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Applications of Shapley-Owen Values and the Spatial Copeland Winner

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The Shapley-Owen value (SOV, Owen and Shapley 1989, Optimal location of candidates in ideological space. *International Journal of Game Theory* 125–42), a generalization of the Shapley-Shubik value applicable to spatial voting games, is an important concept in that it takes us away from *a priori* concepts of power to notions of power that are directly tied to the ideological proximity of actors. SOVs can also be used to locate the spatial analogue to the *Copeland winner*, the *strong point,* the point with smallest win-set, which is a plausible solution concept for games without cores. However, for spatial voting games with many voters, until recently, it was too computationally difficult to calculate SOVs, and thus, it was impossible to find the strong point analytically. After reviewing the properties of the SOV, such as the result proven by Shapley and Owen that size of win sets increases with the square of distance as we move away from the strong point along any ray, we offer a computer algorithm for computing SOVs that can readily find such values even for legislatures the size of the U.S. House of Representatives or the Russian Duma. We use these values to identify the strong point and show its location with respect to the uncovered set, for several of the U.S. congresses analyzed in Bianco, Jeliazkov, and Sened (2004, The limits of legislative actions: Determining the set of enactable outcomes given legislators preferences. *Political Analysis* 12:256–76) and for several sessions of the Russian Duma. We then look at many of the experimental committee voting games previously analyzed by Bianco et al. (2006, A theory waiting to be discovered and used: A reanalysis of canonical experiments on majority-rule decision making. *Journal of Politics* 68:838–51) and show how outcomes in these games tend to be points with small win sets located near to the strong point. We also consider how SOVs can be applied to a lobbying game in a committee of the U.S. Senate.

1 Introduction

There has been a proliferation of solution concepts and geometric constructs that help define the internal structure of majority rule (and quota rule) spatial voting games. Among the most important of these are the *minmax set* (Kramer 1972), the *multidimensional median* (Shepsle and Weingast 1981), the *yolk* (McKelvey 1986), and the *heart* (Schofield 1995). In addition, spatial versions of well-known solution concepts for games with a finite number of alternatives such as the *Borda winner* (Black 1958; Saari 1994), the *Copeland winner* (Straffin 1980), the *uncovered set* (Fishburn 1977; Miller 1980; Moulin 1986; Miller 2007), and the *Banks set* (Banks 1985; Miller, Grofman, and Feld 1990; Banks, Duggan, and Le Breton 2002) have been identified (see also McKelvey, Winer, and Ordeshook 1978; Feld and Grofman 1988a;

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Grofman et al. 1987; Shepsle and Weingast 1984; McKelvey 1986; Cox 1987; Feld et al. 1987; Hartley and Kilgour 1987; Shapley and Owen 1989; Penn 2006a, 2006b).[1] However, until quite recently, most of these concepts lacked direct applicability in games with more than a limited number of (weighted) voters because of the lack of computer algorithms to identify the relevant geometry.

That situation has begun to change dramatically due to the work of Bianco, Sened, and colleagues (Bianco, Jeliazkov, and Sened 2004; Bianco et al. 2006). Building on earlier work of Koehler (1990, 1992, 2001, 2002), these authors have developed algorithms for finding the yolk and the uncovered set for large *n* data sets, and they have illustrated the remarkable power of the uncovered set as a solution concept for experimental majority rule voting games and as a means to elucidate the nature of historical changes in the structure of legislative voting in the U.S. Congress. Their algorithmic approach has been adapted and extended by Joseph Godfrey to create a user friendly computer program (CyberSenate) that can calculate in a two-dimensional space not just the uncovered set but virtually all the standard spatial solution concepts—even for a very large number of spatially embedded voters. We use that program to perform the calculations reported in the figures and tables in this paper.

In this essay, we build on the work of Bianco, Sened, and colleagues and the seminal theoretical essays that preceded them. Here, our focus is on one important theoretical idea with a geometrical basis, the Shapley-Owen value (SOV, Shapley and Owen 1989).[2] The SOV is important for four key reasons:

(1) The SOV is the spatial analogue to the well-known Shapley-Shubik value for cooperative games, that is, it generates a measure of pivotal power for spatial voting games—a fact that is simply not well known. As a solution concept for cooperative games, SOV attempts to address perceived shortcomings of the Shapley-Shubik index as an *a priori* measure by positing a coalition structure that is directly based on the issue proximity of the voters. Given the importance of the concept of pivotality in so many game theoretic models, familiarity with the SOV and how to calculate it for any spatial voting game should prove quite useful.

(2) The SOV is a natural generalization of the notion of the median voter to the multidimensional case because it assigns values to each voter that correspond to the proportion of the possible angles of rotation on which that voter serves as a median voter. When there is a core, and an odd number of voters, there exists a core voter who is the direct analogue of the median voter in a single dimension, with an SOV of 1.

(3) In the absence of a core, however, it is still possible to identify the alternative that loses to the fewest number of other alternatives, although such an alternative will not in general be coterminous with a voter ideal point. In the spatial context, the alternative with smallest win set has been called the *strong poin*t (Shapley and Owen 1989); however, it might better be referred to as the spatial analogue to the *Copeland winner*, one of the most important of the ‘‘almost-core’’ concepts. The SOV is the basis of an elegant algorithm for finding the spatial Copeland winner. As we will see, the strong point can be located as a weighted average of voter ideal points, where the weights correspond to each voter's SOV.

(4) Once we know SOVs, not only can SOVs be used to quickly find the alternative with smallest win set, but once that alternative is identified and its win set calculated, we then can determine the win set areas for any point, because of a little known result that, in two dimensions, the area of win sets increases with the squared distance from the strong point along any ray (Shapley and Owen 1989; see also Feld and Grofman 1990). If we know the win set area of points, we can then test predictions about the likely outcomes of spatial voting games based on positing that, *ceteris paribus*, outcomes with small win sets (even among points in the uncovered set) are more likely to be chosen. However, to do any of these things we must first be able to find SOV values. Although this is trivial for the case of three voters, and relatively straightforward (though tedious) for situations where all voters lie on the boundaries of the Pareto set, in all other cases it is computationally very difficult. Indeed, until the development of the algorithm we describe here—one that was recently added to the CyberSenate program—it was

[1]Cf. Calvert (1985). Other less well-known concepts that are part of the geometric structure of majority rule, such as the Schattschneider set, the locus of all possible multidimensional medians (Feld and Grofman 1988b), and the finagle circle, the smallest circle that is a Von Neumann-Morgenstern externally stable solution set (Wuffle et al. 1989), can be related both to the yolk and to the spatial version of the Banks set (cf. Miller, Grofman, and Feld 1989).

[2]We plan on dealing with other solution concepts in companion essays, but the computational and applied aspects of the SOV are both sufficiently complex and sufficiently important to justify a paper of their own.

essentially impossible to find SOVs for more than a handful of voters in the general case where some voters might be interior to the Pareto set.

Because the Shapley-Owen index is far less familiar to most readers than ''famous'' power indexes such as Penrose/Banzhaf-Coleman or the Shapley-Shubik value, in the next section, we briefly review the definition of the SOV. We then discuss how to calculate SO values for the two-dimensional case by using the computer algorithm based on the results in Shapley and Owen (1989) about the geometry of the SOV that has now been implemented in CyberSenate. This algorithm allows us to deal with cases not encompassed by analytic methods, including relatively large data sets such as those involving NOMINATE scores for the U.S. Congress. We describe the basics of the algorithm; a technical supplementary material with greater detail about some important special cases is posted online. Although all our examples are two dimensional, this algorithm generalizes to arbitrary dimensionality.

In the succeeding section, we turn to four rather different illustrations of the power of the SOV to illuminate actual group decision making.

Our first two applications are to national legislatures:

(1) We replicate six two-dimensional figures from Bianco, Jeliazkov, and Sened (2004) showing the uncovered set in various U.S. congresses, and we add to their analyses by showing where the strong point is in relationship to the uncovered set and the center of the yolk in these congresses, and we also briefly discuss the implications of these empirically observed relationships for understanding the dynamics of legislative voting.

(2) We look at the Russian Duma during several years when Putin was the President of Russia and track the movement of the strong point, 2000–2003, as groups of legislators shifted in their voting patterns in response to events of the time in a weighted voting game based on party groupings.

