Adv. Appl. Prob. **51,** 184–208 (2019)
*doi:*10.1017/apr.2019.8

DEVIATION BOUNDS FOR THE FIRST PASSAGE TIME IN THE FROG MODEL

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Abstract

We consider the so-called frog model with random initial configurations. The dynamics of this model are described as follows. Some particles are randomly assigned to any site of the multidimensional cubic lattice. Initially, only particles at the origin are active and these independently perform simple random walks. The other particles are sleeping and do not move at first. When sleeping particles are hit by an active particle, they become active and start moving in a similar fashion. The aim of this paper is to derive large deviation and concentration bounds for the first passage time at which an active particle reaches a target site.

Keywords: Frog model; egg model; simple random walk; random environment; large deviation inequality; concentration inequality

2010 Mathematics Subject Classification: Primary 60K35
Secondary 82B43

1. Introduction

1.1. The model

For $d \geq 2$, we write $\mathbb{Z}^d$ for the d-dimensional cubic lattice. Let $\omega = (\omega(x))_{x\in\mathbb{Z}^d}$ be independent random variables with a common law on $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, not concentrated at zero. Furthermore, independently of ω, let $(S_k(x, \ell))_{k=0}^{\infty}$, $x \in \mathbb{Z}^d$, $\ell \in \mathbb{N}$, be independent simple random walks on $\mathbb{Z}^d$ with $S_0(x, \ell) = x$. For any $x, y \in \mathbb{Z}^d$, we now introduce the *first passage time* $T(x, y)$ from x to y as follows:

$$T(x, y) := \inf\left\{ \sum_{i=0}^{m-1} \tau(x_i, x_{i+1}) \colon m \geq 1,\ x_0 = x,\ x_m = y,\ x_1, \dots, x_{m-1} \in \mathbb{Z}^d \right\}, \tag{1.1}$$

where

$$\tau(x_i, x_{i+1}) := \inf\{k \geq 0 \colon S_k(x_i, \ell) = x_{i+1} \text{ for some } 1 \leq \ell \leq \omega(x_i)\},$$

with the convention that $\tau(x_i, x_{i+1}) := \infty$ if $\omega(x_i) = 0$. The fundamental object of study is the first passage time $T(0, x)$ conditioned on the event $\{\omega(0) \geq 1\}$. Its intuitive meaning is as follows. We now regard simple random walks as 'frogs' and ω stands for an initial configuration of frogs, i.e. $\omega(y)$ frogs sit on each site y (there is no frog at y if $\omega(y) = 0$). Suppose that the origin 0 is occupied by at least one frog. They are active and independently perform simple random walks, but the other frogs are sleeping and do not move at first.

Received 18 December 2017; revision received 23 December 2018.
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When sleeping frogs are attacked by an active frog, they become active and start doing independent simple random walks. Then $T(0, x)$ describes the first passage time at which an active frog reaches a site x.

The frog model was originally introduced by Ravishankar, and its idea comes from the following information spreading. Consider that every active frog has some information. When it hits sleeping frogs, the information is shared between them. Active frogs move freely and play a role in spreading the information. Recent interests of the frog model are recurrence and transience of frogs (see Section 1.3 for details), and there are few results for the behavior of the first passage time except for [3], [4], and [22]. (Recently, the first passage time is also studied in a Euclidean setting [6].) However, in view of the information spreading, it is important to investigate the behavior of the first passage time precisely. For this purpose, in this paper, we propose nontrivial deviation bounds for the first passage time (see Theorems 1.1, 1.2, and 1.3 below).

1.2. Main results

Let us first mention the results obtained by Alves *et al.* [4] to describe our main results. First, note that the first passage time is subadditive:

$$T(x, z) \le T(x, y) + T(y, z), \qquad x, y, z \in \mathbb{Z}^d.$$

Alves *et al.* [4, Section 2, Theorem 1.1, Steps 1–6] obtained the following asymptotic behavior of the first passage time. There exists a norm $\mu(\,\cdot\,)$ (which is called the *time constant*) on $\mathbb{R}^d$ such that almost surely on the event $\{\omega(0) \ge 1\}$,

$$\lim_{\substack{\|x\|_1 \to \infty \\ x \in \mathbb{Z}^d}} \frac{T(0, x) - \mu(x)}{\|x\|_1} = 0, \tag{1.2}$$

where $\|\cdot\|_1$ is the ℓ^1-norm on $\mathbb{R}^d$. Furthermore, $\mu(\,\cdot\,)$ is invariant under permutations of the coordinates and under reflections in the coordinate hyperplanes, and satisfies

$$\|x\|_1 \le \mu(x) \le \mu(\xi_1)\|x\|_1, \qquad x \in \mathbb{R}^d, \tag{1.3}$$

where ξ_1 is the first coordinate vector of $\mathbb{R}^d$. The first inequality in (1.3) is a consequence of (1.2). Indeed, since $T(0, kx) \ge \|kx\|_1$ for $x \in \mathbb{Z}^d$, we have almost surely on the event $\{\omega(0) \ge 1\}$,

$$\mu(x) = \lim_{k \to \infty} \frac{1}{k} T(0, kx) \ge \|x\|_1, \qquad x \in \mathbb{Z}^d.$$

Hence, the first inequality in (1.3) is valid for $x \in \mathbb{Z}^d$ and we can easily extend it to the case $x \in \mathbb{R}^d$. On the other hand, the second inequality in (1.3) follows from the properties of the time constant.

To prove (1.2), roughly speaking, Alves *et al.* applied the subadditive ergodic theorem to the process $T(ix, jx)$, $0 \le i < j$, with $\omega(ix), \omega(jx) \ge 1$. This method requires the integrability of the first passage time. Thus, in [4, Lemmata 2.2 and 2.3], they first derived the following tail estimate. There exist constants $0 < C_1, C_2 < \infty$ and $0 < \alpha_1 < 1$ such that, for all $x \in \mathbb{Z}^d$ and $t \ge \|x\|_1^4$,

$$\mathbb{P}(T(0, x) \ge t \mid \omega(0) \ge 1) \le C_1 \mathrm{e}^{-C_2 t^{\alpha_1}}. \tag{1.4}$$

https://doi.org/10.1017/apr.2019.8 Published online by Cambridge University Press

Our main results are the following upper and lower large deviations for the first passage time. Throughout this paper, we write $\mathbb{P} := \mathbb{P}(\,\cdot\,|\omega(0) \geq 1)$ to shorten notation.

Theorem 1.1. *There exists a constant* $0 < \alpha_2 < 1$ *such that, for all* $\varepsilon > 0$*,*

$$\limsup_{\substack{\|x\|_1\to\infty \\ x\in\mathbb{Z}^d}} \frac{1}{\|x\|_1^{\alpha_2}} \log \mathbb{P}(T(0, x) \geq (1+\varepsilon)\mu(x)) < 0.$$

Theorem 1.2. *If* $\mathbb{E}[\omega(0)] < \infty$ *then there exists a constant* $0 < \alpha_3 < 1$ *such that, for all* $\varepsilon > 0$*,*

$$\limsup_{\substack{\|x\|_1\to\infty \\ x\in\mathbb{Z}^d}} \frac{1}{\|x\|_1^{\alpha_3}} \log \mathbb{P}(T(0, x) \leq (1-\varepsilon)\mu(x)) < 0.$$

From the above theorems, we can expect the existence of the optimal speeds for the upper and lower large deviations, i.e. there exist exponents $\beta, \beta' \in (0, 1]$ such that

$$\frac{1}{\|x\|_1^{\beta}} \log \mathbb{P}(T(0, x) \geq (1+\varepsilon)\mu(x)) \quad \text{and} \quad \frac{1}{\|x\|_1^{\beta'}} \log \mathbb{P}(T(0, x) \leq (1-\varepsilon)\mu(x)) \tag{1.5}$$

converge to some strictly negative constants as $\|x\|_1 \to \infty$, $x \in \mathbb{Z}^d$. Let us argue here this problem and observe that the optimal speed (if it exists) has to be $\|x\|_1$ to some power in $(0, 1]$ under some assumptions. For simplicity of notation, denote by $\mathcal{I}$ the random set of all sites of $\mathbb{Z}^d$ which frogs initially occupy, i.e.

$$\mathcal{I} := \left\{x \in \mathbb{Z}^d : \omega(x) \geq 1\right\}.$$

First observe the upper large deviation of $T(0, x)$. Let $x \in \mathbb{Z}^d$ with $\|x\|_1 > 1$ and consider the event A that $(S_\cdot(0, \ell))_{1\leq\ell\leq\omega(0)}$ and $(S_\cdot(\xi_1, \ell))_{1\leq\ell\leq\omega(\xi_1)\vee 1}$ stay inside the set $\{0, \xi_1\}$ until time $\lceil(1+\varepsilon)\mu(\xi_1)\rceil \|x\|_1$. We divide $\mathbb{P}(A \cap \{0 \in \mathcal{I}\})$ into two terms:

$$\mathbb{P}\big(A \cap \{0 \in \mathcal{I}\}\big) = \mathbb{P}\big(A \cap \{0 \in \mathcal{I},\ \xi_1 \notin \mathcal{I}\}\big) + \mathbb{P}\big(A \cap \{0, \xi_1 \in \mathcal{I}\}\big). \tag{1.6}$$

Since each simple random walk jumps to one of its nearest neighbors at each step, the first term on the right-hand side of (1.6) is equal to

$$\begin{aligned}&\sum_{L=1}^{\infty} \mathbb{P}(\omega(0) = L) P(\xi_1 \notin \mathcal{I})(2d)^{-(L+1)\lceil(1+\varepsilon)\mu(\xi_1)\rceil\|x\|_1} \\ &\qquad = \mathbb{E}\Big[(2d)^{-(\omega(0)+1)\lceil(1+\varepsilon)\mu(\xi_1)\rceil\|x\|_1} \mathbf{1}_{\{0\in\mathcal{I},\ \xi_1\notin\mathcal{I}\}}\Big].\end{aligned}$$

Similarly, the second term on the right-hand side of (1.6) is equal to

$$\mathbb{E}\Big[(2d)^{-(\omega(0)+\omega(\xi_1))\lceil(1+\varepsilon)\mu(\xi_1)\rceil\|x\|_1} \mathbf{1}_{\{0,\xi_1\in\mathcal{I}\}}\Big].$$

Therefore, we have

$$\mathbb{P}(A) = \mathbb{E}\Big[(2d)^{-(\omega(0)+\omega(\xi_1)+1)\lceil(1+\varepsilon)\mu(\xi_1)\rceil\|x\|_1}\Big].$$

Note that by (1.3), on the event $A \cap \{0 \in \mathcal{I}\}$,

$$T(0, x) \geq \lceil (1+\varepsilon)\mu(\xi_1) \rceil \, \|x\|_1 \geq (1+\varepsilon)\mu(x),$$

and Jensen's inequality yields

$$\liminf_{\substack{\|x\|_1 \to \infty \\ x \in \mathbb{Z}^d}} \frac{1}{\|x\|_1} \log \mathbb{P}(T(0, x) \geq (1+\varepsilon)\mu(x)) \geq -(2\mathbb{E}[\omega(0)] + 1) \lceil (1+\varepsilon)\mu(\xi_1) \rceil \log (2d).$$

This bound does not guarantee the existence of the optimal speed for the upper large deviation and has no meaning if $\mathbb{E}[\omega(0)] = \infty$. However, in the case where $\mathbb{E}[\omega(0)] < \infty$, the above bound combined with Theorem 1.1 shows that the exponent β appearing in (1.5) lies in $[\alpha_2, 1]$ (if the optimal speed exists for the upper large deviation).

