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Periodic solutions for functional differential equations with sign-changing nonlinearities

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We establish some eigenvalue criteria for the existence of non-trivial T-periodic solutions of a class of first-order functional differential equations with a nonlinearity $f(x)$. The nonlinear term $f(x)$ can take negative values and may be unbounded from below. Conditions are determined by the relationship between the behaviour of the quotient $f(x)/x$ for x near 0 and $\pm\infty$, and the smallest positive characteristic value of an associated linear integral operator. This linear operator plays a key role in the proofs of the results and its construction is non-trivial. Applications to related eigenvalue problems are also discussed. The analysis mainly relies on the topological degree theory.

1. Introduction

Functional differential equations with periodic delays appear in a number of applications, such as in the modelling of blood cell production in an animal [6, 17], the control of testosterone levels in the bloodstream [15], and so on. Let $T > 0$ be fixed. We are concerned with the existence of non-trivial T-periodic solutions of the first-order functional differential equation

$$u'(t) = -a(t)u(t) + b(t)f(u(t - \tau(t))), \tag{1.1}$$

where $\tau \in C(\mathbb{R}, \mathbb{R})$ and $a, b \in C(\mathbb{R}, \mathbb{R}^+)$ with $\mathbb{R}^+ = [0, \infty)$ are T-periodic functions and $f \in C(\mathbb{R}, \mathbb{R})$. As by-products of our results, we also derive conditions for the existence of non-trivial T-periodic solutions of the eigenvalue problem

$$u'(t) = -a(t)u(t) + \lambda b(t)f(u(t - \tau(t))), \tag{1.2}$$

where λ is a positive parameter. Here, by a non-trivial T-periodic solution of (1.1) we mean a non-trivial function $u \in C^1(\mathbb{R}, \mathbb{R})$ such that $u(t + T) = u(t)$ for $t \in \mathbb{R}$ and $u(t)$ satisfies (1.1) on $\mathbb{R}$. A similar definition also applies to (1.2). We assume throughout, and without further mention, that the following assumption holds.

(H) The function $g(t) := t - \tau(t)$ is strictly increasing on $\mathbb{R}$,

$$\int_0^T a(v)\,\mathrm{d}v > 0 \quad \text{and} \quad \int_0^T b(v)\,\mathrm{d}v > 0.$$

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In recent years, the existence of periodic solutions of equations (1.1) and (1.2) or their various variations has been investigated by many authors (see, for example, [1–3, 10–13, 16, 20] and the references therein). However, we note that most of the literature only studies the case when the nonlinear term in the differential equation is of one sign, and existence results are rare when the nonlinearity f changes sign. One of the reasons for the lack of results is that the equivalent integral operators for equations (1.1) and (1.2) are not, in general, cone preserving when the nonlinear term f is a sign-changing function and, as a consequence, many fixed-point theorems for cones cannot be directly applied to obtain the existence of solutions.

By means of topological degree theory we derive new criteria for the existence of non-trivial T-periodic solutions of equations (1.1) and (1.2) when f is a sign-changing function and not necessarily bounded from below. Our existence conditions are determined by the relationship between the behaviour of the quotient $f(x)/x$ for x near 0 and $\pm\infty$ and the smallest positive characteristic value (given by (3.2)) of a related linear operator $\mathcal{M}$ defined by (2.18) in § 2. Here, we comment that the linear operator $\mathcal{M}$ plays a very important role in the proofs of our results and that its construction is non-trivial. The techniques of this work are partially motivated by the recent papers [5, 7, 9, 14, 18]. In particular, many kinds of eigenvalue criteria for various second-order boundary-value problems were obtained in [5] and [18] when the nonlinear terms in the differential equations are non-negative. Roughly speaking, [9] and [14] study some second-order boundary-value problems with sign-changing superlinear nonlinearities, while [7] establishes various eigenvalue criteria for a class of periodic boundary-value problems with sign-changing sublinear nonlinearities. All of these papers are very interesting and significant since they reveal some connections between nonlinear problems and some associated linear ones.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, § 3 contains the main results of this paper and two simple examples, and the proofs of the main results are presented in § 4.

2. Preliminary results

We refer the reader to theorem A.3.3(ix) and lemma 2.5.1 of [8], respectively, for the proofs of the following two well-known lemmas. In the rest of this paper, the bold zero denotes the zero element in any given Banach space.

Lemma 2.1. *Let Ω be a bounded open set in a real Banach space X and let $\mathcal{T} : \bar{\Omega} \to X$ be compact. If there exists $u_0 \in X$, $u_0 \neq \mathbf{0}$, such that*

$$u - \mathcal{T}u \neq \tau u_0 \quad \text{for all } u \in \partial\Omega \text{ and } \tau \geqslant 0,$$

then the Leray–Schauder degree

$$\deg(\mathcal{I} - \mathcal{T}, \Omega, \mathbf{0}) = 0.$$

Lemma 2.2. *Let Ω be a bounded open set in a real Banach space X with $\mathbf{0} \in \Omega$ and $\mathcal{T} : \bar{\Omega} \to X$ be compact. If*

$$\mathcal{T}u \neq \tau u \quad \text{for all } u \in \partial\Omega \text{ and } \tau \geqslant 1,$$

then the Leray–Schauder degree

$$\deg(\mathcal{I} - \mathcal{T}, \Omega, \mathbf{0}) = 1.$$

Let $(X, \|\cdot\|)$ be a real Banach space and let $\mathcal{L} : X \to X$ be a linear operator. We recall that λ is an *eigenvalue* of $\mathcal{L}$ with a corresponding *eigenvector* ϕ if ϕ is non-trivial and $\mathcal{L}\phi = \lambda\phi$. The reciprocals of eigenvalues are called the *characteristic values* of $\mathcal{L}$. The *spectral radius* of $\mathcal{L}$, denoted by $r_{\mathcal{L}}$, is given by the well-known spectral radius formula $r_{\mathcal{L}} = \lim_{n\to\infty} \|\mathcal{L}^n\|^{1/n}$. Recall also that a cone P in X is called a *total cone* if $X = \overline{P - P}$.

The following Krein–Rutman theorem can be found in either [4, theorem 19.2] or [19, proposition 7.26].

Lemma 2.3. *Assume that P is a total cone in a real Banach space X. Let $\mathcal{L} : X \to X$ be a compact linear operator with $\mathcal{L}(P) \subseteq P$ and $r_{\mathcal{L}} > 0$. Then $r_{\mathcal{L}}$ is an eigenvalue of $\mathcal{L}$ with an eigenvector in P.*

Let X^* be the dual space of X, let P be a total cone in X and let P^* be the dual cone of P, i.e.

$$P^* = \{l \in X^* : l(u) \geqslant 0 \text{ for all } u \in P\}.$$

Let $\mathcal{L}, \mathcal{M} : X \to X$ be two linear compact operators such that $\mathcal{L}(P) \subseteq P$ and $\mathcal{M}(P) \subseteq P$. If their spectral radii $r_{\mathcal{L}}$ and $r_{\mathcal{M}}$ are positive, then by lemma 2.3 there exist $\phi_{\mathcal{L}}$ and $\phi_{\mathcal{M}} \in P \setminus \{\mathbf{0}\}$ such that

$$\mathcal{L}\phi_{\mathcal{L}} = r_{\mathcal{L}}\phi_{\mathcal{L}} \quad \text{and} \quad \mathcal{M}\phi_{\mathcal{M}} = r_{\mathcal{M}}\phi_{\mathcal{M}}. \tag{2.1}$$

Assume that there exists $h \in P^* \setminus \{\mathbf{0}\}$ such that

$$\mathcal{L}^* h = r_{\mathcal{M}} h, \tag{2.2}$$

where $\mathcal{L}^*$ is the dual operator of $\mathcal{L}$. Choose $\delta > 0$ and define

$$P(h, \delta) = \{u \in P : h(u) \geqslant \delta\|u\|\}. \tag{2.3}$$

Then $P(h, \delta)$ is a cone in X.