Our next application of SOV calculations is to experimental committee majority rule voting games. SOV calculations allow us to find win set sizes in a computationally tractable fashion.

(3) We show that, in virtually all the experimental games reviewed in Bianco et al. (2006), observed outcomes have smaller than average win sets relative to the average size of win sets in the Pareto set, and even relative to the average size of win sets in the uncovered set. We show, too, that the set of points with smallest win sets does a good job of predicting outcomes. We then focus on three of the games in this set and show that, in these games, the strong point/spatial Copeland winner comes close to being the mean outcome location.

Finally,

(4) We offer a somewhat different illustration of how SO values (and the notion which underlies them, namely rotation of median lines) can be used: committee voting in the U.S. Senate about proposals for health care reform during President Clinton's ill-fated attempt to extend health care to a large proportion of the uninsured while simultaneously cutting health care costs. Here we calculate SO values to identify the pivotal members who should have been the target of lobbyists.

2 Calculating SOVs in the General Case

The SOV is an intuitive generalization of the Shapley-Shubik value to the case where voters can be thought of as points in an n-dimensional issue space. The SS value can be thought of as measure of the proportion of cases any given voter is *pivotal* (i.e., turning a winning coalition to a losing coalition, or vice versa) if we assume that all permutations of voter orderings are equally likely. At the heart of the computer program used by the present authors to identify SOVs for the two-dimensional case is the notion of rotating a line over 360 (or 180) degrees to identify which voters are the median (projections) on that line. The proportion of lines over which the voter is pivotal (i.e., at the median) is the direct measure of the SOV. For example, in any polygon where all voters are located on the convex hull, the SOV of any voter is simply the angle subtended by that voter. But, finding SO values gets much more difficult when there are interior points in the Pareto set. Nonetheless, in our algorithm, such problems are straightforward to deal with, though still computationally complicated.

As noted above, SOVs can be used to directly identify the alternative with smallest win set, here denoted the *strong point*, the spatial analogue to the *Copeland winner*. Once we find the strong point we can find the win set of any point based on its distance to the strong point.

More formally:

Following Shapley and Owen (1989), consider a finite set of n voters, N. Introduce a strict order relation on N, <<. Define a set $Q(i, <<)$ to consist of all voters j, such that $j << i$. Finally, let W be a set of subsets of N that are *winning coalitions*. Call a voter, i, a pivot if and only if

$$Q(i, << \notin W \text{ and } Q\ (i, << \cup \{i\} \in W,$$

that is, the pivot splits the set N into two disjoint sets, one of which is winning, namely

$$Q(i, <<) \cup \{i\} \in W.$$

Shapley and Owen note that the Shapley value for voter i, $v_i(N, W)$, can be written

$$v_i = \frac{q_i}{n!}, \tag{1}$$

where q_i is the number of orderings for which voter i is the pivot. Since $n!$ is the total number of all possible orderings, that is, the size of the sample space, v_i has a natural interpretation as the probability that voter i will be the pivot in a random draw from a uniform distribution of orderings. In the spatial context, this translates into a draw such that, as we sweep through a 360-degree rotation, all angles are equally likely.

Shapley and Owen remark that it seems unlikely all possible coalitions of equal size have an equal probability of forming in actual political situations, as required for the probability interpretation of equation (1). Accordingly, each has suggested formal modifications of equation (1), the upshot being to modify equation (1) to reflect a more realistic sample space (see Shapley and Owen 1989, for discussion and references). In the spatial context, the structure of voter proximity shapes the coalition structure.

2.1 *SOV in Proximity Spatial Voting Models*

A particular modification of equation (1) proposed by Shapley involves a spatial voting model (Shapley 1977). Shapley's model consists of a finite set of n voters, N, a set of subsets of N, called winning coalitions, W, and a set of n points $\{P_i\}$, $i \in N$, in an m-dimensional affine (or projective) space, R^m, representing the voters. These points, called *ideal points*, represent the preferred policy outcomes of each voter. The space is assumed to be measurable (Lebesgue) with a Euclidean metric, $d(x,y)$ and inner product, $<x,y>$. Here, $d(x,y) = |<x,y>|^{1/2}$.

Shapley considers unit vectors $U \in R^m$. These vectors lie on the unit sphere H_{m-1}, each vector defining a direction in the space. Furthermore, except for a set of measure 0, each unit vector, U, induces a order relation << as

$$i <<_U j \Leftrightarrow \langle U, P_i \rangle \leqslant \langle U, P_j \rangle.$$

Shapley notes that those U that do not induce an order form an m two-dimensional subspace, and so have measure 0, and thus can be neglected when computing probabilities.

Let U be randomly chosen from a uniform distribution, that is, subsets of H_{m-1} with equal measure have equal probability. Assuming the points $\{P_i\}$ are distinct, the pivot for the order induced by U will be unique almost surely. Let ϕ_i denote the probability that i is the pivot under the ordering induced by U. Note that the sum over i of ϕ_i is unity. Then ϕ_i becomes the modified version of equation (1) for the spatial voting model. In two-dimensional spaces, $m = 2$, it is possible to give an explicit prescription for computing ϕ_i.

2.2 *An Algorithm for Computing SOV*

Our algorithm for computing SOV is a direct translation of Shapley's model discussed above. The only material difference concerns the implementation. In place of the direction unit vectors, we fix a point and rotate a line about that point. Voter ideal points are projected on to the line for each increment of rotation. The pivot is determined as the voter occupying the median position using the natural linear order of the line to order the projected points. Table 1 summarizes the correspondences.

Table 1 Comparison of Shapley's model and algorithm for computing SOV

Shapley model	*Algorithm*
Direction angles: θ_i; $i = 1, 2, \ldots, n - 1$	Rotation angles: θ_i, $i = 1, 2, \ldots, n - 1$
Directional unit vector $U(\theta_i)$	Line vector, $L(\theta_i)$
$\langle U, P_i \rangle$	$\langle L, P_i \rangle$
$i <<_U j \; \langle U, P_i \rangle \leqslant \langle U, P_j \rangle$	$i <<_L j \; P_i \leqslant_L P_j$

Although we could leave our description of the algorithm at this abstract level, it is helpful to consider how one might arrive at it by way of a series of approximations, starting with the Median Voter Theorem.

Consider a finite set of n voters, N, in a one-dimensional proximity spatial voting model, that is, single-peaked preferences, under simple majority rule. Let P_i denote the ideal point for voter i in the issue space, then the linear ordering of points induces a strict order, <<, given by

$$i << j \Leftrightarrow P_i \leqslant P_j.$$

In the case of an odd number of voters, the pivot is the member k occupying the median position for the given order. This position, P_k, determines the policy outcome in the sense that between the status quo and any submitted proposal, the voters will favor the closer of the two to the median voter. If voters are allowed to submit proposals, the median's proposal will defeat all others. The pivot thus determines the outcome of the vote and gains the full value of the outcome when he/she submits the proposal. Considering the particular vote as a game, the full value of the game is thus allocated to the pivot. What this value consists in, that is, what benefit (explicitly) the pivot receives, is exogenously specified.

Now, according to Shapley's model, ϕ_i, is the probability the ith voter is the pivot. If the total value of the game is 1, then ϕ_i, equals the expected payoff voter i receives.[3] Suppose we introduce a second dimension to the issue space. As is well known, there is generally no Condorcet winner in such cases, that is, no spatial median. On the other hand, it may seem evident that even in two dimensions not all voters are equally important, that is, ''centrally'' located voters generally have more opportunities to form winning coalitions. How are we to quantify this?

The suggestion of Shapley and Owen is to assign value according to an angular measure. The key insight is to observe that the distribution of ideal points projected on to a given line is unchanged when projected onto any other parallel line, that is, the projection of ideal points on to a line is invariant under translation of the line. Thus, for any direction in space, we may select a single line to represent the distribution of ideal points for that line and all lines parallel to it, that is, such lines form an equivalence class. A convenient way to find these measures is to pick some arbitrary point in the space as an axis of rotation and represent the equivalence classes of parallel lines by rotating the line about the axis of rotation.

