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FREE AMALGAMATION AND AUTOMORPHISM GROUPS

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Abstract. We show that the class of graded c-nilpotent Lie algebras over a fixed field K is closed under free amalgamation. In [1] this result was applied, but its proof was incorrect. In case of a finite field K we obtain a Fraïssé limit of all finite graded c-nilpotent Lie algebras over K. This gives an example for the following more general considerations. The existence of free amalgamation for the age of a Fraïssé limit implies the universality of its automorphism group for all automorphism groups of substructures of that Fraïssé limit. We use [6] and [5].

§1. Introduction. We consider graded Lie algebras A over a fixed field K in the language L_{Lie} of vector spaces over K extended by a function symbol $[x, y]$ for the Lie multiplication and unary predicates U_i with $1 \leq i < \omega$, such that

$$A = \bigoplus_{1 \leq i < \omega} A_i$$

as a vector space, where A_i is the interpretation of U_i and $[a, b] \in A_{i+j}$, if $a \in A_i$ and $b \in A_j$. We say that the elements of $A_i \setminus \{0\}$ have degree i. A graded Lie algebra A is c-nilpotent, if $A_i = \langle 0 \rangle$ for $c < i$. In this case we use U_i only for $i \leq c$. We show, that the class of c-nilpotent graded Lie algebras over K considered in this language is closed under free amalgamation.

For finite K and c fixed we get the Fraïssé limit of all finite c-nilpotent graded Lie algebras over K. This is an example for the following more general investigations in the paper.

Let L be a countable elementary language. Let M_0 be a Fraïssé limit in L. Eric Jaligot [4] asked whether the group $Aut(M_0)$ of automorphisms of M_0 is universal for all groups $Aut(M)$, where M is a substructure of M_0. He proved this for random tournaments. The first example is the Urysohn space [7]. Also for Fraïssé limits in relational languages it is true [3], if there is free amalgamation for the age.

We introduce the free amalgam $A \otimes_B C$ for a class $\mathcal{J}$ of L-structures, where L is arbitrary (Section 2). In this general situation we use other considerations than in [3].

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If we have free amalgamation in the age of a Fraïssé limit M_0, we can define $A \mathop{\smile\!\!\!\!\!|}_B C$ for finite subsets of M_0 by

$$\langle ABC\rangle = \langle AB\rangle \otimes_{\langle B\rangle} \langle BC\rangle.$$

$\langle X\rangle$ denotes the substructure generated by X. We show that this is a stationary independence relation in M_0 in the sense of K. Tent and M. Ziegler [6]. For relational languages this was an example in [6]. We note, that **Mon** is a consequence of the remaining properties in general. If furthermore the age of M_0 is uniformly locally finite, then we have free amalgamation for the substructures of the monster model $\mathfrak{C}$ of $Th(M_0)$ and it gives a stationary independence relation for the subsets of $\mathfrak{C}$. That means we have all properties of nonforking in a stable theory except that local character and boundedness is replaced by the stronger property stationarity. But the examples we discuss below have the tree property of the second kind.

We use a new idea, developed by Isabel Müller in [5]. Let M_0 be a Fraïssé limit as above. She proved, that the existence of a stationary independence relation for finite subsets of M_0 in the sense of K. Tent und M. Ziegler [6] implies the universality of $Aut(M_0)$ for all $Aut(M)$, where M is a substructure of M_0. The stationary independence relation is used to reconstruct the Fraïssé limit M_0 from a given substructure M using the so-called Katetov extensions. In general the embedding of M in M_0 will change. We apply I. Müller's result to Fraïssé limits M_0 with free amalgamation and obtain the universality of $Aut(M_0)$ for all groups $Aut(M)$, where $M \subseteq M_0$ (Section 3).

In Section 4 we prove the existence of the free amalgam for the class of c-nilpotent graded Lie algebras over a field K in a language L_{Lie} with extra predicates for the graduation. Unfortunately, the proof of this result in [1] is incorrect. The existence of the free amalgam for all graded Lie algebras over a given field follows. We get a Fraïssé limit M_0 of the finitely generated c-nilpotent graded Lie algebras over a finite field K. Then the free amalgam gives a stationary independence relation in M_0 and it follows that $Aut(M_0)$ is universal for all $\{Aut(M) : M \subseteq M_0\}$. For c-nilpotent graded associative algebras even amalgams do not exists in general, as a counterexample in Section 5 shows.

