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Existence and uniqueness of positive solution of a logistic equation with nonlinear gradient term

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The main goal of this work is to study the existence and uniqueness of a positive solution of a logistic equation including a nonlinear gradient term. In particular, we use local and global bifurcation together with some *a priori* estimates. To prove uniqueness, the sweeping method of Serrin is employed.

1. Introduction

We study the existence and possible uniqueness of positive solutions of the problem

$$\left.\begin{aligned} -\Delta u &= \lambda u - u^p \pm |\nabla u|^q \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{$P_\pm$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and regular domain and $1 < p, q$. The constant $\lambda \in \mathbb{R}$ will be regarded as a bifurcation parameter.

When the nonlinear gradient term does not appear, the above equation is the classical logistic one that has been extensively studied in the literature. It is very well known that there exists a positive solution if and only if $\lambda > \lambda_1$, where λ_1 denotes the principal eigenvalue of the Laplacian. In such a case, this solution is unique and stable.

However, when the gradient term is included, the equation is less well known. It was studied in [4,12,18] for the particular case $\lambda = 0$ and under a blow-up Dirichlet condition. In the aforementioned papers, the solution is obtained as the limit of a sequence of solutions of the corresponding non-homogeneous Dirichlet problems. In each step, the authors use [17, theorem 8.3, p. 301], and so $q \leqslant 2$ must be imposed. Some other papers consider similar problems with critical growth ($q = 2$), in which a convenient change of variables works (see, for example, [20, 21, 23]).

Generally speaking, equations of the form

$$-\Delta u = f(x, u, \nabla u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

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have been studied extensively in the literature (see the classical works [3,5,6,8,11,15, 22]). Most of these works are concerned with *a priori* estimates, and existence from those is obtained by using topological methods or sub–supersolution techniques.

In order to obtain our existence results, we use bifurcation methods (local and global) and *a priori* bounds in C^1. It is well known that these *a priori* estimates hold for $q \leqslant 2$. However, as a special feature of our problem, we can even provide a C^1 estimate for (P_-) and any $q > 1$. In order to do that, we first find a supersolution, greater than any solution u of (P_-), and from this we can estimate the maximum of the function $w = \frac{1}{2}|\nabla u|^2$ (see, for example, [26, ch. 5] and [23]).

In general, the uniqueness of elliptic equations is a difficult problem, as remarked on in [10]. For (1.1) one can easily obtain uniqueness from the maximum principle if f is decreasing in u [13, theorem 8.1]. Clearly, the function $f(u) = \lambda u - u^p - |\nabla u|^q$ is not decreasing in u for $\lambda > 0$, and these results cannot be applied. In any case, $u = 0$ is always a solution: we are looking for uniqueness of positive solutions.

We present here a result of the uniqueness of positive solutions for (1.1) which generalizes a classical result for semilinear equations (see, for example, [2,7,14], [16, theorems 7.14 and 7.15] and [24, p. 39] and references therein). Our proof makes use of the sweeping method of Serrin [25, p. 12], as in [16, 24]. To the best of our knowledge, the uniqueness result stated in theorem 4.1 is completely new.

Generally speaking, we show that in the case (P_-) the results about existence and uniqueness are similar to those for the semilinear equation. However, in the case (P_+) the presence of the gradient may have an important influence (depending on the values of p and q; see theorem 3.3). In some cases there is no uniqueness of positive solution.

An outline of the work is as follows: in § 2 we give some preliminaries regarding the maximum principle; § 3 is devoted to existence and non-existence results using bifurcation and *a priori* estimates; and the general result of uniqueness is established in § 4.

2. Preliminaries

In this section we state the exact version of the maximum principle that will be used throughout the paper. The results we present are classical; we include them for the sake of clarity.

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $c_i, d : \Omega \to \mathbb{R}$ be L^∞ functions $(i = 1, \dots, n)$ and consider the inequality

$$-\Delta u + \sum_{i=1}^{N} c_i(x) \frac{\partial u}{\partial x_i} + d(x)u \geqslant 0. \tag{2.1}$$

We begin with a generalized weak maximum principle. A more general statement and the proof can be found in [13, theorem 8.1].