Consider some line and an axis of rotation. Rotate the line by increments θ such that m such increments result in a complete rotation,

$$m\theta = 2\pi.$$

For a given increment $0 < j < m$, there will be some voter who is the median voter. We define a value for the ith voter as

$$v_i(m) = \frac{q_i}{m},$$

where q_i is the number of times i is the median voter and m is the number of increments used to rotate the line through 2π radians. Note that q_i depends on m. In the limit, where the angular increments become infinitesimal, that is, as $m \rightarrow \infty$, we have

[3] A subtle point, however, is that except for odd-number committees in one dimension, the outcome, that is, the adopted proposal, may never reside on the ideal point of a voter. Such voters presumably split the prize in proportion to their influence, the prize itself being a fairly abstract concept. Keeping in mind the largely formal role of ''value'' in spatial voting models, SOV provides a convenient measure for the influence a voter has on the (mean) expected outcome (strong point) in spatial voting.

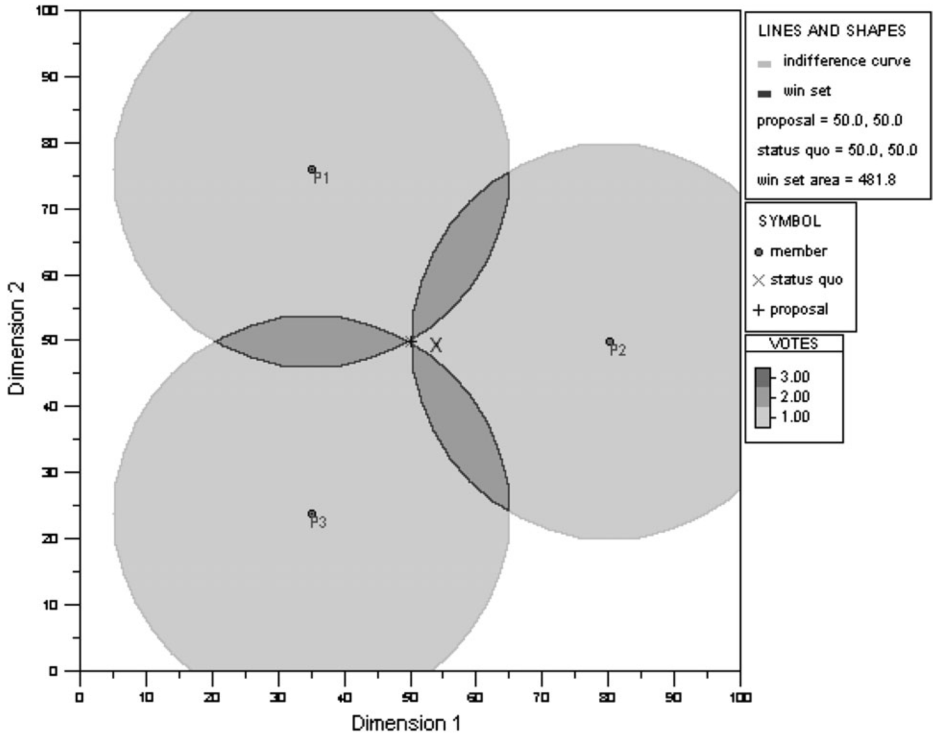

Fig. 1 Elements of proximity spatial models.

$$\phi_i = \lim_{m \to \infty} v_i(m),$$

that is, $v(m)$ approaches the SOV.

Our basic set up involves voters embedded in a two-dimensional space. Figure 1 shows win sets for Euclidean preferences for the simple three-voter case where voters are not all collinear.

Figure 2 illustrates how the algorithm works for three ideal points positioned on an equilateral triangle. The angle measure concentration for each voter's ideal point is indicated. The dashed lines represent transitions from one voter to another being the median voter. The solid line represents one line increment, showing the projection of each voter on to the line. Observe that voter P_1 is the median voter and will remain so, for all lines within the wedge defined by the angle $\pi\phi_1$. Note also that the opposite angle for each concentration is of the same size. The second half of the revolution is identical, except that the projections on the line are in reverse order. The median voters, however, remain invariant. Hence, it is sufficient to consider revolution of the line by π, that is, a half revolution, when computing SOVs, as we noted earlier.

The correspondence of our algorithm with Shapley's model is realized when we replace the rotating line with a directional unit vector. A rotating line, however, is computationally more convenient. By translation invariance, the rotating line can be located anywhere in the plane. Indeed, after each increment of rotation the line can be translated anywhere in the plane without affecting the order of voters projected on the line. It is really only the direction of the line that matters, that is, the directional unit vectors.

Figure 3 illustrates the projection of voters on to a line in a given direction for three distinct spatial translations of the line. Observe that the order of voters projected on the line is identical in all three cases. This is what is meant by translation invariance.

The algorithm has been implemented in a way which allows us it to deal with technical complexities such as identifying median lines for an even number of voters and handling cases where more than one voter lies on a given median line, the general case of which involves weighted voting. In principle, the algorithm can be extended both to supermajoritarian decision making and to forms of voter preferences other than Euclidean. We leave, however, to a technical supplementary material available online at some of the computational issues, such as how to deal with cases where the set of voter ideal points creates special

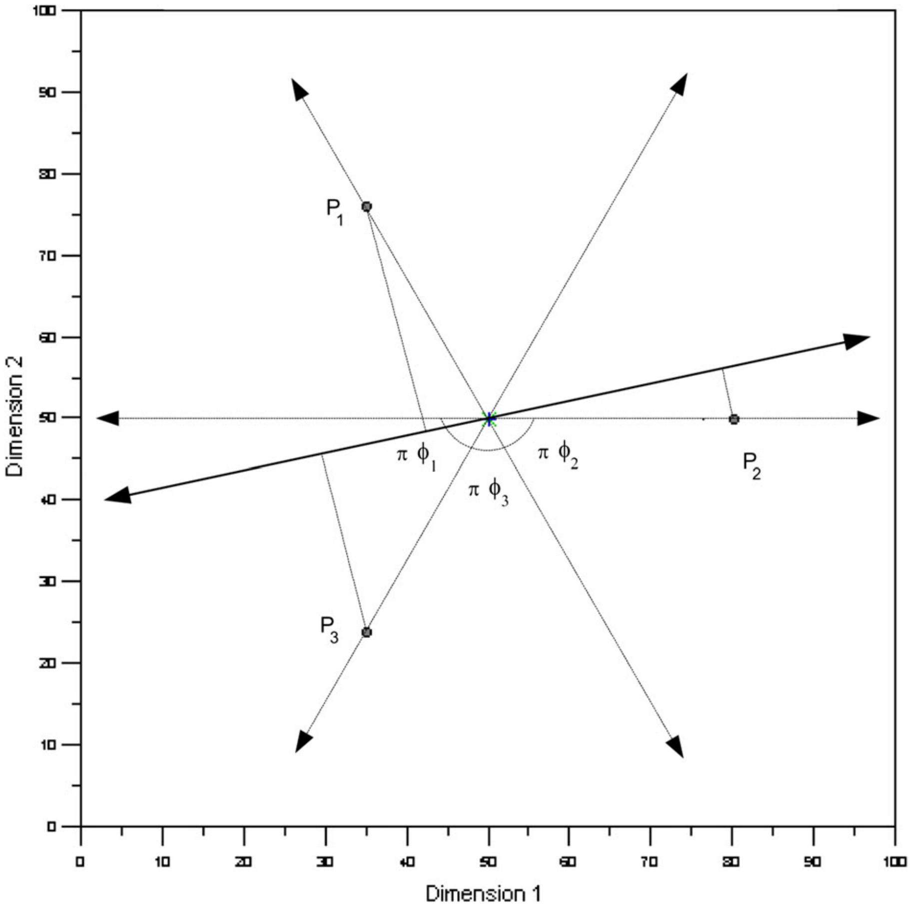

Fig. 2 SOV computation based on rotating line.

difficulties in calculating the angles for which each voter is pivotal because there are multiple voters located at the same point or voters who are "dummies," that is, have zero pivotal power, and cases that give rise to what may be called "Tovey anomalies" (Stone and Tovey 1992), where the minimum circle that touches all median lines is defined by more than three median lines.

Although we have illustrated the algorithm in two dimensions, it can be extended to arbitrary rotations of a line in an n-dimensional space using generalized rotation angles. Such a space in spherical coordinates has $n - 1$ basis angles. Considering independent rotations of each angle, and keeping track for which increments of each angle a voter is pivotal, the generalized SOV can be computed as the sum of fractions of each angle for which the voter is pivotal.

