Robotica (2010) volume 28, pp. 405–411. © Cambridge University Press 2009
doi:10.1017/S0263574709005724

A saturated PD controller for robots equipped with brushless DC-motors

V. M. Hernández-Guzmán†*, V. Santibáñez‡ and A. Zavala-Río§

†*Universidad Autónoma de Querétaro, Facultad de Ingeniería, Apartado Postal 3-24, C.P. 76150, Querétaro, Qro., México*
‡*Instituto Tecnológico de la Laguna, División de Estudios de Posgrado e Investigación, Apartado Postal 49 Adm. 1, C.P. 27001, Torreón, Coahuila, México.*
§*IPICYT, AP 2-66, Lomas 4a, Sección, C.P. 78216, San Luis Potosí, S.L.P., México.*

(Received in Final Form: April 9, 2009. First published online: May 22, 2009)

SUMMARY
In this paper we are concerned with control of rigid robots equipped with brushless DC-motors (BLDC) when the electric dynamics of these actuators is taken into account. We show for the first time that a saturated PD controller suffices to achieve global asymptotic stability. Our controller is the simplest controller proposed until now to solve this problem: it only requires position measurements and linear feedback of electric current.

KEYWORDS: Robot control; Brushless DC motors; PD control; Lyapunov stability.

1. Introduction

Some control schemes have been presented until now for rigid robots actuated by BLDC motors when their electric dynamics is taken into account.[1–4] However, the mathematical complexity of the BLDC motors model has deviated attention of these works towards the design of complicated nonlinear controllers which also include several high-order terms. It is recognized by Ortega *et al.*[5] (pp. 257, 395, 403) that complex control laws increase sensitivity to numerical errors and produce input voltage saturation, further high-order terms amplify noise in practice. Motivated by these observations, a simple PID controller has been proposed recently in ref. [6] to solve this problem ensuring global convergence to the desired positions. This controller includes an adaptive part which is introduced to cope with some cross-terms arising from the bilinear nature of the BLDC motors model. This allows to design a controller which does not require the exact knowledge of any actuator parameter. The problem with this adaptive part is that it demands additional computational effort, includes several third-order terms and requires velocity measurements.

The main contribution of the present paper is introducing a simple saturated PD controller which achieves global asymptotic stability. The main features of our controller, which represent the merit of our contribution, are the following. Velocity measurements are not required. This controller does not have any high-order term and the only nonlinearities being present are saturation functions. Moreover, saturation functions are important to globally dominate some third-order cross-terms arising from the bilinear nature of the BLDC motors model. Torque constant and electric resistance are the only motor parameters to be known exactly and the only variables we have to measure are position and electric current. Linear feedback of electric current is instrumental to prove global asymptotic stability and allows to present a formal justification for the common industrial practice known as torque control.[7,8]

This paper is organized as follows. In Section 2 we present the dynamic model of rigid robots actuated by BLDC motors. Section 3 is devoted to present our main results. Simulation results are presented in Section 4, and some conclusions are given in section 5.

Finally, some remarks on notation. Given some $x \in \mathcal{R}^n$ the Euclidean norm of x is defined as $\|x\| = \sqrt{x^T x}$. We use x_i and A_i to represent, respectively, the i-th component of x and the i-th diagonal entry of $A(x)$ if this is an $n \times n$ diagonal matrix. The 1-norm of x is defined as $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, where $|\cdot|$ stands for the absolute value function. If $A(x)$ is a symmetric positive definite matrix then $\lambda_{\min}(A(x)) > 0$ represents its smallest eigenvalue for any $x \in \mathcal{R}^n$.

2. Dynamic Model of Robots with BLDC Motors

The dynamic model of an n-degree-of-freedom rigid robot equipped with a direct-drive BLDC motor at each joint is given as[1,9]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + F\dot{q} = [K_{T1}I_B + K_{T2}]I_a \quad (1)$$

$$L_a\dot{I}_a + R_a I_a + N_p L_b I_B \dot{q} + K_{T2}\dot{q} = V_a \quad (2)$$

$$L_b\dot{I}_b + R_b I_b - N_p L_a I_A \dot{q} = V_b \quad (3)$$

* Corresponding author. E-mail: vmhg@uaq.mx

where

$$K_{T1} = N_p(L_b - L_a), \quad K_{T2} = \sqrt{\frac{3}{2}} N_p K_B \tag{4}$$
$$V_a = [v_{a1}, v_{a2}, \dots, v_{an}]^T \in \mathcal{R}^n$$
$$V_b = [v_{b1}, v_{b2}, \dots, v_{bn}]^T \in \mathcal{R}^n$$
$$I_a = [i_{a1}, i_{a2}, \dots, i_{an}]^T \in \mathcal{R}^n$$
$$I_b = [i_{b1}, i_{b2}, \dots, i_{bn}]^T \in \mathcal{R}^n$$
$$I_A = \mathrm{diag}\{i_{a1}, i_{a2}, \dots, i_{an}\} \in \mathcal{R}^{n\times n}$$
$$I_B = \mathrm{diag}\{i_{b1}, i_{b2}, \dots, i_{bn}\} \in \mathcal{R}^{n\times n}$$

Link positions are represented by $q \in \mathcal{R}^n$, $M(q)$ is the $n \times n$ symmetric positive definite inertia matrix, $C(q, \dot{q})\dot{q}$ is the centripetal and Coriolis term, $g(q) = \partial U(q)/\partial q$ is the gravity effects term, where $U(q)$ is a scalar-valued function representing the potential energy, and F is an $n \times n$ constant diagonal positive definite matrix representing the viscous friction coefficients at each joint. Throughout the paper we use $\tilde{q} = q_d - q$ to represent the position error where $q_d \in \mathcal{R}^n$ represents the constant desired link positions. We also assume that robot under study is equipped only with revolute joints.

