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# On Suslinian Continua

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*Abstract.* A continuum is said to be Suslinian if it does not contain uncountably many mutually exclusive nondegenerate subcontinua. We prove that Suslinian continua are perfectly normal and rim-metrizable. Locally connected Suslinian continua have weight at most $\omega_1$ and under appropriate set-theoretic conditions are metrizable. Non-separable locally connected Suslinian continua are rim-finite on some open set.

Suslinian continua were introduced by A. Lelek in [7]. There he also gave an example of a (non-locally connected) metrizable Suslinian continuum that is not rim-countable (see *e.g.,*[1]). Some other papers in this area are by Daniel and Treybig [4], Simone [12, 13], Tymchatyn [18] and van Mill and Wattel [19]. It is the purpose of this paper to study further the structure of Suslinian continua, give some interesting examples, and raise problems.

***Example 1*** Note that each metrizable Suslinian continuum $X$ can be "locally connectified" by adding to it a "null-family" of arcs. Let us sketch some details of this reasonably well-known method for the sake of completeness.

Let $X$ be a compact metrizable space. Then there is a continuous map $f\colon C \to X$ of the Cantor set $C$ onto $X$ (see *e.g.,* [5]). We assume that $0, 1 \in C \subset [0, 1]$. Let $G$ denote the decomposition of $[0, 1]$ into the sets $f^{-1}(x)$, $x \in X$, and singletons. The quotient space $Y = [0, 1]/G$ is Hausdorff (see *e.g.,* [5]), and so is metrizable and locally connected. It contains a homeomorphic copy $A$ of $X$ so that $Y - A$ is the union of countably many pairwise disjoint open arcs whose diameters converge to 0. It is easy to see that if $X$ is a Suslinian continuum, then so must be $Y$. Moreover, if $X$ is not rim-countable, $Y$ is not rim-countable either.

***Example 2*** A Souslin line is a linearly ordered continuum $X$ such that $X$ is not separable, and each family of pairwise disjoint open subsets of $X$ is countable (some definitions require Souslin lines to have no separable open subset; this really does not matter much). Under appropriate set-theoretic assumptions (like the Constructibility Axiom) Souslin lines are plentiful, while under other set-theoretic assumptions (such as the Martin Axiom with the negation of the Continuum Hypothesis) Souslin lines do not exist at all (see *e.g.,* [15]). The Souslin Hypothesis is the statement/axiom that "Souslin lines do not exist".

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Obviously, each Souslin line is a locally connected Suslinian continuum. Moreover, each Souslin line is perfectly normal (in particular, first countable) and has weight $\omega_1$.

***Example 3*** The following non-locally connected Suslinian continuum $Y$ is obtained as a part of [3, Theorem 4.4]. Let $X$ be a Souslin line. Then there is a compactification $Y$ of the half-line $[0, \infty)$ whose remainder is homeomorphic to $X$. Thus, $Y$ is the disjoint union of $[0, \infty)$ and a copy of $X$. Roughly speaking, $Y$ looks like the $\sin \frac{1}{x}$ curve condensing on $X$. Of course, $Y$ is a Suslinian continuum that is not locally connected.

A slightly more complicated Suslinian continuum $Z$ can be obtained as follows. Let $x_0$ be an end-point of $X$. Then $X - \{x_0\}$ has a compactification $Z$ whose remainder is homeomorphic to $X$. Of course, $Z$ is not locally connected and it contains no nondegenerate metric subcontinuum.

***Theorem 1*** *Each Suslinian continuum is perfectly normal.*

**Proof** Let $X$ be a Suslinian continuum. First, we shall prove that if $x \in X$, then $\{x\}$ is a $G_\delta$-set in $X$. Choose any $x$ and form collections $\{V_\alpha : \alpha < \alpha_0\}$ of open subsets of $X$ and $\{C_\alpha : \alpha < \alpha_0\}$ of subcontinua of $X$, both labeled by ordinal numbers $\alpha$, as follows.