2.3 *Using SO Values to Find the Strong Point*

The formula in Shapley and Owen (1989) to find the strong point (the center of power) in two dimensions is really quite elegant. For each voter, we calculate that voter's SOV, denoted ϕ_i, which is the proportion of median lines on which this voter is pivotal. Let the vector P_i identify the location of voter i in the two-dimensional space. Let P^* denote the strong point. A key result in Shapley and Owen (1989) is equation (2)

$$P* = \sum_{i \in N} \phi_{\mathrm{i}} P_i. \tag{2}$$

In other words, the strong point is simply the weighted average of the locations of voter ideal points, where the weights are those voter's SOVs. The strong point (center of power) must lie within the convex hull of the points $\{P_i\}$ as each $\phi_i \leqslant 1$.

Below we explicate the geometry that leads to this result.

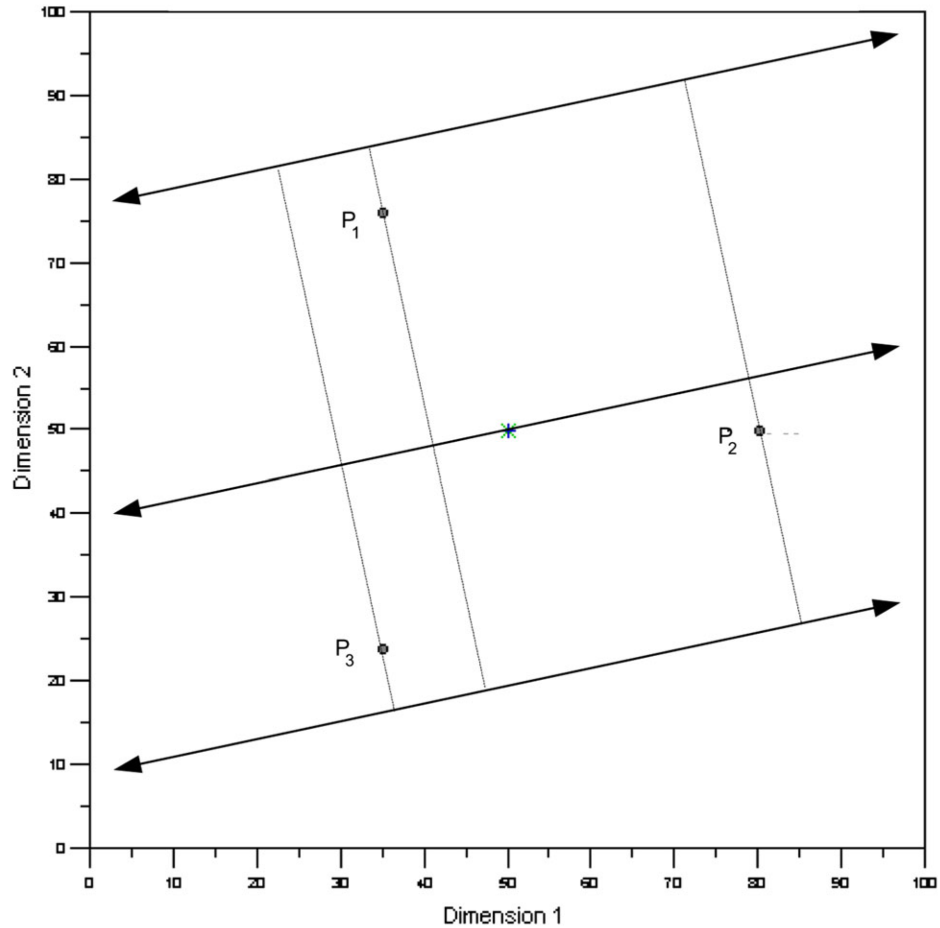

Fig. 3 Translation invariance of projections on to a line.

Consider the simplest example of a proximity spatial voting model consisting of three voters located at points P_1, P_2, and P_3 in a two-dimensional issue space, as well as a distinguished point called the status quo. See Fig. 1 for an illustration. The basic assumption of proximity models is that each voter prefers a proposal "closer" to his/her ideal point than the status quo. Assuming each voter has Euclidean preferences, that is, "closer" is measured using a Euclidean metric, then indifference curves can be drawn as circles centered on each voter's ideal point passing through the status quo, X.[4]

Under simple majority rule, those areas resulting from the intersection of a majority of areas enclosed by indifference curves, often called petals, represent locations where a proposal can be offered that will be majority preferred to the status quo. The union of these areas is generally called the win set. In their 1989 paper, Shapley and Owen refer to this area with the notation $A(X)$; however, they call it the vulnerability of a point.

Shapley and Owen consider whether there exists a point S that minimizes $A(X)$ and whether this point is unique. Shapley and Owen answer in the affirmative, calling S the strong point. The strong point is also known as the spatial version of the Copeland winner as this point, by definition, defeats the greatest number of proposals in pairwise votes. Although SO values are important in and of themselves, they also play an essential role in determining the explicit formula for the strong point.

In order to determine the conditions, minimizing $A(X)$ it is necessary to obtain an analytic expression for $A(X)$. Shapley and Owen accomplish this by developing an expression for computing and aggregating petals. The result is

[4]John Nash introduced the status quo as a formal concept, in bargaining games, calling it the disagreement point, the outcome under no agreement. In voting games, it corresponds to the outcome when a proposal fails to be adopted. The location of the status quo is generally regarded as independent of voter ideal points. Along with the dimensions of the space, it serves to "frame" the decision.

$$A(X) = 2\int_0^\pi \langle U_\theta, P(\theta) - X\rangle^2 d\theta, \tag{3}$$

where $P(\theta) = P_i$, the ideal point of voter i for those θ for which voter i is the pivot and U_θ is a unit vector in the θ direction with respect to some origin. Some intuition for this formula can be gained by considering it from the perspective of polar coordinates, in which case $\sqrt{\langle P(\theta) - X, P(\theta) - X\rangle}$ represents the radial distance between $P(\theta)$ and X, so equation (3) is just an integral for area in polar coordinates. The inner product with the unit directional vector handles the angular variation of the petals.

$A(X)$, as expressed in equation (3), is not readily minimized directly. Instead, Shapley and Owen introduce an auxiliary quantity,

$$B(X) = \frac{1}{\pi}\int_0^\pi \langle P(\theta) - X, P(\theta) - X\rangle d\theta$$

and show that $\pi B(X) - A(X)$ is independent of X. Hence, if we are able to minimize $B(X)$, which is easier to do than minimizing $A(X)$, we will have obtained the value of X we seek.

Observe that $B(X)$ is a radially increasing function for X outside the Pareto set. That is, $\sqrt{\langle P(\theta) - X, P(\theta) - X\rangle}$ increases as X moves away from the Pareto set. The minimization of $B(X)$ is left to the reader by Shapley and Owen; we provide the derivation here.

If we vary $B(X)$ with respect to X, we find

$$\begin{aligned}\frac{\partial B(X)}{\partial x_\alpha} &= \frac{1}{\pi}\int_0^\pi \frac{\partial}{\partial x_\alpha}(\langle P(\theta), P(\theta)\rangle - 2\langle P(\theta), X\rangle + \langle X, X\rangle) d\theta \\ &= \frac{1}{\pi}\int_0^\pi (-2P_\alpha(\theta) + 2X_\alpha) d\theta \\ &= \frac{-1}{\pi}\int_0^\pi P_\alpha(\theta) d\theta + X_\alpha,\end{aligned}$$

where α indicates differentiation with respect to the x and y coordinates individually. Since we are seeking a minimum, we set $\frac{\partial B(X)}{\partial x_\alpha}$ equal to zero for each coordinate. That this represents a minimum and not a maximum is clear from the fact that $B(X)$ increases outside the Pareto set.

$$\mathrm{X}_\alpha = \frac{1}{\pi}\int_0^\pi P_\alpha(\theta) d\theta.$$

As we already noted, the $P(\theta)$ are constant except for a finite number of transitions as the pivotal voter changes for specific values of θ. Thus, for each P_i, we have

$$\phi_i = \frac{1}{\pi}\int_{C_i} d\theta, \tag{4}$$

where C_i denotes the concentration of the angle measure for which $P(\theta) = P_i$. Thus,

$$X = \sum_{i \in N} \phi_i P_i, \tag{5}$$

with

$$\sum_{i \in \mathrm{N}} \phi_i = 1. \tag{6}$$

Hence, as represented in equation (2) earlier, the strong point is simply a weighted sum: the voter ideal points weighted by each's SOV.