In the last section we consider c-nilpotent groups of exponent $p > c$ with extra predicates for a central Lazard series. As shown in [1] the results for graded Lie algebras imply the existence of the free amalgam for all these groups. The Fraïssé limit G_0^U exists for these groups and the free amalgam gives a stationary independence relation. Hence $Aut(G_0^U)$ is universal for $\{Aut(G^U) : G^U \subseteq G_0^U\}$. Let G_0 be the reduct of G_0^U to the language of group theory. Using the lower central series we can transform each c-nilpotent group of exponent $p > c$ to a structure of the extended language. Hence G_0 is universal for all at most countable c-nilpotent groups of exponent $p > c$. Since the upper and lower central series in G_0 coincide, the extra predicates are 0-definable in G_0. Therefore $Aut(G_0)$ is universal for all $Aut(G)$ where G is a subgroup of G_0. Note that the elementary theories of M_0 (Lie algebras), G_0^U, and G_0 have the tree property of the second kind (see [2]).

I would like to thank Martin Ziegler for helpful discussions of the results, especially for a shorter proof of Lemma 3.2.

§**2. Free Amalgamation.** Let $\mathcal{K}$ be a class of finitely generated L-structures. $\mathcal{K}$ is the age (or skeleton) of a L-structure M, if $\mathcal{K}$ is the class of all L-structures that are isomorphic to a finitely generated substructure of M. In this paper L and $\mathcal{K}$ are always countable.

Definition 2.1. M is $\mathcal{K}$-saturated, if $\mathcal{K}$ is the age of M and if for all B, A in $\mathcal{K}$ and all embeddings $f_0 : B \to M$, $f_1 : B \to A$ there is an embedding $g : A \to M$ such that $f_0 = g \circ f_1$.

Then the following is well-known:

Fact 2.2. *Countable $\mathcal{K}$-saturated structures are isomorphic. Let M_0 be a countable $\mathcal{K}$-saturated structure. It is ultrahomogeneous. That means an isomorphism between finitely generated substructures of M_0 can be extended to an automorphism. Conversely countable ultrahomogeneous structures M_0 are $\mathcal{K}$-saturated, where $\mathcal{K}$ is the age of M_0. M_0 is $\mathcal{K}$-universal: Every countable L-structure with an age included in $\mathcal{K}$ can be embedded.*

This fact implies that the quantifier free n-type of an n-tuple implies the full n-type in M_0. But this is not quantifier elimination for $Th(M_0)$.

Fact 2.3. *There is a countable $\mathcal{K}$-saturated L-structure M_0 if and only if $\mathcal{K}$ has the following properties*:

- **HP:** Hereditary Property *For A in $\mathcal{K}$ we have $age(A) \subseteq \mathcal{K}$.*
- **JEP:** Joint Embedding Property *For A and C in $\mathcal{K}$ there are some $D \in \mathcal{K}$ and embeddings $f_0 : A \to D$ and $f_1 : C \to D$.*
- **AP:** Amalgamation Property *Assume $g_0 : B \to A$ and $g_1 : B \to C$ are embeddings for $A, B, C \in \mathcal{K}$. Then there are some D in $\mathcal{K}$ and embeddings $f_0 : A \to D$ and $f_1 : C \to D$ such that $f_0 \circ g_0 = f_1 \circ g_1$ for B.*

M_0 in Fact 2.3 is called the Fraïssé limit of $\mathcal{K}$. By Fact 2.2 it is unique up to isomorphisms.

Definition 2.4. **APS:** We have the strong amalgamation property for $\mathcal{K}$ if in **AP** $f_0(A) \cap f_1(C) = f_0 \circ g_0(B) = f_1 \circ g_1(B)$ holds.

Fact 2.5. *Assume L is finite, $\mathcal{K}$ is uniformly locally finite, and a $\mathcal{K}$-saturated L-structure M_0 exists. Then $Th(M_0)$ is $\aleph_0$-categorical and allows the elimination of quantifiers.*

For the next considerations we assume again, that L is countable and $\mathcal{J}$ is a class of L-structures.

Definition 2.6. Let $A, B, C, D \in \mathcal{J}$ and assume that B is a common substructure of A and C. If D is generated by A and C with $A \cap C = B$, then D is the free amalgam of A and C over B (short $D = A \otimes_B C$) in $\mathcal{J}$, if for all homomorphisms $f : A \to E$ and $g : C \to E$ into some $E \in \mathcal{J}$ with $f(b) = g(b)$ for $b \in B$ there is a homomorphism $h : D \to E$ that extends f and g.

$\mathcal{J}$ is closed under free amalgamation, if for $A, B, C \in \mathcal{J}$ and embeddings $g_0 : B \to A$ and $g_1 : B \to C$, there exists a free amalgam $A' \otimes_{B'} C'$ in $\mathcal{J}$ and isomorphisms $f_0 : A \to A'$ and $f_1 : C \to C'$, such that $f_0 \circ g_0(b) = f_1 \circ g_1(b)$ for $b \in B$ maps B onto B'.

