcal{J}) = \dim_H S_\xi(\mathcal{J}) = Q/\xi. \tag{53}$$

Proof. It is quite easy to obtain that, for every $\xi \geqslant 1$,

$$\dim_H S_\xi(\mathcal{J}) \leqslant Q/\xi \quad \text{and} \quad \dim_H \widetilde{S}_\xi(\mathcal{J}) \leqslant Q/\xi. \tag{54}$$

Indeed, by definition, one has

$$S_\xi(\mathcal{J}) \subset \bigcap_{n\geqslant 1} \bigcup_{j\geqslant n,\ k\in\mathcal{L}_0(j)} \mathcal{B}(x_{j,k}, 2^{-j\xi}).$$

For $d > Q/\xi$ and an arbitrary $\eta > 0$, one chooses n large enough such that $2^{-n\xi} < \eta$. The previous inclusion provides a covering of $S_\xi(\mathcal{J})$ by balls of radius smaller than η; thus

$$\mathcal{H}^d_\eta(S_\xi(\mathcal{J})) \leqslant C \sum_{j\geqslant n} 2^{jQ}(2^{-j\xi})^d \leqslant C2^{n(Q-d\xi)} \underset{n\to+\infty}{\longrightarrow} 0,$$

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which proves the first half. The second half follows by noticing that, as $S_{\xi'}(\mathcal{J})$ is a decreasing family (for inclusion) when ξ' increases,

$$\widetilde{S}_\xi(\mathcal{J}) \subset \bigcap_{\xi'<\xi} S_{\xi'}(\mathcal{J}).$$

The converse inequality to (54) is very difficult, but is contained in [6]. Their main theorem (stated as Theorem 17 below) holds at a great level of generality. It holds in particular on the Heisenberg group, since $\mathbb{H}$ endowed with the metric (1) and the Haar measure $\ell = dp\,dq\,dr$ satisfies the following conditions.

(H1) ℓ is translation-invariant.

(H2) ℓ has a scaling behavior; i.e., there exists a constant $C > 1$ such that

$$\forall x \in \mathbb{H}, \quad \forall r > 0, \quad C^{-1} r^Q \leqslant \ell(\mathcal{B}(x,r)) \leqslant C r^Q,$$

and in particular ℓ is doubling; i.e.,

$$\forall x \in \mathbb{H}, \quad \forall r > 0, \quad \ell(\mathcal{B}(x,2r)) \leqslant C\ell(\mathcal{B}(x,r)).$$

(H3) The dyadic set $\{x_{j,k} : j \in \mathcal{J}, k \in \mathcal{Z}\}$ is discrete.

Together with the covering property (51), the main result of [6], called the *mass transference principle*, implies the following.

Theorem 17 [6, Theorem 2, p. 15]. *For every strictly increasing sequence of integers $\mathcal{J} = (j_m)_{m\geqslant 1}$, and every $\xi \geqslant 1$, one has*

$$\mathcal{H}^{Q/\xi}(S_\xi(\mathcal{J})) = +\infty \quad \text{and} \quad \mathcal{H}^{Q/\xi}(\widetilde{S}_\xi(\mathcal{J})) = +\infty. \tag{55}$$

Statement (55) implies that the Hausdorff dimension of both $S_\xi(\mathcal{J})$ and $\widetilde{S}_\xi(\mathcal{J})$ is greater than or equal to Q/ξ. Combined with (54), this proves (53). □

6.2. Example of a function with the maximal possible spectrum

Recall that, for $x_{j,k} \in [0,1)^3$, we denote by $x_{J,K}$ its irreducible version.

Proposition 18. *Let $\beta = 1/p + 2/q$, and let F be the function whose wavelet coefficients are*

$$F^\varepsilon_{j,k} := d^\varepsilon_{j,k}(F) = \begin{cases} \dfrac{2^{-j(s-Q/p)-JQ/p}}{j^\beta} & \text{if } x_{j,k} \in [0,1)^3, \\ 0 & \text{otherwise.} \end{cases} \tag{56}$$

The function F belongs to $B^s_{p,q}(\mathbb{H})$ and it satisfies (16).

Observe that, by construction, F is essentially supported in $C_0 = [0,1]^3$. Outside C_0, F is as smooth as the mother wavelet and decays rapidly at infinity.