Model (1)–(4) is obtained after a DQ (Park's) transformation is applied on the original Y-connected three-phase model of each motor.[3,9,10] Thus, V_a and V_b represent, respectively, the DQ transformed phase voltages associated with each motor. I_a and I_b are electric currents defined correspondingly. L_a, L_b, R_a, R_b, N_p, K_B are constant, diagonal, positive-definite matrices (see refs. [1,9] for a complete description of these matrices). Finally, K_{T1} and K_{T2} are diagonal torque constant matrices whereas $\tau = [K_{T1} I_B + K_{T2}] I_a$ is torque applied at robot joints. The mechanical dynamics of all motors is included in mechanical subsystem (1).

Let $V_j = [v_{j1}, v_{j2}, v_{j3}]^T \in \mathcal{R}^3$ and $I_j = [i_{j1}, i_{j2}, i_{j3}]^T \in \mathcal{R}^3$ be, respectively, the phase voltages and currents of the Y-connected three-phase BLDC motor placed at the j-th robot joint. Application of a DQ (Park's) transformation means that (see ref. [9, pp. 373])

$$\begin{bmatrix} \zeta_{aj} \\ \zeta_{bj} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \zeta_{j1}\cos(N_{pj}\, q_j) + \zeta_{j2}\cos\left(N_{pj}\, q_j - \frac{2\pi}{3}\right) \\ +\zeta_{j3}\cos\left(N_{pj}\, q_j + \frac{2\pi}{3}\right) \\ \zeta_{j1}\sin(N_{pj}\, q_j) + \zeta_{j2}\sin\left(N_{pj}\, q_j - \frac{2\pi}{3}\right) \\ +\zeta_{j3}\sin\left(N_{pj}\, q_j + \frac{2\pi}{3}\right) \end{bmatrix}$$

$$\begin{bmatrix} v_{j1} \\ v_{j2} \\ v_{j3} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} v_{aj}\cos(N_{pj}\, q_j) + v_{bj}\sin(N_{pj}\, q_j) \\ v_{aj}\cos\left(N_{pj}\, q_j - \frac{2\pi}{3}\right) + v_{bj}\sin\left(N_{pj}\, q_j - \frac{2\pi}{3}\right) \\ v_{aj}\cos\left(N_{pj}\, q_j + \frac{2\pi}{3}\right) + v_{bj}\sin\left(N_{pj}\, q_j + \frac{2\pi}{3}\right) \end{bmatrix}$$

where ζ stands for either v or i (voltages or currents).

On the other hand, as it is by now well known, the following are some important properties of mechanical part (1) when all joints are revolute.

Property 1 (See refs. [11, 12, pp. 96]). Matrices $M(q)$ and $C(q, \dot{q})$ satisfy $0 < \lambda_{\min}(M(q))$, $\forall q \in \mathcal{R}^n$, and

$$\dot{q}^T \left(\frac{1}{2}\dot{M}(q) - C(q, \dot{q}) \right) \dot{q} = 0, \quad \forall \dot{q} \in \mathcal{R}^n. \tag{5}$$

Property 2 (See ref. [12, pp. 101, 102]). There exists a positive constant k' such that for all $q \in \mathcal{R}^n$, we have $\|g(q)\| \le k'$. This means that every element of the gravity effects vector, i.e. $g_i(q)$, $i = 1, \dots, n$, satisfies $|g_i(q)| \le k'_i$ $\forall q \in \mathcal{R}^n$, for some positive constants k'_i, $i = 1, \dots, n$. Further, $\max_{q\in\mathcal{R}^n} |\frac{\partial g_i(q)}{\partial q_j}|$ is bounded for all $i = 1, \dots, n$, and $j = 1, \dots, n$.

Property 3 (See reference [13, pp. 120]). The $n \times n$ Jacobian matrix $\partial g(q)/\partial q$ is symmetric for all $q \in \mathcal{R}^n$ because $g(q) = \partial U(q)/\partial q \in \mathcal{R}^n$ is given as the gradient of the scalar function $U(q)$.

The following class of saturation functions is important for us:

Definition 1. *Given positive constants L and M, with $L < M$, a function $\sigma : \mathcal{R} \to \mathcal{R} : \varsigma \mapsto \sigma(\varsigma)$ is said to be a strictly increasing linear saturation for (L, M) if it is locally Lipschitz, strictly increasing, and satisfies*[14]*:*

$$\sigma(\varsigma) = \varsigma, \text{ when } |\varsigma| \le L,$$
$$|\sigma(\varsigma)| < M, \ \forall \varsigma \in \mathcal{R}$$

Finally, we list some well-known norm properties. Let $w, y \in \mathcal{R}^n$ be two vectors and let $B(x)$ be an $n \times n$ diagonal matrix $\forall x \in \mathcal{R}^n$, then

$$\pm y^T B(x) w \le \max_{\substack{i \in \{1,\dots,n\} \\ x \in \mathcal{R}^n}} |B_i(x)| \, \|y\| \, \|w\| \tag{6}$$

$$y^T B(x) y \ge \lambda_{\min}(B(x)) \|y\|^2 \tag{7}$$

3. Main Result

Proposition 1. *Consider dynamic model (1), (2), (3) together with the following PD controller:*

$$V_a = -r_a I_a + R K_{T2}^{-1}\, s(K_P\, \tilde{q} + g(q_d)) + \overline{K_V}\; Tanh(\vartheta) \tag{8}$$
$$V_b = -r_b I_b \tag{9}$$
$$\vartheta = z - Bq, \quad \dot{z} = -A\; Tanh(z - Bq) \tag{10}$$

where $A = \mathrm{diag}\{a_i\}$, $B = \mathrm{diag}\{b_i\}$, K_P, $\overline{K_V}$, r_a, r_b are $n \times n$ diagonal positive definite matrices, $s(x) = [\sigma_1(x_1), \dots, \sigma_n(x_n)]^T$, $x = K_P\, \tilde{q} + g(q_d)$, with $\sigma_i(x_i)$, $i = 1, \dots, n$ being strictly increasing linear saturation functions for some (L_i, M_i) satisfying (see Definition 1 and Property 2):

$$k'_i < L_i < M_i, \ i = 1, \dots, n \tag{11}$$

https://doi.org/10.1017/S0263574709005724 Published online by Cambridge University Press