Next we treat the lower large deviation of $T(0, x)$. Note that Theorem 1.2 of [4] states that if there exists $\delta \in (0, d)$ such that $\mathbb{P}(\omega(0) \geq t) \geq (\log t)^{-\delta}$ holds for all large t then we have $\mu(x) = \|x\|_1$. Our lower large deviation (Theorem 1.2) does not treat this situation because of the assumption of finite mean of $\omega(0)$. However, for now, we do not know whether $\mu(x) > \|x\|_1$ holds even if $\omega(0)$ has a finite mean, and have to consider the speed of the lower large deviation for several directions $x \in \mathbb{Z}^d$: $\mu(x) < (1-\varepsilon)^{-1}\|x\|_1$ and $\mu(x) \geq (1-\varepsilon)^{-1}\|x\|_1$. In the case where $\mu(x) < (1-\varepsilon)^{-1}\|x\|_1$, we have $T(0, nx) \geq \|nx\|_1 > (1-\varepsilon)\mu(nx)$ and

$$\mathbb{P}(T(0, nx) \leq (1-\varepsilon)\mu(nx)) = 0.$$

Hence, if $\mu(x) < (1-\varepsilon)^{-1}\|x\|_1$ then we do not determine the optimal speed of the lower large deviation for the direction x in the sense of (1.5). On the other hand, in the case where $\mu(x) \geq (1-\varepsilon)^{-1}\|x\|_1$, we have $(1-\varepsilon)\mu(nx) \geq \|nx\|_1$. Fix a self-avoiding nearest-neighbor path $(0 = v_0, v_1, \ldots, v_m = nx)$ with minimal length $m = \|nx\|_1$, and let A' be the event that $S_k(0, 1) = v_k$ for all $0 \leq k \leq m$. It holds that $\mathbb{P}(A') = (2d)^{-\|nx\|_1}$ and $T(0, nx) \leq (1-\varepsilon)\mu(nx)$ on the event $A' \cap \{0 \in \mathcal{I}\}$. Hence,

$$\liminf_{n \to \infty} \frac{1}{n\|x\|_1} \log \mathbb{P}(T(0, nx) \leq (1-\varepsilon)\mu(nx)) \geq -\log (2d).$$

This combined with Theorem 1.2 proves that in the case where $\mu(x) \geq (1-\varepsilon)^{-1}\|x\|_1$, the exponent β' appearing in (1.5) lies in $[\alpha_3, 1]$ (if the optimal speed of the lower large deviation exists for the direction x).

Optimizing the speeds for the above large deviations may be difficult in general because the propagation of active frogs depends on the behavior of the simple random walk and the initial configuration of the frogs. From [21, pp. 333, 338] the average cardinality of the set $\{S_k(0, 1) \colon 0 \leq k \leq n\}$ is of order $n/\log n$ if $d = 2$, and of order n if $d \geq 3$. This means that an active frog wakes more sleeping frogs up in $d \geq 3$ than in $d = 2$. Moreover, without relation to the dimension, active frogs are easily generated if each site of $\mathbb{Z}^d$ first has a lot of sleeping frogs. Thus, the first passage time and the time constant seem to be strongly related to the dimension d and the law of the initial configuration ω. At present we do not have enough information to determine the optimal speed of the large deviations and would like to address these problems in future research.

Our key tool to prove Theorems 1.1 and 1.2 is the modified first passage time defined as follows: For any $x \in \mathbb{Z}^d$, let x^* be the closest point to x in $\mathcal{I}$ for the ℓ^1-norm, with a deterministic rule to break ties. Then, the modified first passage time $T^*(x, y)$ is given by

$$T^*(x, y) := T(x^*, y^*).$$

By definition, the subadditivity is inherited from the original first passage time:

$$T^*(x,z) \le T^*(x,y) + T^*(y,z), \qquad x,y,z \in \mathbb{Z}^d.$$

A particular difference between $T(x,y)$ and $T^*(x,y)$ is that $T(x,y)$ is inevitably equal to ∞ if $\omega(x)=0$, but $T^*(x,y)$ is almost surely finite. Moreover, we can derive the following concentration inequality for $T^*(0,x)$.

Theorem 1.3. *Assume that* $E[\omega(0)] < \infty$. *For all* $\gamma > 0$, *there exist constants* $0 < C_3, C_4, C_5 < \infty$ *and* $0 < \alpha_4 < 1$ *such that, for all* $x \in \mathbb{Z}^d \setminus \{0\}$ *and* $C_3(1+\log\|x\|_1)^{1/\alpha_4} \le t \le \gamma\sqrt{\|x\|_1}$,

$$\mathbb{P}\Big(|T^*(0,x) - \mathbb{E}[T^*(0,x)]| \ge t\sqrt{\|x\|_1}\Big) \le C_4 e^{-C_5 t^{\alpha_4}}.$$

Theorem 1.3 is not only of independent interest in view of the investigation of the modified first passage time, but also plays a key role in obtaining Theorem 1.2 as mentioned in Subsection 1.4 below.

1.3. Earlier literature

There are various models related to the spread of information except for our frog model. Ramírez–Sidoravicius [33] studied a stochastic growth model representing combustion, which is the frog model on $\mathbb{Z}^d$ for continuous-time simple random walks. Although the frog model has active and sleeping frogs, the model in which all frogs are active from the beginning has also been investigated (see, for instance, [24], [25], [27], and [32]). This model is regarded as an infected model and its dynamics are described as follows. We consider continuous-time simple random walks as frogs (which are active from the beginning and never sleep). Initially the frog from the origin is infected, while the other frogs are healthy. Infected frogs transmit the disease to all the frogs they meet without recovery. Furthermore, we can also find the so-called activated random walk model. This is similar to the combustion model, but there are initially some active frogs and any active frog may fall back to sleep randomly; see [5], [8], [34], [35], [36], and the references given therein.

We shall return to the topic of the frog model. The first published result on the frog model is due to Telcs–Wormald [37, Section 2.4] (in their paper, the frog model was called the 'egg model'). They treated the frog model on $\mathbb{Z}^d$ with one-frog-per-site initial configuration, and proved that it is recurrent for all $d \ge 1$, i.e. almost surely, active frogs infinitely often visit the origin. (Otherwise, we say that the frog model is transient.) This result proposed an interesting relationship between the strength of transience for a single random walk and the superior numbers of frogs.

To observe this more precisely, Popov [31] considered the frog model with Bernoulli initial configurations and exhibited phase transitions of its transience and recurrence. After that, Alves *et al.* coped with that kind of problem for the frog model with random initial configuration and random lifetime; see [2] and [32] for more details. In particular, [32] is a nice survey on the frog model and presents several open problems. It has also been a great help to recent progress on recurrence and transience for the frog model: We refer the interested reader to [14], [17], and [26] for the frog model on lattices, [10], [9], and [11] for the frog model with drift on lattices, and [18], [19], and [20] for the frog model on trees.

In this way, recent interest of the frog model seems to concentrate in recurrence and transience problems. On the other hand, as mentioned in Subsection 1.1, there are few results for the behavior of the first passage time in the frog model. We hope that our work is useful for research in the frog model (including the recurrence and transience problems) and the related models mentioned above.

1.4. Organization of the paper

Let us now describe how the present article is organized. In Section 2, for convenience, we summarize some notation and results for supercritical site percolation on $\mathbb{Z}^d$, and provide an upper tail estimate for the first passage time (see Proposition 2.4 below). In addition, Corollary 2.1 tells us that we have to switch frogs frequently to realize the first passage time. More precisely, it guarantees that each frog realizing $T(0, x)$ must find the next one within the ℓ^1-ball of radius $o(\|x\|_1)$. This fact will play an important role in the proofs of Theorems 1.2 and 1.3. It is generally difficult to observe the behavior of the frogs realizing the first passage time. This is because, without loss of the minimality, we have to handle the behavior of the frogs and those initial configurations at the same time. In the proof of Corollary 2.1, we overcome this difficulty by using a large deviation estimate for the simple random walk combined with Proposition 2.4.

In Sections 3 and 4 we prove our main results (Theorems 1.1, 1.2, and 1.3). An early study of large deviation estimates in stochastic growth models was carried out by Grimmett–Kesten [16]. They only treated the large deviation estimates for the first coordinate axis. However, we want to obtain the large deviation estimates for all directions as in Theorems 1.1 and 1.2. Thus, our arguments for these theorems to a large extent is based on the previous work of Garet–Marchand [12], [13] for the chemical distance in the Bernoulli percolation.

For the proof of Theorem 1.1, we basically follow the strategy taken in [12, Subsection 3.3]. Let us give the sketch of the proof here. For simplicity, we consider Theorem 1.1 only for $x = n\xi_1$. Fix $\varepsilon > 0$ and take N large enough. A site y of $\mathbb{Z}^d$ is said to be 'good' if $\|Ny - (Ny)^*\|_1 \le \sqrt{N}$, $\|N(y+\xi) - (N(y+\xi))^*\|_1 \le \sqrt{N}$ and $T^*(Ny, N(y+\xi)) \le (1+\varepsilon)\mu(N\xi_1)$ for any coordinate vector ξ. Note that (1.2) and the independent and identically distributed (i.i.d.) set-up of the configuration imply that each site y of $\mathbb{Z}^d$ is good with high probability. Hence, good sites induce a finitely dependent site percolation on $\mathbb{Z}^d$ with parameter sufficiently close to one (see Lemma 3.3 below). Suppose that an arbitrary integer n is much larger than N. Results in Subsection 2.1 below guarantee that the failure probability of the following event decays exponentially in n: there exist good sites $y_1, \dots, y_Q$ such that

- $Q \approx n/N$ and $\|y_q - y_{q+1}\|_1 = 1$ for all $1 \le q \le Q-1$;
- $\|(Ny_1)^*\|_1$ and $\|n\xi_1 - (Ny_Q)^*\|_1$ are smaller than $n^{1/4}$.

On this event and $\{0 \in \mathcal{I}\}$,

$$\begin{aligned} T(0, n\xi_1) &\le T(0, (Ny_1)^*) + \sum_{q=1}^{Q-1} T^*(Ny_q, Ny_{q+1}) + T((Ny_Q)^*, n\xi_1) \\ &\lesssim T(0, (Ny_1)^*) + (1+\varepsilon)\mu(n\xi_1) + T((Ny_Q)^*, n\xi_1). \end{aligned}$$

Using the upper tail estimate (stated in Proposition 2.4), we can control the first and third terms on the right-hand side, and the desired bound follows in the $x = n\xi_1$ case. We need some more work to carry out the above argument uniformly in any direction x.

In Section 4 we begin with the proof of Theorem 1.2. The lower large deviations have been studied for the first passage time in the first passage percolation and the chemical distance in the Bernoulli percolation, which are the counterparts of the first passage time in the frog model (we refer the interested reader to [1], [12], and [23]). These counterparts are induced by sequences of nearest-neighbor points on $\mathbb{Z}^d$ and depend on only one randomness. On the other hand, the first passage time in the frog model may use sequences of nonnearest-neighbor points on $\mathbb{Z}^d$ (see (1.1)) and depends on two sources of randomness: the simple random walks and the initial configuration. In [1], [12], and [23], the key tool for the lower large deviations is a renormalization procedure combined with a BK-like inequality. Although we also use a renormalization argument to show Theorem 1.1 and Proposition 2.4, due to the difference stated above, a BK-like inequality does not work well for the lower large deviation in the frog model. To overcome this problem, we use the concentration inequality for $T^*(0, x)$ as follows. Divide $T(0, x) - \mu(x)$ into three terms:

$$\begin{aligned} &T(0, x) - \mu(x) \\ &\quad = \big\{T(0, x) - T^*(0, x)\big\} + \big\{T^*(0, x) - \mathbb{E}\big[T^*(0, x)\big]\big\} + \big\{\mathbb{E}\big[T^*(0, x)\big] - \mu(x)\big\}. \end{aligned}$$

From Lemma 3.1 below, $\mathbb{E}[T^*(0, x)] \geq \mu(x)$ holds and the third term is harmless for the lower tail. The second term can be controlled once we get the concentration inequality for $T^*(0, x)$, which is Theorem 1.3. Hence, in the proof of Theorem 1.2 we try to compare $T(0, x)$ and $T^*(0, x)$ on the event $\{\omega(0) \geq 1\}$ by using Corollary 2.1.