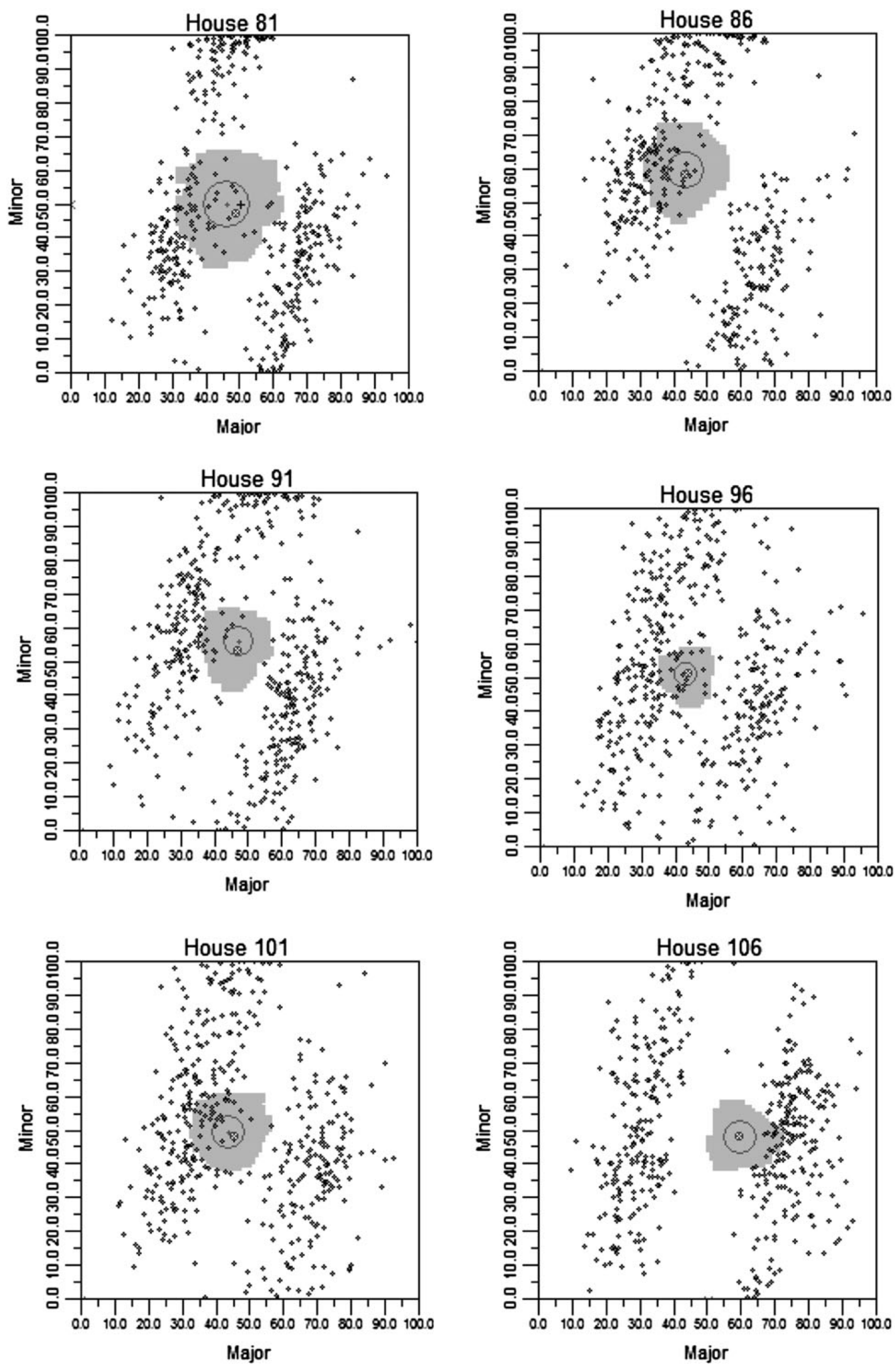

Fig. 4 The uncovered set, yolk, and strong point for selected Congresses from the 81st through the 106th (shaded area is the uncovered set; large circle with center dot is the yolk; *x* within a small circle is the strong point).

2.4 *Using SOVs to find the Size of the Win Set of Any Point*

Equation (3) is the basis for the result that win set area increases as we move away from the strong point along any ray (i.e., in any given direction). In particular, it can be shown that, in two dimensions, the area of the win set of any point is equal to the area of the win set of the strong point + Πd^2, where d is the distance

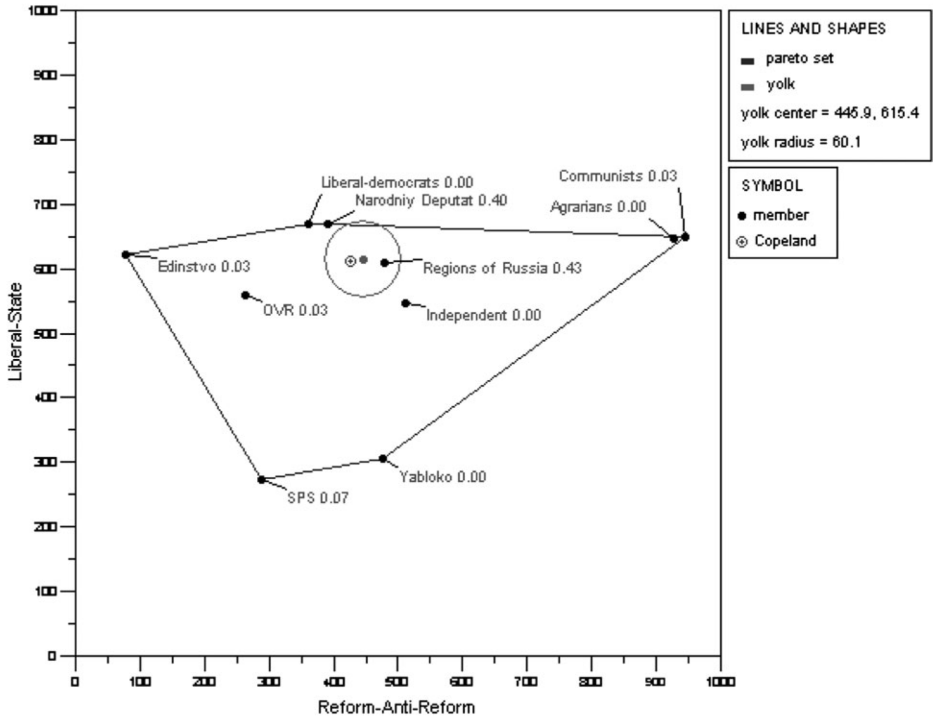

Fig. 5 Average of SOVs for parties in the Russian Duma, 2000–03.

between the point and the strong point. The CyberSenate program can first find the win set of the strong point (using equation 3) and then use the size of that win set to find the sizes of all other win sets.

3 Applications of the SOV to Empirical Data

3.1 *Applications of the SOV to the U.S. Congress*

Bianco, Jeliazkov, and Sened (2004, Fig. 5, p. 268) show the uncovered set in various U.S. congresses, from the 81st to the 106th. We have independently calculated the uncovered set for these congresses using the same first two dimensions of DW-NOMINATE, and we have added to their figures the location of the yolk and the strong point.[5]

McKelvey (1986) provides the simple proof that the Copeland winner must always be in the uncovered set. Furthermore, results in Feld et al. (1987) and Feld and Grofman (1990) and unpublished work by Tovey (personal communication, 2009) show that the spatial Copeland winner, because of its connection to the yolk, must not only lie within the uncovered set but be relatively central within the uncovered set. In particular, Tovey shows that the strong point must lie within 2.16 yolk radii of the center of the yolk. As we see from the above figures, this expectation is confirmed. Indeed, in these real-world examples, the 2.16 yolk radii bound is misleading in being too weak since in fact the strong point is within the yolk circle (i.e., within one yolk radius of the center of the yolk) for all six of the congresses Bianco, Jeliazkov, and Sened (2004, Fig. 5, p. 268) we examine.[6] In fact, in one of these six cases, the two are so close that they cannot be distinguished within the margin of precision of the graphics program we are using (the 106th congress).