Select a nondegenerate continuum $C_0$ such that $x \notin C_0$ and let $V_0$ be an open set containing $x$ so that $C_0 \subset X - V_0$. Suppose that open sets $V_\alpha$ and nondegenerate continua $C_\alpha$ have been chosen for all $\alpha < \gamma$, where $\gamma$ is an ordinal number, so that $C_\alpha$ is contained in $X - V_\alpha$ and $C_\alpha$ meets no $C_\beta$ for $\beta < \alpha$. For ordinal $\gamma$, select nondegenerate continuum $C_\gamma$ so that $x \notin C_\gamma$ and $C_\gamma \cap (\bigcup_{\alpha<\gamma} C_\alpha) = \varnothing$, and let $V_\gamma$ be an open set containing $x$ so that $C_\gamma \subset X - V_\gamma$. Since $X$ is Suslinian and the continua $C_\alpha$ are nondegenerate and pairwise disjoint, this process may only be continued for countably many steps, say until some $\alpha_0 < \omega_1$, and then must cease.

Take a one-to-one correspondence $f$ from $\{\alpha : \alpha < \alpha_0\}$ onto the set $\mathbb{N}$ of natural numbers. This arranges the sets $V_\alpha$ into a sequence $V_{f^{-1}(0)}, V_{f^{-1}(1)}, \dots$. For each $n \in \mathbb{N}$ let $U_n = \bigcap_{i=0}^{n} V_{f^{-1}(i)}$ and select an open set $W_n$ so that $x \in W_n \subset \overline{W}_n \subset U_n$ and $W_{n-1} \supset \overline{W}_n$.

The set $Q = \bigcap_{n=0}^{\infty} W_n = \bigcap_{n=0}^{\infty} \overline{W}_n$ is 0-dimensional. If $y \in Q \setminus \{x\}$, then there exists $n > 0$ so that $x$ is not in the component of $y$ in $\overline{W}_n$. Note that this component is nondegenerate and there are only countably many such components. Hence the set $\{W_n \setminus K : K$ is a component of $\overline{W}_n$ missing $x$, $n = 0, 1, 2, \dots\}$ is countable and

$$\{x\} = \bigcap \{W_n \setminus K : K \text{ is a component of } \overline{W}_n \text{ missing } x, n = 0, 1, 2, \dots\}.$$

Now, suppose that $A$ is a non-empty closed subset of $X$. Let $G$ denote the decomposition of $X$ into singletons and the set $A$. Let $X' = X/G$ and $q\colon X \to X'$ denote the quotient map of $X$ onto the quotient space $X'$. It is easy to see that $X'$ is a Suslinian continuum. By the previous paragraph the singleton $q(A)$ is a $G_\delta$-set in $X'$. Hence, $A$ is a $G_\delta$-set in $X$. ∎

***Theorem 2*** *Let $X$ be a continuum and $A$ a closed $0$-dimensional $G_\delta$-subset such that for each neighbourhood $U$ of $A$, at most countably many components of $U$ meet $A$. Then $A$ is metrizable.*

**Proof** Let $A = \bigcap_{n=0}^{\infty} U_n$ be a 0-dimensional subset of $X$ such that each $U_n$ is open and $\overline{U}_{n+1} \subset U_n$. If $x$ and $y$ are two points in $A$, then for some $n$, $x$ and $y$ lie in distinct components of $\overline{U}_n$, and $\overline{U}_n$ can be separated between $x$ and $y$. The components of $x$ and $y$ in $U_n$ are nondegenerate. By the hypothesis, countably many open sets suffice to separate points of $A$, and therefore $A$ has a countable base. ∎

As a result of this theorem, we have the following corollaries.