The following two lemmas are crucial in the proofs of our theorems. From here on, for any $R > 0$, let $B(\mathbf{0}, R) = \{u \in X : \|u\| < R\}$ be the open ball of X centred at $\mathbf{0}$ with radius R.

Lemma 2.4. *Assume that the following conditions hold:*

(A1) *there exist $\phi_{\mathcal{L}}$, $\phi_{\mathcal{M}} \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ such that (2.1) and (2.2) hold and $\mathcal{L}(P) \subseteq P(h, \delta)$;*

(A2) *$\mathcal{H} : X \to P$ is a continuous operator and satisfies*

$$\lim_{\|u\|\to\infty} \frac{\|\mathcal{H}u\|}{\|u\|} = 0;$$

(A3) *$\mathcal{F} : X \to X$ is a bounded continuous operator and there exists $u_0 \in X$ such that $\mathcal{F}u + \mathcal{H}u + u_0 \in P$ for all $u \in X$;*

(A4) *there exist* $v_0 \in X$ *and* $\varepsilon > 0$ *such that*

$$\mathcal{L}\mathcal{F}u \geqslant r_{\mathcal{M}}^{-1}(1+\varepsilon)\mathcal{L}u - \mathcal{L}\mathcal{H}u - v_0 \quad \textit{for all } u \in X.$$

Let $\mathcal{T} = \mathcal{L}\mathcal{F}$. *Then there exists* $R > 0$ *such that the Leray–Schauder degree*

$$\deg(\mathcal{I} - \mathcal{T}, B(\mathbf{0}, R), \mathbf{0}) = 0.$$

Lemma 2.5. *Assume that (A1) and the following conditions hold:*

(A2*) $\mathcal{H} : X \to P$ *is a continuous operator and satisfies*

$$\lim_{\|u\| \to 0} \frac{\|\mathcal{H}u\|}{\|u\|} = 0;$$

(A3*) $\mathcal{F} : X \to X$ *is a bounded continuous operator and there exists* $r_1 > 0$ *such that*

$$\mathcal{F}u + \mathcal{H}u \in P \quad \textit{for all } u \in X \textit{ with } \|u\| < r_1;$$

(A4*) *there exist* $\varepsilon > 0$ *and* $r_2 > 0$ *such that*

$$\mathcal{L}\mathcal{F}u \geqslant r_{\mathcal{M}}^{-1}(1+\varepsilon)\mathcal{L}u \quad \textit{for all } u \in X \textit{ with } \|u\| < r_2.$$

Let $\mathcal{T} = \mathcal{L}\mathcal{F}$. *Then there exists* $0 < R < \min\{r_1, r_2\}$ *such that the Leray–Schauder degree*

$$\deg(\mathcal{I} - \mathcal{T}, B(\mathbf{0}, R), \mathbf{0}) = 0.$$

Lemma 2.4 is a generalization of [9, theorem 2.1] and it is proved in [14, lemma 2.5] for the case when $\mathcal{L}$ and $\mathcal{M}$ are two specific linear operators, but the proof there also works for any general linear operators $\mathcal{L}$ and $\mathcal{M}$ satisfying (2.1) and (2.2). Lemma 2.5 generalizes [7, lemma 3.5]. In what follows we only give the proof of lemma 2.5.

Proof of lemma 2.5. For any $\nu > 0$ satisfying

$$\nu(\delta^{-1} r_{\mathcal{M}} \|h\| + \|\mathcal{L}\|) < 1, \tag{2.4}$$

from (A2*), there exists $r_3 > 0$ such that

$$\|\mathcal{H}u\| \leqslant \nu\|u\| \quad \text{for all } u \in X \text{ with } \|u\| < r_3. \tag{2.5}$$

We claim that there exists $0 < R < \min\{r_1, r_2, r_3\}$ such that

$$u - \mathcal{T}u \neq \tau\phi_{\mathcal{L}} \quad \text{for all } u \in \partial B(\mathbf{0}, R) \text{ and } \tau \geqslant 0. \tag{2.6}$$

If this is not the case, then, for all $0 < R < \min\{r_1, r_2, r_3\}$, there exist $u_1 \in \partial B(\mathbf{0}, R)$ and $\tau_1 \geqslant 0$ such that

$$u_1 - \mathcal{L}\mathcal{F}u_1 = \tau_1\phi_{\mathcal{L}}. \tag{2.7}$$

Then, from (2.2) and (A4*), we have

$$\begin{aligned} h(u_1) &= h(\mathcal{L}\mathcal{F}u_1) + \tau_1 h(\phi_{\mathcal{L}}) \geqslant h(\mathcal{L}\mathcal{F}u_1) \\ &\geqslant r_{\mathcal{M}}^{-1}(1+\varepsilon)h(\mathcal{L}u_1) = r_{\mathcal{M}}^{-1}(1+\varepsilon)(\mathcal{L}^*h)(u_1) \\ &= r_{\mathcal{M}}^{-1}(1+\varepsilon)r_{\mathcal{M}}h(u_1) = (1+\varepsilon)h(u_1). \end{aligned}$$

Hence, $h(u_1) \leqslant 0$. This, together with (2.2) and (2.5), implies that

$$\begin{aligned} h(u_1 + \mathcal{L}\mathcal{H}u_1) &= h(u_1) + h(\mathcal{L}\mathcal{H}u_1) \\ &= h(u_1) + (\mathcal{L}^* h)(\mathcal{H}u_1) \leqslant (\mathcal{L}^* h)(\mathcal{H}u_1) \\ &= r_{\mathcal{M}} h(\mathcal{H}u_1) \leqslant \nu r_{\mathcal{M}} \|h\| \|u_1\|, \end{aligned} \tag{2.8}$$

From (2.1) and (2.7), we see that

$$\begin{aligned} u_1 + \mathcal{L}\mathcal{H}u_1 &= \mathcal{L}\mathcal{F}u_1 + \mathcal{L}\mathcal{H}u_1 + \tau_1 \phi_{\mathcal{L}} \\ &= \mathcal{L}(\mathcal{F}u_1 + \mathcal{H}u_1) + \tau_1 r_{\mathcal{L}}^{-1} \mathcal{L}\phi_{\mathcal{L}}. \end{aligned}$$

In view of (A1) and (A3*), we see that $u_1 + \mathcal{L}\mathcal{H}u_1 \in P(h, \delta)$. Thus, by (2.3), we have

$$h(u_1 + \mathcal{L}\mathcal{H}u_1) \geqslant \delta \|u_1 + \mathcal{L}\mathcal{H}u_1\| \geqslant \delta \|u_1\| - \delta \|\mathcal{L}\mathcal{H}u_1\|,$$

and so

$$\|u_1\| \leqslant \delta^{-1} h(u_1 + \mathcal{L}\mathcal{H}u_1) + \|\mathcal{L}\mathcal{H}u_1\|.$$

Hence, from (2.5) and (2.8),

$$R = \|u_1\| \leqslant \delta^{-1} \nu r_{\mathcal{M}} \|h\| \|u_1\| + \nu \|\mathcal{L}\| \|u_1\| = \nu(\delta^{-1} r_{\mathcal{M}} \|h\| + \|\mathcal{L}\|) R.$$