The remainder of Section 4 will be devoted to the proof of Theorem 1.3. For the proof, we follow the approach taken by Garet–Marchand [13, Section 3]. In [13, Section 3], they constructed an approximation of the chemical distance with a deterministic upper bound and applied Chebyshev's inequality combined with exponential versions of the Efron–Stein inequality (see (4.3) and (4.4) below) to it. As mentioned above, the chemical distance is induced by sequences of nearest-neighbor points on $\mathbb{Z}^d$, but the first passage time in the frog model may use sequences of nonnearest-neighbor points on $\mathbb{Z}^d$. This difference disturbs the direct use of Garet–Marchand's approximation for our first passage time. To overcome this problem, we define passage times between every sufficiently remote two points x and y to be $C\|x - y\|_1$, where $C \in (0, \infty)$ is a constant much larger than $\mu(\xi_1)$. The first passage time modified in this way is dominated by $C\|\cdot\|_1$. Furthermore, (1.2) says that if $\|x - y\|_1$ is large enough then the first passage time from x to y is approximately equal to $\mu(x - y) < C\|x - y\|_1$. This means that the modified first passage time tends not to use sufficiently remote points, and, hence, it is comparable to the original first passage time. This observation implies our desired approximation of the first passage time.

We close this section with some general notation. Write $\|\cdot\|_1$ and $\|\cdot\|_\infty$ for the ℓ^1- and ℓ^∞-norms on $\mathbb{R}^d$. Denote by $\{\xi_1, \dots, \xi_d\}$ the canonical basis of $\mathbb{R}^d$, and let $\mathcal{E}^d := \{\xi \in \mathbb{Z}^d : \|\xi\|_1 = 1\}$. For $i \in \{1, \infty\}$, $x \in \mathbb{R}^d$, and $r > 0$, $B_i(x, r)$ is the ℓ^i-ball in $\mathbb{R}^d$ of center x and radius r, i.e.

$$B_i(x, r) := \big\{y \in \mathbb{R}^d : \|y - x\|_i \leq r\big\}.$$

Throughout this paper, we use c, c', C, C', C_i, and α_i, $i = 1, 2, \dots$, to denote constants with $0 < c, c', C, C', C_i < \infty$, and $0 < \alpha_i < 1$, respectively.

2. Preliminaries

2.1. Supercritical site percolation

Let $X = (X_v)_{v \in \mathbb{Z}^d}$ be a family of random variables taking values in $\{0, 1\}$. This induces the random set $\{v \in \mathbb{Z}^d : X_v = 1\}$. The *chemical distance* $d_X(v_1, v_2)$ for X between v_1 and v_2 is

https://doi.org/10.1017/apr.2019.8 Published online by Cambridge University Press

defined by

$$d_X(v_1, v_2) := \inf\big\{\#\pi : \pi \text{ is a nearest-neigibor path from } v_1 \text{ to } v_2 \text{ using only sites in } \{v \in \mathbb{Z}^d : X_v = 1\}\big\},$$

where $\#\pi$ is the length of a path π. A connected component of $\{v \in \mathbb{Z}^d : X_v = 1\}$ which contains infinitely many points is called an *infinite cluster* for X. If there exists almost surely a unique infinite cluster for X then we denote it by $\mathcal{C}_\infty(X)$.

For $0 < p < 1$, let $\eta_p = (\eta_p(v))_{v\in\mathbb{Z}^d}$ denote a family of independent random variables satisfying

$$\mathbb{P}(\eta_p(v) = 1) = 1 - \mathbb{P}(\eta_p(v) = 0) = p, \qquad v \in \mathbb{Z}^d.$$

This is called the *independent Bernoulli site percolation* on $\mathbb{Z}^d$ of the parameter p. It is well known that there is $p_c = p_c(d) \in (0, 1)$ such that if $p > p_c$ then the infinite cluster $\mathcal{C}_\infty(\eta_p)$ exists; see, for instance, Theorems 1.10 and 8.1 of [15]. The following proposition presents estimates for the size of the holes in the infinite cluster $\mathcal{C}_\infty(\eta_p)$ and the chemical distance $d_{\eta_p}(\cdot, \cdot)$ (see [13, below Equation (2.2) and Corollary 2.2] for the proof).

Proposition 2.1. *For $p > p_c$, the following results hold:*

(*a*) *there exist constants C_6 and C_7 such that, for all $t > 0$,*

$$\mathbb{P}(\mathcal{C}_\infty(\eta_p) \cap B_1(0, t) = \varnothing) \le C_6 \mathrm{e}^{-C_7 t};$$

(*b*) *there exist constants C_8, C_9, and C_{10} such that, for all $v \in \mathbb{Z}^d$ and $t \ge C_8\|v\|_1$,*

$$\mathbb{P}(t \le d_{\eta_p}(0, v) < \infty) \le C_9 \mathrm{e}^{-C_{10} t}.$$

A part of the proof of Theorem 1.1 relies on the following proposition obtained by Garet–Marchand [12, Theorem 1.4]. (Their argument works not only for bond percolation but also for site percolation.) This tells us that, when p is sufficiently close to 1, the chemical distance looks like the ℓ^1-norm.

Proposition 2.2. *For each $\gamma > 0$, there exists $p'(\gamma) \in (p_c, 1)$ such that, for all $p > p'(\gamma)$,*

$$\limsup_{\|v\|_1\to\infty} \frac{1}{\|v\|_1} \log \mathbb{P}((1+\gamma)\|v\|_1 \le d_{\eta_p}(0, v) < \infty) < 0.$$

Proposition 2.2 gives an estimate of the deviation of $d_{\eta_p}(0, v)$ only around $\|v\|_1$. In the proof of Proposition 2.4, we need an estimate of the upper tail for $d_{\eta_p}(0, v)$ sufficiently away from $\|v\|_1$, and Proposition 2.2 does not work well there. On the other hand, to prove Theorem 1.1, it is necessary that $d_{\eta_p}(0, v)$ is sufficiently close to $\|v\|_1$. The constant C_8 is generally a large constant, and Proposition 2.1(b) is not enough to prove Theorem 1.1. Accordingly, Proposition 2.1(b) and Proposition 2.2 are similar, but we need both results in this paper.

We finally recall the concept of stochastic domination. Let $X = (X_v)_{v\in\mathbb{Z}^d}$ and $Y = (Y_v)_{v\in\mathbb{Z}^d}$ be families of random variables taking values in $\{0, 1\}$. We say that X *stochastically dominates* Y if

$$\mathbb{E}[f(X)] \ge \mathbb{E}[f(Y)]$$

for all bounded, increasing, measurable functions $f\colon \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$. Furthermore, a family $X = (X_v)_{v\in\mathbb{Z}^d}$ of random variables is said to be *finitely dependent* if there exists $L > 0$ such that any two subfamilies $(X_{v_1})_{v_1\in\Lambda_1}$ and $(X_{v_2})_{v_2\in\Lambda_2}$ are independent whenever $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ satisfy $\|v_1 - v_2\|_1 > L$ for all $v_1 \in \Lambda_1$ and $v_2 \in \Lambda_2$.

The following stochastic comparison is useful to compare locally dependent fields with the independent Bernoulli site percolation. For the proof, we refer the reader to [15, Theorem 7.65] or [29, Theorem B26] for instance.

Proposition 2.3. *Suppose that $X = (X_v)_{v\in\mathbb{Z}^d}$ is a finitely dependent family of random variables taking values in $\{0, 1\}$. For a given $0 < p < 1$, X stochastically dominates η_p provided $\inf_{v\in\mathbb{Z}^d} \mathbb{P}(X_v = 1)$ is sufficiently close to 1.*

2.2. Upper tail estimate for the first passage time

As mentioned in Subsection 1.1, the tail estimate (1.4) is established for $x \in \mathbb{Z}^d$ and $t \geq \|x\|_1^4$. The condition $t \geq \|x\|_1^4$ is reasonable to derive the integrability of $T(0, x)$, but it is not enough to study the deviation of $T(0, x)$ around the time constant $\mu(x)$. In fact, since $\mu(x)$ is of order $\|x\|_1$, the bottom $\|x\|_1^4$ of the range of t deviates from $\mu(x)$ too much. Hence, the aim of this subsection is to improve the condition $t \geq \|x\|_1^4$ as in the following proposition.

Proposition 2.4. *There exist constants C_{11}, C_{12}, C_{13}, and α_5 such that, for all $x \in \mathbb{Z}^d$ and $t \geq C_{11}\|x\|_1$,*

$$\mathbb{P}(T(0, x) \geq t) \leq C_{12}\mathrm{e}^{-C_{13}t^{\alpha_5}}. \tag{2.1}$$

Before the proof, we need some preparation. Let N be a positive integer to be chosen large enough later and set $N' := \lfloor N^{1/4}/(4d)\rfloor$. Moreover, set $\Lambda_q := 2N'q + (-N', N']^d$ for $q \in \mathbb{Z}^d$. Then, the boxes Λ_q, $q \in \mathbb{Z}^d$, form a partition of $\mathbb{Z}^d$ and each site in $\mathbb{Z}^d$ is contained in precisely one box. A site v of $\mathbb{Z}^d$ is said to be *white* if the following conditions hold:

- $\Lambda_q \cap \mathcal{I} \neq \varnothing$ for all $q \in \mathbb{Z}^d$ with $\Lambda_q \subset B_\infty(Nv, N)$.
- $T(x, y) \leq N$ for all $x, y \in B_\infty(Nv, N) \cap \mathcal{I}$ with $\|x - y\|_1 \leq N^{1/4}$.

We say that v is *black* otherwise.

Lemma 2.1. *We can find $p \in (p_c, 1)$ and $N \geq 1$ such that $(\mathbf{1}_{\{v \text{ is white}\}})_{v\in\mathbb{Z}^d}$ stochastically dominates η_p and the infinite white cluster $\mathcal{C}_\infty^{\mathrm{w}} := \mathcal{C}_\infty((\mathbf{1}_{\{v \text{ is white}\}})_{v\in\mathbb{Z}^d})$ exists.*

Proof. Let us first check that, for every $v \in \mathbb{Z}^d$, the event $\{v \text{ is white}\}$ depends only on states in $B_1(Nv, 2N)$. It suffices to show that, for all $x, y \in B_1(Nv, N)$, the event $\{T(x, y) \leq N\}$ depends only on states in $B_1(Nv, 2N)$. By the definition of the first passage time, the event $\{T(x, y) \leq N\}$ can be replaced with the event that there exist $m \geq 1$ and $x_0, x_1, \dots, x_m \in \mathbb{Z}^d$, with $x_0 = x$ and $x_m = y$ such that

$$\sum_{i=0}^{m-1} \tau(x_i, x_{i+1}) \leq N.$$

Since every frog can only move to an adjacent site at each step, the above sum is strictly bigger than N provided $\|x_i - x_0\|_1 > N$ for some $1 \leq i \leq m$. Hence, the x_i must satisfy $\|x_i - Nv\|_1 \leq 2N$. This means that the event $\{T(x, y) \leq N\}$ depends only on states in $B_1(Nv, 2N)$.

We next show that $\inf_{v\in\mathbb{Z}^d} \mathbb{P}(v \text{ is white})$ converges to 1 as $N\to\infty$. The union bound proves

$$\mathbb{P}(0 \text{ is black}) \le \sum_{\substack{q\ge 1\\ \Lambda_q\subset B_\infty(0,N)}} \mathbb{P}(\Lambda_q \cap \mathcal{I} = \varnothing) \quad + \sum_{\substack{x,y\in B_\infty(0,N)\\ \|x-y\|_1\le N^{1/4}}} \mathbb{P}(T(0, y-x) > N).$$

The first summation is not larger than $cN^d\mathbb{P}(0\notin\mathcal{I})^{c'N^{d/4}}$ for some constants c and c', and it clearly goes to 0 as $N\to\infty$. By (1.4), we can also see that the second summation vanishes as $N\to\infty$. Therefore, from translation invariance, $\inf_{v\in\mathbb{Z}^d}\mathbb{P}(v \text{ is white})$ converges to 1 as $N\to\infty$.

With these observations, the proof is complete by using Proposition 2.3 and the same strategy taken in the proof of Proposition 5.2 of [30]. □

After the preparation above, we move to the proof of Proposition 2.4.