Further, we also require functions $\sigma_i(x_i)$ to be continuously differentiable such that

$$0 < \frac{\mathrm{d}\sigma_i(x_i)}{\mathrm{d}x_i} \leq 1, \ \forall x_i \in \mathcal{R}, \quad i = 1, \ldots, n \tag{12}$$

We define vector $Tanh(\vartheta) = [\tanh(\vartheta_1), \ldots, \tanh(\vartheta_n)]^T$, with $\tanh(\cdot)$ the hyperbolic tangent function, and $R = R_a + r_a$. There always exist $n \times n$ diagonal positive definite matrices K_P, $\overline{K_V}$, A, B, r_a, r_b such that the closed-loop system has a unique equilibrium point which is globally asymptotically stable. At this equilibrium point we have that $\tilde{q} = 0$.

Proof. Define $\rho = I_a - R^{-1}(RK_{T2}^{-1}s(K_P \ \ \tilde{q} + g(q_d)) + \overline{K_V}\ Tanh(\vartheta))$ and note that $\dot{\vartheta} = -ATanh(\vartheta) - B\dot{q}$, is a realization of (10). Using these expressions and replacing (8) in (2) we can write

$$\begin{aligned} L_a\dot{\rho} &= -R\rho - N_pL_bI_B\dot{q} - K_{T2}\dot{q} + L_aK_{T2}^{-1}\frac{\partial s(x)}{\partial x}K_P\dot{q} \\ &\quad + L_aK_{T2}^{-1}K_V\frac{\partial Tanh(\vartheta)}{\partial \vartheta}[A\ Tanh(\vartheta) + B\dot{q}] \end{aligned} \tag{13}$$

$$K_V = K_{T2}R^{-1}\overline{K_V} \tag{14}$$

Now, define $\delta_a = RK_{T2}^{-1}\ s(K_P\tilde{q} + g(q_d)) + \overline{K_V}\ Tanh(\vartheta)$. Note that we can write

$$\begin{aligned} K_{T2}R^{-1}\delta_a &= s(K_P\ \tilde{q} + g(q_d)) + K_V\ Tanh(\vartheta) = \delta_a^* \\ &= [\delta_{a1}^*, \ldots, \delta_{an}^*]^T \in \mathcal{R}^n \end{aligned} \tag{15}$$

Replacing (9) in (3), adding and subtracting some convenient terms and using definitions of ρ and δ_a^* we can write

$$L_b\dot{I}_b = -\bar{R}I_b + N_pL_a\dot{Q}\rho + N_pL_a\dot{Q}K_{T2}^{-1}\delta_a^* \tag{16}$$

where we have defined $\dot{Q} = \mathrm{diag}\{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\} \in \mathcal{R}^{n\times n}$ and $\bar{R} = R_b + r_b$. On the other hand, (1) can be written as

$$\begin{aligned} &M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + F\dot{q} \\ &\quad = [K_{T1}I_B + K_{T2}]\rho + s(K_P\ \tilde{q} + g(q_d)) \\ &\qquad + K_V\ Tanh(\vartheta) + K_{T1}I_BK_{T2}^{-1}\delta_a^*. \end{aligned} \tag{17}$$

Thus, the closed-loop dynamics is given by (13), (16), (17) together with

$$\dot{\vartheta} = -A\ Tanh(\vartheta) - B\dot{q}. \tag{18}$$

Note that $(\tilde{q}, \dot{q}, \vartheta, \rho, I_b) = (0, 0, 0, 0, 0)$ is the unique equilibrium point of the closed-loop dynamics (13), (16)–(18) if $\tilde{q} = 0$ is the unique solution of

$$g(q) = s(K_P\ \tilde{q} + g(q_d)). \tag{19}$$

According to (11), (19) can be written as

$$K_P\ \tilde{q} = g(q) - g(q_d) \tag{20}$$

which implies

$$\|K_P\ \tilde{q}\|_1 = \|g(q) - g(q_d)\|_1. \tag{21}$$

Expression in (21) can be written as

$$\sum_{i=1}^{n} K_{Pi}\ |\tilde{q}_i| = \sum_{i=1}^{n} |g_i(q) - g_i(q_d)|, \tag{22}$$

where we have used the fact that $|K_{Pi}\ \tilde{q}_i| = K_{Pi}\ |\tilde{q}_i|$ because $K_{Pi} > 0$ for $i = 1, \ldots, n$. A fundamental property of the 1-norm of a vector z is that $\|z\|_1 = 0$ if and only if $z = 0$. Hence, if we prove that $|\tilde{q}_i| = 0$, for $i = 1, \ldots, n$, is the only solution of (22) then $\|K_P\ \tilde{q}\|_1 = 0$ is the only solution of (21), which implies that $\tilde{q} = 0$ is the only solution of (20) because K_P is positive definite. We can use the mean value theorem (see ref. [13, pp. 651]) to write

$$|g_i(q) - g_i(q_d)| \leq \sum_{j=1}^{n}\left[\left(\max_q\left|\frac{\partial g_i(q)}{\partial q_j}\right|\right)\ |\tilde{q}_j|\right].$$

Using this in (22) yields

$$\sum_{i=1}^{n} K_{Pi}\ |\tilde{q}_i| \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\max_q\left|\frac{\partial g_i(q)}{\partial q_j}\right|\right)\ |\tilde{q}_j|\right]. \tag{23}$$

According to property 3 we have

$$\max_q\left|\frac{\partial g_i(q)}{\partial q_j}\right| = \max_q\left|\frac{\partial g_j(q)}{\partial q_i}\right|. \tag{24}$$

