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METAVALUATIONS

ROSS T. BRADY

Abstract. This is a general account of metavaluations and their applications, which can be seen as an alternative to standard model-theoretic methodology. They work best for what are called metacomplete logics, which include the contraction-less relevant logics, with possible additions of Conjunctive Syllogism, (A→B)&(B→C) → .A→C, and the irrelevant, A→ .B→A, these including the logic MC of meaning containment which is arguably a good entailment logic. Indeed, metavaluations focus on the formula-inductive properties of theorems of entailment form A→B, splintering into two types, M1- and M2-, according to key properties of negated entailment theorems (see below). Metavaluations have an inductive presentation and thus have some of the advantages that model theory does, but they represent proof rather than truth and thus represent proof-theoretic properties, such as the priming property, if ⊢A ∨ B then ⊢A or ⊢B, and the negated-entailment properties, not-⊢∼(A→B) (for M1-logics, with M1-metavaluations) and ⊢∼(A→B) iff ⊢A and ⊢∼ B (for M2-logics, with M2-metavaluations). Topics to be covered are their impact on naive set theory and paradox solution, and also Peano arithmetic and Godel's First and Second Theorems. Interesting to note here is that the familiar M1- and M2-metacomplete logics can be used to solve the set-theoretic paradoxes and, by inference, the Liar Paradox and key semantic paradoxes. For M1-logics, in particular, the final metavaluation that is used to prove the simple consistency is far simpler than its correspondent in the model-theoretic proof in that it consists of a limit point of a single transfinite sequence rather than that of a transfinite sequence of such limit points, as occurs in the model-theoretic approach. Additionally, it can be shown that Peano Arithmetic is simply consistent, using metavaluations that constitute finitary methods. Both of these results use specific metavaluational properties that have no correspondents in standard model theory and thus it would be highly unlikely that such model theory could prove these results in their final forms.

Metavaluations are a tool for proving certain kinds of proof-theoretic properties by adding proof into an otherwise semantic valuation for the formulae of a logical system. Such a metavaluation is set up by assigning T or F to each formula by induction. However, we include provability of a formula or formulae into their inductive conditions. Typically, v(A→B) = T iff ⊢A→B and if v(A) = T then v(B) = T. Some inductive conditions are unchanged, e.g., v(A∨B) = T iff v(A) = T or v(B) = T. This would then enable the priming property, if ⊢A∨B then ⊢A or ⊢B, to be shown, once the metacompleteness property, v(A) = T iff ⊢A, is proved. This property is proved by soundness, if ⊢A then v(A) = T, and completeness, if v(A) = T then ⊢A. Completeness is easily shown by formula-induction since

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$v(A \to B) = T$ implies $\vdash A \to B$. Soundness is proved by the usual induction on proof steps, with soundness for Modus Ponens following easily from the above inductive condition for $v(A \to B) = T$. As we can see from the metacompleteness property, the method of metavaluations is really a kind of proof theory. Moreover, variations on the above theme do abound, as we will see in the following pages.

This article is expository, taking one through the history of metavaluations, looking into their applications, both initial and more recent. In the process, we will compare this methodology with that of standard model theory and show that, as well as repeating some results, we are also able to achieve some new results which would, at least, be very difficult to show using standard model theory.

§1. The history. We start by putting metavaluations into their historical context. Although Robert Meyer introduced them in his (1976) article, entitled "Metacompleteness", there were essentially two precursors, both worth noting.

Firstly, the methodology used by Ronald Harrop in (1956) to show the priming and existential properties for sentential intuitionist logic and its arithmetic involved the use of a deletion process on the set of theorems through an enumeration of formulae, ordered according to the number of connectives contained therein. This deletion process is formula-inductive, through its use of formula enumeration in the process of applying it. This process was also designed to ensure that Modus Ponens holds in the newly deleted set, as well as ensuring that the standard conjunction and disjunction properties also hold. (See p. 350 of (1956), where this is set out in the form of three deletion conditions.) He then showed by induction on proof that the original set of theorems are all contained in the final deleted set, and thus that the two sets are the same. He goes on to prove the priming property, if $\vdash A \vee B$ then $\vdash A$ or $\vdash B$, for sentential intuitionist logic and that it and the existential property, if $\vdash \exists x A$ then $\vdash A^n/x$, for some numeral n, where A is a closed formula, both hold in intuitionist arithmetic.