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Proof. Let us fix a generation $j \geqslant 1$ and consider the sequence $a_j = \|2^{j(s-Q/p)} F^{\varepsilon}_{j,k}\|_{\ell^p(k)}$. For a given integer j, one has

$$2^{-j(ps-Q)} a_j^p = \sum_{k \in \mathscr{L}_0(j)} \sum_{1 \leqslant \varepsilon < 2^Q} |F^{\varepsilon}_{j,k}|^p = (2^Q - 1) \sum_{k \in \mathscr{L}_0(j)} |F^1_{j,k}|^p$$
$$\leqslant (2^Q - 1) \sum_{J=0}^{j} \frac{2^{-j(ps-Q)-QJ}}{j^{p\beta}} \#\{K : x_{J,K} \in \mathscr{L}_0(J) \text{ is irreducible}\}. \qquad (57)$$

Consequently, using (50),

$$a_j \leqslant \frac{(2^Q-1)^{1/p}}{j^\beta} \left(1 + (2^Q - 1) \sum_{J=1}^{j} 2^{Q(J-1)} 2^{-QJ}\right)^{1/p} \leqslant \frac{(2^Q-1)^{2/p}}{j^\beta} (1 + j2^{-Q})^{1/p}. \quad (58)$$

Finally, the choice of β in our statement provides $a_j \leqslant (2^Q - 1)^{2/p}/j^{2/q}$; thus the sequence $(a_j)_{j \geqslant 1}$ belongs to ℓ^q and $F \in B^s_{p,q}(\mathbb{H})$.

In order to compute the multifractal spectrum of F one uses the following lemma.

Lemma 19. *For each $x \in [0, 1]^3$, $h_F(x) = s - \frac{Q}{p} + \frac{Q}{p\xi_x}$.*

Here $\xi_x = \xi_x(\mathbb{N})$ is the approximation rate of x by all the dyadic elements. Assume for a while that Lemma 19 holds true; let us explain how to conclude from there. Since $\xi_x \in [1, +\infty]$ for each x, one first observes that $h_F(x)$ belongs necessarily to the interval $[s - Q/p, s]$. If $h \in (s - Q/p, s]$, one can observe further that

$$E_F(h) = \{x \in [0, 1]^3 : h_f(x) = h\} = \left\{x \in [0, 1]^3 : \xi_x = \frac{Q}{ph - ps + Q}\right\}.$$

Applying (53), one deduces that

$$d_F(h) = \dim_H E_F(h) = \dim_H \widetilde{S}_{\frac{Q}{ph-ps+Q}} = \frac{Q}{\frac{Q}{ph-ps+Q}} = ph - ps + Q,$$

which is the expected result. If $h = s - Q/p$, the dimension cannot exceed 0, by the general upper bound given by Theorem 4. The dimension is exactly 0, because one can find $x \in \mathbb{H}$ such that $\xi_x = +\infty$. Such points x are analogs in $\mathbb{H}$ of Liouville numbers in $\mathbb{R}$, which are the irrational real numbers that are the 'closest' to the rationals [20]. □

Proof of Lemma 19. Consider $x \in [0, 1)^3$ with $1 \leqslant \xi_x < +\infty$. By definition, for every $\varepsilon > 0$, one has the following properties.

(i) There exists $J_x > 0$ such that, for every $j \geqslant J_x$ and for every k,

$$\delta(x, x_{j,k}) \geqslant 2^{-j(\xi_x+\varepsilon)};$$

(ii) There exist a strictly increasing sequence of integers $(j_n)_{n \geqslant 1}$ and a sequence $(k_n)_{n \geqslant 1} \in \mathcal{Z}^{\mathbb{N}}$ such that $2^{-j_n} \circ k_n$ is irreducible and

$$\delta(x, x_{j_n,k_n}) \leqslant 2^{-j_n(\xi_x-\varepsilon)}.$$

When $\xi_x = 1$, one may take $\varepsilon = 0$ in the last inequality.

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To get the lower bound for the Hölder exponent, consider dyadic elements $x_{j,k}$ such that their associated irreducible element $x_{J,K}$ satisfy $J \geqslant J_x$. By item (i), one necessarily has $\delta(x, x_{j,k}) = \delta(x, x_{J,K}) \geqslant 2^{-J(\xi_x+\varepsilon)}$. By using that 2^{-j} and $\delta(x, x_{j,k})$ are bounded by above by their sum $2^{-j} + \delta(x, x_{j,k})$, we get that

$$F^{\varepsilon}_{j,k} = \frac{1}{j^\beta} 2^{-j(s-Q/p)-JQ/p} \leqslant (2^{-j} + \delta(x, x_{j,k}))^{s-Q/p} (\delta(x, x_{j,k})^{\frac{1}{\xi_x+\varepsilon}})^{Q/p}$$
$$\leqslant (2^{-j} + \delta(x, x_{j,k}))^{s-Q/p+\frac{Q}{p(\xi_x+\varepsilon)}}.$$

This is equivalent to (14); hence $h_F(x) \geqslant s - \frac{Q}{p} + \frac{Q}{p(\xi_x+\varepsilon)}$. Letting ε tend to zero yields the lower bound in Lemma 19.