Theorem 2.1. *Suppose that $d \geqslant 0$ and that $u \in H^1(\Omega)$ is a weak solution of (2.1) such that $u \geqslant 0$ on $\partial\Omega$. Then, $u \geqslant 0$.*

We now state a version of the strong maximum principle for C^1 weak solutions; throughout the paper, η denotes the unit outward normal to $\partial\Omega$.

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THEOREM 2.2. *Suppose that $d \geqslant 0$ and that $u \in C^1(\bar{\Omega})$ is a weak solution of the inequality (2.1). Assume also that u is not constantly equal to zero, and $u(x) = 0$ for all $x \in \partial\Omega$.*

Then

$$u(x) > 0 \quad \forall x \in \Omega \quad \text{and} \quad \frac{\partial u}{\partial \eta}(x) < 0 \quad \forall x \in \partial\Omega.$$

Proof. The proof follows that of [13, lemma 3.4]. Since the statement we have chosen does not exactly match that in [13], we reproduce the proof in detail. There are several steps.

STEP 1. Define an open ball $B = B(y, R) \subset \mathbb{R}^N$, and suppose that u verifies inequality (2.1) in B, $u(x) > 0$ for all $x \in B$, $u(x_0) = 0$ for some $x_0 \in \partial B$. We claim that $\partial u(x_0)/\partial\eta < 0$, where η is the unit outward normal to B at x_0.

We can assume that $y = 0$. Let $\alpha > 0$ be a constant to be determined later, and define $v(x) = \mathrm{e}^{-\alpha|x|^2} - \mathrm{e}^{-\alpha R^2}$ defined in the annulus $A = \{x \in \mathbb{R}^N : \frac{1}{2}R \leqslant |x| \leqslant R\}$. For any $x \in A$, we compute

$$\begin{aligned} -\Delta v &+ \sum_{i=1}^N c_i(x)\frac{\partial v}{\partial x_i} + d(x)v \\ &= \mathrm{e}^{-\alpha|x|^2}\biggl(-4|x|^2\alpha^2 + 2\alpha N - 2\alpha\sum_{i=1}^N c_i(x)x_i + d(x)[1 - \mathrm{e}^{-\alpha(R^2-|x|^2)}]\biggr) \\ &\geqslant \mathrm{e}^{-\alpha|x|^2}(-4(\tfrac{1}{2}R)^2\alpha^2 + 2\alpha N - 2\alpha MR - M), \end{aligned} \tag{2.2}$$

where M is a constant such that $M > \|d\|_{L^\infty}$, $M > (\sum_{i=1}^N \|c_i\|_{L^\infty}^2)^{1/2}$.

We now choose sufficiently large α such that (2.2) is negative. Clearly, $v(x) = 0 \leqslant u(x)$ for $|x| = R$. Since u is positive in the ball, we can take $\varepsilon > 0$ small enough that $\varepsilon v(x) < u(x)$ for $|x| = \frac{1}{2}R$.

Now, apply theorem 2.1 to the function $u - \varepsilon v$ in A, to conclude that $u(x) \geqslant \varepsilon v(x)$ for all $x \in A$. Recall now that $u(x_0) = v(x_0) = 0$, to conclude that

$$\frac{\partial u}{\partial\eta}(x_0) \leqslant \varepsilon\frac{\partial v}{\partial\eta}(x_0) < 0.$$

STEP 2 ($u(x) > 0$ for any $x \in \Omega$). By theorem 2.1, we have $u \geqslant 0$. In order to prove the strict inequality, we reason by contradiction. Take $\Omega_0 = \{x \in \Omega : u(x) = 0\}$, $\Omega_+ = \{x \in \Omega : u(x) > 0\}$. Observe that both previous sets are non-empty. Choose $y \in \Omega_+$ such that $R = d(y, \Omega_0) < d(y, \partial\Omega)$. Then, $B(y, R) \subset \Omega_+$ and there exists $x_0 \in \partial B \cap \Omega_0$. Obviously, u attains a minimum at x_0, and hence $\nabla u(x_0) = 0$. We then arrive to a contradiction with step 1.