The free amalgam is a strong amalgam by definition. The homomorphism $h : D \to E$ in the definition is unique, since D is generated by A and C. Note that $A \otimes_B C$ is uniquely determined up to isomorphisms, if it exists. If L is relational and $\mathcal{J}$ is the class of all L-structures, then the free amalgam exists. Its domain is the union of A and C with intersection B and the only relations are the old relations from A and C. In this paper we will consider free amalgams in the class graded Lie algebras over fields and in the class of c-nilpotent groups of exponent p $(c < p)$ with extra predicates for a central Lazard series.

We add new constant symbols e_a for $a \in A \setminus B$ e_b for $b \in B$ and e_c for $c \in C \setminus B$ to the language L and assume that we have the same symbols for the elements of B as a substructure of A and of C, respectively. Using these constant symbols we define the diagrams $Dia(A)$ and $Dia(C)$–the sets of all atomic sentences and negated atomic sentences in this enriched language that are true in A respectively in C, if we interpret the new constant symbols by the elements they represent.

We say that $\mathcal{J}$ is $\forall$-elementary, if it is elementary and its elementary theory is universal. It is equivalent to say that $\mathcal{J}$ is elementary and closed under substructures.

Definition 2.7. Let $\Sigma_{\mathcal{J}}(A, B, C)$ be the union of $Dia(A)$ and $Dia(C)$ with all negated atomic sentences $e_a \neq e_c$ for $a \in (A \setminus B)$ and $c \in (C \setminus B)$ and all negated atomic sentences $\neg\phi(\bar{e}_{\bar{a}}, \bar{e}_{\bar{b}}, \bar{e}_{\bar{c}})$, where $\bar{a} \subseteq A$, $\bar{b} \subseteq B$, and $\bar{c} \subseteq C$ and there are homomorphisms f and g of A and C respectively into some $E \in \mathcal{J}$ with $f(b) = g(b)$ for $b \in B$, such that $E \models \neg\phi(\bar{f}(\bar{a}), \bar{f}(\bar{b}), \bar{g}(\bar{c}))$.

Lemma 2.8. (1) *Assume $\mathcal{J}$ is closed under substructures. For $A, B, C \in \mathcal{J}$ the free amalgam $A \otimes_B C$ exists in $\mathcal{J}$ if and only if $\Sigma_{\mathcal{J}}(A, B, C)$ has a model in $\mathcal{J}$.*

(2) *Let $\mathcal{J}$ be an $\forall$-elementary class such that substructures of finitely generated structures in $\mathcal{J}$ are again finitely generated. Then $\mathcal{J}$ is closed under free amalgamation if and only if the finitely generated structures in $\mathcal{J}$ are closed under free amalgamation.*

(3) *Let L be finite and $\mathcal{K}$ be a countable class of finitely generated L-structures that are uniformly locally finite. Assume a $\mathcal{K}$-saturated model M_0 exists. Let $\mathcal{J}$ be the class of the substructures of the models of $Th(M_0)$. If $\mathcal{K}$ is closed under free amalgamation, then $\mathcal{J}$ is closed under free amalgamation.*

Proof. (1) $A \otimes_B C$ models $\Sigma_{\mathcal{J}}(A, B, C)$. If M is a model of $\Sigma_{\mathcal{J}}(A, B, C)$ in $\mathcal{J}$, then let D be the substructure of M generated by the interpretations of the constant symbols e_a, e_b, e_c. D is in $\mathcal{J}$ by assumption. D is a strong amalgam of A and C over B. D is free, since $\Sigma_{\mathcal{J}}(A, B, C)$ contains the set of conditions we need to extend every given pair of homomorphisms.

(2) To show the nontrivial direction it is sufficient to prove that $\Sigma_{\mathcal{J}}(A, B, C)$ is consistent with $Th(\mathcal{J})$, since $\mathcal{J}$ is closed under substructures, as it is $\forall$-elementary. Because $\mathcal{J}$ is elementary we can use compactness. Let Σ_0 be a finite subset of $\Sigma_{\mathcal{J}}(A, B, C)$. Let A^0 be the substructure of A generated by all elements of A, that occur in a formula of Σ_0 and C^0 the substructure of C generated by all elements of C, that occur in a formula of Σ_0. Let B^1 be $\langle (B \cap A^0), (B \cap C^0) \rangle$. $B^1 \subseteq B$. By assumption B^1 is finitely generated. Then $A^1 = \langle A^0, B^1 \rangle$ and $C^1 = \langle C^0, B^1 \rangle$ are finitely generated. B^1 is a common substructure of A^1 and C^1. Since $\mathcal{J}$ is $\forall$- elementary A^1, B^1, and C^1 are in $\mathcal{J}$.