Let us bound by above the Hölder exponent of F at x, by using item (ii). Assume that $1 < \xi_x < +\infty$, and fix $\varepsilon > 0$ such that $\xi_x - \varepsilon > 1$. For any integer $n \geqslant 1$, let $\widetilde{j}_n = [j_n(\xi_x - \varepsilon)]$. Consider the unique dyadic element $x_{\widetilde{j}_n,\widetilde{k}_n}$ such that $x_{\widetilde{j}_n,\widetilde{k}_n} = x_{j_n,k_n}$. Using that $2^{-j_n} \circ k_n$ is irreducible, one sees that

$$F^{\varepsilon}_{\widetilde{j}_n,\widetilde{k}_n} = \frac{1}{(\widetilde{j}_n)^\beta} 2^{-\widetilde{j}_n(s-Q/p)-j_nQ/p} \geqslant \frac{1}{(\xi_x-\varepsilon)^\beta j_n^\beta} 2^{-j_n(\xi_x-\varepsilon)\left(s-\frac{Q}{p}+\frac{Q}{p(\xi_x-\varepsilon)}\right)}.$$

Hence, since $\log(j_n(\xi_x - \varepsilon))$ is negligible with respect to j_n when $n \to +\infty$, one has

$$F^{\varepsilon}_{\widetilde{j}_n,\widetilde{k}_n} \geqslant 2^{-j_n(\xi_x-\varepsilon)(s-\frac{Q}{p}+\frac{Q}{p(\xi_x-\varepsilon)}+\varepsilon)} \geqslant d(x, x_{j_n,k_n})^{s-\frac{Q}{p}+\frac{Q}{p(\xi_x-\varepsilon)}+\varepsilon} = d(x, x_{\widetilde{j}_n,\widetilde{k}_n})^{s-\frac{Q}{p}+\frac{Q}{p(\xi_x-\varepsilon)}+\varepsilon}.$$

This proves that $h_F(x) \leqslant s - \frac{Q}{p} + \frac{Q}{p(\xi_x-\varepsilon)} + \varepsilon$, for every $\varepsilon > 0$. Letting ε tend to zero yields the upper bound in Lemma 19. The cases $\xi_x = 1$ and $\xi_x = +\infty$ are dealt with similarly. □

6.3. The residual set in $B^s_{p,q}(\mathbb{H})$

Let us define the wavelet version of local spaces. Recall that the set of indices $\mathscr{L}_0(j)$ was defined in (48).

Definition 8. The space $B^s_{p,q}([0,1)^3)$ is the closed subspace of $B^s_{p,q}(\mathbb{H})$ defined by

$$k \notin \mathscr{L}_0(j) \Longrightarrow d^{\varepsilon}_{j,k}(f) = 0.$$

It is equipped with norm $\|f\|_{B^s_{p,q}(\mathbb{H})} = \|f\|_\infty + \|(a_j)_{j\geqslant 1}\|_{l^q}$, where the a_j are the Besov coefficients of f given by (22).

Since $B^s_{p,q}([0,1)^3)$ is separable, one can consider a countable sequence $(f_n)_{n\geqslant 1}$ dense in $B^s_{p,q}([0,1)^3)$. Let us consider the sequence $(g_n)_{n\geqslant 1}$ built as follows.

Definition 9. For every $n \geqslant 1$, the wavelet coefficients of g_n up to the generation $j = n - 1$ are those of f_n; for $j \geqslant n$, the wavelet coefficients of generation j of g_n are those of the function F, which are prescribed by equation (56).

Since $\|f_n - g_n\|_{B^s_{p,q}(\mathbb{H})}$ tends to zero when $n \to +\infty$, $(g_n)_{n\geqslant 1}$ is also dense in $B^s_{p,q}([0,1)^3)$.

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Definition 10. Let $r_n = n^{-\beta} 2^{-nQ/p}/2$ with β given by (56). One defines the set $\widetilde{\mathcal{R}}$

$$\widetilde{\mathcal{R}} = \bigcap_{N \geqslant 1} \bigcup_{n \geqslant N} \mathscr{B}(g_n, r_n),$$

where $\mathscr{B}(g, r) = \{f \in B^s_{p,q}([0, 1)^3) : \|f - g\|_{B^s_{p,q}(\mathbb{H})} < r\}$.

The set $\widetilde{\mathcal{R}}$ is an intersection of dense open sets, hence is a residual set in $B^s_{p,q}([0, 1)^3)$. The choice for the radius r_n is small enough to ensure that any function f in $\mathscr{B}(g_n, r_n)$ has its wavelet coefficients at generation n close to those of g_n (and thus to those of F).

Lemma 20. *If $f \in \mathcal{B}(g_n, r_n)$, then $|d^\varepsilon_{n,k}(f) - d^\varepsilon_{n,k}(g_n)| \geqslant |d^\varepsilon_{n,k}(g_n)|/2\cdot$*

Proof. By definition, one has $d^\varepsilon_{n,k}(g_n) = F^\varepsilon_{n,k}$, $\forall k$. Hence, by definition of the Besov norm and the inclusion $\ell^q \subset \ell^\infty$,

$$\left(\sum_k 2^{pn(s-Q/p)} |d^\varepsilon_{n,k}(f) - F^\varepsilon_{n,k}|^p \right)^{1/p} < r_n.$$