Thus,

$$\nu(\delta^{-1} r_{\mathcal{M}} \|h\| + \|\mathcal{L}\|) \geqslant 1,$$

which contradicts (2.4). Therefore, there exists $0 < R < \min\{r_1, r_2, r_3\}$ such that (2.6) holds. Note that the operator $\mathcal{T}$ is compact. The conclusion now readily follows lemma 2.1, and this completes the proof of the lemma. □

Now we define

$$\left.\begin{aligned} G(t,s) &= \frac{\exp(\int_t^s a(v)\,\mathrm{d}v)}{\exp(\int_0^T a(v)\,\mathrm{d}v) - 1}, \\ c &= \frac{1}{\exp(\int_0^T a(v)\,\mathrm{d}v) - 1}, \\ d &= \frac{\exp(2\int_0^T a(v)\,\mathrm{d}v)}{\exp(\int_0^T a(v)\,\mathrm{d}v) - 1}. \end{aligned}\right\} \tag{2.9}$$

Then it is easy to see that $G(t+T, s+T) = G(t,s)$, $d > c > 0$ and

$$c \leqslant G(t,s) \leqslant d \quad \text{if } t \leqslant s \leqslant t + T, \tag{2.10}$$

$$c \leqslant G(t,s) \leqslant d \quad \text{if } -\tau(0) \leqslant t \leqslant s \leqslant T - \tau(0), \tag{2.11}$$

$$c \leqslant G(t,s) \leqslant d \quad \text{if } -\tau(0) \leqslant t \leqslant T - \tau(0) \leqslant s \leqslant 2T - \tau(0). \tag{2.12}$$

The following lemma can be directly verified.

Lemma 2.6. *The function $u(t)$ is a T-periodic solution of the equation*

$$u' = -a(t)u + k(t) \tag{2.13}$$

if and only if

$$u(t) = \int_t^{t+T} G(t,s)k(s)\,\mathrm{d}s,$$

where $k \in C(\mathbb{R},\mathbb{R})$ is a T-periodic function.

In the remainder of the paper, let the Banach space X be defined by

$$X = \{u \in C(\mathbb{R},\mathbb{R}) : u(t+T) = u(t) \text{ for } t \in \mathbb{R}\} \tag{2.14}$$

equipped with the norm $\|u\| = \sup_{t\in\mathbb{R}} |u(t)|$. Define a cone P in X by

$$P = \{u \in X : u(t) \geqslant 0 \text{ on } \mathbb{R}\}, \tag{2.15}$$

and a subcone K of P by

$$K = \{u \in P : u(t) \geqslant \sigma\|u\| \text{ on } \mathbb{R}\}, \tag{2.16}$$

where $\sigma = c/d$. Let the linear operators $\mathcal{L}, \mathcal{M} : X \to X$ be defined by

$$\mathcal{L}u(t) = \int_t^{t+T} G(t,s)b(s)u(g(s))\,\mathrm{d}s \tag{2.17}$$

and

$$\mathcal{M}u(t) = \int_0^{g^{-1}(t)} G(g(s),t)b(s)u(s)\,\mathrm{d}s + \int_{g^{-1}(t)}^{T} G(g(s),t+T)b(s)u(s)\,\mathrm{d}s, \tag{2.18}$$

where $g^{-1}(t)$ is the inverse function of $g(t)$.

The next two lemmas provide some useful information about the operators $\mathcal{L}$ and $\mathcal{M}$.

Lemma 2.7. *The operators $\mathcal{L}$ and $\mathcal{M}$ map P into K and are compact.*

Proof. We first show that $\mathcal{L}(P) \subseteq K$. For $u \in P$ and $t \in \mathbb{R}$, from (2.10) and (2.17) we have

$$c\int_0^T b(s)u(g(s))\,\mathrm{d}s \leqslant \mathcal{L}u(t) \leqslant d\int_0^T b(s)u(g(s))\,\mathrm{d}s.$$

As a result, $\mathcal{L}u(t) \geqslant (c/d)\|\mathcal{L}u\| = \sigma\|\mathcal{L}u\|$. Thus, $\mathcal{L}(P) \subseteq K$.

Next, we show that $\mathcal{M}$ maps P into K. To this end, first we prove that

$$\mathcal{M}u(t+T) = \mathcal{M}u(t) \quad \text{for any } u \in X \text{ and } t \in \mathbb{R}. \tag{2.19}$$

In fact, for any $u \in X$ and $t \in \mathbb{R}$, from (2.18),

$$\begin{aligned}
&\mathcal{M}u(t+T) \\
&\quad = \int_0^{g^{-1}(t+T)} G(g(s),t+T)b(s)u(s)\,\mathrm{d}s + \int_{g^{-1}(t+T)}^{T} G(g(s),t+2T)b(s)u(s)\,\mathrm{d}s \\
&\quad = I_1(u(t)) + I_2(u(t)),
\end{aligned}$$

where

$$I_1(u(t)) = \int_0^{g^{-1}(t+T)} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s$$

and

$$I_2(u(t)) = \int_{g^{-1}(t+T)}^{T} G(g(s), t+2T)b(s)u(s)\,\mathrm{d}s.$$

Since $g^{-1}(t+T) = g^{-1}(t)+T$, $g(s+T) = g(s)+T$ and $G(t+T, s+T) = G(t,s)$ for any $t, s \in \mathbb{R}$, we have

$$\begin{aligned}
I_1(u(t)) &= \int_0^{g^{-1}(t)+T} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s \\
&= \int_0^{g^{-1}(t)} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s \\
&\quad + \int_{g^{-1}(t)}^{T} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s \\
&\quad + \int_T^{g^{-1}(t)+T} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s \\
&= \int_0^{g^{-1}(t)} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s \\
&\quad + \int_{g^{-1}(t)}^{T} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s \\
&\quad + \int_0^{g^{-1}(t)} G(g(s)+T, t+T)b(s+T)u(s+T)\,\mathrm{d}s \\
&= \int_0^{g^{-1}(t)} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s \\
&\quad + \int_{g^{-1}(t)}^{T} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s \\
&\quad + \int_0^{g^{-1}(t)} G(g(s), t)b(s)u(s)\,\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
I_2(u(t)) &= \int_{g^{-1}(t)+T}^{T} G(g(s), t+2T)b(s)u(s)\,\mathrm{d}s \\
&= \int_{g^{-1}(t)}^{0} G(g(s)+T, t+2T)b(s+T)u(s+T)\,\mathrm{d}s \\
&= \int_{g^{-1}(t)}^{0} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s.
\end{aligned}$$

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Thus,

$$\begin{aligned}\mathcal{M}u(t+T) &= I_1(u(t)) + I_2(u(t)) \\ &= \int_{g^{-1}(t)}^{T} G(g(s), t+T)b(s)u(s)\,\mathrm{d}s + \int_0^{g^{-1}(t)} G(g(s), t)b(s)u(s)\,\mathrm{d}s \\ &= \mathcal{M}u(t),\end{aligned}$$

i.e. (2.19) holds. Hence, $\mathcal{M}(X) \subseteq X$. Consequently, for $u \in P$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{M}u(t) &\geqslant \min_{v\in[-\tau(0),T-\tau(0)]} \mathcal{M}u(v) \\ &= \min_{v\in[-\tau(0),T-\tau(0)]} \biggl(\int_0^{g^{-1}(v)} G(g(s), v)b(s)u(s)\,\mathrm{d}s \\ &\qquad\qquad + \int_{g^{-1}(v)}^{T} G(g(s), v+T)b(s)u(s)\,\mathrm{d}s\biggr)\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\mathcal{M}u(t) &\leqslant \max_{v\in[-\tau(0),T-\tau(0)]} \mathcal{M}u(v) \\ &= \max_{v\in[-\tau(0),T-\tau(0)]} \biggl(\int_0^{g^{-1}(v)} G(g(s), v)b(s)u(s)\,\mathrm{d}s \\ &\qquad\qquad + \int_{g^{-1}(v)}^{T} G(g(s), v+T)b(s)u(s)\,\mathrm{d}s\biggr).\end{aligned} \tag{2.21}$$