STEP 3 ($\partial u(x)/\partial\eta < 0$ for all $x \in \partial\Omega$). Take $x_0 \in \partial\Omega$, and let B be an interior sphere at x_0. By step 2, $u(x) > 0$ for $x \in B$, and $u(x_0) = 0$. We conclude by applying step 1. □

In the previous theorem, the condition $d(x) \geqslant 0$ is needed. In next theorem we consider any L^∞ function $d(x)$ but assume that the function u is non-negative (which is no longer given by theorem 2.1).

THEOREM 2.3. *Suppose that $u \in C^1(\bar{\Omega})$ is a weak solution of the inequality (2.1). Assume also that $u(x) \geqslant 0$ but is not constantly equal to zero, and that $u(x) = 0$ for all $x \in \partial\Omega$.*

Then, $u(x) > 0$ for all $x \in \Omega$ and $\partial u(x)/\partial\eta < 0$ for all $x \in \partial\Omega$.

Proof. We define $d^+(x) = \max\{d(x), 0\}$. Since u is non-negative, it follows that

$$-\Delta u + \sum_{i=1}^{N} c_i(x)\frac{\partial u}{\partial x_i} + d^+(x)u \geqslant -\Delta u + \sum_{i=1}^{N} c_i(x)\frac{\partial u}{\partial x_i} + d(x)u \geqslant 0.$$

Now we conclude the proof by applying theorem 2.2 to the operator:

$$L[z] = -\Delta z + \sum_{i=1}^{N} c_i(x)\frac{\partial z}{\partial x_i} + d^+(x)z.$$

□

3. Bifurcation of positive solution

First, we need some notation: λ_1 and φ_1 denote the principal eigenvalue of $-\Delta$ subject to the homogeneous Dirichlet boundary condition and its positive associated eigenfunction, respectively.

We denote by $E := C^1(\bar{\Omega})$ and $P = \{u \in E : u(x) \geqslant 0\ \forall x \in \Omega\}$ its positive cone. We look for u solution belonging to P. Observe that if $u \in E$ is a solution of $(P_\pm)$, by the elliptic regularity we have $u \in C^{3,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$. On the other hand, if u is a non-trivial solution in P, we have

$$-\Delta u \mp |\nabla u|^{q-2}\nabla u \cdot \nabla u - (\lambda - u^{p-1})u = 0,$$

and therefore theorem 2.3 implies that $u(x) > 0$ for any $x \in \Omega$, and also that $\partial u(x)/\partial\eta < 0$. That is to say, $u \in \mathrm{Int}(P)$; in such a case we say that u is positive.

Our first result provides us with the existence of a continuum of positive solutions of $(P_\pm)$. We also obtain an *a priori* bound in L^∞ for any solution of $(P_\pm)$ in E.

PROPOSITION 3.1. *There exists an unbounded continuum of positive solution $\mathcal{C} \subset \mathbb{R} \times E$ bifurcating from the trivial solution at $\lambda = \lambda_1$.*

Moreover, if (λ, u) is a solution of $(P_\pm)$, $u \neq 0$, then

$$\lambda > 0 \quad \text{and} \quad \|u\|_\infty \leqslant \lambda^{1/(p-1)}. \tag{3.1}$$

Proof. Observe that $(P_\pm)$ can be written as

$$u = \mathcal{K}(\lambda u + \mathcal{N}(u)) \quad \text{in } E,$$

where $\mathcal{K} := (-\Delta)^{-1} : C(\bar{\Omega}) \to E$ under homogeneous Dirichlet boundary conditions, and

$$\mathcal{N} : E \to C(\bar{\Omega}), \qquad \mathcal{N}(u) := (-u^p \pm |\nabla u|^q).$$

Observe that if $f \in C(\bar{\Omega})$, then $u = \mathcal{K}(f)$ belongs to $W^{2,p}$ for all $p > 1$ [1], and hence $u \in C^{1,\gamma}$. Thus, the operator $\mathcal{K}$ is compact. The strong maximum principle implies also that is strongly positive.