***Corollary 1*** *If $X$ is a continuum such that each open subset of $X$ has at most countably many components, then each $0$-dimensional, closed $G_\delta$-subset of $X$ is metrizable.*

***Corollary 2*** (Mardešić [10]) *If $X$ is a locally connected continuum, then each closed, $0$-dimensional, $G_\delta$-subset of $X$ is metrizable.*

***Corollary 3*** *If $X$ is a Suslinian continuum then each $0$-dimensional closed set in $X$ is metrizable.*

***Theorem 3*** *Each Suslinian continuum is rim-metrizable. In fact, it has a basis of open sets with metrizable and zero-dimensional boundaries.*

**Proof** Let $X$ be a Suslinian continuum and $x_o \in X$. By Theorem 1, there is a continuous surjection $f\colon X \to [0,1]$ such that $f^{-1}(0) = x_o$. Since $X$ is a continuum, all fibers $f^{-1}(t)$, $t \in\, ]0,1[$ are non-empty. Since $X$ is Suslinian, only countably many of those fibers of $f$ can have a nondegenerate component. Each fiber of $f$ is a $G_\delta$-set. So for all but countably many $t > 0$, $f^{-1}([0,t))$ is a neighbourhood of $x_o$ with metrizable and 0-dimensional boundary. By choosing $t$ small enough, the neighbourhood $f^{-1}([0,t))$ can be made arbitrarily small. ∎

Daniel and Treybig [4] have shown that if there is an example of a locally connected Suslinian continuum which is not an IOK (the continuous image of an ordered compactum), then there is such a continuum $X$ which is separable. In this connection we prove Theorems 4, 5 and 6.

***Theorem 4*** *Let $X$ denote a nondegenerate, separable, Suslinian continuum which contains no nondegenerate metric subcontinuum. Then, there is a collection $G$ of mutually exclusive nondegenerate subcontinua of $X$ such that*

(1) *$G' = \bigcup\{g : g \in G\}$ is dense in $X$,*
(2) *$X$ fails to be locally rim-finite at each point of $G'$, and*
(3) *$G$ is maximal with respect to having properties* (1) *and* (2).

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**Proof** Let $U$ be a nonempty open subset of $X$. Then $K = \overline{U}/Bd(U)$ is a separable continuum which is not an IOK, otherwise by [16] $K$ would be metrizable and by the boundary bumping theorem $U$ would contain a metrizable nondegenerate subcontinuum. By [11], $K$ is not hereditarily locally connected. Thus by [14, Theorem 3] $K$ and therefore $U$ contains a continuum of convergence $C$. Clearly $X$ is not rim-finite at each point of $C$.

Now consider the collection $T$ of all sets $g$ such that $g$ is a collection of mutually exclusive nondegenerate subcontinua of $X$, where $g$ contains $C$, and where each continuum $C'$ in $g$ has the property that $X$ is not rim-finite at any point of $C'$. We now partially order $T$ by set inclusion and let $G$ denote a maximal element of $T$.

If $G' = \bigcup\{g : g \in G\}$ is not dense in $X$, then there is a open set $W$ so that $\overline{W} \cap G' = \varnothing$. As above we may find a continuum like $C$ in $\overline{W}$. Thus $G$ is not maximal, a contradiction. ■

***Theorem 5*** *Suppose that X is a locally connected Suslinian continuum containing no nondegenerate metrizable continuum. The following are equivalent:*

(1) *X is the continuous image of an ordered continuum,*
(2) *X is rim-finite,*
(3) *X contains no nondegenerate continuum that lies in the closure of a countable set.*

**Proof** By Theorem 3, $X$ is rim-metrizable. Then by Theorem 1 of [2] $X$ is rim-finite if and only if $X$ contains no nondegenerate continuum that lies in the closure of a countable set. By Theorems 6 and 10 of [13] and by [17], $X$ is the continuous image of an ordered continuum if and only if $X$ is rim-finite. ■

***Remark*** There is an example of a metric Suslinian continuum which is nowhere locally connected. Fitzpatrick and Lelek [6] have constructed a nondegenerate rational dendroid $X$ in the plane such that each nonempty connected open subset of $X$ is dense in $X$ and is therefore nowhere locally connected. As described in Example 1, it can be embedded in a rim-countable, locally connected, metric continuum of rim-type $\omega$. Therefore, the assumption that $X$ does not contain any nondegenerate metrizable continuum in the hypothesis of Theorem 5 is necessary.