Note that

$$0 \leqslant g^{-1}(v) \leqslant T \iff -\tau(0) = g(0) \leqslant v \leqslant g(T) = T - \tau(0), \tag{2.22}$$

$$0 \leqslant s \leqslant g^{-1}(v) \iff -\tau(0) = g(0) \leqslant g(s) \leqslant v, \tag{2.23}$$

$$g^{-1}(v) \leqslant s \leqslant T \iff v \leqslant g(s) \leqslant g(T) = T - \tau(0). \tag{2.24}$$

Then, for $u \in P$ and $t \in \mathbb{R}$, from (2.11), (2.12), (2.20) and (2.21), it follows that

$$\begin{aligned}\mathcal{M}u(t) &\geqslant \min_{v\in[-\tau(0),T-\tau(0)]} \mathcal{M}u(v) \\ &\geqslant c\int_0^{g^{-1}(v)} b(s)u(s)\,\mathrm{d}s + c\int_{g^{-1}(v)}^{T} b(s)u(s)\,\mathrm{d}s = c\int_0^T b(s)u(s)\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}u(t) &\leqslant \max_{v\in[-\tau(0),T-\tau(0)]} \mathcal{M}u(v) \\ &\leqslant d\int_0^{g^{-1}(v)} b(s)u(s)\,\mathrm{d}s + d\int_{g^{-1}(v)}^{T} b(s)u(s)\,\mathrm{d}s = d\int_0^T b(s)u(s)\,\mathrm{d}s,\end{aligned}$$

from which we have $\mathcal{M}u(t) \geqslant (c/d)\|\mathcal{M}u\| = \sigma\|\mathcal{M}u\|$. Therefore, $\mathcal{M}(P) \subseteq K$.

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Finally, standard arguments can be used to show that $\mathcal{L}$ and $\mathcal{M}$ are compact and we omit the details here. This completes the proof of the lemma. □

LEMMA 2.8. *We have the following.*

(i) *The spectral radius, $r_{\mathcal{L}}$, of $\mathcal{L}$ satisfies $r_{\mathcal{L}} > 0$. Moreover, $r_{\mathcal{L}}$ is an eigenvalue of $\mathcal{L}$ with an eigenvector $\phi_{\mathcal{L}} \in P$.*

(ii) *The spectral radius, $r_{\mathcal{M}}$, of $\mathcal{M}$ satisfies $r_{\mathcal{M}} > 0$. Moreover, $r_{\mathcal{M}}$ is an eigenvalue of $\mathcal{M}$ with an eigenvector $\phi_{\mathcal{M}} \in P$.*

Proof. The ideas of the proof for parts (i) and (ii) are essentially the same. In the following, we only prove part (ii). Let $u \in K$ and $t \in \mathbb{R}$. Noting (2.22)–(2.24), from (2.11), (2.12) and (2.20) we see that

$$\begin{aligned}\mathcal{M}u(t) &\geqslant \min_{v\in[-\tau(0),T-\tau(0)]} Mu(v)\\ &\geqslant c\int_0^{g^{-1}(v)} b(s)u(s)\,\mathrm{d}s + c\int_{g^{-1}(v)}^T b(s)u(s)\,\mathrm{d}s\\ &= c\int_0^T b(s)u(s)\,\mathrm{d}s \geqslant \sigma\|u\|c\int_0^T b(s)\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}^2u(t) &= \mathcal{M}(\mathcal{M}u(t))\\ &\geqslant \min_{v\in[-\tau(0),T-\tau(0)]} \mathcal{M}(\mathcal{M}u(v))\\ &\geqslant c\int_0^{g^{-1}(v)} b(s)\left(c\sigma\|u\|\int_0^T b(s)\,\mathrm{d}s\right)\mathrm{d}s\\ &\quad + c\int_{g^{-1}(v)}^T b(s)\left(c\sigma\|u\|\int_0^T b(s)\,\mathrm{d}s\right)\mathrm{d}s\\ &= \sigma\|u\|\left(c\int_0^T b(s)\,\mathrm{d}s\right)^2.\end{aligned}$$

By induction, we obtain that

$$\mathcal{M}^n u(t) \geqslant \sigma\|u\|\left(c\int_0^T b(s)\,\mathrm{d}s\right)^n.$$

Then,

$$\|\mathcal{M}^n\|\|u\| \geqslant \|\mathcal{M}^n u\| \geqslant \mathcal{M}^n u(t) \geqslant \sigma\|u\|\left(c\int_0^T b(s)\,\mathrm{d}s\right)^n,$$

and so

$$\|\mathcal{M}^n\| \geqslant \sigma\left(c\int_0^T b(s)\,\mathrm{d}s\right)^n.$$

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Hence,

$$r_{\mathcal{M}} = \lim_{n\to\infty} \|\mathcal{M}^n\|^{1/n} \geqslant c\int_0^T b(s)\,\mathrm{d}s > 0.$$

Now, in view of the fact that the cone P defined by (2.15) is a total cone and that $r_{\mathcal{M}} > 0$, the 'moreover' part of part (ii) readily follows from lemmas 2.3 and 2.7. This completes the proof of the lemma. □

3. Main results

For convenience, we introduce the following notation:

$$f_0 = \liminf_{x\to 0^+} \frac{f(x)}{x}, \qquad f_\infty = \liminf_{x\to\infty} \frac{f(x)}{x},$$
$$F_0 = \limsup_{x\to 0} \left|\frac{f(x)}{x}\right|, \qquad F_\infty = \limsup_{|x|\to\infty} \left|\frac{f(x)}{x}\right|,$$
$$\xi = \left(d\int_0^T b(s)\,\mathrm{d}s\right)^{-1}, \qquad \eta = \left(c\sigma\int_0^T b(s)\,\mathrm{d}s\right)^{-1}. \tag{3.1}$$

In the rest of this paper, we also let

$$\mu_{\mathcal{M}} = \frac{1}{r_{\mathcal{M}}}, \tag{3.2}$$

where $r_{\mathcal{M}}$ is given in lemma 2.8(ii). Clearly, $\mu_{\mathcal{M}}$ is the smallest positive characteristic value of $\mathcal{M}$ satisfying $\phi_{\mathcal{M}} = \mu_{\mathcal{M}} M\phi_{\mathcal{M}}$, and as we will see by lemma 4.1, $\xi \leqslant \mu_{\mathcal{M}} \leqslant \eta$.

We need the following assumptions.