Using this we can write

$$\begin{aligned} &\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\max_q\left|\frac{\partial g_i(q)}{\partial q_j}\right|\right)\ |\tilde{q}_j|\right] \\ &\quad = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\max_q\left|\frac{\partial g_i(q)}{\partial q_j}\right|\right)\ |\tilde{q}_i|\right]. \end{aligned}$$

Hence, (23) is equivalent to

$$\sum_{i=1}^{n}\left[\left(K_{Pi} - \sum_{j=1}^{n}\max_q\left|\frac{\partial g_i(q)}{\partial q_j}\right|\right)\ |\tilde{q}_i|\right] \leq 0. \tag{25}$$

Thus, if we choose K_{Pi} such that

$$K_{Pi} > \sum_{j=1}^{n}\max_q\left|\frac{\partial g_i(q)}{\partial q_j}\right|, \quad i = 1, \ldots, n, \tag{26}$$

the only manner to satisfy the equality in (25), i.e. to satisfy (22), is $|\tilde{q}_i| = 0, \quad i = 1, \ldots, n$. We conclude that the only solution of (19), (20), (22) is $\tilde{q} = 0$, i.e. $(\tilde{q}, \dot{q}, \vartheta, \rho, I_b) = (0, 0, 0, 0, 0)$ is the unique equilibrium point of the closed-loop dynamics (13), (16)–(18).

https://doi.org/10.1017/S0263574709005724 Published online by Cambridge University Press

The following scalar function:

$$V_1(\tilde{q},\dot{q},\vartheta) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \mathcal{U}_{cl}(\tilde{q}) + \sum_{i=1}^{n} \frac{K_{Vi}}{b_i}\ln(\cosh(\vartheta_i)) \tag{27}$$

$$\mathcal{U}_{cl}(\tilde{q}) = \sum_{i=1}^{n}\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i) + \sigma_i(K_{Pi}\ r_i + g_i(q_d))]\mathrm{d}r_i$$

where

$$\begin{aligned}
\overline{g}_1(r_1) &= g_1(q_{d1} - r_1, q_{d2}, \ldots, q_{dn}) \\
\overline{g}_2(r_2) &= g_2(q_1, q_{d2} - r_2, q_{d3}, \ldots, q_{dn}) \\
&\ \ \vdots \\
\overline{g}_n(r_n) &= g_n(q_1, q_2, \ldots, q_{n-1}, q_{dn} - r_n)
\end{aligned} \tag{28}$$

is positive definite and radially unbounded if K_V and B are positive definite and (26) is satisfied (see appendix). We stress that all variables different from r_i remain constant as integral with respect to $\tilde{q}_i$ is computed. We propose the following positive definite and radially unbounded scalar function as Lyapunov function candidate:

$$V(\tilde{q},\dot{q},\vartheta,\rho,I_b) = V_1(\tilde{q},\dot{q},\vartheta) + V_2(\rho) + V_3(I_b)$$

$$V_2(\rho) = \frac{1}{2}\rho^T L_a\rho, \quad V_3(I_b) = \frac{1}{2}I_b^T L_b I_b$$

where V_1 is given in (27). Using (4) and the diagonal property of all the involved matrices, we have $\dot{q}^T K_{T1} I_B\rho - \rho^T N_p L_b I_B\dot{q} + I_b^T N_p L_a\dot{Q}\rho = 0$ and $\dot{q}^T K_{T1} I_B K_{T2}^{-1}\delta_a^* + I_b^T N_p L_a\dot{Q}K_{T2}^{-1}\delta_a^* = I_b^T N_p L_b\dot{Q}K_{T2}^{-1}\delta_a^*$. These facts as well as (5), $g(q) = \partial U(q)/\partial q$, and $\frac{\mathrm{d}}{\mathrm{d}t}\ln(\cosh(u)) = \tanh(u)\frac{\mathrm{d}u}{\mathrm{d}t}$, $u\in\mathcal{R}$, allow one to find the following time derivative along the trajectories of dynamics (13), (16), (17), (18):

$$\begin{aligned}
\dot{V} = &-\dot{q}^T F\dot{q} - Tanh^T(\vartheta)K_V AB^{-1}Tanh(\vartheta) \\
&- \rho^T R\rho - I_b^T\bar{R}I_b + \dot{q}^T N_p L_b I_B K_{T2}^{-1}\delta_a^* \\
&+ \rho^T L_a K_{T2}^{-1}\frac{\partial s(x)}{\partial x}K_P\dot{q} \\
&+ \rho^T L_a K_{T2}^{-1}K_V\frac{\partial Tanh(\vartheta)}{\partial\vartheta}(A\ Tanh(\vartheta) + B\dot{q}).
\end{aligned} \tag{29}$$

We recall that[15] $\frac{\mathrm{d}\tanh(u)}{\mathrm{d}u} = \frac{1}{\cosh^2(u)} = \frac{4}{(\mathrm{e}^u+\mathrm{e}^{-u})^2}$ with $\cosh(\cdot)$ the hyperbolic cosinus function. Hence, the reader can verify easily that this means that $0 < \frac{\mathrm{d}\tanh(u)}{\mathrm{d}u} \le 1$, $\forall u\in\mathcal{R}$. According to this and (12), both $\frac{\partial s(x)}{\partial x}$ and $\frac{\partial Tanh(\vartheta)}{\partial\vartheta}$ are diagonal matrices whose entries are positive and smaller than or equal to 1. Thus, we can use (6) and (7) to obtain

$$\dot{V} \le -\begin{bmatrix}\|\dot{q}\| \\ \|Tanh(\vartheta)\| \\ \|\rho\| \\ \|I_b\|\end{bmatrix}^T P \begin{bmatrix}\|\dot{q}\| \\ \|Tanh(\vartheta)\| \\ \|\rho\| \\ \|I_b\|\end{bmatrix} \tag{30}$$