[5]As Bianco, Jeliazkov, and Sened (2004) point out, we do not, in general, know how to find the uncovered set analytically. Indeed, the only analytical results for the usual majority rule spatial voting game are for the three-voter case (Feld et al. 1987; Hartley and Kilgour 1987).

[6]Indeed, in all the empirical cases that one of the present authors has examined, the strong point lies within the one radius yolk circle. The only possible counterexamples we have been able to think of are ones where we are in a near core situation, so that the yolk radius itself is very small relative to the Pareto set.

Relatedly, if we look at the six plots in Fig. 1, we see that the center of the yolk is, due to symmetry, very close to the center of gravity of the uncovered set—although it is possible to construct majority rule voting games with sufficient skewness that this is less true. Nonetheless, this observed close proximity between the strong point and the center of the yolk shows that it will be difficult in practice in these real world settings to distinguish between the hypothesis that outcomes will tend to center around the strong point, and the hypothesis that outcomes will tend to center around the center of the yolk.

We also see from Fig. 4 that the uncovered set in these six congressional voting games is closer to the center of the yolk than the well-known four yolk radii bound in McKelvey (1986), or even than the 3.7 three yolk radii bound identified in Feld et al. (1987; see also Miller 2007). In fact, the average radius of the uncovered set in these 6 cases (as a multiple of the radius of the yolk) is around three yolk radii.

Moreover, we can see from Fig. 4 how remarkably small the yolk is relative to the Pareto set in the U.S. Congress in the time period including the 81st congress (1949–50) and the 106th congress (1999–2000). This suggests that political outcomes of standard agenda voting processes, while technically indeterminate, since there is no core, are likely to be confined to a relatively small portion of the Pareto set. On the other hand, although the yolk is small relative to the Pareto set, and thus, the uncovered set must also be reasonably small relative to the Pareto set, neither the yolk nor the strong point nor the uncovered set are located in the ‘‘center’’ of the space between the two parties (nor is there that much overlap between the parties over this period, especially in the most recent congresses: see, e.g., Brunell and Grofman 2008).

In these congresses, there is a largely bipolar form of competition and these constructs are all closer to the majority party's preferences than to the mean location of legislator ideal points, but with their exact location a function of the dispersion (and degree of relative dispersion of) minority and majority blocs. As Bianco, Jeliazkov, and Sened (2004: 270) point out, the non-centrality of the yolk in situations of polarized politics has important implications for expected legislative outcomes since the uncovered set indicates what outcomes are feasible results of voting processes that use standard amendment procedure.[7] Thus, with polarized politics, even without any form of sophisticated voting, we would expect that outcomes have a considerable bias in favor of the majority party, rather than simply reflecting the mean of legislator ideal points. In particular, as we see in comparing the 101st and the 106th congresses, there is a dramatic movement in the yolk and in the location of the strong point when the majority party in the House shifts after 1994![8] Since the strong point reflects the weighted average of voter ideal points, with weights indicating the proportion of votes on which that legislator can be expected to be pivotal, the shift of the location of the strong point between the 101st and 106th congresses directly tells us about the magnitude of the power shift between the two parties that takes place during this period in terms of the key dimensions of political conflict that are reflected in Poole-Rosenthal NOMINATE scores.

3.2 *SOV in the Russian Duma*

Figure 5 is a spatial map for the parties in the Russian Duma in 2000–2003 showing SOVs for what we treat as a weighted majority rule voting game. The horizontal axis is a Reform ↔ Anti-Reform dimension, the vertical axis is a Liberal ↔ Statist dimension. The scales on each dimension run from 0 to 1000. We take the coding of party locations (based on votes in the legislature over this period) from a data set provided us by a Russian scholar, Fuad Aleskerov (Aleskerov and Otchour, n.d.). Preferences are assumed to be Euclidean. Although we take the results shown in Fig. 5 to be merely illustrative, they do appear to demonstrate the considerable power of Regions of Russia, a party whose politics vacillated but was sometimes in opposition to President Putin.[9] In particular, looking at the location of the strong point, we see

[7]Matters become more complicated when we shift to other forms of amendment procedure (Ordeshook and Schwartz 1987).

[8]Of course, however, to understand outcomes in the U.S. Congress, we would also need to take into account the actions of the president.

[9]Regions of Russia is probably best viewed as a collection of regional notables and not really a coherent party. After 2003, it joined with other parties to form the Rodina (Motherland) party but then subsequently splintered. Changes made in the electoral rules for regional legislatures and governors in 2003 are viewed by many Russian specialists as in part directed to eliminating the power base of such regional notables (Golosov 2006).

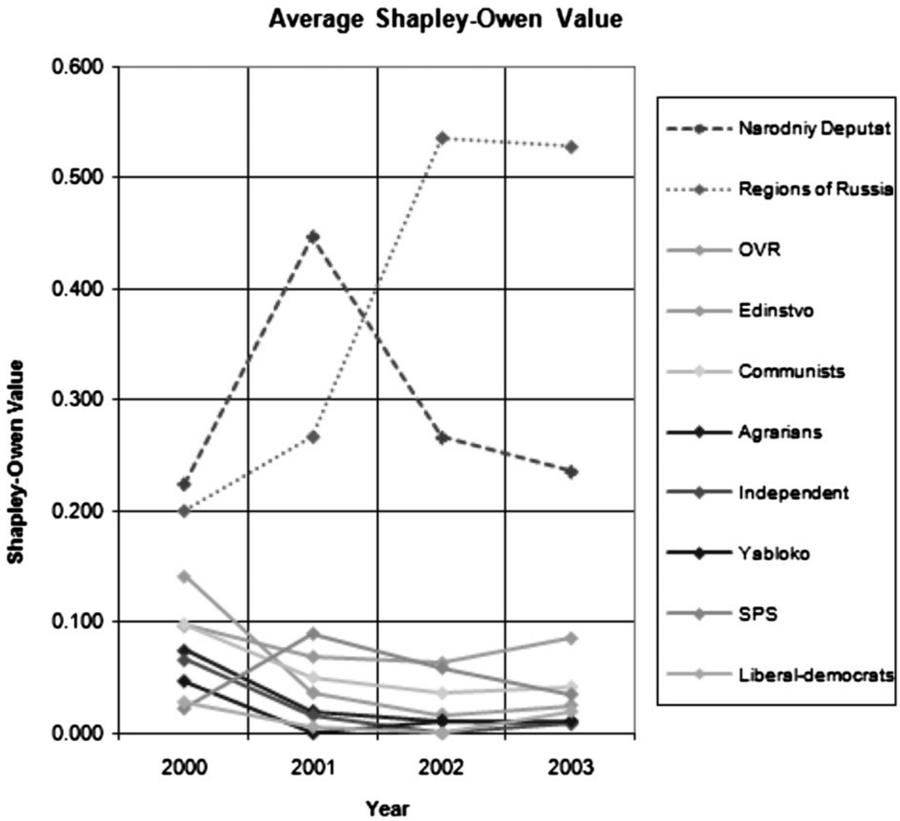

Fig. 6 Change in SOVs for parties in the Russian Duma 2000–03. *Note.* Parties appear in the legend ordered by their SOV value in the year 2000.

that, despite its relatively small size**,** Regions of Russia was very close to a core location, were there to be a core in this voting game.[10]

A more comprehensive picture of what is happening in the Duma during this period is shown in Fig. 6. Note how Regions of Russia grows in power over this time period.

3.3 *SOV and Win Set Area in Experimental Voting Games*

Bianco et al. (2006) review data from a number of classic experiments on committee voting and demonstrate that a very high proportion of all outcomes in these games lie in the game's uncovered set (Miller 1980; Feld et al. 1987). But, the reason why the uncovered set is such a good prediction of committee choice processes remains unclear. Moreover, despite its really strong predictive powers, because the uncovered set is usually a substantial proportion of the Pareto set in these experimental voting games with only five players (unlike what we observed for the 435 member U.S. House), and because superimposition of outcomes in these experimental games on the uncovered set suggests that the outcomes are not uniformly distributed within the uncovered set, it would seem plausible that either there might be some "refinement" of the uncovered set which would (in error reduction terms) do better as a predictor or that there might be some other feature of spatial voting games that can "explain" why the uncovered set works so well.