Proof of Proposition 2.4. Without loss of generality, we can assume that $\|x\|_1\ge d^4$. Let p and N be the constants appearing in Lemma 2.1. Consider the events

$$\Gamma_1 := \{\text{there exists } v_1\in\mathcal{C}^{\mathrm{w}}_\infty\cap B_1(0,t^{1/4}) \text{ and there exists } v_2\in\mathcal{C}^{\mathrm{w}}_\infty\cap B_1(v(x),t^{1/4}) \text{ such that } d^{\mathrm{w}}(v_1,v_2) < 4C_8t\},$$

$$\Gamma_2 := \{T(0,y) < (3N)^4t \text{ and } T(z,x) < (3N)^4t \text{ for all } y\in B_1(0,2Nt^{1/4}) \text{ and } z\in B_1(Nv(x),2Nt^{1/4})\cap\mathcal{I}\},$$

where $d^{\mathrm{w}}(\cdot,\cdot)$ is the chemical distance for $(\mathbf{1}_{\{v \text{ is white}\}})_{v\in\mathbb{Z}^d}$ and $v(x)$ is the site v of $\mathbb{Z}^d$ minimizing $\|Nv-x\|_\infty$ with a deterministic rule to break ties. Note that, on the event $\Gamma_1\cap\Gamma_2\cap\{0\in\mathcal{I}\}$,

$$T(0,x) < \{2(3N)^4 + 4C_8N^2\}t, \qquad t\ge\|x\|_1.$$

To complete the proof, we shall estimate $\mathbb{P}(\Gamma_1^{\mathrm{c}})$ and $\mathbb{P}(\Gamma_2^{\mathrm{c}})$. Lemma 2.1 implies that $\mathbb{P}(\Gamma_1^{\mathrm{c}})$ is bounded from above by

$$\begin{aligned}\mathbb{P}\Big(d_{\eta_p}(v_1,v_2)\ge 4C_8t \text{ for all } v_1\in\mathcal{C}_\infty(\eta_p)\cap B_1\Big(0,t^{1/4}\Big) \text{ and } v_2\in\mathcal{C}_\infty(\eta_p)\cap B_1\big(v(x),t^{1/4}\big)\Big)\\ \le 2\mathbb{P}\big(\mathcal{C}_\infty(\eta_p)\cap B_1(0,t^{1/4})=\varnothing\big) + \sum_{\substack{v_1\in B_1(0,t^{1/4})\\ v_2\in B_1(v(x),t^{1/4})}} \mathbb{P}(4C_8t\le d_{\eta_p}(v_1,v_2)<\infty).\end{aligned} \tag{2.2}$$

From Proposition 2.1(a), the first term on the right-hand side of (2.2) is not larger than $2C_6\mathrm{e}^{-C_7t^{1/4}}$. Note that, for $t\ge\|x\|_1$, $v_1\in B_1(0,t^{1/4})$, and $v_2\in B_1(v(x),t^{1/4})$,

$$\|v_1-v_2\|_1 \le 2t^{1/4} + \frac{1}{N}\|Nv(x)-x\|_1 + \frac{\|x\|_1}{N} \le 4t.$$

This combined with Proposition 2.1(b) shows that the second term on the right-hand side of (2.2) is exponentially small in t. Consequently, $\mathbb{P}(\Gamma_1^{\mathrm{c}})$ decays faster than $\mathrm{e}^{-C_7t^{1/4}}$. On the other hand, we have, for $t\ge\|x\|_1$ and $z\in B_1(Nv(x),2Nt^{1/4})$,

$$\|x-z\|_1 \le \|x-Nv(x)\|_1 + \|Nv(x)-z\|_1 \le 3Nt^{1/4}.$$

This together with (1.4) proves that $\mathbb{P}(\Gamma_2^c)$ is bounded from above by a multiple of $t^{d/2}\exp\{-C_2(3N)^{4\alpha_1}t^{\alpha_1}\}$. Therefore, (2.1) immediately follows from the above bounds for $\mathbb{P}(\Gamma_1^c)$ and $\mathbb{P}(\Gamma_2^c)$. □

We close this section with the corollary of Proposition 2.4.

Corollary 2.1. *Suppose that $\mathbb{E}[\omega(0)] < \infty$. Then there exist constants C_{14}, C_{15}, and α_6 such that, for all $x \in \mathbb{Z}^d$ and $t > 0$,*

$$\begin{aligned}&\mathbb{P}(\text{there exists } v_1, v_2 \in \mathcal{I} \text{ with } \|v_1 - v_2\|_1 \geq t \text{ such that } T(0, x) = T(0, v_1) + \tau(v_1, v_2) + T(v_2, x))\\&\quad \leq C_{14}\|x\|_1^{2d} \mathrm{e}^{-C_{15}t^{\alpha_6}}.\end{aligned} \tag{2.3}$$

Proof. Since the left-hand side of (2.3) is smaller than or equal to $\mathbb{P}(T(0, x) \geq t)$, the corollary immediately follows from Proposition 2.4 provided $t \geq C_{11}\|x\|_1$.

Assume that $t < C_{11}\|x\|_1$. We use Proposition 2.4 to show that the left-hand side of (2.3) is bounded from above by

$$\begin{aligned}&C_{12}\exp\{-C_{13}(C_{11}\|x\|_1)^{\alpha_5}\} + \sum_{\substack{v_1, v_2 \in B_1(0, C_{11}\|x\|_1)\\ \|v_1 - v_2\|_1 \geq t}} \mathbb{P}(\tau(0, v_2 - v_1) = T(0, v_2 - v_1))\\&\quad \leq C_{12}\mathrm{e}^{-C_{13}t^{\alpha_5}} + \sum_{\substack{v_1, v_2 \in B_1(0, C_{11}\|x\|_1)\\ \|v_1 - v_2\|_1 \geq t}} \{I_1(v_2 - v_1) + I_2(v_2 - v_1)\},\end{aligned} \tag{2.4}$$

where, for $z \in \mathbb{Z}^d$,

$$I_1(z) := \mathbb{P}\Bigg(\max_{\substack{0 \leq k \leq C_{11}\|z\|_1\\ 1 \leq \ell \leq \omega(0)}} \|S_k(0, \ell)\|_1 \geq \|z\|_1\Bigg),$$

$$I_2(z) := \mathbb{P}\Bigg(\max_{\substack{0 \leq k \leq C_{11}\|z\|_1\\ 1 \leq \ell \leq \omega(0)}} \|S_k(0, \ell)\|_1 < \|z\|_1,\ \tau(0, z) = T(0, z)\Bigg).$$

To estimate $I_1(v_2 - v_1)$, we rely on the following simple large deviation estimate for the simple random walk; see [28, Lemma 1.5.1]. For any $\gamma > 0$, there exists a constant c (which may depend on γ) such that, for all $n, u \geq 0$,

$$\mathbb{P}\Big(\max_{0 \leq k \leq n} \|S_k(0, 1)\|_1 \geq \gamma u \sqrt{n}\Big) \leq c\mathrm{e}^{-u}.$$

Fix $v_1, v_2 \in B_1(0, C_{11}\|x\|_1)$ with $\|v_1 - v_2\|_1 \geq t$, and set $\gamma = C_{11}{}^{-1/2}$, $n = C_{11}\|v_2 - v_1\|_1$, and $u = \|v_2 - v_1\|_1^{1/2}$. Then,

$$\begin{aligned}I_1(v_2 - v_1) &\leq \sum_{L=1}^{\infty} \mathbb{P}(\omega(0) = L) \sum_{\ell=1}^{L} \mathbb{P}\Big(\max_{0 \leq k \leq C_{11}\|v_2 - v_1\|_1} \|S_k(0, \ell)\|_1 \geq \|v_2 - v_1\|_1\Big)\\&\leq \mathbb{E}[\omega(0)]c\mathrm{e}^{-t^{1/2}}.\end{aligned}$$

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We again use Proposition 2.4 to obtain, for $v_1, v_2 \in B_1(0, C_{11}\|x\|_1)$ with $\|v_1 - v_2\|_1 \geq t$,

$$\begin{aligned} I_2(v_2 - v_1) &\leq \mathbb{P}(C_{11}\|v_2 - v_1\|_1 < \tau(0, v_2 - v_1) = T(0, v_2 - v_1)) \\ &\leq C_{12} \exp\{-C_{13}(C_{11}t)^{\alpha_5}\}. \end{aligned}$$

Therefore, (2.3) follows from (2.4) and these bounds for $I_1(v_2 - v_1)$ and $I_2(v_2 - v_1)$. □

3. Upper large deviation bound

In this section we give the proof of Theorem 1.1. We basically follow the approach taken in [12, Subsection 3.3]. Let us first prepare some notation and lemmata.

Lemma 3.1. *For each $x \in \mathbb{Z}^d$, $\mathbb{P}$-a.s. and in L^1,*

$$\mu(x) = \lim_{k\to\infty} \frac{1}{k} T^*(0, kx) = \lim_{k\to\infty} \frac{1}{k} \mathbb{E}[T^*(0, kx)] = \inf_{k\geq 1} \frac{1}{k} \mathbb{E}[T^*(0, kx)]. \tag{3.1}$$

Proof. From (1.2), we have, on the event $\{0 \in \mathcal{I}\}$ of positive probability,

$$\mu(x) = \lim_{k\to\infty,\, kx\in\mathcal{I}} \frac{1}{k} T(0, kx) = \lim_{k\to\infty,\, kx\in\mathcal{I}} \frac{1}{k} T^*(0, kx).$$

Therefore, once the integrability of $T^*(0, x)$ is proved, (3.1) follows from the subadditive ergodic theorem for the process $T^*(ix, jx)$, $0 \leq i < j$, $i, j \in \mathbb{N}_0$.

For the integrability,

$$\begin{aligned} \mathbb{E}[T^*(0, x)] \leq{}& \int_0^\infty \mathbb{P}\Big(\|0^*\|_1 > \frac{t}{3C_{11}}\Big)\,\mathrm{d}t + \int_0^\infty \mathbb{P}\Big(\|x - x^*\|_1 > \frac{t}{3C_{11}}\Big)\,\mathrm{d}t \\ &+ \int_0^\infty \mathbb{P}\Big(T^*(0, x) \geq t,\ \|0^*\|_1 \leq \frac{t}{3C_{11}},\ \|x - x^*\|_1 \leq \frac{t}{3C_{11}}\Big)\,\mathrm{d}t. \end{aligned}$$

It is clear that the first and second terms on the right-hand side are finite. Moreover, the third term is not larger than

$$3C_{11}\|x\|_1 + \sum_{\substack{y\in B_1(0,t/(3C_{11}))\\ z\in B_1(x,t/(3C_{11}))}} \int_{3C_{11}\|x\|_1}^\infty \mathbb{P}(T(0, z - y) \geq t)\,\mathrm{d}t,$$

and the integrability of $T^*(0, x)$ follows by using Proposition 2.4. □

We denote by $\mathcal{S}_d$ the symmetric group on $\{1, \ldots, d\}$. For each $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\sigma \in \mathcal{S}_d$, and $\varepsilon \in \{+1, -1\}^d$, we define

$$\Psi_{\sigma,\varepsilon}(x) := \sum_{i=1}^d \varepsilon(i) x_{\sigma(i)} \xi_i.$$

Then, $\mathcal{O}(\mathbb{Z}^d) := \{\Psi_{\sigma,\varepsilon} : \sigma \in \mathcal{S}_d,\ \varepsilon \in \{+1, -1\}^d\}$ is the group of orthogonal transformations that preserve the grid $\mathbb{Z}^d$. Consequently, its elements also preserve the ℓ^1-norm $\|\cdot\|_1$ and

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the time constant $\mu(\cdot)$. For $x \in \mathbb{R}^d$ and $(g_1, \dots, g_d) \in (\mathcal{O}(\mathbb{Z}^d))^d$, the linear map $L_x^{g_1,\dots,g_d}$ is defined by

$$L_x^{g_1,\dots,g_d}(y) := \sum_{i=1}^{d} y_i g_i(x), \qquad y = (y_1, \dots, y_d) \in \mathbb{R}^d.$$

To study the first passage time in each direction x, we want to find a basis of $\mathbb{R}^d$ adapted to the studied direction, i.e. made of images of x by elements of $\mathcal{O}(\mathbb{Z}^d)$. The following technical lemma, which is obtained by Garet–Marchand [12, Lemma 2.2], gives the existence of such a basis.