(B1) There exist a constant $M \geqslant 0$ and a function $\alpha \in C(\mathbb{R}, \mathbb{R}^+)$ such that α is even and non-decreasing on $\mathbb{R}^+$,

$$f(x) \geqslant -M - \alpha(x) \quad \text{for all } x \in \mathbb{R} \tag{3.3}$$

and

$$\lim_{x\to\infty} \frac{\alpha(x)}{x} = 0. \tag{3.4}$$

(B2) There exist a constant $0 < r < 1$ and a function $\beta \in C(\mathbb{R}, \mathbb{R}^+)$ such that β is even and non-decreasing on $\mathbb{R}^+$,

$$f(x) \geqslant -\beta(x) \quad \text{for all } x \in [-r, 0] \tag{3.5}$$

and

$$\lim_{x\to 0} \frac{\beta(x)}{x} = 0. \tag{3.6}$$

Remark 3.1. Here, we wish to emphasize that, in (B1), we assume that $f(x)$ is bounded from below by $-M - \alpha(x)$ for all $x \in \mathbb{R}$. However, in (B2), we only require that $f(x)$ is bounded from below by $-\beta(x)$ for x in a small left-neighbourhood of 0.

We first state our existence results for equation (1.1).

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Theorem 3.2. *Assume that (B1) holds. If*

$$F_0 < \mu_{\mathcal{M}} < f_\infty,$$

then equation (1.1) has at least one non-trivial T-periodic solution.

Theorem 3.3. *Assume that (B2) holds. If*

$$F_\infty < \mu_{\mathcal{M}} < f_0,$$

then equation (1.1) has at least one non-trivial T-periodic solution.

Corollary 3.4. *Assume that (B1) holds. If*

$$\frac{F_0}{\xi} < 1 < \frac{f_\infty}{\eta},$$

then equation (1.1) has at least one non-trivial T-periodic solution.

Corollary 3.5. *Assume that (B2) holds. If*

$$\frac{F_\infty}{\xi} < 1 < \frac{f_0}{\eta},$$

then equation (1.1) has at least one non-trivial T-periodic solution.

Next, we state our existence results for equation (1.2); they are immediate consequences of the above results.

Corollary 3.6. *Assume that (B1) holds. If*

$$\frac{\mu_{\mathcal{M}}}{f_\infty} < \lambda < \frac{\mu_{\mathcal{M}}}{F_0},$$

then equation (1.2) has at least one non-trivial T-periodic solution.

Corollary 3.7. *Assume that (B2) holds. If*

$$\frac{\mu_{\mathcal{M}}}{f_0} < \lambda < \frac{\mu_{\mathcal{M}}}{F_\infty},$$

then equation (1.2) has at least one non-trivial T-periodic solution.

Corollary 3.8. *Assume that (B1) holds. If*

$$\frac{\eta}{f_\infty} < \lambda < \frac{\xi}{F_0},$$

then equation (1.2) has at least one non-trivial T-periodic solution.

Corollary 3.9. *Assume that (B2) holds. If*

$$\frac{\eta}{f_0} < \lambda < \frac{\xi}{F_\infty},$$

then equation (1.2) has at least one non-trivial T-periodic solution.

We conclude this section with the following two simple examples.

Example 3.10. Let

$$f(x)=\begin{cases}\displaystyle\sum_{i=1}^{n}\gamma_i x^i, & x\in[-1,\infty),\\ \displaystyle\sum_{i=1}^{n}(-1)^i\gamma_i-|x|^{\theta}\ln(1+|x|)+\ln 2, & x\in(-\infty,-1],\end{cases}\tag{3.7}$$

where $n\geqslant 1$ is an integer, $\gamma_i\in\mathbb{R}$ with $0\leqslant\gamma_1<(\mathrm{e}^{2\pi}-1)/2\pi\mathrm{e}^{4\pi}$ and $\gamma_n>0$, and $0<\theta<1$. Clearly, $f\in C(\mathbb{R},\mathbb{R})$.

We claim that functional differential equation

$$u'(t)=-u(t)+f(u(t-\sin t))\tag{3.8}$$

has a non-trivial 2π-periodic solution.

In fact, with $a(t)=b(t)=1$, $T=2\pi$, $\tau(t)=\sin t$ and $g(t)=t-\sin t$ it is easy to see that equation (3.8) is of the form of equation (1.1) and assumption (H) is satisfied. Moreover, from (3.1), we have

$$\xi=\frac{\mathrm{e}^{2\pi}-1}{2\pi\mathrm{e}^{4\pi}}\quad\text{and}\quad\eta=\frac{\mathrm{e}^{4\pi}(\mathrm{e}^{2\pi}-1)}{2\pi}.\tag{3.9}$$

Let

$$M=\sum_{i=1}^{n}|\gamma_i|+\ln 2\quad\text{and}\quad\alpha(x)=|x|^{\theta}\ln(1+|x|).$$

Then, in view of (3.7), we have

$$f(x)\geqslant -M-\alpha(x)\quad\text{for all } x\in\mathbb{R}$$

and

$$\lim_{x\to\infty}\frac{\alpha(x)}{x}=\lim_{x\to\infty}\frac{\ln(1+x)}{x^{1-\theta}}=0.$$

Thus, (B1) holds. From (3.7), we have

$$F_0=\limsup_{x\to 0}\left|\frac{f(x)}{x}\right|=\gamma_1<\xi\quad\text{and}\quad f_\infty=\liminf_{x\to\infty}\frac{f(x)}{x}=\infty.$$

Hence, $F_0/\xi<1<f_\infty/\eta$. The conclusion then follows from corollary 3.4.

Example 3.11. Let f be defined as in example 3.10. Then we claim that, for any λ satisfying $0<\lambda<(\mathrm{e}^{2\pi}-1)/2\pi\gamma_1\mathrm{e}^{4\pi}$, the eigenvalue problem

$$u'(t)=-u(t)+\lambda f(u(t-\sin t))$$

has a non-trivial 2π-periodic solution.

In fact, as in example 3.10, (B1) holds. Moreover, ξ and η are given by (3.9), $F_0=\gamma_1$ and $f_\infty=\infty$. Note that

$$0<\lambda<\frac{\mathrm{e}^{2\pi}-1}{2\pi\gamma_1\mathrm{e}^{4\pi}}\quad\Longleftrightarrow\quad\frac{\eta}{f_\infty}<\lambda<\frac{\xi}{F_0}.$$

The conclusion then readily follows from corollary 3.8.

Note that, in these examples, for x negative, the function f is negative and unbounded from below.

4. Proofs of the main results

Let X, P, K, $\mathcal{L}$ and $\mathcal{M}$ be defined by (2.14)–(2.18), respectively. By lemma 2.7, $\mathcal{L}$ and $\mathcal{M}$ map P into K and are compact. Define operators $\mathcal{F}$ and $\mathcal{T}: X \to X$ by

$$\mathcal{F}u(t) = f(u(t)) \tag{4.1}$$

and

$$\mathcal{T}u(t) = \mathcal{L}\mathcal{F}u(t) = \int_t^{t+T} G(t,s)b(s)f((u(g(s))))\,\mathrm{d}s. \tag{4.2}$$

Then $\mathcal{F}: X \to X$ is bounded and $\mathcal{T}: X \to X$ is compact. Moreover, by lemma 2.6, a T-periodic solution of equation (1.1) is equivalent to a fixed point of the operator $\mathcal{T}$ in X.

Proof of theorem 3.2. We first verify that conditions (A1)–(A4) of lemma 2.4 are satisfied.