***Theorem 6*** *Let X denote a locally connected Suslinian continuum which is not separable. Then there is a hereditarily locally connected subcontinuum Y with nondegenerate interior in X such that X is rim-finite at each point of Y.*

**Proof** *Case 1:* There is a nondegenerate connected open set $U$ in $X$ such that no nondegenerate subcontinuum of $X$ lying in $U$ lies in the closure of a countable set. Let $U_1$ denote a nondegenerate connected open subset of $U$ such that $\overline{U}_1 \subset U$, and $Bd(U_1)$ is a totally disconnected metric set.

We find from the proof of Theorem 1 of [2] that if $\overline{U}_1$ fails to be connected im kleinen at some point, then $\overline{U}_1$ fails to be connected im kleinen at each point of a nondegenerate subcontinuum $L$ of $\overline{U}_1$. Since $L$ would be a subcontinuum of $Bd\overline{U}_1$,

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which is totally disconnected, we have a contradiction. Since no nondegenerate subcontinuum of $\overline{U}_1$ lies in the closure of a countable set, then $\overline{U}_1$ contains no nondegenerate metric continuum. By Theorem 5, $\overline{U}_1$ is rim-finite. Let $U_2$ denote a connected open set such that $\overline{U}_2 \subset U_1$. Then $X$ is rim-finite at each point of $\overline{U}_2$.

*Case 2:* If $U$ is any nondegenerate connected open subset of $X$, then $U$ contains a nondegenerate subcontinuum $g$ of $X$ such that $g$ lies in the closure of a countable set $C(g)$.

Let $g_1$ denote a nondegenerate subcontinuum of $X$ which lies in the closure of a countable set $C(g_1)$, and let $T$ denote the set of all collections $Q$ of nondegenerate continua such that

(1) $g_1 \in Q$,

(2) each element $g$ of $Q$ lies in the closure of a countable set, $C(g)$ and

(3) if $g, g'$ are distinct elements of $Q$, then $g \cap g' = \varnothing$.

Partially order $T$ by set inclusion and pick a maximal element $Q'$ of $T$. If $Q'' = \bigcup\{g : g \in Q'\}$ is not dense in $X$, then there is a connected open set $W$ such that $W \cap Q'' = \varnothing$. There is a nondegenerate subcontinuum $g$ of $X$ lying in $W$ such that $g$ lies in the closure of a countable set $C(g)$. Thus $Q'$ is not maximal. Since $X$ is Suslinian $Q'$ is countable, and $\bigcup\{C(g') : g' \in Q'\}$ is dense in $X$, and so $X$ is separable, a contradiction. Thus Case 1 must hold. ■

The proofs of the following two results are substantially different in nature from the arguments we have used above. They require the use of transfinite inverse sequences and then exploit a certain tree structure of subsets of the space under investigation that is introduced using an inverse limit description of the space. They use and extend some methods of reasoning that were already employed in [3].