The similarity with Meyer's metavaluations can be seen as follows. The standard conjunction and disjunction properties of the deleted set follow his metavaluations and Harrop's ensuring that Modus Ponens holds in the deleted set can be similarly achieved through use of Meyer's metavaluation for $\to$. Also, the alignment of the deleted set with the original set is similar to that achieved by soundness and completeness of the metavaluations, yielding what is called metacompleteness. (See also Brady (2006), p. 155)

Secondly, Meyer in (1971) introduces the notion of *metavaluation* v for a classical modal logic L as a function on formulae taking them to $\{0,1\}$, satisfying the following conditions:

(i) $v(\Box A) = 1$ iff $\vdash \Box A$ in L, and
(ii) $v(A \to B) = v(A) \to v(B)$, $v({\sim}A) = {\sim}v(A)$, and similarly for the other nonmodal connectives, the operations $\sim$ and $\to$ on the metavaluations being understood classically.

[Here, the metavaluation regards the valuation of necessitated formulae atomically, along with the usual atomic formulae.]

A formula A is true on a metavaluation v iff v(A) = 1.

Then, a *coherent* modal logic is defined as one for which every theorem is metavalid, where a formula A is metavalid iff it is true on all metavaluations of the logic. He goes on to show that a wide range of modal logics are coherent, and hence have the S4-property: $\vdash\Box A \vee \Box B$ iff $\vdash\Box A$ or $\vdash\Box B$. Meyer in (1972–1975) started a manuscript on coherence but, unfortunately, did not finish it, as far as the author is aware.

Thus, metavaluations are defined inductively on formulae, but do embody elements of proof. The 'meta-' refers to the fact that the valuation is dependent on such proof, differentiating it from a valuation in standard model theory, which is inductively built up from the independent valuations of atoms with help from semantic primitives. As in the above example, metavaluations can therefore provide a shortcut method of proving properties about proof itself, that might be more circuitous using algebraic or purely semantic methods. However, later on, we include applications which replace such theoremhood by elements of a theory, commonly used in completeness arguments but still based on proof theory.

Following up the historical theme, Meyer added at the proof stage of his (1976), on p. 514:

> Since the ideas here are very simple, it is not surprising, as Professor Kripke has pointed out to me, that they have occurred in a number of related forms to other authors, e.g., to Harrop, Rasiowa, Dwyer, and Fine, the first two of whom, with Kleene and others, have been interested in them in particular with intuitionist logics and mathematics in mind, while the latter two (and I) have been more interested in modal and relevant applications.

We will briefly clarify some of this work. Harrop in (1956) does refer to a number of authors, including some of the above, who have proved the priming property for intuitionist logic using a variety of methods, including Gentzen and algebraic methods. Further, Harrop in his (1960), extends the use of his method of (1956), which is sketched above, to show, for intuitionistic propositional calculus, that

(*) if $A \to B \vee C$ is a theorem then either $A \to B$ or $A \to C$ is also a theorem, provided A contains no relevant occurrence of $\vee$.

Harrop also shows, for intuitionist elementary number theory, that (*) also holds, subject to the above proviso but with the added condition that A, B and C are closed formulae, and further that if $A \to \exists xB$ is a theorem then so is $A \to B^n/x$, for some numeral n, with A satisfying the above proviso together with the condition that both A and $\exists xB$ are closed formulae. Kleene, in his (1962) and (1963), takes Harrop's work on intuitionist propositional logic, predicate logic and elementary number theory a step further, by finding necessary and sufficient conditions to replace Harrop's sufficient conditions, which are set out above.

Fine has recently made the author aware of his unpublished contribution (1971–3) which, similar to that of Meyer in (1971), uses a metavaluational method to prove a disjunctive property for some modal logics. Indeed, Fine replaces Meyer's metavaluation for $\Box$A, viz. $v(\Box A) = 1$ iff $\vdash\Box A$, by the valuation $v(\Box A) = T$ iff $\vdash A$, though provability here is generalized to theories which contain the modal logic K and are closed under Modus Ponens. Logics are then closed under the Necessitation rule as well. Fine then shows, for the modal logics K, KT, K4, and S4, that they satisfy the following Separation Property:

whenever $\Box B_1 \vee \Box B_2 \vee \cdots \vee \Box B_n \vee C$ is a theorem, where C is nonmodal, either C is a theorem or $\Box B_i$ is a theorem, for some $i = 1, 2, \ldots, n$.