The nonlinear operator $\mathcal{N}$ is continuous and bounded. Moreover, since $p > 1$, $q > 1$, we have $\mathcal{N}(u) = o(\|u\|_E)$ as $u \to 0$; we can apply, for example, [19, theorem 6.5.5] and conclude the existence of an unbounded continuum $\mathcal{C}$ in $\mathbb{R} \times P$ of positive solution of $(P_\pm)$ emanating from $(\lambda_1, 0)$.

On the other hand, suppose that $\lambda \leqslant 0$ and that $u \in P$ is a solution of $(P_\pm)$. Then $L[u] = 0$, where L is defined as

$$L[z] = -\Delta z \mp |\nabla u|^{q-2}\nabla u \cdot \nabla z - (\lambda - u^{p-1})z = 0.$$

Since $(\lambda - u^{p-1}) \leqslant 0$, we can apply theorem 2.1 to $-u$ to conclude that $u \leqslant 0$. Therefore, $u = 0$.

Finally, if $x_M \in \Omega$ is such that $u(x_M) = \max_{x\in\bar{\Omega}} u(x)$, then

$$\lambda u(x_M) - u(x_M)^p \pm |\nabla u(x_M)|^q \geqslant 0,$$

and now, taking into account that $\nabla u(x_M) = 0$, we get (3.1). □

The next result characterizes the existence and uniqueness of a positive solution of (P_-).

THEOREM 3.2. *Consider the case* (P_-). *For all* $p, q > 1$ *there exists a positive solution if and only if* $\lambda > \lambda_1$. *Moreover, if* $\lambda > \lambda_1$, *there exists a unique positive solution of* (P_-) *that is linearly asymptotically stable.*

Proof. If u is a positive solution of (P_-), then, multiplying the equation by φ_1 and integrating by parts, we have

$$(\lambda_1 - \lambda)\int_\Omega u\varphi_1 = \int(-u^p - |\nabla u|^q)\varphi_1,$$

which implies that $\lambda > \lambda_1$.

Now, by proposition 3.1, the proof of existence for $\lambda > \lambda_1$ concludes if we find *a priori* estimates. Specifically, we claim that if (λ, u) is a positive solution of (P_-) and $\lambda \in I \subset \mathbb{R}$ is compact, then there exists $C > 0$ such that

$$\|u\|_E \leqslant C. \tag{3.2}$$

We prove the result for all $q > 1$ by estimating the derivative first in the boundary, and then in the interior of Ω.

First of all, we claim that if u is a solution of (P_-), then

$$u \leqslant \theta_\lambda, \tag{3.3}$$

where θ_λ is the unique positive solution of the logistic equation

$$\left.\begin{aligned} -\Delta u &= \lambda u - u^p \quad \text{in } \Omega,\\ u &= 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.4}$$

Indeed, if u is a solution of (P_-), then u is a subsolution of (3.4). Clearly, a large positive constant is a supersolution of (3.4). The sub–supersolution method yields a solution of (3.4) greater than u and, since θ_λ is the unique positive solution of (3.4), (3.3) is verified.

Hence, it follows that

$$\frac{\partial \theta_\lambda}{\partial \eta} \leqslant \frac{\partial u}{\partial \eta} < 0 \quad \text{on } \partial\Omega. \tag{3.5}$$

This gives us an *a priori* bound of ∇u on the boundary.

In order to estimate the gradient in the interior of Ω, define

$$w := \tfrac{1}{2}|\nabla u|^2.$$

It is not hard to show that

$$\Delta w = 2w(pu^{p-1} - \lambda) + q|\nabla u|^{q-2}\nabla u \cdot \nabla w + \sum_{i,j=1}^{N} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2.$$

Assume that the maximum of w is attained at $x_M \in \Omega$. Then using $\Delta w(x_M) \leqslant 0$ and $\nabla w(x_M) = 0$, we get

$$2w(x_M)(\lambda - pu^{p-1}(x_M)) \geqslant \sum_{i,j=1}^{N} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_M)\right)^2. \tag{3.6}$$

On the other hand, there exists a $C > 0$ such that

$$\sum_{i,j=1}^{N} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_M)\right)^2 \geqslant C(\Delta u(x_M))^2 = C(-\lambda u(x_M) + u(x_M)^p + |\nabla u(x_M)|^q)^2.$$

By taking into account the L^∞ bound of u, there exist positive constants $C_1, C_2 > 0$ such that

$$2w(x_M)(\lambda - pu^{p-1}(x_M)) \geqslant C_1 w(x_M)^q - C_2. \tag{3.7}$$

Inequalities (3.5) and (3.7) imply the *a priori* estimate in C^1 of the solutions of (P_-) for any λ fixed.