***Theorem 7*** *The weight of a non-metrizable locally connected Suslinian continuum is $\omega_1$.*

**Proof** Let $X$ be a locally connected Suslinian continuum. Suppose that $X$ is not metrizable, *i.e.,* $w(X) > \omega$. Combining two results of S. Mardešić [8, 9], one obtains an inverse system $\mathcal{S} = (X_\alpha, f_\alpha^\beta, \kappa)$ such that $X = \lim \operatorname{inv} \mathcal{S}$, each $X_\alpha$ is a locally connected continuum with $w(X_\alpha) < w(X)$, each $f_\alpha^\beta$ is a monotone and surjective map and $\kappa$ is a cardinal number so that $\kappa \leq w(X)$ (as usual, $\kappa$ is treated as the first ordinal number of the given cardinality). Because of the properties of the bonding maps $f_\alpha^\beta$, all projections $f_\alpha \colon X \to X_\alpha$ are monotone and surjective, too. Therefore, each $X_\alpha$ is a (locally connected) Suslinian continuum. We may assume that for each ordinal number $\alpha < \kappa$, $f_\alpha^{\alpha+1} \colon X_{\alpha+1} \to X_\alpha$ has a nondegenerate fiber, that is, it is not a homeomorphism. It follows that for each $\alpha < \kappa$ there is $x_\alpha \in X_\alpha$ so that $(f_\alpha^{\alpha+1})^{-1}(x_\alpha)$ is a nondegenerate subcontinuum of $X_{\alpha+1}$. Then $f_\alpha^{-1}(x_\alpha)$ is a nondegenerate continuum in $X$. When $\alpha < \beta < \kappa$ then either $f_\beta^{-1}(x_\beta) \subset f_\alpha^{-1}(x_\alpha)$ or the two fibers are disjoint.

Now, assume that $\kappa > \omega_1$ and consider the family $\mathcal{A} = \{f_\alpha^{-1}(x_\alpha) : \alpha < \kappa\}$. When ordered by reverse inclusion, $\mathcal{A}$ becomes a tree (in the set-theoretic sense, see *e.g.,* [15]). Each level (even more: each antichain) of the tree $\mathcal{A}$ is a family of pairwise disjoint nondegenerate subcontinua of $X$ and, therefore, is countable. By

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[15, Theorem 2.7], $\mathcal{A}$ has an uncountable chain $\mathcal{C}$. We may assume that $\mathcal{C}$ consists of $\omega_1$ sets and label them as $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$ so that $C_\beta \subset C_\alpha$ and $C_\beta \neq C_\alpha$ whenever $\alpha < \beta < \omega_1$. Since all sets $C_\alpha$ are continua, for each $\alpha < \omega_1$, $C_\alpha - C_{\alpha+1}$ is a nondegenerate subset of $C_\alpha$. Hence, $C_\alpha - C_{\alpha+1}$ contains a nondegenerate continuum $B_\alpha$. It follows that $\{B_\alpha : \alpha < \omega_1\}$ is a family of pairwise disjoint nondegenerate continua in $X$. This contradiction shows that $\kappa \leq \omega_1$.

Now, suppose that $X$ was chosen so that $w(X) > \omega_1$ and there is no locally connected Suslinian continuum $Y$ with $\omega_1 < w(Y) < w(X)$. Then $w(X_\alpha) \leq \omega_1$ for each $\alpha < \kappa$. Since $\kappa \leq \omega_1$, it follows that $w(X) \leq \omega_1$ which contradicts the assumption that $w(X) > \omega_1$. Thus, we must have that $w(X) = \omega_1$. ■

***Theorem 8*** *If the Souslin Hypothesis holds, then each locally connected Suslinian continuum is metrizable.*

**Proof** Let $X$ be a locally Suslinian continuum. Suppose that $X$ is non-metrizable. By Theorem 7, $w(X) = \omega_1$. As already used in the proof of Theorem 7, there exists a transfinite inverse sequence $\mathcal{S} = (X_\alpha, f_\alpha^\beta, \omega_1)$ such that $X = \lim \operatorname{inv} \mathcal{S}$, each $X_\alpha$ is metrizable and each $f_\alpha^\beta$ is monotone and surjective. Then, moreover, each projection $f_\alpha : X \to X_\alpha$ is monotone and surjective, and each $X_\alpha$ is a locally connected Suslinian continuum.