By assumption $A^1 \otimes_{B^1} C^1 = D^1$ exists in $\mathcal{J}$. We claim that D^1 is a model of Σ_0. The formulas from $Dia(A)$ and $Dia(C)$ in Σ_0 are satisfied in D^1. Assume we have $\neg\phi(\bar{e}_{\bar{a}}, \bar{e}_{\bar{b}}, \bar{e}_{\bar{c}})$ in Σ_0, where $\bar{a} \subseteq A \setminus B$, $\bar{b} \subseteq B$, and $\bar{c} \subseteq C \setminus B$ and furthermore homomorphisms f and g of A and C respectively into some $E \in \mathcal{J}$ with $f(b) = g(b)$ for $b \in B$, such that $E \models \neg\phi(\bar{f}(\bar{a}), \bar{f}(\bar{b}), \bar{g}(\bar{c}))$. If we consider the restriction of f to A^1 and of g to C^1, then $f(b) = g(b)$ for $b \in B^1$. By the definition of the free amalgam

$$D^1 \models \neg\phi(\bar{e}_{\bar{a}}, \bar{e}_{\bar{b}}, \bar{e}_{\bar{c}}),$$

as desired. Formulas $e_a \neq e_c$ from Σ_0 are satisfied in D^1.

(3) $\mathcal{J}$ is the class of the models of the $\forall$ - elementary theory of M_0. Since $\mathcal{K}$ is uniformly locally finite and L is finite, $\mathcal{K}$ is the class of finite structures in $\mathcal{J}$. We apply (2). ⊣

§3. Stationary independence and universal automorphism groups.

Let L be countable. K. Tent and M. Ziegler defined a stationary independence relation for the investigation of automorphism groups in [6]. We consider finite subsets A, B, C, D of a L-structure M.

DEFINITION 3.1. A relation $A \perp_B C$ for finite subsets of M is called a stationary independence relation in M if it fulfils the following properties.

Inv: *Invariance* $A \perp_B C$ depends only on the type of A, B, C.
Mon: *Monotonicity* $A \perp_B CD$ implies $A \perp_B C$ and $A \perp_{BC} D$.
Trans: *Transitivity* $A \perp_B C$ and $A \perp_{BC} D$ imply $A \perp_B CD$.
Sym: *Symmetry* $A \perp_B C$ if and only if $C \perp_B A$.
Ex: *Existence* For A, B, C there is some A' in M such that $\mathrm{tp}(A/B) = \mathrm{tp}(A'/B)$ and $A' \perp_B C$.
Stat: *Stationarity* If $\mathrm{tp}(A/B) = \mathrm{tp}(A'/B)$, $A \perp_B C$, and $A' \perp_B C$, then $\mathrm{tp}(A/BC) = \mathrm{tp}(A'/BC)$.

LEMMA 3.2. *Let $A \perp_B C$ be a relation on finite subsets of M, that satisfies all properties of a stationary independence relation except* **Mon**. *Then* **Mon** *follows.*

PROOF. We assume $A \perp_B CD$. Applying **Ex** we get A', such that $A' \perp_B C$ and $tp(A'/B) = tp(A/B)$. Again by **Ex** there is some A'' such that $tp(A''/BC) = tp(A'/BC)$ and $A'' \perp_{BC} D$. By **Inv** $A'' \perp_B C$. By **Trans** $A'' \perp_B CD$. Since $tp(A''/B) = tp(A/B)$ **Stat** implies $tp(A''/BCD) = tp(A/BCD)$. The assertion follows from **Inv**. ⊣

Note that **Sym** is not used in the proof above, as mentioned by the referee.

I. Müller combined the existence of a stationary independence relation with Katětov's construction [5]. She proved:

THEOREM 3.3. *If M_0 is a Fraïssé limit and there exists a stationary independence relation in M_0, then $Aut(M_0)$ is universal for all $Aut(N)$, where N is a substructure of M_0.*

We will see that free amalgams provide a stationary independence relation.

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THEOREM 3.4. *Let M_0 be a Fraïssé limit. We assume that the free amalgam of finitely generated substructures of M_0 exists in M_0 and define for finite subsets A, B, C of M_0:*

$$A \underset{B}{\downarrow} C$$

if and only if

$$\langle ABC\rangle = \langle AB\rangle \otimes_{\langle B\rangle} \langle BC\rangle.$$

Then $\downarrow$ is a stationary independence relation in M_0.

PROOF. By definition $A \downarrow_B C$ if and only if $\langle AB\rangle \downarrow_{\langle B\rangle} \langle BC\rangle$. Hence we assume w.l.o.g. that A, B, C are finitely generated substructures.

Inv: It is clear since the free amalgam is uniquely determined by its isomorphism type.

Sym: It follows directly from the definition.

Ex: Since the class of finitely generated substuctures of M_0 is closed under free amalgamation and M_0 is age(M_0) - saturated we get **Ex**.

Stat: It is a consequence of the uniqueness of the free amalgam and ultrahomogeneity.