where entries of matrix P are

$$\begin{aligned}
P_{11} &= \lambda_{\min}(F) \\
P_{22} &= \lambda_{\min}(K_V AB^{-1}) \\
P_{33} &= \lambda_{\min}(R) \\
P_{44} &= \lambda_{\min}(\bar{R}) \\
P_{12} &= P_{21} = P_{42} = P_{24} = P_{43} = P_{34} = 0 \\
P_{13} &= P_{31} = -\frac{1}{2}\max_i(L_{ai}K_{T2i}^{-1}K_{Pi}) \\
&\quad -\frac{1}{2}\max_i(L_{ai}K_{T2i}^{-1}K_{Vi}b_i) \\
P_{23} &= P_{32} = -\frac{1}{2}\max_i(L_{ai}K_{T2i}^{-1}K_{Vi}a_i) \\
P_{14} &= P_{41} = -\frac{1}{2}\max_i(N_{pi}L_{bi}K_{T2i}^{-1}(M_i + K_{Vi}))
\end{aligned}$$

where M_i comes from (11) and Definition 1. Matrix P is positive definite if and only if

$$P_{11} > 0, \quad P_{22} > 0 \tag{31}$$

$$\delta_3 = P_{33}P_{22}P_{11} - P_{13}P_{22}P_{31} - P_{23}P_{11}P_{32} > 0 \tag{32}$$

$$P_{44}\delta_3 - P_{14}P_{41}(P_{22}P_{33} - P_{23}P_{32}) > 0 \tag{33}$$

According to (14), given any R we can always adjust $\overline{K_V}$ to maintain the desired value for K_V. Hence, P_{33} is the only entry in the third principal minor (i.e. in (32)) which grows as R grows. Also note that P_{44} is the only entry in the fourth principal minor (i.e. in (33)) which grows as $\bar{R}$ grows. Thus, conditions in (31)–(33) are always satisfied by choosing positive definite matrices K_V, A, B (F is a positive definite matrix) and matrices R and $\bar{R}$ large enough, i.e. large r_a and r_b. Hence, we conclude that $\dot{V}$, bounded in (30), can always be rendered globally negative semidefinite. Use of the LaSalle invariance principle ensures global asymptotic stability of $(\tilde{q},\dot{q},\vartheta,\rho,I_b) = (0,0,0,0,0)$ if (26) is satisfied. This completes the proof of Proposition 1.

We remark that conditions ensuring result in Proposition 1 are summarized in (26) and (31)–(33). Joint-wise selection of proportional gains as stated in (26) was proposed first time by the Hernández *et al.*[16]. Further, it was shown by Hernández *et al.*[17] that criterion introduced by Tomei[18] (which is commonly used to tune both PD and PID robot controllers) results in excessively large proportional gains for most joints. This is because Tomei's criterion requires the smallest proportional gain to dominate the gravity effect on the whole robot. Finally, note that according to (15) and the fact that[15] $|\tanh(u)| \le 1$, $\forall u\in\mathcal{R}$, we can write

$$|\delta_{ai}^*| \le M_i + K_{Vi}, \quad \forall\tilde{q}_i,\vartheta_i\in\mathcal{R}, \quad i = 1,\ldots,n$$

where, we recall, M_i comes from (11) and Definition 1. This fact is instrumental to globally dominate third-order term $\dot{q}^T N_p L_b I_B K_{T2}^{-1}\delta_a^*$ in (29).

Remark 1. In industrial practice it is common to consider that the torque applied by BLDC motors to robot joints is

https://doi.org/10.1017/S0263574709005724 Published online by Cambridge University Press

proportional to current. Further, the drives for those motors include some current controllers ensuring the generation of the desired torque. This is known as torque control or current control.[8] In the following we recall the procedure presented by Campa *et al.*[7] to implement this strategy for controlling BLDC motors under the assumption that $L_a = L_b$. In such case torque applied by motors to robot joints is given as $\tau = K_{T2} I_a$ and torque control can be written as

$$V_a = K_d (I_a^* - I_a), \tag{34}$$

where K_d is a diagonal positive definite matrix and I_a^* represents the value of the electric current I_a necessary to generate the desired torque τ^*, i.e.

$$I_a^* = K_{T2}^{-1} \tau^*. \tag{35}$$

Suppose that a PD control law is used as the desired torque

$$\tau^* = s(\kappa_p\, \tilde{q} + g(q_d)) + \kappa_v\, Tanh(\vartheta). \tag{36}$$

We stress that, in practice[7], it is always chosen r_a much larger than R_a and, hence $R \approx r_a$. Thus, V_a given in (8) is retrieved from (34), (35), (36) by setting $r_a = K_d$ and $\overline{K_V} = K_d K_{T2}^{-1} \kappa_v$. Note that this relaxes the requirement on the exact knowledge of R_a and also implies that $K_P \approx \kappa_p$ and $K_V \approx \kappa_v$. Aside from these facts, it is important to stress that our result is valid even if $L_a \neq L_b$.