Drawing on the result in Shapley and Owen (1989) showing that the size of win sets increases monotonically as we move along any ray from the spatial Copeland winner, we can demonstrate that a very plausible reason why the uncovered set does so well as a predictor of outcomes in experimental committee

[10]Data not shown provide a similar mapping for the votes each month during this period—allowing us to see the movement of parties over the issue space.

Table 2 Comparison of predictive power of uncovered set and smallest win-set predictions for various experimental voting games

Study and condition	*Pareto set* Radius mean	*Uncovered set-radius mean* Radius mean	*Outcomes* Radius mean	*Uncovered set* Outcomes IN	*Smallest win set area* Outcomes IN	*N*	*Uncovered set* Proportion correct prediction	*Small win set area* Proportion correct prediction
Endersby PHR 2 Open	60.6	48.2	38.2	10	10	10	1.00	1.00
Endersby PH 1 Closed	60.6	48.2	39.7	10	10	10	1.00	1.00
Endersby PH 2 Closed	60.6	48.2	33.6	10	10	10	1.00	1.00
Endersby PH 2 Open	60.6	48.2	41.7	10	10	10	1.00	1.00
Endersby PHR 1 Closed	60.6	48.2	39	10	10	10	1.00	1.00
Fiorina Plott—NOT core	90.4	29.9	28.1	12	13	15	0.80	0.87
Laing Olmstead—two insiders	103.2	72.5	75.5	18	18	19	0.95	0.95
Laing Olmstead—the house	111.4	87.8	95	14	15	19	0.74	0.79
Laing Olmstead—the bear	110.1	84.1	78.4	17	17	18	0.94	0.94
Laing Olmstead—skew star	111.9	95.6	86.2	14	13	18	0.78	0.72
McKelvey Ordeshook PHR Open	60.6	48.2	36.1	18	18	18	1.00	1.00
McKelvey Ordeshook PHR Closed	60.6	48.2	34.7	15	15	15	1.00	1.00
McKelvey Ordeshook PH Open	60.6	48.2	35.9	16	16	16	1.00	1.00
McKelvey Ordeshook PH Closed	60.6	48.2	33.5	17	17	17	1.00	1.00
McKelvey Ordeshook Winer	120.6	93.1	81	8	8	8	1.00	1.00
Total				199	200	213		
Average	79.5	59.8	51.8	13.3	13.3	14.2	0.93	0.93

voting games is that outcomes in the uncovered set have small win sets, that is, it is less likely that another alternative will defeat them. Results for a large proportion of the experimental games Bianco et al. (2006) review are reported in Table 21. From Table 2 we see that, in 15 of 15 instances, the mean win set radius within the uncovered set (i.e., the mean distance to the furthest element of a point's win set for the set of points that lie in the uncovered set) is smaller than the mean win set radius of points in the Pareto set (overall, an average win set radius size of 59.8 for the uncovered set, as compared to 79.5 for the Pareto set). Moreover, we also see that, in 13 of 15 instances, the mean win set radius size of the actual outcomes is even smaller than the mean win set radius of outcomes in the uncovered set (overall, an average win set size of 51.8 for the actual outcomes, as compared to 59.8 for the uncovered set).

The importance of win-set size for predicting outcomes can be made even clearer by using distance from the spatial Copeland winner to construct a "smallest win set" portion of the space that has exactly the same area as the uncovered set.[11] If it is actually size of win set that is the missing factor in explaining why the uncovered set works so well as an outcome predictor, then predicting outcomes lie in the Pareto set in an area equal to that of the uncovered set, but defined instead directly on "smallest win sets," should be just as good in predicting experimental game outcomes as the uncovered set. Indeed, as we see from Table 2,

[11]The points, however, are restricted to those in the Pareto set. If we were to look simply at the set of points with smallest win sets at a given radius from the strong point, some of those points would lie outside the Pareto set.

that is the case. On average 93% of the outcomes are correctly predicted by the uncovered set and 93% by the smallest win set construction.[12]

We can also talk about results directly in terms of the strong point, that is, the spatial Copeland winner. We have calculated the strong point for some of the classic experiments reported in Table 2, one conducted by Fiorina and Plott (1978) and two conducted by McKelvey and Ordeshook (1984). In the basic Fiorina and Plott game, the strong point offers a very good fit to the location of the average outcome; in the two of the McKelvey and Ordeshook series of games we consider more closely, it offers a reasonably good fit.

In the Fiorina and Plott (1978) basic experiment involving a five voter game without a core,[13] the set of eight outcomes had a mean location of (45, 61) with a SD of 16.4 on the x dimension and 18.0 on the y dimension. The strong point (spatial Copeland winner) in this game is at (49, 62). Clearly the Copeland winner is a good predictor of outcomes in this experimental game.[14] Moreover, weighting voter locations by their SOVs improves predictive power for outcomes since the unweighted average of the voter ideal points is (66, 65).[15]

In all the games in the McKelvey and Ordeshook (1984) series of experiments, the strong point (spatial Copeland winner) is always roughly (58, 36). We will look at two of these games more closely. The outcome in one of the series, the open rule game, is (70, 35), with a SD of 7.5 on the x dimension and 5.7 on the y dimension; in the closed rule game it is (64, 36), with a SD of 4.7 on the x dimension and 5.0 on the y dimension. The fit of the spatial Copeland winner for these games is still within 2 SDs on the x dimension and a near perfect fit on the y dimension.

We are not claiming that the strong point is necessarily the best single predictor of committee outcomes in situations where members use a sequential pairwise voting rule (king of the hill) to choose among alternatives, although it does reduce to the core when there is a core, but we do anticipate that it, or some point near it, will be a good predictor in most such voting games because of its connection to win set size. Of course, as we previously observed, in practice, the strong point may lie very close to the center of the yolk, and the center of the yolk may closely coincide with the center of gravity of the uncovered set, so determining which concepts provide the most explanatory power is, in our view, still an open one. In the future, it is clear that experiments need to be run with games where the center of the yolk, the center of gravity of the uncovered set, and the strong point are not located in almost identical locations in the space.

3.4 *SOV Values and Pivotal Power in the 1993 U.S. Senate Finance Committee*

In 1993, Bill (and Hillary) Clinton offered a major health care reform proposal that was drafted without the participation of the Democratic congressional leadership and which went down to rather ignominious defeat, with no bill even being reported out of the Senate. In Fig. 7, we show the locations of members of the Senate Finance Committee considering health care reform proposals at the time, using a two-dimensional representation based on Americans for Democratic Action scores (a measure of general left-right orientation) and NFIB scores (compiled by the National Federation of Independent Businesses,

[12]Although win set areas drop of with squared distance to the strong point the ratio of the area of the win set of the strong point, which we will denote, Winset(P^*) to the area of any other point's win set to will be
Winset(P^*)/(Winset(P^*) + Πd^2).
Thus, when the strong point has a very large win set, points close to it will not differ as much in win set area from the strong point as when the strong point has a very small win set, since in the latter case the second term in the denominator will assume more importance. This result has implications for comparisons of results across experimental voting games with different sized strong point win set areas, but exploring this point is beyond the scope of the present essay.

[13]The five voter ideal points in this experiment were considerably dispersed:

x	51	30	25	62	165
y	59	52	72	109	32

Thus, the range of voter ideal points on the x dimension was from 25 to 165 and on the y dimension was from 32 to 109.

[14]For the Fiorina and Plott (1978) non-core game, this result was demonstrated by Grofman et al. (1987), but the result for the McKelvey and Ordeshook (1984) game is new.

[15]This game is further discussed in Grofman et al. (1987) and in Bianco et al. (2006).