Trans: By assumption $\langle ABCD\rangle = (\langle AB\rangle \otimes_B \langle BC\rangle) \otimes_{\langle BC\rangle} \langle BCD\rangle$. We show that this structure is the free amalgam of $\langle AB\rangle$ and $\langle BCD\rangle$ over B. Let G be a structure in age(M_0). Let f_0 be a homomorphism of $\langle AB\rangle$ into G and f_1 be a homomorphism of $\langle BCD\rangle$ into G such that $f_0(b) = f_1(b)$ for $b \in B$. By $A \downarrow_B C$ there is a homomorphism g of $\langle ABC\rangle$ into G, that extends f_0 and f_1 restricted to $\langle BC\rangle$. Since $g(e) = f_1(e)$ for $e \in \langle BC\rangle$, there is a homomorphism h of $\langle ABCD\rangle$ into G, that extends g, f_1, and therefore f_0. We get $A \downarrow_B CD$, as desired.

Mon: It follows by Lemma 3.2. ⊣

Using the Theorem of I. Müller we obtain:

COROLLARY 3.5. *Let M_0 be a Fraïssé limit. Assume that the free amalgam of finitely generated substructures of M_0 exists. Then $Aut(M_0)$ is universal for all substructures $N \subseteq M_0$.*

DEFINITION 3.6. A relation $A \downarrow_B C$ for small subsets of a monster model $\mathfrak{C}$ is a stationary independence relation in $\mathfrak{C}$, if it fulfils **Inv, Mon, Trans, Sym, Ex, Stat** and

Fin: *Finite Character* $A \downarrow_B C$ if and only if $\bar{a} \downarrow_B \bar{c}$ for all finite tuple $\bar{a}$ in A and $\bar{c}$ in C.

A stationary independence relation in $\mathfrak{C}$ has all properties of nonforking in a stable theory except *Local Character*. Furthermore *Boundedness* is replaced by the stronger property **Stat**. In the next chapter there are examples with the tree property of the second kind.

COROLLARY 3.7. *Let L be finite and $\mathcal{K}$ be a countable class of finitely generated L-structures that are uniformly locally finite. Assume a $\mathcal{K}$-saturated model M_0 exists and $\mathfrak{C}$ is a monster model of $Th(M_0)$. If $\mathcal{K}$ is closed under free amalgamation, then the free amalgam for small subsets of $\mathfrak{C}$ exists and defines a stationary independence relation in $\mathfrak{C}$.*

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PROOF. Let $\mathcal{J}$ be the class of all substructures of models of $Th(M_0)$. Then $\mathcal{K}$ is the class of the finitely generated structures in $\mathcal{J}$. These structures are finite. By Lemma 2.8(3) the free amalgam of substructures of $\mathfrak{C}$ exists. The independence relation for subsets of $\mathfrak{C}$ is defined as above. All properties except **Fin** are shown in the same way as in Theorem 3.4. **Mon** implies the assertion from the left to the right of **Fin**. By Lemma 2.8(1) the other direction follows from the consistency of $Th(M_0) \cup \Sigma_{\mathcal{J}}(A, B, C)$. We use compactness and the consistency of all $Th(M_0) \cup \Sigma_{\mathcal{J}}(\langle \bar{a} B\rangle, B, \langle \bar{c} B\rangle)$ for all finite $\bar{a}$ and $\bar{c}$, similarly as in the proof of Lemma 2.8 (2). ⊣

We will apply the results of this section to the Fraïssé limits of graded Lie algebras over finite fields and of c-nilpotent groups of exponent p $(c < p)$ with extra predicates for a central Laszard series.

§4. Graded Lie algebras over fields.

We consider graded Lie algebras A over a fixed field K in the language L_{Lie} as described in the introduction.

THEOREM 4.1. *The class of c-nilpotent graded Lie algebras over a field K is closed under free amalgamation.*

PROOF. Let $\mathcal{J}$ be the class of c-nilpotent graded Lie algebras over K. It is $\forall$-elementary and subalgebras of finitely generated algebras in $\mathcal{J}$ are again finitely generated. By Lemma 2.8(2) it is sufficient to give a construction of a free amalgam of A and C over B, where A, B, C are finitely generated c-nilpotent graded Lie algebras over K and B is a common subalgebra.

We choose a vector space basis

$$X_B = \{b_{i,j} : 1 \leq i \leq c, j < \beta_i\}$$

of B with $U_i(b_{i,j})$. Then we extend X_B by

$$X_A = \{a_{i,j} : 1 \leq i \leq c, j < \alpha_i\}$$

and

$$X_C = \{c_{i,j} : 1 \leq i \leq c, j < \gamma_i\}$$

with $U_i(a_{i,j})$, $U_i(c_{i,j})$ and $X_A \cap X_C = \emptyset$, such that $X_A X_B$ is a vector space basis for A and $X_B X_C$ is a vector space basis of C. Let X be $X_A X_B X_C$. We use the graded set X as a set of free generators of the free c-nilpotent graded Lie algebra $F(X)$. The elements of X are in U_i according to the definition above. Let J_A be the ideal in $F(X_B X_A)$ generated by all equations $[x, y] = z$ in A where $x, y \in X_B X_A$ and z is a linear combination of elements in $X_A X_B \cap U_{i+j}$, if $U_i(x)$ and $U_j(y)$. Then $F(X_A X_B)/J_A$ is isomorphic to A. Analogously we define J_C in $F(X_B X_C)$, such that $F(X_B X_C)/J_C$ is isomorphic to C. Let J be the ideal in $F(X)$ generated by J_A and J_C.