Lemma 3.2. *For each $x \in \mathbb{R}^d$, there exists a family $(g_{1,x}, g_{2,x}, \dots, g_{d,x}) \in (\mathcal{O}(\mathbb{Z}^d))^d$ with $g_{1,x} = \mathrm{Id}_{\mathbb{R}^d}$ such that the linear map $L_x := L_x^{g_{1,x},\dots,g_{d,x}}$ satisfies*

$$C_{16}\|x\|_1\|y\|_1 \le \|L_x(y)\|_1 \le \|x\|_1\|y\|_1, \qquad y \in \mathbb{R}^d,$$

where C_{16} is a universal constant not depending on x, y, and $(g_{1,x}, g_{2,x}, \dots, g_{d,x})$.

Proof of Theorem 1.1. We fix an arbitrary $\varepsilon > 0$ and break the proof into three steps.

Step 1. In this step we choose appropriate constants for our proof. By (1.3), $\mu(y) \ge 1$ holds for all $y \in \mathbb{R}^d$ with $\|y\|_1 = 1$. Hence, there exists $\delta > 0$ such that, for all $y \in \mathbb{R}^d$ with $\|y\|_1 = 1$,

$$\left(1 + \frac{3\delta}{2C_{11}}\right)(1+\delta)^2\mu(y) + 2\delta < \mu(y)(1+\varepsilon) \tag{3.2}$$

and

$$\delta < \frac{C_{11}}{2}.$$

To shorten notation, write

$$\beta := \frac{\delta}{2C_{11}} < \frac{1}{4}.$$

Take $M \in \mathbb{N}$ large enough such that

$$M \ge \frac{d}{\delta}\max\left\{\frac{\mu(\xi_1)}{2}, \frac{8C_{11}}{C_{16}}\right\} \ge 4, \tag{3.3}$$

and choose $p \in (0, 1)$ to satisfy

$$p > p'\left(\frac{\beta}{1+2\beta}\right) > p_c, \tag{3.4}$$

where $p'(\cdot)$ is the parameter appearing in Proposition 2.2.

Step 2. In this step we tackle the construction of the renormalization procedure. Let N be a positive integer to be chosen large enough later. A site $v \in \mathbb{Z}^d$ is said to be *good* if the following conditions hold, for all $y \in \mathbb{Z}^d/M$ with $\|y\|_1 = 1$:

(1) $T^*(NL_{My}(v), NL_{My}(v+\xi)) \leq MN\mu(y)(1+\delta)$ for all $\xi \in \mathcal{E}^d$.

(2) $(NL_{My}(v))^*$ is included in $B_1(NL_{My}(v), \sqrt{N})$, and $(NL_{My}(v+\xi))^*$ belongs to $B_1(NL_{My}(v+\xi), \sqrt{N})$ for all $\xi \in \mathcal{E}^d$.

Otherwise, v is called *bad*.

Lemma 3.3. *There exists $N \in \mathbb{N}$ such that $(\mathbf{1}_{\{v \text{ is good}\}})_{v\in\mathbb{Z}^d}$ stochastically dominates η_p.*

Proof. Since the set $\{y \in \mathbb{Z}^d/M \colon \|y\|_1 = 1\}$ is finite, Lemma 3.1 implies that $\mathbb{P}(v \text{ is bad})$ converges to 0 as $N \to \infty$. Therefore, due to Proposition 2.3, our task is now to show the finite dependence of $(\mathbf{1}_{\{v \text{ is good}\}})_{v\in\mathbb{Z}^d}$. Condition (ii) depends only on the configuration in $B_1(NL_{My}(v), \sqrt{N})$ and $B_1(NL_{My}(v+\xi), \sqrt{N})$. We have $\|NL_{My}(v) - NL_{My}(v+\xi)\|_1 \leq MN$ from Lemma 3.2, and condition (ii) particularly depends only on the configuration in $B_1(NL_{My}(v), 2MN)$. The same argument as in the proof of Lemma 2.1 implies that condition (i) depends only on the configuration in $B_1(NL_{My}(v), 2MN\mu(\xi_1)(1+\delta))$. Note that Lemma 3.2 ensures that if $\|v-w\|_1 > (2/C_{16})\mu(\xi_1)(1+\delta)$, then, for all $y \in \mathbb{Z}^d/M$ with $\|y\|_1 = 1$,

$$\|NL_{My}(v) - NL_{My}(w)\|_1 > 2MN\mu(\xi_1)(1+\delta).$$

With these observations, $(\mathbf{1}_{\{v \text{ is good}\}})_{v\in\mathbb{Z}^d}$ is finitely dependent. □

For a given $x \in \mathbb{Z}^d \setminus \{0\}$, we set $x' := x/\|x\|_1$. Then, there exists $\widehat{x} \in \mathbb{Z}^d/M$ such that $\|\widehat{x}\|_1 = 1$ and $\|x' - \widehat{x}\|_1 \leq d/(2M)$. Note that, by (1.3) and (3.3),

$$|\mu(x') - \mu(\widehat{x})| \leq \mu(\xi_1)\|x' - \widehat{x}\|_1 \leq \mu(\xi_1)\,\frac{d}{2M} \leq \delta \tag{3.5}$$

and

$$\|x' - \widehat{x}\|_1 \leq \frac{C_{16}\beta}{8}. \tag{3.6}$$

The definition of $L_{M\widehat{x}}$ and Lemma 3.2 tell us that, for all $1 \leq i \leq d$,

$$\mu(L_{M\widehat{x}}(\xi_i)) = M\mu(\widehat{x}), \qquad \|L_{M\widehat{x}}(\xi_i)\|_1 = M,$$

and, for all $y \in \mathbb{R}^d$,

$$C_{16}M\|y\|_1 \leq \|L_{M\widehat{x}}(y)\|_1 \leq M\|y\|_1.$$

Denote by $d^{\mathrm{g}}(\cdot,\cdot)$ the chemical distance for $(\mathbf{1}_{\{v \text{ is good}\}})_{v\in\mathbb{Z}^d}$. We now consider the event

$$G := \{\text{there exists } v \in \mathcal{A}(0, \beta\|\bar{x}\|_1),\ \text{there exists } w \in \mathcal{A}(\bar{x}, \beta\|\bar{x}\|_1) \text{ such that } d^{\mathrm{g}}(v,w) < (1+3\beta)\|\bar{x}\|_1\},$$

where

$$\bar{x} := \left\lfloor \frac{\|x\|_1}{MN} \right\rfloor \xi_1$$

and

$$\mathcal{A}(z,r) := \left\{ y \in \mathbb{Z}^d \colon \frac{r}{2} \leq \|y-z\|_1 \leq r \right\}, \qquad z \in \mathbb{Z}^d,\ r > 0.$$

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It is easy to see that, on the event G, for some $v \in \mathcal{A}(0, \beta\|\overline{x}\|_1)$ and $w \in \mathcal{A}(\overline{x}, \beta\|\overline{x}\|_1)$,

$$T^*(NL_{M\widehat{x}}(v), NL_{M\widehat{x}}(w)) < MN\mu(\widehat{x})(1+\delta)(1+3\beta)\|\overline{x}\|_1. \tag{3.7}$$

Furthermore, Lemma 3.3 proves

$$\begin{aligned} P(G^{\mathrm{c}}) &\le P(d_{\eta_p}(v, w) \ge (1+3\beta)\|\overline{x}\|_1 \text{ for all } v \in \mathcal{A}(0, \beta\|\overline{x}\|_1) \text{ and } w \in \mathcal{A}(\overline{x}, \beta\|\overline{x}\|_1)) \\ &\le 2\mathbb{P}\bigg(B_1\bigg(0, \frac{\beta\|\overline{x}\|_1}{2}\bigg) \cap \mathcal{C}_\infty(\eta_p) = \varnothing\bigg) \\ &\quad + \sum_{\substack{v \in B_1(0, \beta\|\overline{x}\|_1) \\ w \in B_1(\overline{x}, \beta\|\overline{x}\|_1)}} \mathbb{P}\bigg(\frac{1+3\beta}{1+2\beta}\|v-w\|_1 \le d_{\eta_p}(v, w) < \infty\bigg). \end{aligned}$$

Thanks to $\beta < \frac{1}{4}$ and (3.4), Proposition 2.1(a) and Proposition 2.2 imply that, for some constants c and c',

$$\mathbb{P}(G^{\mathrm{c}}) \le c\mathrm{e}^{-C_7\beta\|x\|_1/(2MN)} + c\bigg(\frac{\beta\|x\|_1}{MN}\bigg)^{2d} \mathrm{e}^{-c'(1-2\beta)\|x\|_1/(MN)}. \tag{3.8}$$

Step 3. Finally, we complete the proof. There is no loss of generality in assuming that

$$\|x\|_1 \ge \frac{4MN}{\beta C_{16}}. \tag{3.9}$$

By the definition of x' and (3.2),

$$\begin{aligned} \mathbb{P}(T(0, x) \ge (1+\varepsilon)\mu(x)) &= \mathbb{P}(T(0, x) \ge \mu(x')(1+\varepsilon)\|x\|_1) \\ &\le \mathbb{P}\bigg(T(0, x) > \bigg(1 + \frac{3\delta}{2C_{11}}\bigg)(1+\delta)^2\mu(x')\|x\|_1 + 2\delta\|x\|_1\bigg). \end{aligned} \tag{3.10}$$

Let A be the event that $T(0, y) < \delta\|x\|_1$ for all $y \in NL_{M\widehat{x}}(\mathcal{A}(0, \beta\|\overline{x}\|_1)) + B_1(0, \sqrt{N})$ and $T(z, x) < \delta\|x\|_1$ for all $z \in [NL_{M\widehat{x}}(\mathcal{A}(\overline{x}, \beta\|\overline{x}\|_1)) + B_1(0, \sqrt{N})] \cap \mathcal{I}$. By (3.5) and (3.7), on the event $G \cap A \cap \{0 \in \mathcal{I}\}$, there exist $v \in \mathcal{A}(0, \beta\|\overline{x}\|_1)$ and $w \in \mathcal{A}(\overline{x}, \beta\|\overline{x}\|_1)$ such that

$$\begin{aligned} T(0, x) &\le T(0, (NL_{M\widehat{x}}(v))^*) + T^*(NL_{M\widehat{x}}(v), NL_{M\widehat{x}}(w)) + T((NL_{M\widehat{x}}(w))^*, x) \\ &\le MN\mu(\widehat{x})(1+\delta)(1+3\beta)\|\overline{x}\|_1 + 2\delta\|x\|_1 \\ &\le \bigg(1 + \frac{3\delta}{2C_{11}}\bigg)(1+\delta)^2\mu(x')\|x\|_1 + 2\delta\|x\|_1. \end{aligned}$$

This means that the right-hand side of (3.10) is bounded from above by $\mathbb{P}(G^{\mathrm{c}}) + \mathbb{P}(A^{\mathrm{c}})$. Due to (3.8), our task is to estimate $\mathbb{P}(A^{\mathrm{c}})$. We use Lemma 3.2 and (3.9) to obtain, for $y \in NL_{M\widehat{x}}(\mathcal{A}(0, \beta\|\overline{x}\|_1)) + B_1(0, \sqrt{N})$,

$$\|y\|_1 \le MN\beta\|\overline{x}\|_1 + \sqrt{N} \le \beta\|x\|_1 + \sqrt{N} \le \frac{17}{16}\beta\|x\|_1 \le \frac{\delta}{C_{11}}\|x\|_1$$

and

$$\|y\|_1 \ge MNC_{16}\frac{\beta\|\overline{x}\|_1}{2} - \sqrt{N} \ge C_{16}\frac{\beta\|x\|_1}{2} - MN\frac{C_{16}\beta}{2} - \sqrt{N} \ge \frac{13}{32}C_{16}\beta\|x\|_1.$$