In particular, for any ε and k,

$$|d^\varepsilon_{n,k}(f) - F^\varepsilon_{n,k}| \leqslant r_n 2^{-n(s-Q/p)} \leqslant= 2^{-ns} n^{-\beta}/2.$$

The inequality $J \leqslant j$ in (56) reads $|F^\varepsilon_{j,k}| \geqslant 2^{-js}/j^\beta\cdot$ Combining both inequalities ensures the result. □

Lemma 21. *If $f \in \widetilde{\mathcal{R}}$, then its multifractal spectrum d_f satisfies (16).*

Proof. Given a function $f \in \widetilde{\mathcal{R}}$, there exists a strictly increasing sequence $(n_m)_{m \geqslant 1}$ of integers such that $f \in \mathscr{B}(g_{n_m}, r_{n_m})$. Lemma 20 provides a precise estimate of the wavelet coefficients of f, namely, for any $m \geqslant 1$,

$$\frac{1}{2} F^\varepsilon_{n_m,k} \leqslant |d^\varepsilon_{n_m,k}(f)| \leqslant \frac{3}{2} F^\varepsilon_{n_m,k}\cdot$$

The same proof as the one developed for Lemma 19 ensures that, for any $x \in [0, 1)^3$,

$$s - Q/p \leqslant h_f(x) \leqslant s - Q/p + Q/(p\xi_x(\mathcal{J})) \leqslant s,$$

where $\xi_x(\mathcal{J})$ is the approximation rate by the family $\mathcal{J} = (n_m)_{m \geqslant 1}$.

Given $h \in [s - Q/p, s]$ and the unique ξ such that $h = s - Q/p + Q/(p\xi)$, one introduces the set (see Definition 7 and Lemma 14)

$$\mathcal{E} = S_\xi(\mathcal{J}) \Big\backslash \bigcup_{i=1}^{+\infty} E^{\leqslant}_f (h - 1/i)\,.$$

By (43), one knows that $\dim_H E^{\leqslant}_f(h') \leqslant p(h' - s - Q/p)$ for any $h' < h$. In particular, for every $i \geqslant 1$, one has

$$\dim_H E^{\leqslant}_f (h - 1/i) \leqslant p\,(h - 1/i - s - Q/p) < p\,(h - s - Q/p) = Q/\xi\cdot$$

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But according to (55), one has $\mathcal{H}^{Q/\xi}(S_\xi(\mathcal{J})) = +\infty$; thus $\mathcal{H}^{Q/\xi}(\mathcal{E}) = +\infty$ and

$$\dim_H \mathcal{E} \geqslant Q/\xi\cdot$$

Next, one observes that $\mathcal{E} \subset E_f(h)$, since every $x \in S_\xi(\mathcal{J})$ satisfies $h_f(x) \leqslant h$ and, by definition, $\mathcal{E}$ does not contains those elements x which have a local exponent strictly smaller than h. One can thus finally infer that

$$\dim_H E_f(h) \geqslant \dim_H \mathcal{E} \geqslant Q/\xi = p\,(h - s - Q/p)\,.$$

The converse inequality is provided by Theorem 4, because $f \in B^s_{p,q}([0,1)^3)$. Consequently, the identity (16) is satisfied. □

To conclude the proof of Theorem 5, let us go back to the initial remarks of § 6. The subset $\mathcal{R}_0$ of $B^s_{p,q}(\mathbb{H})$ whose (wavelet) restriction to $[0,1)^3$ satisfies (16) and is generic in $B^s_{p,q}([0,1)^3)$ is simply

$$\mathcal{R}_0 = \pi^{-1}(\widetilde{\mathcal{R}}),$$

where $\pi : B^s_{p,q}(\mathbb{H}) \to B^s_{p,q}([0,1)^3)$ is the projection defined in wavelet coefficients by $d^\varepsilon_{j,k}(\pi(f)) = d_{j,k}(f)\cdot \mathbf{1}_{\mathscr{L}_0(j)}(k)$, $\mathbf{1}_A(x)$ being equal to 1 if $x \in A$, and 0 otherwise.

7. Generalization to stratified nilpotent groups

A Carnot group G is a connected, simply connected, and nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification; i.e., for some integer $N_G \geqslant 1$,

$$\mathfrak{g} = \bigoplus_{k=1}^{N_G} \mathfrak{n}_k \quad \text{where } [\mathfrak{n}_1, \mathfrak{n}_k] = \mathfrak{n}_{k+1}$$

with $\mathfrak{n}_{N_G} \neq \{0\}$ but $\mathfrak{n}_{N_G+1} = \{0\}$. Let us denote the dimensions $q_k = \dim \mathfrak{n}_k$:

$$d = \sum_{k=1}^{N_G} q_k \quad \text{and} \quad Q_G = \sum_{k=1}^{N_G} k q_k. \tag{59}$$

Given a basis $(X_i)_{i=1,\dots,d}$ of $\mathfrak{g}$ adapted to the stratification, each index $i \in \{1,\dots,d\}$ can be associated to a unique $\sigma_i = j \in \{1,\dots,N_G\}$ such that $X_i \in \mathfrak{n}_j$.