By lemma 2.8, there exist $\phi_{\mathcal{L}}, \phi_{\mathcal{M}} \in P \setminus \{\mathbf{0}\}$ such that (2.1) holds. To show that (2.2) holds, we let

$$h(u) = \int_0^T b(t)\phi_{\mathcal{M}}(t)u(g(t))\,\mathrm{d}t, \quad u \in X. \tag{4.3}$$

Then $h \in P^* \setminus \{\mathbf{0}\}$ and

$$\begin{aligned}(L^*h)(u) = h(Lu) &= \int_0^T b(t)\phi_{\mathcal{M}}(t)\mathcal{L}u(g(t))\,\mathrm{d}t \\ &= \int_0^T b(t)\phi_{\mathcal{M}}(t)\biggl(\int_{g(t)}^{g(t)+T} G(g(t),s)b(s)u(g(s))\,\mathrm{d}s\biggr)\mathrm{d}t \\ &= \int_0^T \int_{g(t)}^{g(t)+T} G(g(t),s)b(s)u(g(s))b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Interchanging the order of integration and noting that $g(T) = g(0) + T$, we have

$$\begin{aligned}(\mathcal{L}^*h)(u) = &\int_{g(0)}^{g(T)} b(s)u(g(s))\biggl(\int_0^{g^{-1}(s)} G(g(t),s)b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}t\biggr)\mathrm{d}s \\ &+ \int_{g(0)+T}^{g(T)+T} b(s)u(g(s))\biggl(\int_{g^{-1}(s-T)}^{T} G(g(t),s)b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}t\biggr)\mathrm{d}s.\end{aligned}$$

Letting $s = v + T$ in the second term and noting that

$$g(v+T) = g(v) + T,$$

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we obtain

$$
\begin{aligned}
(\mathcal{L}^*h)(u) &= \int_{g(0)}^{g(T)} b(s)u(g(s))\biggl(\int_0^{g^{-1}(s)} G(g(t),s)b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}t\biggr)\,\mathrm{d}s \\
&\quad + \int_{g(0)}^{g(T)} b(v+T)u(g(v)+T) \\
&\qquad \times \biggl(\int_{g^{-1}(v)}^{T} G(g(t),v+T)b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}t\biggr)\,\mathrm{d}v \\
&= \int_{g(0)}^{g(T)} b(s)u(g(s))\biggl(\int_0^{g^{-1}(s)} G(g(t),s)b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}t\biggr)\,\mathrm{d}s \\
&\quad + \int_{g(0)}^{g(T)} b(v)u(g(v))\biggl(\int_{g^{-1}(v)}^{T} G(g(t),v+T)b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}t\biggr)\,\mathrm{d}v \\
&= \int_{g(0)}^{g(T)} b(s)u(g(s))\biggl(\int_0^{g^{-1}(s)} G(g(t),s)b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}t \\
&\qquad\qquad + \int_{g^{-1}(s)}^{T} G(g(t),s+T)b(t)\phi_{\mathcal{M}}(t)\,\mathrm{d}t\biggr)\,\mathrm{d}s.
\end{aligned}
$$

Thus, in view of (2.1), (2.18), (4.3) and the fact that $g(T) = g(0) + T$, we have

$$
\begin{aligned}
(\mathcal{L}^*h)(u) &= \int_{g(0)}^{g(T)} b(s)u(g(s))\mathcal{M}\phi_{\mathcal{M}}(s)\,\mathrm{d}s = r_{\mathcal{M}}\int_{g(0)}^{g(T)} b(s)\phi_{\mathcal{M}}(s)u(g(s))\,\mathrm{d}s \\
&= r_{\mathcal{M}}\int_0^T b(s)\phi_{\mathcal{M}}(s)u(g(s))\,\mathrm{d}s = r_{\mathcal{M}}h(u),
\end{aligned}
$$

i.e. h satisfies (2.2). Since $\phi_{\mathcal{M}}(s) \geqslant \sigma\|\phi_{\mathcal{M}}\| > 0$ on $\mathbb{R}$, there exists $\delta_1 > 0$ such that

$$
\phi_{\mathcal{M}}(s) \geqslant \delta_1 G(t,s) \quad \text{for } t,s \in \mathbb{R}. \tag{4.4}
$$

Let $\delta = r_{\mathcal{M}}\delta_1$. For any $u \in P$ and $t \in \mathbb{R}$, from (2.17), (4.3) and (4.4), it follows that

$$
\begin{aligned}
h(\mathcal{L}u) &= r_{\mathcal{M}}h(u) \\
&= r_{\mathcal{M}}\int_0^T b(s)\phi_{\mathcal{M}}(s)u(g(s))\,\mathrm{d}s \\
&= r_{\mathcal{M}}\int_t^{t+T} b(s)\phi_{\mathcal{M}}(s)u(g(s))\,\mathrm{d}s \\
&\geqslant r_{\mathcal{M}}\delta_1\int_t^{t+T} G(t,s)b(s)u(g(s))\,\mathrm{d}s \\
&= \delta\mathcal{L}u(t).
\end{aligned}
$$

Hence, $h(\mathcal{L}u) \geqslant \delta\|\mathcal{L}u\|$, i.e. $\mathcal{L}(P) \subseteq P(h,\delta)$. Therefore, condition (A1) of lemma 2.4 holds.

Since α is non-decreasing on $\mathbb{R}^+$, we have

$$\alpha(u) \leqslant \alpha(\|u\|) \quad \text{for all } u \in P.$$

Then, from the fact that α is even, it follows that

$$\alpha(u) \leqslant \alpha(\|u\|) \quad \text{for all } u \in X.$$

Thus,

$$\|\alpha(u)\| \leqslant \alpha(\|u\|) \quad \text{for all } u \in X.$$

From (3.4), we see that

$$\lim_{\|u\|\to\infty} \frac{\|\alpha(u)\|}{\|u\|} = 0 \quad \text{for any } u \in X.$$

Letting $\mathcal{H}u = \alpha(u)$ for $u \in X$, we show that condition (A2) in lemma 2.4 holds.

With $\mathcal{F}$ defined by (4.1) and $u_0 = M$, from (3.3), we have $\mathcal{F}u + \mathcal{H}u + u_0 \in P$ for all $u \in X$. Hence, (A3) of lemma 2.4 holds.

Since $f_\infty > \mu_{\mathcal{M}}$, there exist $\varepsilon > 0$ and $N > 0$ such that

$$f(x) \geqslant \mu_{\mathcal{M}}(1+\varepsilon)x \quad \text{for } x \geqslant N.$$

Then, in view of (3.3), there exists $\rho > 0$ such that

$$f(x) \geqslant \mu_{\mathcal{M}}(1+\varepsilon)x - \alpha(x) - \rho \quad \text{for all } x \in \mathbb{R}.$$

From (3.2) and (4.1), we have

$$\mathcal{F}u \geqslant \mu_{\mathcal{M}}(1+\varepsilon)u - \alpha(u) - \rho = r_{\mathcal{M}}^{-1}(1+\varepsilon)u - \mathcal{H}u - \rho \quad \text{for all } u \in X.$$

Thus,

$$\mathcal{L}\mathcal{F}u \geqslant r_{\mathcal{M}}^{-1}(1+\varepsilon)\mathcal{L}u - \mathcal{L}\mathcal{H}u - \mathcal{L}\rho \quad \text{for all } u \in X.$$

Then (A4) of lemma 2.4 holds with $v_0 = \mathcal{L}\rho$.