The uniqueness of positive solution follows from theorem 4.1, whose statement and proof are postponed to the next section.

Let $u_0 > 0$ be a positive solution of (P_-). We now plan to prove that u_0 is asymptotically stable. It is well known (see, for example, [24]) that for stability it suffices to show that the first eigenvalue of the problem linearized around u_0 is positive, i.e. the first eigenvalue of the problem

$$\left.\begin{aligned} L[v] &= \sigma v \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{3.8}$$

is positive, where

$$L[v] := -\Delta v + q|\nabla u_0|^{q-2}\nabla u_0 \cdot \nabla v + (pu_0^{p-1} - \lambda)v.$$

Thus, it suffices to find a positive supersolution $\bar{u}$ of L, that is, a positive function $\bar{u}$ such that $L[\bar{u}] \geqslant 0$ in Ω and $\bar{u} \geqslant 0$ on $\partial\Omega$ with some strict inequality. Take $\bar{u} = u_0$. Then

$$L[\bar{u}] = (q-1)|\nabla u_0|^q + (p-1)u_0^p > 0 \text{ in } \Omega \quad \text{and} \quad \bar{u} = 0 \text{ on } \partial\Omega.$$

The proof is completed. □

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Observe that, for (P_-), the bifurcation is always *supercritical* for any values of p and q; that is, there exists a neighbourhood $\mathcal{V}$ of $(\lambda_1, 0)$ such that every positive solution $(\lambda, u) \in \mathcal{V}$ of (P_-) satisfies $\lambda > \lambda_1$ (we can define *subcritical* similarly). The case for (P_+) is different.

THEOREM 3.3. *Consider the case* (P_+).

(i) *With respect to the local bifurcation, we find that*

(a) *if* $p > q$ *(respectively,* $p < q$*) the bifurcation is subcritical (respectively, supercritical),*

(b) *if* $p = q$*, and* (λ_n, u_n) *is a solution of* (P_+) *such that* $\lambda_n \to \lambda_1$ *and* $\|u_n\|_E \to 0$ *as* $n \to \infty$*, then*

$$0 < (\lambda_n - \lambda_1)\biggl(\int_\Omega \varphi_1^{p+1} - \int_\Omega |\nabla \varphi_1|^q \varphi_1\biggr).$$

whenever the last term is non-zero,

(ii) *If* $q \leqslant 2$*, then there exists at least a positive solution for* $\lambda > \lambda_1$.

Proof. It is clear that solutions of (P_+) are the zeros of the regular operator $\mathcal{F} : E \times \mathbb{R} \mapsto E$ defined by

$$\mathcal{F}(u, \lambda) := -\Delta u - \lambda u + u^p - |\nabla u|^q.$$

Observe that $\mathcal{F}(0, \lambda) = 0$. Denoting by $N[\cdot]$ and $R[\cdot]$ the kernel and rank of the operator, respectively, we can can show that

$$N[D_u \mathcal{F}(0, \lambda_1)] = [\operatorname{span} \varphi_1].$$

We claim that

$$D_{\lambda u} \mathcal{F}(0, \lambda_1)\varphi_1 = -\varphi_1 \notin R[D_u \mathcal{F}(0, \lambda_1)].$$