Similarly to (18), the horizontal derivatives are the derivatives of the first layer:

$$\nabla_G f = (X_1 f, \dots, X_{q_1} f).$$

The stratification hypothesis ensures that each derivative $X_i f$ can be expressed as at most $\sigma_i - 1$ commutators of horizontal derivatives.

A Carnot group is naturally endowed with a family of algebra homomorphisms called dilations $\{D_\lambda\}_{\lambda>0}$ that are defined by

$$\forall i \in \{1,\dots,d\}, \quad D_\lambda(X_i) = \lambda^{\sigma_i} X_i.$$

The exponential map $\exp : \mathfrak{g} \to G$ is a global analytic diffeomorphism, and one can identify G and $\mathfrak{g}$ equipped with the (non-commutative if $N \geqslant 2$) law:

$$X * Y = \exp^{-1}(\exp(X)\cdot \exp(Y)).$$

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Finally, as was the case for $\mathbb{H}$, one can identify $\mathfrak{g}$ to $\mathbb{R}^d$ through the basis $(X_i)_{i=1,\dots,d}$. The gage distance is then defined by

$$\delta(x, y) = \left\| x^{-1} * y \right\|_G, \quad \text{where } \|x\|_G = \left(\sum_{i=1}^{d} |x_i|^{2\sigma/\sigma_i} \right)^{1/2\sigma}$$

and $\sigma = \operatorname{lcm}\{\sigma_1, \dots, \sigma_d\}$. This distance is left invariant and homogeneous of degree 1 with respect to the dilations. The triangular inequality stated in Proposition 17 and Corollary 7 still hold.

With those identifications, the Haar measure ℓ_G on G coincides with the Lebesgue measure on $\mathbb{R}^d$ and the volume of the ball $\mathcal{B}(x, r)$ is $r^{Q_G} \operatorname{Vol} \mathcal{B}(0, 1)$, where $Q_G \geqslant d$ is defined by (59). Hausdorff measures can be defined in a similar fashion to $\mathbb{H}$, and the Hausdorff dimension of G is Q_G. See [27] for further references.

One can wonder whether the results of multifractal analysis obtained in the present paper still hold in any Carnot group. We claim that the answer is positive. Let us list the modifications that are necessary to deal with a Carnot group G.

7.1. Families of wavelets

Wavelets have been constructed by Lemarié [24] on any Carnot group, but these wavelets have the inconvenience that the number of mother wavelets to be used Ψ^ε may be infinite, and that the discrete lattice $\mathcal{Z} = \mathbb{Z}^d$ may not be a subgroup any more (which complicates the notion of decomposition of a function on the wavelet basis, and the analysis of wavelet coefficients as well).

One is naturally led to the wavelet construction proposed by Führ and Mayeli in [17]. Let us recall the part of their results adapted to our context.

Definition 11. Let G be a Carnot group. A discrete subset $\Gamma \subset G$ is a regular sampling set if there exists a relatively compact Borel set $W \subset G$, neighborhood of the identity, satisfying

$$G = \bigcup_{\gamma \in \Gamma} \gamma W, \tag{60}$$

where the equality holds up to a set of ℓ_G-measure 0, and such that there is almost no covering in this union; i.e., for all $\alpha \neq \gamma \in \Gamma$, $\ell_G(\alpha W \cap \gamma W) = 0$.

The role of the lattice $\mathcal{Z}$ in $\mathbb{H}$ is now played by Γ in G.

Definition 12. For every function Ψ, every $j \in \mathbb{Z}$, and every $\gamma \in \Gamma$, we set

$$\Psi_{j,\gamma}(x) = \Psi(\gamma^{-1} * D_{2^j} x) = \Psi(D_{2^j}(x_{j,\gamma}^{-1} * x))$$

with $x_{j,\gamma} = D_{2^{-j}} \gamma$. For every function $f \in B^s_{p,q}(G)$, the wavelet coefficients of f are

$$d_{j,\gamma}(f) = 2^{jQ_G} \int_G f(x) \Psi_{j,\gamma}(x)\, dx. \tag{61}$$

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Existence of admissible wavelets (i.e., such that any smooth enough function can be reconstructed from its wavelet coefficients) belonging to the Schwartz class on G and having vanishing moments of arbitrary order is proved for instance in [17, Theorem 4.2].

In particular, when adapting our proofs to general Carnot groups, one chooses ψ such that the estimates of the tail of the wavelets (10) and the vanishing moments (11) remain unchanged.