4. Simulation Results

In this section we present some simulation results to study performance of controller in Proposition 1. We use the numerical values of the rigid robot reported by Kelly *et al.*[12] (Ch. 5) and Campa *et al.*[19]. This is a two-degrees-of-freedom rigid robot with both revolute joints moving on a vertical plane. Position $[q_1, q_2] = [0, 0]$ corresponds to configuration where both links are parallel and downwards. We also consider that this robot is equipped with two BLDC motors whose numerical parameters are those identified in ref. [7], i.e. $N_p = \text{diag}\{120, 120\}$, $R_a = R_b = \text{diag}\{1.9, 1.9\}$ (Ohm), $L_b = \text{diag}\{0.00636, 0.00636\}$ (H), $L_a = \text{diag}\{0.00672,\ 0.00672\}$ (H), $K_B = \text{diag}\{0.0106, 0.0106\}$ (Wb), $J = \text{diag}\{0.0025, 0.0025\}$ (kg m^2). Finally, the viscous friction coefficients matrix is selected as $F = \text{diag}\{0.203, 0.203\}$ (Nm/(rad/s)) which was also identified experimentally in ref. 7. In all simulations we choose all initial conditions equal to zero and desired link positions as $q_d = [\pi/9, \pi/30]^T$ (rad). Controller gains are chosen such that all of conditions (26), (31)–(33) are satisfied: $K_P = \text{diag}\{18, 14\}$, $\overline{K_V} = \text{diag}\{3144.9, 1347.8\}$, $A = \text{diag}\{200, 200\}$, $B = \text{diag}\{200, 200\}$. We choose $r_a = r_b = \text{diag}\{698, 698\}$ (Ohm) because Campa *et al.*[7] found that this value (i.e. $K_d = \text{diag}\{698, 698\}$ (Ohm)) is used in the actual commercial drive provided together with the BLDC motor identified in that work. Inspired by Zavala and Santibáñez[14] we used the following saturation function

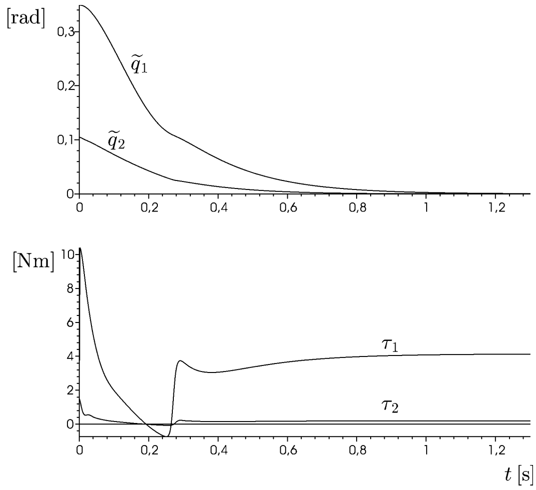

Fig. 1. Simulation results. Position errors and applied torques. Robot response is fast and well damped.

for $i = 1, 2$:

$$\sigma_i(\varsigma) = \begin{cases} -L_i + (M_i - L_i)\tanh\left(\frac{\varsigma + L_i}{M_i - L_i}\right), & \text{if } \varsigma < -L_i \\ \varsigma, & \text{if } |\varsigma| \leq L_i \\ L_i + (M_i - L_i)\tanh\left(\frac{\varsigma - L_i}{M_i - L_i}\right), & \text{if } \varsigma > L_i \end{cases}$$

where $L_1 = 12.1,\ L_2 = 0.7,\ M_1 = 12.5,\ M_2 = 2$. These values satisfy (11) because, according to numerical parameters reported by Kelly *et al.*[12] (Ch. 5), $k_1' = 11.9674$ and $k_2' = 0.4596$. We are not interested in showing that the applied voltages are bounded: from (8) and (14) we realize that large values of R may render V_a very large in spite of saturation functions. Note that conditions (31)–(33) require their first terms to dominate the remaining terms. These first terms involve small values of the viscous friction coefficients matrix F. Hence, the reader may wonder whether this may require an excessively large matrix K_V, i.e. degrading performance (if we choose $A = B$ they cancel in P_{22}). What we want to show is that this is not the case: we only have to choose large values for r_a and r_b which is a common selection in practice (see Remark 3.1). Moreover, although the derivative gain $\overline{K_V} = \text{diag}\{3144.9, 1347.8\}$ may seem to be very large, we used (14) to find that this corresponds to $K_V = \text{diag}\{7, 3\}$, which is the common selection in experimental tests reported in ref. [12, Ch. 8].

In Figs. 1 and 2 we present the simulation results that we obtained. We set all initial conditions to zero. Note that the desired positions are reached in approximately 1 s without overshoot. Also note that torques remain within limits of the experimental platform reported in ref. [19], i.e. 15 (Nm) for joint 1 and 4 (Nm) for joint 2. Although we observe in Fig. 2 that voltages do not exceed 15 (V) for motor 1 and 2 (V) for motor 2, it is important to say that very large peak values appear on both voltage signals which disappear in about 10^{-4} s. Reason for this is that large voltage values are computed by controller when large electric current errors appear, i.e. when discontinuous desired positions are commanded, because of the large proportional gain in the electric current loop (see Remark 3.1). Input voltages decrease very fast as the electric

https://doi.org/10.1017/S0263574709005724 Published online by Cambridge University Press

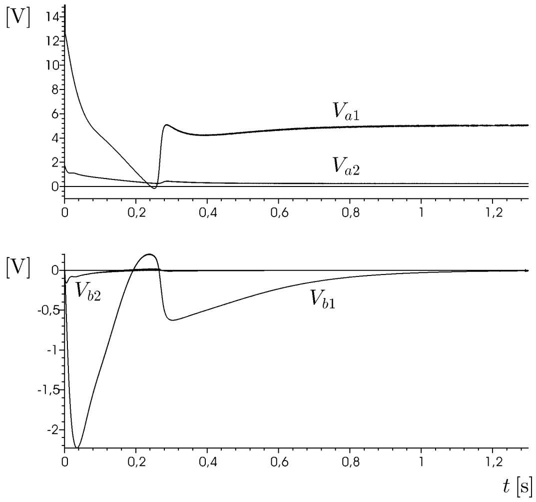

Fig. 2. Simulation results. Applied voltages. Only V_a contributes to steady-state torques since V_b converges to zero.

current error decreases. However, such large peak voltages are not possible in practice. Thus, we saturated voltage at ± 25 (V) for motor 1 and ± 5 (V) for motor 2 and identical results were observed, i.e. limitations of practical power amplifiers do not degrade performance. We stress that these events occur in practice whenever torque control is used, i.e. whenever we use in practice any controller designed under the assumption that torques are the control inputs. Finally, let us remark that torques and voltages in Figs. 1 and 2 are not so oscillatory as torques and voltages in simulations reported in ref. [6] which were performed under the same desired positions and the same initial positions and velocities. This represents an important advantage of controller in the present paper in any practical application. Of course, this is due to the fact that controller in the present paper is much simpler than the controller in ref. [6].