We have verified that all the conditions of lemma 2.4 hold, so there exists $R_1 > 0$ such that

$$\deg(\mathcal{I} - \mathcal{T}, B(\mathbf{0}, R_1), \mathbf{0}) = 0. \tag{4.5}$$

Next, since $F_0 < \mu_{\mathcal{M}}$, there exist $0 < \nu < 1$ and $0 < R_2 < R_1$ such that

$$|f(x)| \leqslant \mu_{\mathcal{M}}(1-\nu)|x| \quad \text{for } |x| < R_2. \tag{4.6}$$

We claim that

$$\mathcal{T}u \neq \tau u \quad \text{for all } u \in \partial B(\mathbf{0}, R_2) \text{ and } \tau \geqslant 1. \tag{4.7}$$

If this is not the case, then there exist $\bar{u} \in \partial B(\mathbf{0}, R_2)$ and $\bar{\tau} \geqslant 1$ such that $\mathcal{T}\bar{u} = \bar{\tau}\bar{u}$. It follows that $\bar{u} = \bar{s}\mathcal{T}\bar{u}$, where $\bar{s} = 1/\bar{\tau}$. Clearly, $\bar{s} \in (0, 1]$. Then, from (2.17), (4.2) and (4.6), we have

$$\begin{aligned}|\bar{u}(t)| &= \bar{s}|\mathcal{T}\bar{u}(t)| \\ &\leqslant \int_t^{t+T} G(t,s)b(s)|f(\bar{u}(g(s)))|\,\mathrm{d}s \\ &\leqslant \mu_{\mathcal{M}}(1-\nu)\int_t^{t+T} G(t,s)b(s)|\bar{u}(g(s))|\,\mathrm{d}s \\ &= \mu_{\mathcal{M}}(1-\nu)\mathcal{L}|\bar{u}(t)|.\end{aligned}$$

Consequently,

$$\begin{aligned}h(|\bar{u}|) &\leqslant \mu_{\mathcal{M}}(1-\nu)h(\mathcal{L}|\bar{u}|) \\ &= \mu_{\mathcal{M}}(1-\nu)(\mathcal{L}^*h)(|\bar{u}|) \\ &= r_{\mathcal{M}}^{-1}(1-\nu)r_{\mathcal{M}}h(|\bar{u}|) \\ &= (1-\nu)h(|\bar{u}|).\end{aligned}$$

Thus, $h(|\bar{u}|) \leqslant 0$. On the other hand, in view of the fact that $\phi_{\mathcal{M}}(t) > 0$ and $\|\bar{u}\| = R_2 > 0$, by (4.3), $h(|\bar{u}|) > 0$. This contradiction implies that (4.7) holds. Now lemma 2.2 implies

$$\deg(\mathcal{I}-\mathcal{T}, B(\mathbf{0},R_2),\mathbf{0}) = 1, \tag{4.8}$$

so, by the additivity property of the Leray–Schauder degree, (4.5) and (4.8), we have

$$\deg(\mathcal{I}-\mathcal{T}, B(\mathbf{0},R_1)\setminus\overline{B(\mathbf{0},R_2)}) = -1.$$

Then, from the solution property of the Leray–Schauder degree, $\mathcal{T}$ has at least one fixed point u in $B(\mathbf{0},R_1)\setminus\overline{B(\mathbf{0},R_2)}$. Clearly, $u(t)$ is a non-trivial solution of equation (1.1). This completes the proof of the theorem. □

Proof of theorem 3.3. We first verify that conditions (A1) and (A2*)–(A4*) of lemma 2.5 are satisfied.

As in the proof of theorem 3.2, there exist $\phi_{\mathcal{L}}$, $\phi_{\mathcal{M}} \in P\setminus\{\mathbf{0}\}$ and $h \in P^*\setminus\{\mathbf{0}\}$ defined by (4.3) such that (A1) holds.

From the fact that β is even and non-decreasing on $\mathbb{R}^+$, it is easy to see that

$$\beta(u) \leqslant \beta(\|u\|) \quad \text{for all } u \in X.$$

Thus,

$$\|\beta(u)\| \leqslant \beta(\|u\|) \quad \text{for all } u \in X.$$

This, together with (3.6), implies that

$$\lim_{\|u\|\to 0}\frac{\|\beta(u)\|}{\|u\|} = 0 \quad \text{for any } u \in X.$$

Let $\mathcal{H}u = \beta(u)$ for $u \in X$. Then (A2*) of lemma 2.5 holds.

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Since $f_0 > \mu_{\mathcal{M}}$, there exist $\varepsilon > 0$ and $0 < \zeta_1 < 1$ such that

$$f(x) \geqslant \mu_{\mathcal{M}}(1+\varepsilon)x = r_{\mathcal{M}}^{-1}(1+\varepsilon)x \geqslant 0 \quad \text{for } x \in [0, \zeta_1]. \tag{4.9}$$

Let r be given as in (B2) and let F be defined by (4.1). Now, in view of (3.5) and (4.9), we see that (A3*) of lemma 2.5 holds with $r_1 = \min\{r, \zeta_1\}$.

From (3.6), there exists $0 < \zeta_2 < \min\{r, \zeta_1\}$ such that

$$-\beta(x) \geqslant r_{\mathcal{M}}^{-1}(1+\varepsilon)x \quad \text{for } x \in [-\zeta_2, 0].$$

Then, from (3.5),

$$f(x) \geqslant r_{\mathcal{M}}^{-1}(1+\varepsilon)x \quad \text{for } x \in [-\zeta_2, 0]. \tag{4.10}$$

From (4.9) and (4.10), we have

$$f(x) \geqslant r_{\mathcal{M}}^{-1}(1+\varepsilon)x \quad \text{for } x \in [-\zeta_2, \zeta_2], \tag{4.11}$$

which clearly implies that

$$\mathcal{L}\mathcal{F}u \geqslant r_{\mathcal{M}}^{-1}(1+\varepsilon)\mathcal{L}u \quad \text{for all } u \in X \text{ with } \|u\| < \zeta_2.$$

Hence, (A4*) of lemma 2.5 holds with $r_2 = \zeta_2$.

We have verified that all the conditions of lemma 2.5 hold, so there exists $R_3 > 0$ such that

$$\deg(\mathcal{I} - \mathcal{T}, B(\mathbf{0}, R_3), \mathbf{0}) = 0. \tag{4.12}$$

Next, since $F_\infty < \mu_{\mathcal{M}}$, there exist $0 < \tilde{\nu} < 1$ and $\bar{R} > R_3$ such that

$$|f(x)| \leqslant \mu_{\mathcal{M}}(1-\tilde{\nu})|x| = r_{\mathcal{M}}^{-1}(1-\tilde{\nu})|x| \quad \text{for } |x| \in (\bar{R}, \infty). \tag{4.13}$$

Let

$$\mathcal{C} = \max_{|x| \leqslant \bar{R}} |f(x)| \sup_{t \in \mathbb{R}} \int_t^{t+T} G(t,s)b(s)\,\mathrm{d}s. \tag{4.14}$$

Then $0 < \mathcal{C} < \infty$. Choose R_4 large enough so that

$$R_4 > \max\{\bar{R}, \tilde{\nu}^{-1}\mathcal{C}\}. \tag{4.15}$$

We claim that

$$\mathcal{T}u \neq \tau u \quad \text{for all } u \in \partial B(\mathbf{0}, R_4) \text{ and } \tau \geqslant 1. \tag{4.16}$$

If this is not the case, then there exist $\tilde{u} \in \partial B(\mathbf{0}, R_4)$ and $\tilde{\tau} \geqslant 1$ such that $\mathcal{T}\tilde{u} = \tilde{\tau}\tilde{u}$. It follows that $\tilde{u} = \tilde{s}\mathcal{T}\tilde{u}$, where $\tilde{s} = 1/\tilde{\tau}$. Clearly, $\tilde{s} \in (0,1]$. Since $\tilde{u}$ is T-periodic, we may assume that $R_4 = \|\tilde{u}\| = |\tilde{u}(\tilde{t})|$ for some $\tilde{t} \in [0,T]$. Let

$$J_1(\tilde{u}) = \{t \in [\tilde{t}, \tilde{t}+T] : |\tilde{u}(g(t))| > \bar{R}\},$$

$$J_2(\tilde{u}) = [\tilde{t}, \tilde{t}+T] \setminus J_1(\tilde{u})$$

and

$$p(\tilde{u}(t)) = \min\{|\tilde{u}(g(t))|, \bar{R}\} \quad \text{for } t \in [\tilde{t}, \tilde{t}+T].$$