7.2. Taylor polynomials

Taylor polynomials and estimates of the error term in a Taylor expansion (Theorem 9) hold on stratified groups (see [16]). For general homogeneous groups, a weaker estimate is given in [16], and explicit Taylor formulas with various remainder terms can also be found in [8]. In particular, formula (19) above remains valid to compute the (right) Taylor polynomial of order $N \geqslant 1$ at $x_0 \in G$ on any Carnot group G:

$$P_{x_0}(y) = \sum_{|\alpha|=|\beta|\leqslant N} c_{\alpha,\beta} \nabla_G^\beta f(x_0) y^\alpha.$$

The formula only involves 'horizontal' derivatives through $\nabla_G f = (X_1 f, \ldots, X_{q_1} f)$ but contains homogeneous monomials $y_1^{\alpha_1} \ldots y_d^{\alpha_d}$ of all the coordinates. Homogeneity of a multi-index is defined by the proper weights $|\alpha| = \sum_{i=1}^d \sigma_i \alpha_i$.

7.3. Hausdorff dimension

Throughout the generalization, one substitutes the new value of the homogeneous dimension Q_G. The numerical value of the constants related to the numeration of neighboring cubes or balls will also have to be modified. Apart from this, the definitions and methods are the same.

7.4. Relations with Besov spaces

A characterization of Besov spaces in G in terms of (discrete) wavelet coefficients is similar to the one we used, except that the regular sample is not $\mathcal{Z} = \mathbb{Z}^3$ any more, but rather the regular sampling Γ. This is a consequence of [17, Theorems 5.4, 6.1 and 6.7], which can be restated in the following form which suits our context.

Theorem 22 [17]. *There exist an admissible wavelet Ψ belonging to the Schwartz class of G and having infinitely many vanishing moments, and a regular sampling set Γ, such that*

$$f \in B^s_{p,q}(G) \Rightarrow \sum_{j\in\mathbb{Z}} \left(2^{j(ps-Q_G)} \left(\sum_{\gamma\in\Gamma} |d_{j,\gamma}(f)|^p \right) \right)^{q/p} < +\infty. \tag{62}$$

Reciprocally, if a sequence of coefficients $(c_{j,\gamma})_{j\in\mathbb{Z},\gamma\in\Gamma}$ satisfies

$$\sum_{j\in\mathbb{Z}} \left(2^{j(ps-Q_G)} \left(\sum_{\gamma\in\Gamma} |c_{j,\gamma}|^p \right) \right)^{q/p} < +\infty, \tag{63}$$

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then the function

$$f = \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma} c_{j,\gamma} \Psi_{j,\gamma}$$

belongs to $B^s_{p,q}(G)$*, and the norm* $\|f\|_{B^s_{p,q}(G)}$ *is equivalent to the sum (63).*

Finally, there exists another admissible wavelet $\widetilde{\Psi}$ *in the Schwartz class of* G*, called the* dual to Ψ*, (depending on the Besov space* $B^s_{p,q}(G)$*) such that any function* $f \in B^s_{p,q}(G)$ *can be decomposed as*

$$f = \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma} \widetilde{d}_{j,\gamma}(f)\Psi_{j,\gamma} \quad \text{with } \widetilde{d}_{j,\gamma}(f) = 2^{jQ} \int_{\mathbb{H}} f(x)\widetilde{\Psi}_{j,\gamma}(x)\,dx. \tag{64}$$

The presence of the pair of bi-orthogonal wavelets $(\Psi, \widetilde{\Psi})$ implies that the coefficients involved in equation (62) can either be $d_{j,\gamma}(f)$ or $\widetilde{d}_{j,\gamma}(f)$. Since Ψ and $\widetilde{\Psi}$ enjoy exactly the same regularity properties, we replace the notation $\widetilde{d}_{j,\gamma}(f)$ in (64) by $d_{j,\gamma}(f)$, by a slight abuse of notation.

In particular, all the methods we developed can easily be adapted using the wavelet coefficients $d_{j,\gamma}(f)$ for all $j \geqslant 1$ and $\gamma \in \Gamma$ instead of the family $d_{j,k}(f)$, $j \geqslant 1$, and $k \in \mathcal{Z}$. Let us now quickly review the adaptations of each proof.

7.5. Results about Hölder regularity

Theorem 23. *Theorems 1, 2, and 3 remain valid on any Carnot group.*

Proof of Theorem 1. In § 3.1, the only modification consists in naming ϑ a solution of $\mathcal{L}^M\vartheta = \psi$ for some arbitrarily large integer M, and where $\mathcal{L} = -(X_1^2 + \cdots + X_{q_1}^2)$ denotes the hypoelliptic Laplace operator on G. Thanks to [17], one can choose ψ properly to ensure that ϑ has at least one vanishing moment and fast decay at infinity, which is all that is required for the proof to work. When $[s]$ is odd, the last integration by parts with respect to (X, Y) is obviously replaced by an integration by parts against each $(X_1, \ldots, X_{q_1})$ and produces q_1 terms, all dealt with in a similar way. Section 3.2 remains exactly unchanged.