5. Conclusions

We have shown that a saturated PD controller plus linear feedback of electric current suffices to ensure global asymptotic stability of rigid robots when the electric dynamics of the brushless DC motors used as actuators is taken into account in the stability analysis. Further, velocity measurements are not required. This is the simplest control strategy that has been presented until now to solve this problem. Although saturation functions are used in the controller, we do not try to ensure that applied voltages (control input) remain bounded. This is because the large proportional gain used in the electric current loop produces large peak voltages which, however, disappear very fast. We stress that this feature is common when torque control is used in practice, i.e. when torques are assumed to be the control input.

Appendix A: Function $\mathcal{U}_{cl}(\tilde{q})$ in (27)

Function $\mathcal{U}_{cl}(\tilde{q})$ in (27) was introduced by Zavala and Santibáñez[14]. However, in this part we show that (26) is a novel condition to ensure positive definiteness and radial unboundedness of $\mathcal{U}_{cl}(\tilde{q})$. We prove that $\mathcal{U}_{cl}(\tilde{q})$ is positive definite and radially unbounded if we prove that so are $\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i$, for $i = 1, \ldots, n$. Recall that all variables different from r_i remain constant as integral with respect to $\tilde{q}_i$ is computed. We divide study of these integrals in two parts:

1. Assume that $\tilde{q}_i > 0$. Define a constant $\tilde{q}_i^* > 0$ such that $x_i^* = K_{Pi}\,\tilde{q}_i^* + g_i(q_d) = L_i$, i.e. $\sigma_i(x_i^*) = L_i$. First suppose that $\tilde{q}_i \le \tilde{q}_i^*$, i.e. $x_i = K_{Pi}\,\tilde{q}_i + g_i(q_d) \le L_i$. Note that integral $\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i$ is positive definite if

$$\frac{\partial^2}{\partial \tilde{q}_i^2}\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i = \frac{\partial \overline{g}_i(q_i)}{\partial q_i} + \frac{\mathrm{d}\sigma_i(x_i)}{\mathrm{d}x_i}K_{Pi} > 0,\ \forall q, q_d \in \mathcal{R}^n,\ x_i \le x_i^* \tag{A1}$$

where $\overline{g}_i(q_i)$ is obtained from (28) just by using $q_i = q_{di} - r_i$ and $r_i = \tilde{q}_i$. We stress that all variables different from $\tilde{q}_i$, or equivalently r_i, remain constant as (A1) is computed. According to Definition 1, $\frac{\mathrm{d}\sigma_i(x_i)}{\mathrm{d}x_i} = 1$ in (A1) because $x_i \le x_i^* = L_i$ implies that $\sigma_i(x_i) = x_i$. Hence, (A1) is true if

$$\frac{\partial \overline{g}_i(q_i)}{\partial q_i} + K_{Pi} > 0,\ \forall q, q_d \in \mathcal{R}^n,\ x_i \le x_i^*.$$

We conclude that $\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i$ is positive definite for all $\tilde{q}_i \le \tilde{q}_i^*$ if (26) is satisfied.
Now consider the case when $\tilde{q}_i > \tilde{q}_i^*$. Note that $x_i = K_{Pi}\,\tilde{q}_i + g_i(q_d) > L_i$, i.e. $\sigma_i(x_i) > L_i$, and we can write

$$\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i = \int_0^{\tilde{q}_i^*}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i + \int_{\tilde{q}_i^*}^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i. \tag{A2}$$

According to (11), $[-\overline{g}_i(\tilde{q}_i)+\sigma_i(x_i)] > L_i - k_i' > 0$ because $\sigma_i(x_i) > L_i$ in this case. Thus, we can write

$$\int_{\tilde{q}_i^*}^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i > \int_{\tilde{q}_i^*}^{\tilde{q}_i}(L_i - k_i')\mathrm{d}r_i = (L_i - k_i')(\tilde{q}_i - \tilde{q}_i^*) > 0.$$

Since we have proven that first right-hand integral in (A2) is positive, this proves that $\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\, r_i + g_i(q_d))]\mathrm{d}r_i$ is positive definite and unbounded if $\tilde{q}_i > 0$.

2. Assume that $\tilde{q}_i < 0$. Define a constant $\tilde{q}_i^* < 0$ such that $x_i^* = K_{Pi}\,\tilde{q}_i^* + g_i(q_d) = -L_i$, i.e. $\sigma_i(x_i^*) = -L_i$. First suppose that $\tilde{q}_i \ge \tilde{q}_i^*$, i.e. $x_i = K_{Pi}\,\tilde{q}_i + g_i(q_d) \ge -L_i$.

https://doi.org/10.1017/S0263574709005724 Published online by Cambridge University Press

Note that integral $\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))]\mathrm{d}r_i$ is positive definite if

$$\frac{\partial^2}{\partial \tilde{q}_i^2}\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))]\mathrm{d}r_i = \frac{\partial \overline{g}_i(q_i)}{\partial q_i}+\frac{\mathrm{d}\sigma_i(x_i)}{\mathrm{d}x_i}K_{Pi}>0,\ \forall q, q_d\in\mathcal{R}^n,\ x_i\geq x_i^*. \tag{A3}$$

We stress that all variables different from $\tilde{q}_i$, or equivalently r_i, remain constant as (A3) is computed. According to Definition 1, $\frac{\mathrm{d}\sigma_i(x_i)}{\mathrm{d}x_i}=1$ in (A3) because $x_i\geq x_i^*=-L_i$ implies that $\sigma_i(x_i)=x_i$. Hence, (A3) is true if

$$\frac{\partial \overline{g}_i(q_i)}{\partial q_i}+K_{Pi}>0,\ \forall q, q_d\in\mathcal{R}^n,\ x_i\geq x_i^*.$$

We conclude that $\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))]\mathrm{d}r_i$ is positive definite for all $\tilde{q}_i\geq\tilde{q}_i^*$ if (26) is satisfied.