CLAIM 1. *A strong amalgam of A and C over B exists.*

CLAIM 2. *$F(X)/J$ is the free amalgam of A and C over B.*

First we show that Claim 1 implies Claim 2. Let G^0 be a strong amalgam of A^0 and C^0 over B^0, such that there are isomorphisms $h_A : A \to A^0$ and $h_C : C \to C^0$ with $h_A(b) = h_C(b)$ for $b \in B$. h_A and h_C give us a map h^0 of X into G^0. We can extent h^0 to a homomorphism h of $F(X)$ onto G^0. The kernel of h contains J. Hence G^0 is a homomorphic image of $F(X)/J$. Therefore $F(X)/J$ contains isomorphic images A', B', C' of A, B, C respectively, such that $A' \cap C' = B'$ and $\langle A', C' \rangle = F(X)/J$. Let f_A and f_C be any pair of homomorphisms of A' and C' respectively into a c-nilpotent graded Lie algebra G over K, such that $f_A(b) = f_C(b)$ for $b \in B'$. If we map the elements of X onto their f_A- respectively f_C-images in G, then we get an homomorphism f of $F(X)$ into G. The kernel of f contains J by the definition of J_A and J_C. Hence f induces the desired homomorphism of $F(X)/J$ into G.

To prove Claim 1 we construct a strong amalgam directly step by step. We use again X_A, X_B, and X_C, where the α_i , β_i, and γ_i are finite. Now the underlying vector space of the amalgam D is a vector space where $X = X_A X_B X_C$ is part of a basis of this space. Note that the role of X has changed. Above it was a graded set of free generators. Now X_B is a vector space basis for the image of B in D, $X_B X_A$ is a vector space basis for the image of A in D, and $X_B X_C$ is a vector space basis of the image of C in D. For $x \in X$ we have $U_i(x)$ if and only if $x = a_{i,j}$ or $x = b_{i,j}$ or $x = c_{i,j}$ for some j. i is the degree of x. The only problem is the definition of the Lie multiplication for the elements of a vector space basis of D. Since multiplication with elements from U_c gives 0, we can put all elements of $U_c(X)$ into X_B. Therefore we assume w.l.o.g. that

$$\alpha_c = \gamma_c = 0.$$

First we solve the following essential case:

MAJOR CASE: $X_A = \{a\}$ and $X_C = \{e\}$ with $U_i(a)$, $U_j(e)$, and $i, j < c$.

Let H be the free c-nilpotent graded Lie algebra over K freely generated by a and e. The graduation is given by $degree(a) = i$, $degree(e) = j$, and $degree([y_1, y_2]) = degree(y_1) + degree(y_2)$ for monomials y_1 and y_2 in a, e , as usual. The elements of $U_k(H)$ are linear combinations of monomials of degree k.

FACT 4.2. *Since H is freely generated by a and e there is a subset Y of the set of monomials over a and e such that*:

(1) *a and e are in Y.*
(2) *For $y \in Y$ with $y \neq a$ and $y \neq e$ there are y_1 and y_2 in Y such that $y = [y_1, y_2]$.*
(3) *Y is a vector space basis of H.*

This fact is well known, even in the case with graduation. You can choose Y as a set of basic monomials. For us it is important that for every $y \in Y$ there is a unique way of construction starting with a and e. Hence we can use induction on the number $n(y)$ of Lie multiplications in the monomial y. Now $X_B Y$ will be a vectorspace basis of the amalgam D. Finally we have to extend the Lie multiplication for X_B and for Y to $X_B Y$, such that D becomes a Lie algebra with the given graduation. We have only to consider $[y, b] = -[b, y]$ for $y \in Y$ and $b \in X_B$. By induction on the number $n(y)$ of Lie multiplications in the monomial y we define $[y, b]$ for all $b \in X_B$. We will have $[y, b] \in B$.

If $y = a$, then $[y, b]$ is defined in A. Since $A_k = B_k$ for $i < k$ we have $[y, b] \in B$. Analogously we get $[y, b] \in B$ for $y = e$ using the multiplication in C. For the induction step we consider $y = [y_1, y_2]$, where the definition for y_1 and y_2 is given and the products are in B. With respect to the Jacobi identity we define

$$[[y_1, y_2], b] = [[y_1, b], y_2] + [y_1, [[y_2, b]].$$

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By induction $[y_1, b] = b_1 \in B$ and $[y_2, b] = b_2 \in B$ are defined. By the same argument we get $[b_1, y_2] \in B$ and $[y_1, b_2] \in B$.