Proof of Theorem 2. Section 4.1 also remains unchanged because, as noticed above, Corollary 7 is still valid on G. The conclusive statement (14) should obviously read

$$\delta(x_{j,\gamma}, x_0) < R \Longrightarrow |d_{j,\gamma}(f)| \leqslant C2^{-js}\left(1 + 2^j\delta(x_{j,\gamma}, x_0)\right)^s.$$

In § 4.2, the reconstruction of f from its wavelet coefficients to be used is (64), and it rewrites as

$$f(x) = f^\flat(x) + \sum_{j=1}^{\infty} \sum_{\gamma\in\Gamma} d_{j,\gamma}(f)\Psi_{j,\gamma}(x),$$

with $f^\flat(x)$ being a smooth function.

The polynomial P suitable for the pointwise Hölder estimate is $P = P^\flat + \sum_{j=1}^{\infty} P_j$, where P_j is the Taylor expansion of $\sum_{\gamma\in\Gamma} d_{j,\gamma}(f)\Psi_{j,\gamma}$. As noticed above, the Taylor expansion formula (19) remains valid on G, so the rest of the section is unchanged.

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Proof of Theorem 3. Section 4.3 is an abstract game of seeking wavelet coefficients of the proper order and rounding them up to the closest dyadic integer. It only connects to the ambient space through the application of Theorems 1 and 2 that we now know to hold true on G. The sole modification is the notation $d_{j,\gamma}(f)$ instead of $d^{\varepsilon}_{j,k}(f)$. □

7.6. Results about Besov spaces and diophantine approximation in G

One powerful property of the mass transference principle by Beresnevich, Dickinson, and Velani [6], and similar results in heterogeneous situations [3–5], is that these theorems not only apply to approximation by dyadics or rationals in Euclidean settings but also to all sufficiently well distributed systems of points in doubling metric spaces.

The definition of the regular sampling Γ and its associated tile W such that (60) holds true implies the two following properties.

(C1) Since W is compact, there exists a sufficiently large $M_G > 0$ such that any ball $\mathcal{B}(x, M_G)$ contains at least one point $\gamma \in \Gamma$.

(C2) Since the union (60) is constituted by sets whose intersections are always of ℓ_G-measure 0 and W is bounded, there exists another constant $N_G > 0$ such that, for every $x \in G$, the ball $\mathcal{B}(x, M_G)$ contains at most N_G points belonging to Γ.

These properties are analogs in G to Lemma 15 in the Heisenberg group $\mathbb{H}$. One concludes that

$$G = \bigcup_{\gamma\in\Gamma} \mathcal{B}(\gamma, M_G),$$

and that there is almost no redundancy in the covering; i.e., for every $x \in G$, the cardinality of those $\gamma \in \Gamma$ such that $x \in \mathcal{B}(\gamma, M_G)$ is bounded from above by N_G uniformly in $x \in G$. Immediately, one also deduces an analog of (51):

$$W = \limsup_{j\to+\infty} \bigcup_{\gamma\in\mathcal{L}_0(j_m)} \mathcal{B}(x_{j,\gamma}, 2^{-j} M_G),$$

where $\mathcal{L}_0(j) = \{\gamma \in \Gamma : x_{j,\gamma} = D_{2^{-j}}\gamma \in W\}$. The compact tile W is a natural candidate to replace $[0, 1)^3$ on $\mathbb{H}$. The notion of approximation rate and the sets $\mathcal{S}_\xi(\mathcal{J})$ and $\widetilde{\mathcal{S}}_\xi(\mathcal{J})$ are perfectly defined (recall Definition 7), and have the same interpretation as in the case of $\mathbb{H}$.

We are now ready to state our last result.

Theorem 24. *Theorems 4 and 5 remain valid on any Carnot group.*

Proof of Theorem 4. A careful reading of § 5 shows that the arguments go through by simply replacing $\mathcal{Z}$ by Γ. Indeed, Lemma 13 is a general counting argument for convergent series, and Lemma 14 requires only counting and coverings arguments that are exactly items (C1) and (C2) explained a few lines above. One deduces that every functions $f \in B^s_{p,q}(G)$ satisfies

$$\forall h \geqslant s - Q_G/p, \quad d_f(h) \leqslant \min(Q_G, ph - ps + Q_G),$$

as was the case for $\mathbb{H}$.

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Proof of Theorem 5. Let us carefully go through § 6. As mentioned above, W replaces $[0,1)^3$. The first change concerns the cardinality of $\mathcal{L}_0(j)$. As W is a neighborhood of the origin, there exists $\varepsilon > 0$ such that $\mathcal{B}(\mathrm{Id}, \varepsilon) \subset W$. By (17), there also exists a constant C such that $\ell_G(\bigcup_{w\in W} \mathcal{B}(w, \varepsilon 2^{-j})) \leqslant C$ uniformly in $j \geqslant 0$. If $\gamma_1 \neq \gamma_2 \in \mathcal{L}_0(j)$, then $\ell_G\big(\mathcal{B}(x_{j,\gamma_1}, \varepsilon 2^{-j}) \cap \mathcal{B}(x_{j,\gamma_2}, \varepsilon 2^{-j})\big) = 2^{-jQ_G}\ell_G\big(\mathcal{B}(\gamma_1, \varepsilon) \cap \mathcal{B}(\gamma_2, \varepsilon)\big)$, which is zero because $\mathcal{B}(\gamma_i, \varepsilon) \subset \gamma_i W$ and $\ell_G(\gamma_1 W \cap \gamma_2 W) = 0$. Therefore

$$\#\mathcal{L}_0(j) \leqslant \frac{C}{\ell_G\big(\mathcal{B}((\mathrm{Id}, \varepsilon 2^{-j}))\big)} \leqslant \widetilde{C} 2^{jQ_G}, \tag{65}$$

which replaces (49).