(Nonmodal formulae do not contain any modal operators at all.)

He also goes on to prove, as corollaries of this property, that these logics are decidable and complete.

We continue with the main results of Meyer (1976), where he introduces metavaluations for positive logics, both sentential and quantified. This extends to a very wide range of sentential logics, including the positive relevant logic R_+, with possible addition of $A \to .B\to A$ to form RK_+, and full intuitionist logic, with the defined negation $\neg A =$df $A\to\mathbf{F}$, where $\mathbf{F}\to A$ is included as an axiom. There is imposed a very minimal logical requirement that **t** be an axiom and that Modus Ponens ($A, A\to B \Rightarrow B$), Adjunction ($A, B \Rightarrow A\&B$) and Disjunction Introduction ($A \Rightarrow A\vee B$; $B \Rightarrow A\vee B$) be rules, though even these can be reduced in some cases. Standard quantificational axioms and rules can be added for the quantified logics.

Meyer introduces what he calls the *canonical metavaluation* V from formulae to {T, F}, defined inductively for sentential logic L, containing the following connectives and constants[1]:

(i) $V(p) = F$, for each sentential variable p;
(ii) $V(\mathbf{t}) = T$;
(iii) $V(\mathbf{F}) = F$;
(iv) $V(A\&B) = T$ iff $V(A) = T$ and $V(B) = T$;
(v) $V(A\vee B) = T$ iff $V(A) = T$ or $V(B) = T$;
(vi) $V(A\to B) = T$ iff $\vdash A\to B$;
(vii) $V(\neg A) = T$ iff $\vdash\neg A$.

[Note that (iii) and (vii) are just included for the intuitionist logic.]

Since the atomic formulae are all given fixed values, there is a single metavaluation defined, and called 'canonical' as Meyer will go on to prove that $V(A) = T$ iff $\vdash A$, through the intermediate use of a canonical quasi-valuation.

A quasi-valuation V' adds 'and either $V'(A) = F$ or $V'(B) = T$' to the RHS of (vi) and 'and $V'(A) = F$' to the RHS of (vii). He then shows that $V(A) = T$ iff $V'(A) = T$.

[1]The use of metavaluations of sentential constants in general for classical logics was introduced in Brady (2010). This work could also be included as part of this survey of metavaluations, but we feel the article is long enough already.

He then defines a logic L to be *metacomplete* when this property, $V(A) = T$ iff $\vdash A$, holds for the logic. Then, by (v), the Priming Property, if $\vdash A \vee B$ then $\vdash A$ or $\vdash B$, follows.

Meyer extended this process to their quantified logics through the addition of standard quantificational axioms and rules and the following inductive steps for the metavaluation V:

(i) $V(A) = F$, for atomic formulae A; [extending the previous (i)]
(viii) $V(\forall xA) = T$ iff $V(A^t/x) = T$, for all terms t;
(ix) $V(\exists xA) = T$ iff $V(A^t/x) = T$, for some term t.

The Existential Property, if $\vdash \exists xA$ then $\vdash A^t/x$, for some term t, then follows by metacompleteness and (ix). Meyer also added identity = with the usual axiom and rule [$t{=}t$; $t{=}u \Rightarrow A(t) \leftrightarrow A(u)$], and extended V to include identity statements:

(x) $V(t{=}t) = T$;
(xi) $V(t{=}u) = F$, if t and u are distinct terms of the quantified logic.

The key remaining problem concerns the addition of a primitive (nonintuitionist) negation, to round out the logics. It was left to Slaney in (1984) and (1987) to provide the answers. In (1984), he introduced the *-metavaluation $v^*(A)$, which will ultimately mean 'not-$\vdash\sim A$', in contrast to metavaluation $v(A)$ meaning '$\vdash A$', for the logic concerned. Importantly, the range of logics for which this pair of metavaluations will work is significantly reduced from that of Meyer in (1976). Such logics are to be called 'metacomplete logics'.