We have to check the Jacobi identity. By the definition above we have only to consider the case $y \in Y$ and $b, d \in X_B$. Again we use induction on $n(y)$ to show the Jacobi identity for $y \in Y$ and all $b, d \in X_B$. If $y = a$ or $y = e$, then we are in A or C respectively. Hence the Jacobi identity is true.

Now we assume that y_1 with any two elements of X_B and y_2 with any two elements of X_B satisfy the Jacobi identity. We have to show:

$(l = r)$:

$$[[[y_1, y_2], b], d] = [[[y_1, y_2], d], b] + [[y_1, y_2], [b, d]].$$

Using the inductive definition of $[y, x]$ for $y \in Y$ and $x \in X_B$ the right side can be written as:

(r):

$$[[[y_1, d], y_2], b] + [[y_1, [y_2, d]], b] + [[y_1, [b, d]], y_2] + [y_1, [y_2, [b, d]]].$$

Now we apply the definitions and the induction to the left side and obtain the following identities. We use that $[y_1, b], [y_2, b], [y_1, d], [y_2, d] \in B$.

(l_1):

$$[[[y_1, b], y_2], d] + [[y_1, [y_2, b]], d] =$$

(l_2):

$$[[[y_1, b], d], y_2] + [[y_1, b], [y_2, d]] + [[y_1, d], [y_2, b]] + [y_1, [[y_2, b], d]] =$$

(l_3):

$$[[[y_1, d], b], y_2] + [[y_1, [b, d]], y_2] + [[y_1, [y_2, d]], b] + [y_1, [b, [y_2, d]]] +$$
$$[[[y_1, d], y_2], b] + [y_2, [[y_1, d], b]]] + [y_1, [[y_2, d], b] + [y_1, [y_2, [b, d]]].$$

After cancellation in (l_3) we see that it is equal to (r) as desired. The proof for the Major Case is finished. In fact we have constructed a free amalgam.

Reduction to $X_C = \{e\}$: If the strong amalgam exists for all A, B and $C = \langle X_B e \rangle$, then it exists for all A, B, and C.

We assume that the assertion is true for $X_C = \{e\}$. We show by induction on $c - i$ that the strong amalgam exists for all X_C with $\gamma_j = 0$ for $j < i$. The case $c = i$ is clear, since we have assumed that $\alpha_c = \gamma_c = 0$, as discussed above.

We fix i and assume that the assertion is true for $i + 1$. By a second induction on the size of γ_i we reduce the problem to the case $\gamma_i = 1$. For induction step of this induction let $e = c_{\gamma_i - 1}$ and C^- be the subalgebra of C generated by $X_B X_C \setminus \{e\}$. By the second induction there is a strong amalgam D^- of A and C^- over B. Now we have to amalgamate D^- and C over C^-. This is the case $X_C = \{e\}$ and we can apply the assumption in the claim.

Reduction to the Major Case: The Major Case implies the Reduction to $X_C = \{e\}$.

We can apply the same arguments to all situations A, B and $X_C = \{e\}$ and come to the Major Case. The assertion of Claim 2 and the theorem follow.

Note that the proof of Claim 2 is a direct construction of a free amalgam. We only have to check that all amalgams constructed are free. For this we use **Trans**. ⊣

By compactness and Lemma 2.8(1) we obtain:

COROLLARY 4.3. *The class of all graded Lie algebras over a given field K is closed under free amalgamation.*

Note that the class of finitely generated c-nilpotent graded Lie algebras over a finite field is uniformly locally finite. Therefore it is countable. The size of the free object with n free generators is an upper bound for the size of all c-nilpotent graded Lie algebras over K with n generators.

COROLLARY 4.4. *Let K be a finite field and $\mathcal{K}$ be the class of finitely generated c-nilpotent graded Lie algebras over K. Then the following is true.*

(1) *A countable $\mathcal{K}$-saturated structure M_0 exists.*
(2) *In M_0 the free amalgam of finitely generated substructures exists and is a stationary independence relation.*
(3) *$Aut(M_0)$ is universal for all $Aut(M)$, where M is an at most countable c-nilpotent graded Lie algebra over K.*

PROOF. We use Fact 2.3, Theorem 4.1, Theorem 3.4, and Corollary 3.5. All at most countable c-nilpotent graded Lie algebras over K can be embedded in M_0. ⊣

Corollary 3.7 implies:

COROLLARY 4.5. *If K is a finite field, M_0 is the Fraïssé limit of all finitely generated c-nilpotent graded Lie algebras and $\mathfrak{C}$ is a monster model of $Th(M_0)$, then the free amalgam defines a stationary independence relation in $\mathfrak{C}$. The theory has the tree property of the second kind.*

For the tree property of the second kind see [2].