In order to prove the optimality of the upper bound for the multifractal spectrum of functions in $B^s_{p,q}(G)$, an 'optimal' function F was built in Proposition 18. Here, the new function to be studied is called F_G, and it is defined as the sum

$$F_G = \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma} F^G_{j,\gamma} \Psi_{j,\gamma},$$

where the wavelet coefficients are

$$F^G_{j,\gamma} := \begin{cases} \dfrac{2^{-j(s-Q_G/p)-JQ_G/p}}{j^\beta} & \text{if } x_{j,\gamma} \in W, \\ 0 & \text{otherwise,} \end{cases} \tag{66}$$

where J is the minimal positive integer such that $D_{2^J} x_{j,\gamma} = D_{2^{J-j}}\gamma \in \Gamma$. This naturally replaces the notion of irreducibility of dyadics given in Definition 6: for any positive integer j' such that $\gamma' = D_{2^{j'}}\gamma \in \Gamma$, one has $x_{j,\gamma} = x_{j+j',\gamma'}$; thus $x_{j,\gamma}$ is irreducible if it cannot be written as $x_{j'',\gamma''}$ with $0 \leqslant j'' < j$. One checks easily that F_G belongs to $B^s_{p,q}(G)$. The main estimate that replaces (57) and (58) is

$$\|2^{j(s-Q_G/p)} F_{j,\gamma}\|_{\ell^p(\Gamma)} \leqslant \frac{1}{j^{2/q}} \left(\frac{1}{j} \sum_{J=0}^{j} 2^{-Q_G J} \#\{\gamma \in \mathcal{L}_0(J); x_{J,\gamma} \text{ irreducible}\} \right)^{1/p} \leqslant \frac{\widetilde{C}^{1/p}}{j^{2/q}}$$

which, as expected, belongs to $\ell^q(j \in \mathbb{N})$. The last inequality results from (65), which is slightly rougher than the right-hand side of (50) but still sufficient for our purpose.

In the proof of Proposition 16, the constant $\widetilde{C}$ of (65) also appears in the upper bound for the Hausdorff pre-measure that now reads

$$\mathcal{H}^d_\eta(S_\xi(\mathcal{J})) \leqslant C \sum_{j\geqslant n} (\widetilde{C} 2^{jQ_G})(2^{-j\xi})^d \leqslant C\widetilde{C} 2^{n(Q_G - d\xi)}$$

and still tends to zero as $n \to +\infty$ when $d > Q_G/\xi$. Conversely, let us observe that the techniques we used to find lower bounds for the Hausdorff dimensions of sets extend to Carnot groups, thus ensuring the second half of Proposition 16. It is an easy matter to check that the Haar measure ℓ_G satisfies the three conditions (H1), (H2), and (H3), where the set $\{x_{j,k} : j \in \mathcal{J}, k \in \mathcal{Z}\}$ is replaced by the discrete set $\{x_{j,\gamma} = D_{2^{-j}}\gamma : j \in \mathcal{J}, \gamma \in \Gamma\}$. This implies that the mass transference principle (Theorem 17) holds true on G as it did in the Heisenberg group $\mathbb{H}$. Hence, all the arguments developed to find lower bounds for the Hausdorff multifractal spectrum of typical functions in $B^s_{p,q}(\mathbb{H})$ can be extended without alteration to the Carnot group G and its Besov space $B^s_{p,q}(G)$.

The last alteration consists in defining the wavelet-local space $B^s_{p,q}(W)$ by the criterion

$$x_{j,\gamma} \notin W \Longrightarrow d_{j,\gamma}(f) = 0.$$

The rest of § 6.3 remains unchanged. □

Taking those remarks in consideration, one can assert that Theorems 1 to 5 remain valid on any Carnot group.

7.7. Open problems

Further generalizations (e.g., to the realm of homogeneous groups) are not straightforward. For example, even though the metric structure of homogeneous groups is still defined in a similar fashion to the gage distance on Carnot groups, the notion of horizontal derivative ceases to exist, which changes deeply the nature of the Taylor formula and its remainder [8], and thus the subsequent analysis. The construction and analysis of wavelets in such a general setting is also an active area of mathematics.

The reader might also be interested in works concerning wavelets on compact Lie groups [30], on general Lie groups [32], on homogeneous spaces [14], and even Riemannian manifolds [19].

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