Slaney's metavaluations are as follows, with my above terminology, a slightly simplified layout, and a small v:

(i) $v(p) = F$; $v^*(p) = T$, for sentential variables p.
(ii) $v(A\&B) = T$ iff $v(A) = T$ and $v(B) = T$;
$v^*(A\&B) = T$ iff $v^*(A) = T$ and $v^*(B) = T$.
(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$;
$v^*(A \vee B) = T$ iff $v^*(A) = T$ or $v^*(B) = T$.
(iv) $v(\sim A) = T$ iff $v^*(A) = F$;
$v^*(\sim A) = T$ iff $v(A) = F$.
(v) $v(A \rightarrow B) = T$ iff $\vdash A \rightarrow B$, if $v(A) = T$ then $v(B) = T$, and if $v^*(A) = T$ then $v^*(B) = T$.
$v^*(A \rightarrow B) = T$, for M1-logics.
$v^*(A \rightarrow B) = T$ iff, if $v(A) = T$ then $v^*(B) = T$, for M2-logics.

Note that Slaney's metavaluation v is like Meyer's quasi-valuation V', but (legitimately) drops $\vdash\sim A$ from the valuation of $v(\sim A) = T$. Slaney's metavaluation will be the primary focus in what follows.

The distinction between M1- and M2-logics is made on pp. 396–397 of Slaney's (1987) article, and will divide the metacomplete logics into two types: M1-metacomplete and M2-metacomplete. He considers the following axioms and rules, upon which M1- and M2-logics are defined:

Axioms:

1. $A \rightarrow A$.
2. $A\&B \rightarrow A$.

3. $A\&B \to B$.
4. $(A{\to}B)\&(A{\to}C) \to .A \to B\&C$.
5. $A \to A{\vee}B$.
6. $B \to A{\vee}B$.
7. $(A{\to}C)\&(B{\to}C) \to .A{\vee}B \to C$.
8. $A\&(B{\vee}C) \to (A\&B) \vee (A\&C)$. (Slaney has '$\ldots \vee C$'.)
9. $\sim\sim A \to A$.

Rules:

1. $A, A{\to}B \Rightarrow B$.
2. $A, B \Rightarrow A\&B$.
3. $A{\to}B, C{\to}D \Rightarrow B{\to}C \to .A{\to}D$.
4. $A{\to}{\sim}B \Rightarrow B{\to}{\sim}A$.

The above axioms and rules give us the basic system B of Routley-Meyer. The system B can be weakened significantly for the results to go through, as Meyer in (1976) considers, but we think B is basic enough for the time being. Nevertheless, we will later need to remove the distribution axiom, A8.

We give additional axioms and rules for the construction of stronger logics:

Additional axioms:

10. $A{\to}{\sim}B \to .B{\to}{\sim}A$.
11. $A{\to}B \to .C{\to}A \to .C{\to}B$.
12. $A{\to}B \to .B{\to}C \to .A{\to}C$.
13. $(A{\to}B)\&(B{\to}C) \to .A{\to}C$.
14. $A \to .A{\to}A$.
15. $A \to .B{\to}A$.
16. $(A \to B{\vee}C)\&(A\&B \to C) \to .A{\to}C$.
17. $A \to .A{\to}B \to B$.
18. $(A \to .B{\to}C) \to .B \to .A{\to}C$.
19. $A\&(A{\to}B) \to B$.

Additional rules:

5. $A \Rightarrow B{\to}A$.
6. $A \Rightarrow A{\to}B \to B$.
7. $A \Rightarrow {\sim}(A{\to}{\sim}A)$. [Or, equivalently, $A, {\sim}B \Rightarrow {\sim}(A{\to}B)$.[2]] (R7 is added on for M2-logics. See also the last paragraph of Section 2.)
8. ${\sim}A, A{\vee}B \Rightarrow B$. [Added for future reference.]

Some familiar logics:

DW: B + A10.
TW: DW + A11 + A12.
DJ: DW + A13.
TJ: TW + A13.
EW: TW + R6.
RW: TW + A17.
RWK (or BCK): RW + A15 (or R5).

[2]This equivalence is a so-called 'deductive equivalence', which not only allows for the use of axioms and rules (and meta-rules) in its establishment, but also the use of the uniform substitution rule applied to sentential variables.

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R: RW + A19 (or A13). [N.B. R is not metacomplete in either sense.]
RM: R + A14. [RM is not metacomplete either.]

An M1-*logic* is any logic axiomatizable as B, plus zero or more of A10-16, R5. [So, TJ + A15 + A16 is the maximal M1-logic formed from the above axioms and rules.]
An M2-logic is one axiomatizable as B plus R7, plus zero or more of A10-12,14-15,17-18,R5-6. (Slaney uses R6 here instead of my weaker R7.) [So, RWK is the maximal M2-logic formed from the above.]
[N.B. A19 is not included in either metacomplete logic, nor is the LEM, $A\vee\sim A$.]