§5. c-nilpotent graded associative algebras.

Every Lie algebra has an universal enveloping associative algebra. It was a first idea to prove that the amalgamation property for associative algebras. The next example shows, that this is not true.

LEMMA 5.1. *c-nilpotent graded associative algebras do not have the amalgamation property for $2 < c$.*

PROOF. We use the language from graded Lie algebras over K, but for the multiplication of x and y we write xy. Let $c = 3$. We consider the free 3-nilpotent graded associative algebras F freely generated by $a_0, a_1, b_0, b_1, b_2, b_3, b_4, b_5, c$, the subalgebra F_A freely generated by $a_0, a_1, b_0, b_1, b_2, b_3, b_4, b_5$, the subalgebra F_B freely generated by $b_0, b_1, b_2, b_3, b_4, b_5$, and the subalgebra F_C freely generated by $b_0, b_1, b_2, b_3, b_4, b_5, c$. The set of all xy where x and y are elements of the generating set above is a vector space basis of F_2 and the set of all xyz where x, y, z are elements of the generating set are a vector space basis for F_3. To obtain A we factorize F_A by the ideal J_A generated by $a_0b_0+a_1b_1$. C is obtained from F_C using the ideal generated by $b_0c+b_2b_4$ and $b_1c+b_3b_5$. The images of the b_i's generate in A and C a subalgebra isomorphic to F_B. We call it B. An amalgam would be isomorphic to F/J, where J is the ideal generated by J_A and J_C. But this contains a new relation in A:

$$(a_0b_0 + a_1b_1)c - a_1(b_1c + b_3b_5) - a_0(b_0c + b_2b_4) = -a_1b_3b_5 - a_0b_2b_4$$

a contradiction. ⊣

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§6. c-nilpotent groups of exponent $p > c$**.** As in [1] we consider the class c-nilpotent groups $\mathfrak{G}^U_{c,p}$ of exponent $p > c$ with extra unary predicates $U_1, \dots, U_c$ for a central Lazard series. That means we have

$$G = U_1(G) \supseteq \cdots \supseteq U_c(G)$$

and

$$\langle \bigcup_{l+k=n} [U_l(G), U_k(G)] \rangle \subseteq U_n \subseteq Z_{c+1-n}.$$

Examples for central Lazard series are the lower central series $\Gamma_i(G)$ and the upper central series $Z_i(G)$. Then we have $\Gamma_i(G) \subseteq U_i(G) \subseteq Z_{c+1-i}(G)$. Without the extra predicates this class of groups is denoted by $\mathfrak{G}_{c,p}$. It is well known that the amalgamation property fails for $\mathfrak{G}_{c,p}$. In [1] strong amalgamation for $\mathfrak{G}^U_{c,p}$ is shown. A careful analysis of the proof shows that it is a free amalgam. The free amalgamation of c-nilpotent graded Lie algebras over $\mathbb{F}_p$ is used (see Theorem 4.1 for a correct proof). Therefore we get:

Theorem 6.1. *It holds*:

(1) $\mathfrak{G}^U_{c,p}$ *has* (HEP), (JEP), *and the free amalgamation property.*
(2) *The Fraïssé limit* G^U_0 *of the finite groups in* $\mathfrak{G}^U_{c,p}$ *exists.*
(3) $Th(G^U_0)$ *is* $\aleph_0$ *- categorical and allows the elimination of quantifiers.*

Let G_0 be the reduct of G^U_0 to the language of group theory. Since $\Gamma_n(G^U_0) = Z_{c+1-n}(G^U_0) = U_n(G^U_0)$, $U_n(G^U_0)$ is in G_0 0-definable. Furthermore every $G \in \mathfrak{G}_{c,p}$ becomes a structure in $\mathfrak{G}^U_{c,p}$, if we define $U_n(G) = \Gamma_n(G)$. Hence we obtain:

Corollary 6.2. $Th(G_0)$ *is* $\aleph_0$ *- categorical and universal for all at most countable* $G \in \mathfrak{G}_{c,p}$.

By Theorem 3.4 and Corollary 3.5 we obtain

Theorem 6.3.

(a) *In the monster model of* $Th(G^U_0)$ *the free amalgam exists and is a stationary independence relation.*
(b) $Aut(G^U_0)$ *is universal for all* $Aut(N)$, *where* (N) *is a substructure of* G^U_0.

The same arguments as for Corollary 6.2 imply

Corollary 6.4. $Aut(G_0)$ *is universal for all* $Aut(G)$ *for all at most countable groups* $G \in \mathfrak{G}_{c,p}$.

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