Indeed, if there exists $u \in E$ satisfying

$$D_u \mathcal{F}(0, \lambda_1) u = -\varphi_1 \quad \iff \quad -\Delta u - \lambda_1 u = -\varphi_1,$$

then, by multiplying this equation by φ_1 and integrating, we obtain

$$\int_\Omega \varphi_1^2 = 0,$$

which is a contradiction. Now we are in a position to apply the Crandall–Rabinowitz theorem [9], and so, if Y is any closed subspace of E such that $E = [\operatorname{span} \varphi_1] \oplus Y$, there exist $\varepsilon > 0$ and two continuous functions

$$\lambda : (-\varepsilon, \varepsilon) \mapsto \mathbb{R} \quad \text{and} \quad u : (-\varepsilon, \varepsilon) \mapsto Y$$

with

$$\lambda(s) = \lambda_1 + \mu(s), \quad u(s) = s(\varphi_1 + v(s)), \quad s \in (-\varepsilon, \varepsilon) \tag{3.9}$$

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and $\mu(0) = v(0) = 0$ and, in a neighbourhood of $(\lambda_1, 0)$, all the solutions are of the form $(\lambda(s), u(s))$. Introducing this expression into the equation for (P_+), taking into account the fact that that $-\Delta\varphi_1 = \lambda_1\varphi_1$ and dividing by s, we obtain

$$(-\Delta - \lambda_1)v(s) = \mu(s)(\varphi_1 + v(s)) - s^{p-1}(\varphi_1 + v(s))^p + s^{q-1}|\nabla(\varphi_1 + v(s))|^q.$$

Finally, applying the Fredholm alternative, we get

$$\mu(s) = \frac{s^{p-1}\int_\Omega(\varphi_1 + v(s))^p\varphi_1 - s^{q-1}\int_\Omega(\nabla\varphi_1 + \nabla v(s))^q\varphi_1}{\int_\Omega(\varphi_1 + v(s))\varphi_1},$$

whence we deduce (i).

In order to prove (ii), recall that (P_+) has no non-zero solutions if $\lambda \leqslant 0$ (see theorem 3.1).

Moreover, since $q \leqslant 2$, the nonlinearity $f(x, \xi, \eta) = \lambda\xi - \xi^p + |\eta|^q$ satisfies

$$|f(x, \xi, \eta)| \leqslant c(|\xi|)(1 + |\eta|^2) \quad \text{for } (x, \xi, \eta) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N,$$

for some increasing function $c : \mathbb{R}_+ \mapsto \mathbb{R}_+$, and so, by [3, proposition 2], it follows that

$$\|u\|_E \leqslant \gamma(\|u\|_\infty) \leqslant C,$$

where $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}$ is an increasing function and C a positive constant. Thus, the result follows by proposition 3.1 and the fact that there exists no solution for $\lambda \leqslant 0$ (see (3.1)). □

Corollary 3.4. *If $q \leqslant 2$ and $p > q$, then there exist at least two positive solutions of (P_+) in $(\lambda_1 - \delta, \lambda_1)$ for some small $\delta > 0$.*

Proof. Define $\mathcal{C} \subset \mathbb{R}^+ \times \mathrm{Int}(P)$, the connected set of solutions bifurcating from $(\lambda_1, 0)$. As stated in the proof of theorem 3.3, in a neighbourhood $\mathcal{U}$ of $(\lambda_1, 0)$, all solutions of (P_+) are of the form $(\lambda(s), u(s))$, where $\lambda(s) < \lambda_1$ and $u(s)$ are defined in (3.9). We may assume that this neighbourhood is given by

$$\mathcal{U} = (\lambda_1 - \delta, \lambda_1 + \delta) \times (B(0, \varepsilon) \cap P)$$

for some $\varepsilon > 0$, $\delta > 0$. By taking smaller δ if necessary, we may also assume that (P_+) has no solutions (λ, u) such that $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, $\|u\|_E = \varepsilon$.

Recall that $\mathcal{C}$ does not cross the $\lambda = 0$ line, and cannot blow up for finite λ, since we have obtained *a priori* estimates. Thus, the unbounded continuum $\mathcal{C}$ has an unbounded projection $[\underaccent{\bar}{\lambda}, +\infty)$ on the real λ-axis with $\underaccent{\bar}{\lambda} > 0$.