Let $\mathcal{T} = \{f_\alpha^{-1}(x) : \alpha < \omega_1, x \in X_\alpha \text{ and } f_\alpha^{-1}(x) \text{ is nondegenerate}\}$. It was noticed in [3] that $\mathcal{T}$ ordered by reverse inclusion is a tree. The elements of $\mathcal{T}$ are nondegenerate subcontinua of $X$. Since $X$ is non-metrizable and each $X_\alpha$ is metrizable, $\mathcal{T}$ is uncountable. The argument used in the proof of Theorem 7 for $\mathcal{A}$ works for $\mathcal{T}$ as well, showing that $\mathcal{T}$ can not have an uncountable antichain and it can not have an uncountable chain. This means that $\mathcal{T}$ is a Souslin tree. However, existence of Souslin trees is equivalent to existence of Souslin lines (see *e.g.,* [15]). We have obtained a contradiction with the assumption that the Souslin Hypothesis holds. Hence, $X$ can not be non-metrizable. ■

Several nice-looking open problems remain to be settled (preferably) without using extra set-theoretic assumptions.

***Problem 1*** Is local connectivity essential in Theorems 7, 8?

***Problem 2*** Is it true that a (hereditarily) separable locally connected Suslinian continuum must be metrizable?

***Problem 3*** Is it true that a separable locally connected Suslinian continuum must be hereditarily separable?

In particular:

***Problem 4*** Is it possible to embed the space $Y$ of Example 3 in a locally connected Suslinian continuum?

***Problem 5*** Let $X$ be a locally connected Suslinian continuum. Is it true that $X$ is connected by ordered continua (that is, generalized arcs)?

***Problem 6*** Given a locally connected Suslinian continuum $X$, is it true that $X$ is either rim-finite or contains a nondegenerate metrizable continuum?

## References

[1] D. Daniel, J. Nikiel, L. B. Treybig, H. M. Tuncali and E. D. Tymchatyn, *On rim-properties of continua.* Questions Answers Gen. Topology **19**(2001), 187–193.
[2] ______, *Concerning continua that contain no metric subcontinua.* preprint, 2002 (to appear in Houston J. Math.).
[3] ______, *On perfectly normal compacta*, preprint, 2002, Submitted for publication.
[4] D. Daniel and L. B. Treybig, *A decomposition theorem for locally connected Suslinian continua*, Topology Proc. **23**(1998), 93–105.
[5] R. Engelking, *General topology.* Heldermann Verlag, Berlin, 1989.
[6] B. Fitzpatrick, Jr. and A. Lelek, *Some local properties of Suslinian compacta.* Colloq. Math. **31**(1974), 189–197.
[7] A. Lelek, *On the topology of curves, II.* Fund. Math. **70**(1971), 131–138.
[8] S. Mardešić, *On covering dimension and inverse limits of compact spaces.* Illinois J. Math. **4**(1960), 278–291.
[9] ______, *Locally connected, ordered and chainable continua.* Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **319**(1961), 147–166.
[10] ______, *Images of ordered compacta are locally peripherally metric.* Pacific J. Math. **23**(1967), 557–568.
[11] J. Nikiel, *The Hahn-Mazurkiewicz theorem for hereditarily locally connected continua.* Topology Appl. **32**(1989), 307–323.
[12] J.N. Simone, *Metric components of continuous image of ordered compacta.* Pacific J. Math. **69**(1977), 269–274.
[13] ______, *Continuous images of ordered compacta and hereditarily locally connected continua.* Coll. Math. **40**(1978), 77–84.
[14] ______, *Concerning hereditarily locally connected continua.* Coll. Math. **39**(1978), 243–251.
[15] S. Todorčević, *Trees and linearly ordered sets.* In: Handbook of set-theoretic topology, (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 235–293.
[16] L. B. Treybig, *Concerning continua which are continuous images of compact ordered spaces.* Duke Math. J. **32**(1965), 417–422.
[17] ______, *Arcwise connectivity in continuous images of ordered compacta.* Glas. Mat. Ser. III **21(41)**(1986), 201–211.
[18] E. D. Tymchatyn, *The Hahn-Mazurkiewicz theorem for finitely Suslinian continua.* General Topology Appl. **7**(1977), 1123–127.
[19] J. van Mill and E. Wattel, *Souslin dendrons.* Proc. Amer. Math. Soc. **72**(1978), 545–555.

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