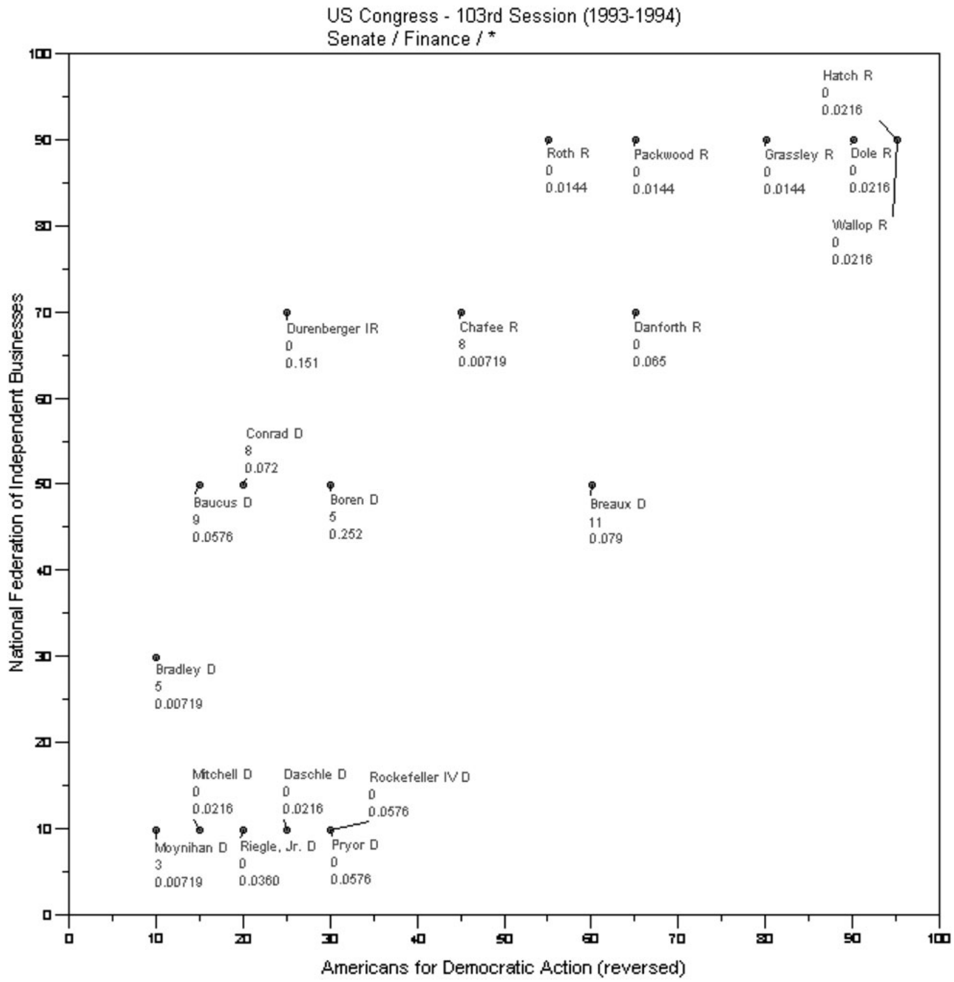

Fig. 7 Senate Finance Committee: Member locations in two-dimensional ADA and NFIB space, lobbying efforts vis-à-vis Health Care Reform and Shapley-Owen scores. *Source:* Godfrey and Grofman (2008, Fig. 3). For each legislator, the upper value is number of firms mobilizing, the lower number is his/her SO value.

and here intended as a proxy for a key dimension of health care reform controversy, namely the degree to which coverage obligations for workers would extend down into smaller businesses). The figure also shows the number of lobbying contacts of each Senator on health care reform (data collected by Goldstein 1999). What we see from this figure is that the SOVs of the legislators do a reasonable job in predicting which will be lobbied.[16]

4 Discussion

The SOV can be interpreted as the probability a given voter is the median voter when all angles of rotation are equally likely. This value can be computed using basic Monte Carlo techniques, for example, randomly generating lines in the policy space and computing the number of times each voter is median and dividing

[16]The linkage is far from perfectly proportional, however. In particular, legislators who are very far away from the strong point are not lobbied at all, unless they are Senators who are regarded as generally influential, such as committee chair Daniel Moynihan. For additional details, see Godfrey and Grofman (2008). Of course, the two dimensions shown in Fig. 3 are only rough approximations for member locations on the key dimensions of health care reform. We should also note that, in one congressional committee considering health care reform in 1993, SOV scores were not as good predictors of which members were lobbied because members who looked peripheral in terms of centrality, but who nonetheless had high SOV scores, were not lobbied.

by the number of lines considered. But this approach can be improved substantially by observing that for any given line and all lines parallel to it, the order of voters projected on to all these lines is the same between lines, that is, the projection of voter ideal points on to an arbitrary line is invariant under parallel translation of that line. This suggests fixing a point in space and rotating a line about that point to represent each class of parallel lines. Rotating this line through finite but arbitrarily small increments, the SOV for each voter is just the fraction of increments for which that voter is the median voter.

We have looked at SOV results for a number of voter ideal point distributions in a 2-dimensional proximity spatial voting model. These examples confirm the intuition that, in a spatial context, SOV measures one key idea about the ''pivotality'' of a voter, that is, the ability of a voter to participate in winning coalitions.[17]

We then showed how the SOV was linked to win-set areas in a surprising and powerful fashion. The point with smallest win-set, the strong point, the spatial analogue to the Copeland winner can be located as the weighted sum of voter ideal points where the weights are each voter's SOVs. Moreover, win set sizes increase along any ray from the strong point (Shapley and Owen 1989). These too little known results are quite elegant ones.

What makes the SOV computer algorithm useful is its applicability to both experimental and real-world legislative and committee voting games. We have provided four examples. One application shows how the strong point, the spatial analogue to the Copeland winner that is calculated as a weighted average of voter ideal point locations weighted by each voter's SOVs, can be used to predict the mean location of outcomes in experimental spatial majority rule committee voting games by using theoretical results about proximity to the strong point to assess the claim that outcomes of committee experiments will tend to be points with relatively small win sets. Another example showed how polarized politics in the U.S. congress can lead the strong point to be very far away from the mean location of legislator ideal points but instead lying close to the preferences of members of the majority party. A third showed SOV scores of Russian political parties in the Russian Duma as a function of party weights and ideological locations to see how political coalitions shifted over time. And a final example showed how SOV weights can be taken to be indicators of how deserving of lobbying efforts each member of the U.S. Senate Finance Committee might have been on proposed 1993 legislation involving health care reform.

We should also note that the general computer methodology for angles of rotation and median lines exposited in this paper has many applications beyond those specific to the SOV. For example, it can be applied in the study of multicameral voting games (Hammond and Miller 1987; Miller, Hammond, and Kile 1996; Brauninger 2003) or the study of optimal candidate strategies in spatial voting games over multidimensional issues where some dimensions are weighted more heavily than others by some or all voters (Feld, Grofman, and Godfrey 2007). Perhaps most importantly, the ideas laid out above help provide a basic foundation for developing computer solutions for the entire panoply of game-theoretic solution concepts for committee voting outcomes in the context of (large n) spatial voting games where analytic solutions are infeasible or simply unknown by making use of the underlying geometry of win sets.

[17]Although not discussed in the paper, the algorithm generalizes naturally to n-dimensions. Using a spherical coordinate system, there will be $n - 1$ angular coordinates. Consider rotating an $n - 1$ dimensional hyperplane about this point by incrementing independently each of the $n - 1$ angles. The SOV is computed as the sum of the fractions of these angles for which a voter is the median voter. It would also be useful to know whether the result of equation (5), which states the formula for the strong point for majority rule games with Euclidean preferences, can be generalized to the case of non-Euclidean preferences and/or to the more general case of k of N rules. Observe that the minimization of $B(X)$ leading to equation (4) in the supplementary material depends only on there being an inner product, not the specific form of the metric. The Euclidean form of the metric enters the picture in deriving equation (5), that is, establishing that is independent of X. We suspect that an extension to at least some kinds of non-Euclidean preferences can be generated by taking advantage of the fact that we can think of some non-Euclidean preferences as based on dimensional rescaling weights for a starting set of Euclidean preferences (Feld, Grofman, and Godfrey 2007). We suspect, too, that an extension of equation (5) to the general class of k of n rules can be generated by thinking in terms of ''inner'' and ''outer'' median lines (Grofman and Feld, 1992), where the analogue of the two types of lines in the voting power literature is to the two basic kinds of power, the power to block and the power to implement. Such inner and outer median lines are already being made use of in the computer program use used for the data analyses in this paper, when that program is used to find win-sets for super-majoritarian games. In effect, in k of n games, each point can be thought of as having two ''win-sets,'' one for the set of points that it beats and one for the set of points that it is not beaten by. Inner and outer median lines define each of the two types of ''win-sets,'' respectively, and give rise to inner and outer yolks.

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