Reasoning by contradiction, suppose that there exists $\delta_0 \in (0, \delta)$ such that (P_+) has a unique positive solution for $\lambda = \lambda_1 - \delta_0$. Define

$$\begin{aligned}
\mathcal{S}_1 &= \{(\lambda, u) \in \mathbb{R}^+ \times P : \lambda \in [\lambda_1 - \delta_0, \lambda_1],\ \|u\|_E = \varepsilon\},\\
\mathcal{S}_2 &= \{(\lambda, u) \in \mathbb{R}^+ \times P - \{0\} : \lambda = \lambda_1,\ \|u\|_E \leqslant \varepsilon\},\\
\mathcal{S}_3 &= \{(\lambda, u) \in \mathbb{R}^+ \times P : \lambda = \lambda_1 - \delta_0,\ \|u\|_E \geqslant \varepsilon\},\\
\mathcal{S} &= \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3.
\end{aligned}$$

Clearly, $\mathcal{S}$ separates $\mathbb{R}^+ \times P - \{0\}$ into two connected components, and $\mathcal{C} \cap \mathcal{S} = \emptyset$.

Moreover, we find that the branch $(\lambda(s), u(s))$ is in one of these components, whereas any solution $(\lambda, u) \in \mathcal{C}$ with $\lambda > \lambda_1$ is in the other component. This is a contradiction of the fact that $\mathcal{C}$ is connected. □

4. A uniqueness result

In this section we prove the uniqueness of positive solution of (1.1) under a certain condition on f, extending the result of [2, 7, 14, 16] to the case of quasi-linear equations.

THEOREM 4.1. *Consider the problem*

$$-\Delta u = f(x, u, \nabla u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.1}$$

where $f : \bar{\Omega} \times \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function that is locally Lipschitz with respect to $(u, \eta) \in \mathbb{R}_0^+ \times \mathbb{R}^n$. Suppose that f also verifies the following condition:

$$\begin{gathered}\text{for any } x \in \Omega,\ (u, \eta) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \text{the function } t \mapsto \frac{f(x, tu, t\eta)}{t} \text{ is strictly decreasing in } t \in \mathbb{R}^+. \end{gathered} \tag{4.2}$$

There then exists at most one non-negative non-zero solution of problem (4.1) in E.

Proof. First, let us show that (4.2) implies that $f(x, 0, 0) \geqslant 0$ for any $x \in \Omega$. Otherwise, if there exists $x_0 \in \Omega$ such that $f(x_0, 0, 0) < 0$, we would have

$$\lim_{t \to 0^+} \frac{f(x_0, tu, t\eta)}{t} = -\infty$$

for any $(u, \eta) \in \mathbb{R}^+ \times \mathbb{R}^n$. This would contradict (4.2).

We now prove that any non-negative non-zero solution of (4.1) must be positive, that is, it must belong to $\operatorname{Int}(P)$. Given any $u \in P$ a solution, we can write

$$\begin{aligned} -\Delta u &= f(x, u, \nabla u) - f(x, u, 0) + f(x, u, 0) - f(x, 0, 0) + f(x, 0, 0) \\ &= \frac{f(x, u, \nabla u) - f(x, u, 0)}{|\nabla u|^2} \nabla u \cdot \nabla u + \frac{f(x, u, 0) - f(x, 0, 0)}{u} u + f(x, 0, 0). \end{aligned}$$

Since u and ∇u are both bounded and f is locally Lipschitz, the functions

$$\frac{f(x, u, \nabla u) - f(x, u, 0)}{|\nabla u|^2} \nabla u \quad \text{and} \quad \frac{f(x, u, 0) - f(x, 0, 0)}{u}$$

are uniformly bounded. Therefore, u is a solution of the problem $L[u] = f(x, 0, 0)$, where L is a linear operator defined as

$$L[z] = -\Delta z - \frac{f(x, u, \nabla u) - f(x, u, 0)}{|\nabla u|^2} \nabla u \cdot \nabla z - \frac{f(x, u, 0) - f(x, 0, 0)}{u} z.$$

Recall now that u vanishes in $\partial\Omega$ and is non-negative and non-zero. Theorem 2.3 implies that $u \in \operatorname{Int}(P)$.