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Similarly, for $z \in NL_{M\widehat{x}}(\mathcal{A}(\overline{x}, \beta\|\overline{x}\|_1)) + B_1(0, \sqrt{N})$,

$$\frac{13}{32}C_{16}\beta\|x\|_1 \le \|z - NL_{M\widehat{x}}(\overline{x})\|_1 \le \frac{17}{16}\beta\|x\|_1.$$

In addition, by (3.6), we have, for $z \in NL_{M\widehat{x}}(\mathcal{A}(\overline{x}, \beta\|\overline{x}\|_1)) + B_1(0, \sqrt{N})$,

$$\begin{aligned}\|x - z\|_1 &\le \|x - NL_{M\widehat{x}}(\overline{x})\|_1 + \|NL_{M\widehat{x}}(\overline{x}) - z\|_1 \\ &\le \frac{3}{8}C_{16}\beta\|x\|_1 + \frac{17}{16}\beta\|x\|_1 \\ &\le 2\beta\|x\|_1 \\ &= \frac{\delta}{C_{11}}\|x\|_1\end{aligned}$$

and

$$\begin{aligned}\|x - z\|_1 &\ge \|z - NL_{M\widehat{x}}(\overline{x})\|_1 - \|NL_{M\widehat{x}}(\overline{x}) - x\|_1 \\ &\ge \frac{13}{32}C_{16}\beta\|x\|_1 - \frac{3}{8}C_{16}\beta\|x\|_1 \\ &= \frac{C_{16}}{32}\beta\|x\|_1.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}(A^c) \le &\sum_{(13/32)C_{16}\beta\|x\|_1 \le \|y\|_1 \le (17/16)\beta\|x\|_1} \mathbb{P}(T(0, y) > C_{11}\|y\|_1) \\ &+ \sum_{(13/32)C_{16}\beta\|x\|_1 \le \|z - NL_{M\widehat{x}}(\overline{x})\|_1 \le (17/16)\beta\|x\|_1} \mathbb{P}(T(0, x - z) > C_{11}\|x - z\|_1),\end{aligned}$$

and this combined with Proposition 2.4 completes the proof. □

4. Lower large deviation and concentration bounds

The aim of this section is to prove Theorems 1.2 and 1.3. Let us first show Theorem 1.2 by using Theorem 1.3. The proof of Theorem 1.3 will be given after that of Theorem 1.2.

Proof of Theorem 1.2. Let $v(x)$ denote a site of $\mathcal{I}$ satisfying

$$T(0^*, x) = T(0^*, v(x)) + \tau(v(x), x).$$

We first prove that, for all $\varepsilon > 0$, there exist constants C_{17} and C_{18} such that

$$\mathbb{P}(T(v(x), x^*) \ge \varepsilon\|x\|_1) \le C_{17}\exp\left\{-C_{18}\|x\|_1^{\alpha_5 \wedge \alpha_6}\right\}. \tag{4.1}$$

Corollary 2.1 tells us that there exist constants c and c' such that

$$\begin{aligned}\mathbb{P}(\text{there exists } v_1, v_2 \in \mathcal{I} \text{ with } \|v_1 - v_2\|_1 \ge \varepsilon(2C_{11})^{-1}\|x\|_1 \text{ such that } T(0^*, x) = T(0^*, v_1) \\ + \tau(v_1, v_2) + T(v_2, x)) \\ \le c\mathrm{e}^{-c'\|x\|_1^{\alpha_6}}.\end{aligned}$$

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It follows that

$$\begin{aligned}
&\mathbb{P}(T(v(x), x^*) \geq \varepsilon\|x\|_1) \\
&\leq c\mathrm{e}^{-c'\|x\|_1^{\alpha_6}} + \mathbb{P}\bigg(\|x - x^*\|_1 \geq \frac{\varepsilon\|x\|_1}{2C_{11}}\bigg) \\
&\quad + \mathbb{P}\bigg(T(v(x), x^*) \geq \varepsilon\|x\|_1,\ \|v(x) - x\|_1 < \frac{\varepsilon\|x\|_1}{2C_{11}},\ \|x - x^*\|_1 < \frac{\varepsilon\|x\|_1}{2C_{11}}\bigg).
\end{aligned}$$

Since the second term has the desired form, our task is to bound the last probability. To this end, we use Proposition 2.4 to obtain, for some constants C and C',

$$\begin{aligned}
&\mathbb{P}\bigg(T(v(x), x^*) \geq \varepsilon\|x\|_1,\ \|v(x) - x\|_1 < \frac{\varepsilon\|x\|_1}{2C_{11}},\ \|x - x^*\|_1 < \frac{\varepsilon\|x\|_1}{2C_{11}}\bigg) \\
&\leq \sum_{\substack{y \in \mathbb{Z}^d \\ \|y-x\|_1 < (2C_{11})^{-1}\varepsilon\|x\|_1}} \sum_{\substack{z \in \mathbb{Z}^d \\ \|x-z\|_1 < (2C_{11})^{-1}\varepsilon\|x\|_1}} \mathbb{P}(T(0, z - y) \geq \varepsilon\|x\|_1) \\
&\leq C\mathrm{e}^{-C'\|x\|_1^{\alpha_5}}.
\end{aligned}$$

Hence, (4.1) follows.

Take $t = \varepsilon\sqrt{\|x\|_1}$ and recall that $0^* = 0$ under $\mathbb{P} = \mathbb{P}(\,\cdot \mid 0 \in \mathcal{I})$. Then, by (1.3) and (3.1),

$$\mathbb{P}(T(0, x) \leq (1 - \varepsilon)\mu(x)) \leq \mathbb{P}(0 \in \mathcal{I})^{-1}\mathbb{P}\Big(T(0^*, x) - \mathbb{E}[T^*(0, x)] \leq -t\sqrt{\|x\|_1}\Big).$$

The last probability is bounded from above by

$$\mathbb{P}\Big(T(0^*, x) - T^*(0, x) \leq -\frac{t}{2}\sqrt{\|x\|_1}\Big) + \mathbb{P}\Big(T^*(0, x) - \mathbb{E}[T^*(0, x)] \leq -\frac{t}{2}\sqrt{\|x\|_1}\Big).$$

Note that

$$T^*(0, x) \leq T(0^*, v(x)) + T(v(x), x^*) + \tau(v(x), x) \leq T(0^*, x) + T(v(x), x^*),$$

and (4.1) implies that

$$\begin{aligned}
\mathbb{P}\Big(T(0^*, x) - T^*(0, x) \leq -\frac{t}{2}\sqrt{\|x\|_1}\Big) &\leq \mathbb{P}\Big(T(v(x), x^*) \geq \frac{\varepsilon}{2}\|x\|_1\Big) \\
&\leq C_{17}\exp\big\{-C_{18}\|x\|_1^{\alpha_5 \wedge \alpha_6}\big\}.
\end{aligned}$$

Furthermore, Theorem 1.3 proves that

$$\mathbb{P}\Big(T^*(0, x) - \mathbb{E}[T^*(0, x)] \leq -\frac{t}{2}\sqrt{\|x\|_1}\Big) \leq C_4\exp\Big\{-C_5\Big(\frac{\varepsilon}{2}\sqrt{\|x\|_1}\Big)^{\alpha_4}\Big\},$$

and, therefore, the theorem follows. □

Proof of Theorem 1.3. For each $t > 0$, define the two-point function $\sigma_t(\cdot, \cdot)$ as follows. Take $K > d(C_{11} + \gamma + 1)$. First, if $\|x - y\|_\infty \leq t$ and $\tau(x, y) > 4Kt$, then set $\sigma_t(x, y) := 4Kt$. Next, if $\|x - y\|_\infty > t$ then set $\sigma_t(x, y) := 4K\|x - y\|_\infty$. Otherwise, set $\sigma_t(x, y) := \tau(x, y)$. By definition, for any $x, y \in \mathbb{Z}^d$,

$$\|x - y\|_1 \leq \sigma_t(x, y) \leq 4K(t \vee \|x - y\|_\infty). \tag{4.2}$$

We write $T_t(x, y)$ for the first passage time from x to y corresponding to $\sigma_t(\cdot, \cdot)$, i.e.,

$$T_t(x, y) := \inf\left\{\sum_{i=0}^{m-1} \sigma_t(x_i, x_{i+1})\colon m \geq 1,\ x_0 = x,\ x_m = y,\ x_1, \ldots, x_{m-1} \in \mathbb{Z}^d\right\}.$$

Proposition 4.1. *There exist constants C_{19}, C_{20}, and α_7 such that, for all $x \in \mathbb{Z}^d \setminus \{0\}$ and $0 \leq t \leq \|x\|_1$,*

$$\max\{\mathbb{P}(T_t(0^*, x^*) \neq T^*(0, x)), \mathbb{E}[(T_t(0^*, x^*) - T^*(0, x))^2]^{1/2}\} \leq C_{19}\|x\|_1^{4d} \mathrm{e}^{-C_{20}t^{\alpha_7}}.$$

Proposition 4.2. *For all $\gamma > 0$, there exists a constant C_{21} such that, for all $x \in \mathbb{Z}^d \setminus \{0\}$ and $0 \leq t \leq \gamma\sqrt{\|x\|_1}$,*

$$\mathbb{P}\Big(|T_t(0, x) - \mathbb{E}[T_t(0, x)]| \geq t\sqrt{\|x\|_1}\Big) \leq 2\mathrm{e}^{-C_{21}t}.$$

Let us postpone the proofs of these propositions to the end of this section, and continue the proof of Theorem 1.3. To this end, without loss of generality, we can assume that $\|x\|_1 \geq (32K\mathbb{E}[1 \vee \|0^*\|_\infty])^2$. Take $c \geq 1$ large enough such that, for all $t \geq c(1 + \log \|x\|_1)^{1/\alpha_7}$,

$$C_{19}\|x\|_1^{4d}\mathrm{e}^{-C_{20}t^{\alpha_7}} \leq C_{19}\mathrm{e}^{-C_{20}t^{\alpha_7}/2} \leq \frac{t}{4}.$$

From (4.2) and Proposition 4.1, we have

$$\begin{aligned}|\mathbb{E}[T^*(0, x)] - \mathbb{E}[T_t(0, x)]| &\leq \mathbb{E}[|T^*(0, x) - T_t(0^*, x^*)|] + 2\mathbb{E}[T_t(0, 0^*) \vee T_t(0^*, 0)] \\ &\leq C_{19}\|x\|_1^{4d}\mathrm{e}^{-C_{20}t^{\alpha_7}} + 8K\mathbb{E}[t \vee \|0^*\|_\infty].\end{aligned}$$

Hence, for all $t \geq c(1 + \log \|x\|_1)^{1/\alpha_7}$,

$$|\mathbb{E}[T^*(0, x)] - \mathbb{E}[T_t(0, x)]| \leq \frac{t}{2}\sqrt{\|x\|_1}.$$

This together with Proposition 4.1 leads to

$$\begin{aligned}&\mathbb{P}\Big(|T^*(0, x) - \mathbb{E}[T^*(0, x)]| \geq t\sqrt{\|x\|_1}\Big) \\ &\qquad \leq C_{19}\mathrm{e}^{-C_{20}t^{\alpha_7}/2} + \mathbb{P}\bigg(|T_t(0^*, x^*) - \mathbb{E}[T_t(0, x)]| \geq \frac{t}{2}\sqrt{\|x\|_1}\bigg).\end{aligned}$$

For the second term on the right-hand side,

$$\begin{aligned}|T_t(0^*, x^*) - \mathbb{E}[T_t(0, x)]| &\leq |T_t(0^*, x^*) - T_t(0, x)| + |T_t(0, x) - \mathbb{E}[T_t(0, x)]| \\ &\leq T_t(0, 0^*) \vee T_t(0^*, 0) + T_t(x, x^*) \vee T_t(x^*, x) \\ &\quad + |T_t(0, x) - \mathbb{E}[T_t(0, x)]|.\end{aligned}$$

Using (4.2) again we obtain

$$\begin{aligned}&\mathbb{P}\bigg(|T_t(0^*, x^*) - E[T_t(0, x)]| \geq \frac{t}{2}\sqrt{\|x\|_1}\bigg) \\ &\quad \leq 2\mathbb{P}\bigg(4K(t \vee \|0^*\|_\infty) \geq \frac{t}{6}\sqrt{\|x\|_1}\bigg) + \mathbb{P}\bigg(|T_t(0, x) - E[T_t(0, x)]| \geq \frac{t}{6}\sqrt{\|x\|_1}\bigg),\end{aligned}$$

and the theorem is a consequence of Proposition 4.2. □

In the rest of this section we shall prove Propositions 4.1 and 4.2.