Now consider the case when $\tilde{q}_i<\tilde{q}_i^*$. Note that $x_i=K_{Pi}\ \tilde{q}_i+g_i(q_d)<-L_i$, i.e. $\sigma_i(x_i)<-L_i$, and we can write

$$\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))]\mathrm{d}r_i = \int_0^{\tilde{q}_i^*}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))]\mathrm{d}r_i + \int_{\tilde{q}_i^*}^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))]\mathrm{d}r_i. \tag{A4}$$

According to (11), $[-\overline{g}_i(\tilde{q}_i)+\sigma_i(x_i)]<-L_i+k_i'<0$, i.e. $|-\overline{g}_i(\tilde{q}_i)+\sigma_i(x_i)|>|-L_i+k_i'|>0$, because $\sigma_i(x_i)<-L_i$ in this case. Thus, we can write

$$\int_{\tilde{q}_i^*}^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))]\mathrm{d}r_i = \int_{\tilde{q}_i}^{\tilde{q}_i^*}|-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))|\mathrm{d}r_i > \int_{\tilde{q}_i}^{\tilde{q}_i^*}|-L_i+k_i'|\mathrm{d}r_i = |-L_i+k_i'|(\tilde{q}_i^*-\tilde{q}_i)>0$$

because $\tilde{q}_i^*-\tilde{q}_i>0$ in this case. Since we have proven that first right-hand integral in (A4) is positive, this proves that $\int_0^{\tilde{q}_i}[-\overline{g}_i(r_i)+\sigma_i(K_{Pi}\ r_i+g_i(q_d))]\mathrm{d}r_i$ is positive definite and unbounded if $\tilde{q}_i<0$.

Thus, $\mathcal{U}_{cl}(\tilde{q})$ in (27) is positive definite and radially unbounded if (26) is satisfied.

Acknowledgments

A. Zavala was partially supported by CONACYT, México, under grant 48281 S-3119.

References

1. M. M. Bridges and D. M. Dawson, "Adaptive Control of Rigid-Link Electrically-Driven Robots Actuated with Brushless DC Motors," *Proceedings of 33rd Conference on Decision and Control*, Lake Buena Vista, FL (1994), pp. 1284–1289.
2. J. Hu, D. M. Dawson, T. Burg and P. Vedagarbha, "An Adaptive Tracking Controller for a Brushless DC Motor with Reduced Overparametrization Effects," *Proceedings of 33rd Conference on Decision and Control*, Lake Buena Vista, FL (1994), pp. 1850–1855.
3. H. Melkote and F. Khorrami, "Nonlinear adaptive control of direct-drive brushless DC motors and applications to robotic manipulators," *IEEE/ASME Trans. Mechatron.* **4**(1), 71–81 (1999).
4. N. Hemati, J. Thorp and M. C. Leu, "Robust nonlinear control of brushless DC motors for direct-drive robotic applications," *IEEE Trans. Indus. Electr.* **37**(6), 460–468 (1990).
5. R. Ortega, A. Loría, P. Nicklasson and H. Sira-Ramírez, *Passivity-Based Control of Euler–Lagrange Systems* (Springer, London, 1998).
6. V. M. Hernández-Guzmán, V. Santibáñez and R. Campa, "PID control of robot manipulators equipped with brushless DC motors," *Robotica* **27**, 225–233 (2009).
7. R. Campa, E. Torres, V. Santibáñez and R. Vargas, "Electromechanical Dynamics Characterization of Brushless Direct-Drive Servomotor," *Proceedings of VII Mexican Congress on Robotics, COMRob 2005*, México, D.F. (2005).
8. Parker Automation, "Compumotor's Virtual Classroom," Position Systems and Controls, Training and Product Catalog, CD-ROM (1998).
9. D. M. Dawson, J. Hu and T. C. Burg, *Nonlinear control of electric machinery* (Marcel Dekker, New York, 1998).
10. P. C. Krause, O. Wasynczuk and S. D. Sudhoff, *Analysis of Electric Machinery and Drive Systems* (IEEE Press, Hoboken, New Jersy, 2002).
11. R. Kelly, "A tuning procedure for stable PID control of robot manipulators," *Robotica* **13**, 141–148 (1995).
12. R. Kelly, V. Santibáñez and A. Loría. *Control of Robot Manipulators in Joint Space* (Springer, London, 2005).
13. H. K. Khalil, *Nonlinear Systems* (Prentice Hall, Upper Saddle River, 2002).
14. A. Zavala-Río and V. Santibañez, "A natural saturating extension of the PD-with-desired-gravity-compensation control law for robot manipulators with bounded inputs," *IEEE Trans. Rob.* 23 (2), 386–391 (2007).
15. M. R. Spiegel. *Mathematical Handbook of Formulas and Tables* (McGraw-Hill, Schaum, New York, 1974).
16. V. M. Hernández-Guzmán, V. Santibáñez and R. Silva-Ortigoza, "A new tuning procedure for PID control of rigid robots," *Adv. Rob.* **22**, 1007–1023 (2008).
17. V. M. Hernández, V. Santibáñez, R. V. Carrillo, J. Molina and J. J. López, "PD control of robots: Actuator dynamics and a new tuning procedure," (in Spanish), *Revista Iberoamericana de Automática e Informática Industrial* **5**(4), 62–68 (2008).
18. P. Tomei, "Adaptive PD controller for robot manipulators," *IEEE Trans. Rob. Automat.* **7**(4), 565–570 (1991).
19. R. Campa, R. Kelly and V. Santibañez, "Windows-based real-time control of direct-drive mechanisms: platform description and experiments," *Mechatronics* **14**, 1021–1036 (2004).