We now prove uniqueness by using Serrin's sweeping method. Suppose that u and v are two non-negative solutions of (4.1), both non-zero. By direct computation (using (4.2)), we conclude that the functions su, with $s \in (0,1)$, are strict subsolutions of (4.1). Define the set

$$D = \{s \in [0,1] : su(x) \leqslant v(x) \text{ for any } x \in \Omega\}.$$

From the definition, it is obvious that D is closed. Since $u, v \in \mathrm{Int}(P)$, there exists $\varepsilon > 0$ belonging to D. Take $\gamma = \max D > 0$: we claim that $\gamma = 1$. Reasoning by contradiction, suppose that $\gamma \in (0,1)$. Let us set $w = \gamma u$, $w \leqslant v$, and recall that w is a subsolution of (4.1).

Then we have

$$\begin{aligned}
-\Delta(v-w) &\geqslant f(x,v,\nabla v) - f(x,w,\nabla w) \\
&= f(x,v,\nabla v) - f(x,v,\nabla w) + f(x,v,\nabla w) - f(x,w,\nabla w) \\
&= \frac{f(x,v,\nabla v) - f(x,v,\nabla w)}{|\nabla v - \nabla w|^2} \nabla(v-w)\cdot\nabla(v-w) \\
&\quad + \frac{f(x,v,\nabla w) - f(x,w,\nabla w)}{v-w}(v-w).
\end{aligned}$$

We now argue as above. Since u and v belong to E, the functions

$$\frac{f(x,v,\nabla v) - f(x,v,\nabla w)}{|\nabla v - \nabla w|^2}\nabla(v-w), \qquad \frac{f(x,v,\nabla w) - f(x,w,\nabla w)}{v-w}$$

are uniformly bounded. Therefore, we find that $\bar{L}[v-w] \geqslant 0$, where $\bar{L}$ is defined by

$$\bar{L}[z] = -\Delta z - \frac{f(x,v,\nabla v) - f(x,v,\nabla w)}{|\nabla v - \nabla w|^2}\nabla(v-w)\cdot\nabla z - \frac{f(x,v,\nabla w) - f(x,w,\nabla w)}{v-w} z.$$

Recall now that $v-w$ vanishes in the boundary and $v - w \geqslant 0$. Therefore, the maximum principle (theorem 2.3) yields either $v-w=0$ or $v-w \in \mathrm{Int}(P)$. Observe now that the first possibility does not hold, since v is a solution of (4.1) and $w = \gamma u$ is not. Therefore, $v - \gamma u \in \mathrm{Int}(P)$. However, this implies that $v - (\gamma+\varepsilon)u \geqslant 0$ for $\varepsilon > 0$ small enough. Thus, $\gamma + \varepsilon \in D$, contradicting the definition of γ.

Then, $u \leqslant v$. We can also apply the preceding argument, but change the roles of u and v, to obtain $v \leqslant u$. This concludes the proof. □

Remark 4.2. In the case in which f does not depend on ∇u, condition (4.2) is the same as that given in [7, 16].

Remark 4.3. Condition (4.2) can be relaxed in different ways. For instance, theorem 4.1 is also true if (4.2) is replaced with the following condition:

$$\begin{aligned}
&\text{for any } (u,\eta) \in \mathbb{R}^+ \times \mathbb{R}^n, \text{ the function } t \mapsto \frac{f(x,tu,t\eta)}{t} \\
&\qquad \text{is decreasing for any } x \in \Omega \text{ and strictly decreasing for any } x \in \Omega',
\end{aligned} \tag{4.3}$$

where $\Omega' \subset \Omega$ is a subset with non-zero measure.

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Furthermore, if (4.2) is relaxed to the condition that

$$\text{For any } x \in \Omega, (u, \eta) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \text{the function } t \mapsto \frac{f(x, tu, t\eta)}{t} \text{ is decreasing in } t \in \mathbb{R}^+, \tag{4.4}$$

then the arguments laid out in the proof of theorem 4.1 imply the following result.

Suppose that u and v are non-negative solutions of (4.1) in E, and assume that (4.4) holds. Then, u and v are proportional functions.

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