Proof of Proposition 4.1. Let $0 \le t \le \|x\|_1$. We first estimate $\mathbb{P}(T_t(0^*, x^*) \neq T^*(0, x))$. To this end, consider the following events Γ_j, $1 \le j \le 5$:

$$\Gamma_1 := \left\{ \left\|0^*\right\|_1 \le \frac{t}{8},\ \left\|x - x^*\right\|_1 \le \frac{t}{8} \right\},$$

$$\Gamma_2 := \bigcap_{y,z \in B_1(0, 7K\|x\|_1)} \left\{ y \in \mathcal{I} \text{ implies that } T(y, z) \neq T(y, v_1) + \tau(v_1, v_2) + T(v_2, z) \right.$$
$$\left. \text{for all } v_1, v_2 \in \mathcal{I} \text{ with } \|v_1 - v_2\|_1 \ge \frac{t}{2} \right\},$$

$$\Gamma_3 := \bigcap_{\substack{y,z \in B_1(0, 7K\|x\|_1) \\ \|y-z\|_\infty \le 2t}} \{ y \in \mathcal{I} \text{ implies that } T(y, z) \le 2Kt \},$$

$$\Gamma_4 := \bigcap_{\substack{y,z \in B_1(0, 7K\|x\|_1) \\ \|y-z\|_\infty \ge t/2}} \{ y \in \mathcal{I} \text{ implies that } T(y, z) \le K\|y - z\|_\infty \},$$

$$\Gamma_5 := \bigcap_{y,z \in B_1(0, 7K\|x\|_1)} \{ y \in \mathcal{I} \text{ implies that } T(y, z) \ge T_t(y, z) \}.$$

We shall observe that $T_t(0^*, x^*) = T^*(0, x)$ holds on the event $\bigcap_{j=1}^5 \Gamma_j$. Denote by $(x_i)_{i=0}^m$ a finite sequence of $\mathbb{Z}^d$ satisfying that $x_0 = 0^*$, $x_m = x^*$, and $T_t(0^*, x^*) = \sum_{i=0}^{m-1} \sigma_t(x_i, x_{i+1})$. Moreover, the index i_0 is defined by

$$i_0 := \max\{0 \le i \le m \colon T(0^*, x_i) = T_t(0^*, x_i)\}.$$

On the event Γ_1, we have $\|0^*\|_1 \le K\|x\|_1$ and it holds by (4.2) that

$$T_t(0^*, x^*) \le 4K(t \vee \|0^* - x^*\|_\infty) \le 4K\left\{ t \vee \left(\frac{t}{4} + \|x\|_\infty \right) \right\} \le 5K\|x\|_1.$$

This proves that the x_i are included in $B_1(0, 6K\|x\|_1)$ on the event Γ_1. Let x'_{i_0} denote a site of $\mathcal{I}$ satisfying $T(0^*, x_{i_0}) = T(0^*, x'_{i_0}) + \tau(x'_{i_0}, x_{i_0})$. Note that $\|x'_{i_0} - x_{i_0}\|_1 \le t/2$ and $\|x'_{i_0}\|_1 \le 7K\|x\|_1$ on the event $\Gamma_1 \cap \Gamma_2$. Assume that $i_0 < m$. Then, on the event $\Gamma_1 \cap \Gamma_5$,

$$T(0^*, x_{i_0+1}) > T_t(0^*, x_{i_0+1}) = T_t(0^*, x_{i_0}) + \sigma_t(x_{i_0}, x_{i_0+1}) = T(0^*, x_{i_0}) + \sigma_t(x_{i_0}, x_{i_0+1}).$$

We now consider the following three cases:

1. $\|x_{i_0} - x_{i_0+1}\|_\infty \le t$ and $\tau(x_{i_0}, x_{i_0+1}) > 4Kt$;
2. $\|x_{i_0} - x_{i_0+1}\|_\infty > t$;
3. $\|x_{i_0} - x_{i_0+1}\|_\infty \le t$ and $\tau(x_{i_0}, x_{i_0+1}) \le 4Kt$.

Case 1: On the event $\Gamma_1 \cap \Gamma_2$,

$$\|x'_{i_0} - x_{i_0+1}\|_\infty \le \|x'_{i_0} - x_{i_0}\|_\infty + \|x_{i_0} - x_{i_0+1}\|_\infty \le \frac{t}{2} + t \le 2t.$$

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Therefore, on the event $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \cap \Gamma_5$,

$$\begin{aligned} T(0^*, x_{i_0+1}) &> T(0^*, x_{i_0}) + \sigma_t(x_{i_0}, x_{i_0+1}) \\ &= T(0^*, x_{i_0}) + 4Kt \\ &\geq T(0^*, x'_{i_0}) + T(x'_{i_0}, x_{i_0+1}) \\ &\geq T(0^*, x_{i_0+1}). \end{aligned}$$

This is a contradiction.

Case 2: On the event $\Gamma_1 \cap \Gamma_2$,

$$\|x'_{i_0} - x_{i_0+1}\|_\infty \geq \|x_{i_0+1} - x_{i_0}\|_\infty - \|x_{i_0} - x'_{i_0}\|_\infty \geq \frac{t}{2},$$

and on the event $\Gamma_1 \cap \Gamma_2 \cap \Gamma_5$,

$$\begin{aligned} T(0^*, x_{i_0+1}) &> T(0^*, x_{i_0}) + \sigma_t(x_{i_0}, x_{i_0+1}) \\ &= T(0^*, x_{i_0}) + 4K\|x_{i_0} - x_{i_0+1}\|_\infty \\ &\geq T(0^*, x'_{i_0}) + 2Kt + 2K\|x_{i_0} - x_{i_0+1}\|_\infty. \end{aligned}$$

It follows that on the event $\Gamma_1 \cap \Gamma_2 \cap \Gamma_4 \cap \Gamma_5$,

$$\begin{aligned} T(0^*, x_{i_0+1}) &> T(0^*, x'_{i_0}) + 2K\|x'_{i_0} - x_{i_0}\|_\infty + 2K\|x_{i_0} - x_{i_0+1}\|_\infty \\ &\geq T(0^*, x'_{i_0}) + 2K\|x'_{i_0} - x_{i_0+1}\|_\infty \\ &\geq T(0^*, x'_{i_0}) + T(x'_{i_0}, x_{i_0+1}) \geq T(0^*, x_{i_0+1}), \end{aligned}$$

and this leads to another contradiction.

Case 3: Since $\sigma_t(x_{i_0}, x_{i_0+1}) = \tau(x_{i_0}, x_{i_0+1})$, on the event $\Gamma_1 \cap \Gamma_5$,

$$T(0^*, x_{i_0+1}) > T(0^*, x_{i_0}) + \sigma_t(x_{i_0}, x_{i_0+1}) = T(0^*, x_{i_0}) + \tau(x_{i_0}, x_{i_0+1}) \geq T(0^*, x_{i_0+1}),$$

which is also a contradiction.

With these observations, on the event $\bigcap_{j=1}^5 \Gamma_j$, $i_0 = m$ must hold and $T_t(0^*, x^*) = T^*(0, x)$ is valid. It remains to estimate the probability of $\bigcup_{j=1}^5 \Gamma_j^c$. Obviously, $\mathbb{P}(\Gamma_1^c)$ is exponentially small in t. The following bound is an immediate consequence of Proposition 2.4 and Corollary 2.1: For some constants c and c',

$$\mathbb{P}(\Gamma_2^c) + \mathbb{P}(\Gamma_3^c) + \mathbb{P}(\Gamma_4^c) \leq c\|x\|_1^{4d} \exp\{-c' t^{\alpha_5 \wedge \alpha_6}\}.$$

To estimate $\mathbb{P}(\Gamma_5^c)$, let us introduce the event $\Gamma_6(w)$ that $T(0, w) \neq T(0, v_1) + \tau(v_1, v_2) + T(v_2, w)$ for all $v_1, v_2 \in \mathcal{I}$ with $\|v_1 - v_2\|_\infty \geq t$. Since $T(0, w) \geq T_t(0, w)$ on the event $\Gamma_6(w) \cap \{0 \in \mathcal{I}\}$, we have

$$\begin{aligned} \mathbb{P}(\Gamma_5^c) &\leq \sum_{y,z \in B_1(0,7K\|x\|_1)} \mathbb{P}(T(0, z-y) - T_t(0, z-y) < 0) \\ &\leq \sum_{y,z \in B_1(0,7K\|x\|_1)} \mathbb{P}(\Gamma_6(z-y)^c). \end{aligned}$$

From Corollary 2.1, this is bounded from above by a multiple of $\|x\|_1^{4d} \mathrm{e}^{-C_{15} t^{\alpha_6}}$. Therefore, we get the desired bound for $\mathbb{P}(T_t(0^*, x^*) \neq T^*(0, x))$.

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We next estimate $\mathbb{E}[(T_t(0^*, x^*) - T^*(0, x))^2]^{1/2}$. Schwarz's inequality implies that

$$\begin{aligned}
&\mathbb{E}\big[(T_t(0^*, x^*) - T^*(0, x))^2\big] \\
&\quad = \mathbb{E}\big[(T_t(0^*, x^*) - T^*(0, x))^2 \mathbf{1}_{\{T_t(0^*, x^*) \neq T^*(0,x)\}}\big] \\
&\quad \leq \big(\mathbb{E}\big[T_t(0^*, x^*)^4\big]^{1/2} + \mathbb{E}\big[T^*(0, x))^4\big]^{1/2}\big)\mathbb{P}(T_t(0^*, x^*) \neq T^*(0, x))^{1/2}.
\end{aligned}$$

By (4.2),

$$\mathbb{E}\big[T_t(0^*, x^*)^4\big] \leq (4K)^4 \mathbb{E}\big[(t \vee \|0^* - x^*\|_\infty)^4\big] \leq (12K)^4 (2\mathbb{E}[\|0^*\|_1^4] + 1)\|x\|_1^4.$$

On the other hand, letting $r(s) := s^{1/4}/(3C_{11})$, we have

$$\begin{aligned}
\mathbb{E}\big[T^*(0, x))^4\big] &\leq (3C_{11}\|x\|_1)^4 + \int_{(3C_{11}\|x\|_1)^4}^{\infty} \mathbb{P}\big(T^*(0, x)^4 \geq s\big)\, \mathrm{d}s \\
&\leq (3C_{11}\|x\|_1)^4 + 2\int_{(3C_{11}\|x\|_1)^4}^{\infty} \mathbb{P}(\|0^*\|_1 \geq r(s))\, \mathrm{d}s \\
&\quad + \int_{(3C_{11}\|x\|_1)^4}^{\infty} \sum_{\substack{y \in B_1(0, r(s)) \\ z \in B_1(x, r(s))}} \mathbb{P}\big(T(y, z) \geq s^{1/4},\ y \in \mathcal{I}\big)\, \mathrm{d}s.
\end{aligned}$$

It follows from Proposition 2.4 that $\mathbb{E}[T^*(0, x))^4]$ is not greater than a multiple of $\|x\|_1^4$. Combining these bounds with that for $\mathbb{P}(T_t(0^*, x^*) \neq T^*(0, x))$, we can derive the desired bound for $\mathbb{E}[(T_t(0^*, x^*) - T^*(0, x))^2]^{1/2}$, and the proof is complete. □