Then, from (2.17), (4.2), (4.13) and (4.14), we have

$$\begin{aligned}
R_4 &= |\tilde{u}(\tilde{t})| = \tilde{s}|\mathcal{T}\tilde{u}(\tilde{t})| \\
&\leqslant \int_{\tilde{t}}^{\tilde{t}+T} G(\tilde{t},s)b(s)|f(\tilde{u}(g(s)))|\,\mathrm{d}s \\
&= \int_{J_1(\tilde{u})} G(\tilde{t},s)b(s)|f(\tilde{u}(g(s)))|\,\mathrm{d}s + \int_{J_2(\tilde{u})} G(\tilde{t},s)b(s)|f(\tilde{u}(g(s)))|\,\mathrm{d}s \\
&\leqslant r_{\mathcal{M}}^{-1}(1-\tilde{\nu})\int_{J_1(\tilde{u})} G(\tilde{t},s)b(s)|\tilde{u}(g(s))|\,\mathrm{d}s + \int_{J_2(\tilde{u})} G(\tilde{t},s)b(s)|f(p(\tilde{u}(s)))|\,\mathrm{d}s \\
&\leqslant r_{\mathcal{M}}^{-1}(1-\tilde{\nu})\int_{\tilde{t}}^{\tilde{t}+T} G(\tilde{t},s)b(s)|\tilde{u}(g(s))|\,\mathrm{d}s + \int_{\tilde{t}}^{\tilde{t}+T} G(\tilde{t},s)b(s)|f(p(\tilde{u}(s)))|\,\mathrm{d}s \\
&\leqslant r_{\mathcal{M}}^{-1}(1-\tilde{\nu})\mathcal{L}|u(\tilde{t})| + \mathcal{C} \\
&= r_{\mathcal{M}}^{-1}(1-\tilde{\nu})\mathcal{L}R_4 + \mathcal{C}.
\end{aligned}$$

Hence, for h defined by (4.3),

$$\begin{aligned}
h(R_4) &\leqslant r_{\mathcal{M}}^{-1}(1-\tilde{\nu})h(\mathcal{L}R_4) + h(\mathcal{C}) \\
&= r_{\mathcal{M}}^{-1}(1-\tilde{\nu})(\mathcal{L}^*h)(R_4) + h(\mathcal{C}) \\
&= r_{\mathcal{M}}^{-1}(1-\tilde{\nu})r_{\mathcal{M}}h(R_4) + h(\mathcal{C}) \\
&= (1-\tilde{\nu})h(R_4) + h(\mathcal{C}),
\end{aligned}$$

which implies that

$$(\tilde{\nu}R_4 - \mathcal{C})h(1) \leqslant 0.$$

In view of the fact that $h(1) > 0$, it follows that $R_4 \leqslant \tilde{\nu}^{-1}\mathcal{C}$. This contradicts (4.15) and so (4.16) holds. By lemma 2.2, we have

$$\deg(\mathcal{I} - \mathcal{T}, B(\mathbf{0}, R_4), \mathbf{0}) = 1. \tag{4.17}$$

By the additivity property of the Leray–Schauder degree, (4.12) and (4.17), we obtain

$$\deg(\mathcal{I} - \mathcal{T}, B(\mathbf{0}, R_4) \setminus \overline{B(\mathbf{0}, R_3)}) = 1.$$

Thus, from the solution property of the Leray–Schauder degree, $\mathcal{T}$ has at least one fixed point u in $B(\mathbf{0}, R_4) \setminus \overline{B(\mathbf{0}, R_3)}$. Clearly, $u(t)$ is a non-trivial solution of equation (1.1), and this completes the proof of the theorem. □

Lemma 4.1. *Let $\mu_{\mathcal{M}}$ be defined by (3.2). Then $\xi \leqslant \mu_{\mathcal{M}} \leqslant \eta$, where ξ and η are given by (3.1).*

Proof. Let $\phi_{\mathcal{M}}$ be given as in lemma 2.8(ii). Then $\phi_{\mathcal{M}}(t) = \mu_{\mathcal{M}}\mathcal{M}\phi_{\mathcal{M}}(t)$, i.e.

$$\phi_{\mathcal{M}}(t) = \mu_{\mathcal{M}}\bigg(\int_0^{g^{-1}(t)} G(g(s),t)b(s)\phi_{\mathcal{M}}(s)\,\mathrm{d}s + \int_{g^{-1}(t)}^{T} G(g(s),t+T)b(s)\phi_{\mathcal{M}}(s)\,\mathrm{d}s\bigg).$$

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Thus, in view of (2.22)–(2.24) and from (2.11) and (2.12), it follows that

$$\begin{aligned}\phi_{\mathcal{M}}(t) &\leqslant \mu_{\mathcal{M}} \max_{v\in[-\tau(0),T-\tau(0)]} \mathcal{M}\phi_{\mathcal{M}}(v)\\ &\leqslant \mu_{\mathcal{M}}\left(d\|\phi_{\mathcal{M}}\| \int_0^{g^{-1}(v)} b(s)\,\mathrm{d}s + d\|\phi_{\mathcal{M}}\| \int_{g^{-1}(v)}^T b(s)\,\mathrm{d}s\right)\\ &= \mu_{\mathcal{M}} d\|\phi_{\mathcal{M}}\| \int_0^T b(s)\,\mathrm{d}s \quad \text{on } \mathbb{R}.\end{aligned}$$

Hence,

$$\mu_{\mathcal{M}} \geqslant \left(d\int_0^T b(s)\,\mathrm{d}s\right)^{-1} = \xi.$$

On the other hand, since $\phi_{\mathcal{M}} \in K$ we have $\phi_{\mathcal{M}}(t) \geqslant \sigma\|\phi_{\mathcal{M}}\|$ on $\mathbb{R}$, and then again from (2.11) and (2.12), we have

$$\begin{aligned}\phi_{\mathcal{M}}(t) &\geqslant \mu_{\mathcal{M}} \min_{v\in[-\tau(0),T-\tau(0)]} \mathcal{M}\phi_{\mathcal{M}}(v)\\ &\geqslant \mu_{\mathcal{M}}\left(c\sigma\|\phi_{\mathcal{M}}\| \int_0^{g^{-1}(v)} b(s)\,\mathrm{d}s + c\sigma\|\phi_{\mathcal{M}}\| \int_{g^{-1}(v)}^T b(s)\,\mathrm{d}s\right)\\ &= \mu_{\mathcal{M}} c\sigma\|\phi_{\mathcal{M}}\| \int_0^T b(s)\,\mathrm{d}s \quad \text{on } \mathbb{R}.\end{aligned}$$

Thus,

$$\mu_{\mathcal{M}} \leqslant \left(c\sigma\int_0^T b(s)\,\mathrm{d}s\right)^{-1} = \eta.$$

This completes the proof of the lemma. □

Proof of corollary 3.4. The conclusion follows from theorem 3.2 and lemma 4.1. □

Proof of corollary 3.5. The conclusion follows from theorem 3.3 and lemma 4.1. □

Finally, by virtue of lemma 4.1, corollaries 3.6–3.9 are direct applications of theorems 3.2 and 3.3 and corollaries 3.4 and 3.5 with f in equation (1.1) replaced by λf. We omit the proofs here.

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