Before starting the proof of Proposition 4.2, let us prepare some notation and lemmata. For a given $x \in \mathbb{Z}^d \setminus \{0\}$ and $t > 0$, tile $\mathbb{Z}^d$ with copies of $(-t/2, t/2]^d$ such that each box is centered at a point in $\mathbb{Z}^d$ and each site in $\mathbb{Z}^d$ is contained in precisely one box. We denote these boxes by Λ_q, $q \in \mathbb{N}$, and consider the random variables

$$U_q := ((\omega(z))_{z \in \Lambda_q}, (S_\cdot(z, \ell))_{z \in \Lambda_q, \ell \in \mathbb{N}}), \qquad q \in \mathbb{N}.$$

Note that $(U_q)_{q=1}^{\infty}$ are independent and identically distributed. Due to (4.2), $T_t(0, x)$ depends only on states in some finite boxes $\Lambda_1, \ldots, \Lambda_Q$, and $T_t(0, x)$ can be regarded as a function of $(U_q)_{q=1}^{Q}$:

$$Z := T_t(0, x) = T_t(0, x, U_1, \ldots, U_Q).$$

In addition, let $(U'_q)_{q=1}^{Q}$ be independent copies of $(U_q)_{q=1}^{Q}$ and define

$$Z'_q := T_t(0, x, U_1, \ldots, U_{q-1}, U'_q, U_{q+1}, \ldots, U_Q), \qquad 1 \leq q \leq Q.$$

Our main tools for the proof of Proposition 4.2 are Chebyshev's inequality and the following exponential versions of the Efron–Stein inequality; we refer the reader to [7, Theorem 6.16] and [13, Lemma 3.2]. For any $\lambda, \theta > 0$ with $\lambda\theta < 1$,

$$\log \mathbb{E}[\exp\{-\lambda(Z - \mathbb{E}[Z])\}] \leq \frac{\lambda\theta}{1 - \lambda\theta} \log \mathbb{E}\left[\exp\left\{\frac{\lambda V_-}{\theta}\right\}\right], \tag{4.3}$$

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where

$$V_- := \sum_{q=1}^{Q} \mathbb{E}\big[(Z - Z'_q)_-^2 \,\big|\, U_1, \ldots, U_Q\big].$$

Furthermore, if there exist $\delta > 0$ and functions $(\phi_q)_{q=1}^Q$, $(\psi_q)_{q=1}^Q$, and $(g_q)_{q=1}^Q$ such that, for all $1 \le q \le Q$,

$$(Z - Z'_q)_- \le \psi_q(U'_q), \qquad (Z - Z'_q)_-^2 \le \phi_q(U'_q) g_q(U_1, \ldots, U_Q),$$

and $\mathbb{E}[\mathrm{e}^{\delta \psi_q(U_q)} \phi_q(U_q)] < \infty$, then, for any $\lambda, \theta > 0$ with $\lambda < \delta \wedge \theta^{-1}$,

$$\log \mathbb{E}[\, \exp\{\lambda(Z - \mathbb{E}[Z])\}] \le \frac{\lambda\theta}{1 - \lambda\theta} \log \mathbb{E}\bigg[\exp\bigg\{\frac{\lambda W}{\theta}\bigg\}\bigg], \tag{4.4}$$

where

$$W := \sum_{q=1}^{Q} \mathbb{E}\big[\mathrm{e}^{\delta \psi_q(U_q)} \phi_q(U_q)\big] g_q(U_1, \ldots, U_Q).$$

We use the following lemmata to estimate the right-hand sides of (4.3) and (4.4).

Lemma 4.1. *Write $\pi_t(0, x) = (0 = x_0, x_1, \ldots, x_m = x)$ for the finite sequence of $\mathbb{Z}^d$ that has $T_t(0, x) = \sum_{i=0}^{m-1} \sigma_t(x_i, x_{i+1})$, chosen with a deterministic rule to break ties. Moreover, let R_q be the event that $\pi_t(0, x)$ intersects Λ_q. Then we have, for $1 \le q \le Q$,*

$$(Z - Z'_q)_- \le 8Kt\mathbf{1}_{R_q}. \tag{4.5}$$

Proof. Since $(Z - Z'_q)_- = (Z'_q - Z)\mathbf{1}_{\{Z \le Z'_q\} \cap R_q}$, we focus on the event $\{Z \le Z'_q\} \cap R_q$ from now on. Let us first treat the case where $x \in \Lambda_q$. Define $i_0 := \min\{0 \le i \le m \colon x_i \in \Lambda_q\}$ and set $a := x_{i_0}$. Then, since $\|a - x\|_\infty \le t$,

$$Z'_q - Z \le T_t(a, x, U_1, \ldots, U_{q-1}, U'_q, U_{q+1}, \ldots, U_Q) \le 4Kt.$$

For the case where $x \notin \Lambda_q$, let us introduce the indices i_1 and i_2 as follows:

$$i_1 := \min\{0 \le i \le m - 1 \colon x_i \in \Lambda_q\}, \qquad i_2 := \max\{0 \le i \le m - 1 \colon x_i \in \Lambda_q\}.$$

In addition, write $a := x_{i_1}$, $b := x_{i_2}$, and $c := x_{i_2+1}$. If $\|a - c\|_\infty \le t$, then

$$Z'_q - Z \le T_t(a, c, U_1, \ldots, U_{q-1}, U'_q, U_{q+1}, \ldots, U_Q) \le 4Kt.$$

If $\|a - c\|_\infty > t$ and $\|b - c\|_\infty \le t$, then

$$\begin{aligned} Z'_q - Z &\le T_t(a, c, U_1, \ldots, U_{q-1}, U'_q, U_{q+1}, \ldots, U_Q) \\ &\le 4K\|a - c\|_\infty \\ &\le 4K(\|a - b\|_\infty + \|b - c\|_\infty) \\ &\le 8Kt. \end{aligned}$$

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Otherwise (i.e. $\|a-c\|_\infty > t$ and $\|b-c\|_\infty > t$),

$$\begin{aligned}Z'_q - Z &\le T_t(a, c, U_1, \dots, U_{q-1}, U'_q, U_{q+1}, \dots, U_Q) - \sigma_t(b, c)\\ &\le 4K(\|a-c\|_\infty - \|b-c\|_\infty)\\ &\le 4K\|a-b\|_\infty\\ &\le 4Kt.\end{aligned}$$

With these observations, $Z'_q - Z \le 8Kt$ is valid in the case where $x \notin \Lambda_q$, and (4.5) follows. □

Lemma 4.2. *There exists a constant $C_{22} \ge 1$ such that*

$$\sum_{q=1}^{Q} \mathbf{1}_{R_q} \le C_{22} K\Big(1 \vee \frac{\|x\|_\infty}{t}\Big).$$

Proof. Let $\pi_t(0, x) = (0 = x_0, x_1, \dots, x_m = x)$. For each $z \in \mathbb{Z}^d$, write $w(z)$ for the center of the box Λ_q containing z. Then, define $\rho_0 := 0$ and, for $j \ge 1$,

$$\rho_{j+1} := \min\Big\{\rho_j < i \le m \colon x_i \notin w(x_{\rho_j}) + \Big(-\frac{3t}{2}, \frac{3t}{2}\Big]^d\Big\},$$

with the convention that $\min\varnothing := \infty$. Define $J := \max\{j \ge 1 \colon \rho_j < \infty\}$ and assume that

$$J > 4K\Big(1 \vee \frac{\|x\|_\infty}{t}\Big).$$

By definition, we have $T_t(0, x) \ge Jt$ and, hence,

$$T_t(0, x) \ge Jt > 4K(t \vee \|x\|_\infty),$$

which contradicts (4.2). Therefore,

$$J \le 4K\Big(1 \vee \frac{\|x\|_\infty}{t}\Big),$$

and the proof is complete since $\pi_t(0, x)$ intersects at most $3^d(J+1)\Lambda_q$s. □

We are now in a position to prove Proposition 4.2.

Proof of Proposition 4.2. Fix arbitrary $\gamma > 0$, $x \in \mathbb{Z}^d \setminus \{0\}$, and $0 \le t \le \gamma\sqrt{\|x\|_1}$. We use Chebyshev's inequality to obtain, for all $u, \lambda \ge 0$,

$$\begin{aligned}\mathbb{P}(|T_t(0, x) - \mathbb{E}[T_t(0, x)]| \ge u) &\le e^{-\lambda u}\mathbb{E}[\exp\{\lambda|Z - \mathbb{E}[Z]|\}]\\ &\le e^{-\lambda u}(\mathbb{E}[\exp\{-\lambda(Z - \mathbb{E}[Z])\}] + \mathbb{E}[\exp\{\lambda(Z - \mathbb{E}[Z])\}]).\end{aligned} \tag{4.6}$$

On the other hand, Lemmata 4.1 and 4.2 show that

$$V_- \le (8Kt)^2 \sum_{q=1}^{Q} \mathbf{1}_{R_q} \le C_{22}(8Kt)^2\Big(1 \vee \frac{\|x\|_\infty}{t}\Big).$$

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Moreover, taking $\delta := 1/t$, $\phi_q := (8Kt)^2$, $\psi_q := 8Kt$, and $g_q := \mathbf{1}_{R_q}$ (see the notation above (4.4)), we use Lemmata 4.1 and 4.2 again to obtain

$$W \le (8Kt)^2 e^{8K} \sum_{q=1}^{Q} \mathbf{1}_{R_q} \le C_{22}(8Kt)^2 e^{8K} \bigg(1 \vee \frac{\|x\|_\infty}{t}\bigg).$$

These bounds combined with (4.3), (4.4), and (4.6) prove that, for all $u \ge 0$ and for all $\lambda, \theta > 0$ with $0 < \lambda < t^{-1} \wedge (2\theta)^{-1}$,

$$\begin{aligned}\mathbb{P}(|T_t(0, x) - \mathbb{E}[T_t(0, x)]| \ge u) &\le 2e^{-\lambda u} \exp\bigg\{\frac{\lambda^2}{1-\lambda\theta} C_{22}(8K)^2 e^{8K} t(t \vee \|x\|_\infty)\bigg\} \\ &\le 2 \exp\big\{2C_{22}(8K)^2 e^{8K} t(t \vee \|x\|_\infty)\lambda^2 - u\lambda\big\}.\end{aligned}$$

Substitute $u = t\sqrt{\|x\|_1}$ for

$$\mathbb{P}(|T_t(0, x) - \mathbb{E}[T_t(0, x)]| \ge t\sqrt{\|x\|_1}) \le 2 \exp\{2C_{22}(8K)^2 e^{8K} t(t \vee \|x\|_\infty)\lambda^2 - t\sqrt{\|x\|_1}\lambda\}.$$

To minimize the right-hand side, we choose

$$\lambda = \frac{\sqrt{\|x\|_1}}{4C_{22}(8K)^2 e^{8K}(t \vee \|x\|_\infty)}.$$

Since $t \le \gamma\sqrt{\|x\|_1}$, $C_{22} \ge 1$, and $K \ge \gamma$,

$$\lambda \le \frac{\sqrt{\|x\|_1}}{2K^2\|x\|_\infty} \le \frac{\sqrt{\|x\|_1}}{2K\|x\|_1} = \frac{1}{2K\sqrt{\|x\|_1}} < \frac{1}{t}.$$

In addition, taking $\theta = (3\lambda)^{-1}$ leads to $0 < \lambda < t^{-1} \wedge (2\theta)^{-1}$. Therefore,

$$\mathbb{P}\Big(|T_t(0, x) - \mathbb{E}[T_t(0, x)]| \ge t\sqrt{\|x\|_1}\Big) \le 2 \exp\bigg\{-\frac{t}{8C_{22}(8K)^2 e^{8K}(1+\gamma)}\bigg\},$$

which proves the proposition. □

Acknowledgements

The author was supported by JSPS Grant-in-Aid for Young Scientists (B) 16K17620. The author would also like to express his profound gratitude to the reviewer for a very careful reading of the manuscript.

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