As in Slaney (1984), it can be shown for M1- and M2-logics that if $v(A) = T$ then $\vdash A$ and if $v^*(A) = F$ then $\vdash\sim A$ (completeness theorem), by induction on formulae, and conversely if $\vdash A$ then $v(A) = T$ (soundness theorem), by induction on proof steps. The consistency theorem in the form, if $v(A) = T$ then $v^*(A) = T$, can easily be shown, by induction on formulae. For some cases in the proof of the soundness theorem, completeness and/or consistency is needed for soundness to be proved, and thus completeness is done first, followed by consistency. A logic is called *metacomplete* when both these soundness and completeness theorems are shown. With use of metavaluation condition (iii), the Priming Property, if $\vdash A\vee B$ then $\vdash A$ or $\vdash B$, can be shown. And, by (v), the negated entailment properties, not-$\vdash\sim(A\to B)$ (for M1-logics) and $\vdash\sim(A\to B)$ iff $\vdash A$ and $\vdash\sim B$ (for M2-logics), can easily be shown.

For the quantified logics, we separate the free and bound variables, for ease of operation:
a,b,c, ... range over free variables.
x,y,z, ... range over bound variables.
Terms s,t,u, ... can be individual constants (when introduced) or free variables, and just $\forall$ and $\exists$ are primitive.
We just add the standard axioms and rules that apply to any of the sentential systems:
Quantificational axioms:

1. $\forall xA \to A^t/x$, for any term t.
2. $\forall x(A\to B) \to .A \to \forall xB$.
3. $\forall x(A\vee B) \to A\vee\forall xB$.
4. $A^t/x \to \exists xA$, for any term t.
5. $\forall x(A\to B) \to .\exists xA \to B$.
6. $A\&\exists xB \to \exists x(A\&B)$.

Quantificational rule:

1. $A^a/x \Rightarrow \forall xA$, where a is not free in A.

As in Meyer (1976), the above axioms and rules can be weakened considerably for metacompleteness to still apply. Later, we will consider the removal of the distribution axioms QA3 and QA6, together with the sentential distribution A8.

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The additional conditions for metavaluations v are:

(vi) $v(\forall xA) = T$ iff $v(A^t/x) = T$, for all terms t;
(vii) $v(\exists xA) = T$ iff $v(A^t/x) = T$, for some term t.

The completeness, consistency and soundness results can be shown in a similar manner to before, and these will yield, using (vii), the Existential Property: if $\vdash \exists xA$ then $\vdash A^t/x$, for some term t.

§2. Examining metacomplete logics. Before going further with applications of metavaluations, we should examine the particular features of these logics. The Priming Property, $\vdash A \vee B$ then $\vdash A$ or $\vdash B$, ensures that these logics are *disjunctively constructive*, which is appropriate for logics based on proof. That these logics are based on proof is seen from the soundness and completeness theorems, yielding $v(A) = T$ iff $\vdash A$. Note that there is no variety of models used here, though we did see this with Meyer's earlier concept of coherence in (1971).

Where these logics differ from truth-based models is that the Priming Property does not in general extend to the interiors of $\to$-formulae, of form $A \to B$, i.e., not to the A and not to the B. However, there are exceptions for formulae of form $A \to B \vee C$, which, with application of Modus Ponens, becomes $A \Rightarrow B \vee C$, whereupon either B or C must be established upon the assumption of A. In the case of distribution, this antecedent A is itself disjunctive, in the form of $D \& (E \vee F)$, and upon choice of disjunct, either the B or the C can be proved. [However, with the contraposed Modus Ponens Axiom, $\sim B \to \sim A \vee \sim (A \to B)$, this creates a problem, which is indeed why A19 is neither an M1- nor an M2-logic.] These exceptions can be clearly extended to formulae of the form, $A_1 \to .A_2 \to \ldots . A_n \to B \vee C$.

These logics are also *existentially constructive* in that the Existential Property holds for these logics with the above addition of quantifiers. Again, because these logics are based on proof, this is appropriate. The above point about the interiors of $\to$-formulae can be made here as well.

Due to metavaluation condition (v), these logics have an entailment focus, since all their theorems are inductively built up from provable entailments, together with sentential variables. So, the entailments are seen to be like atoms, though the same point can be made for the conclusions of their rule-forms. Indeed, it is reasonable to say that metacomplete logics are entailment logics and further that a good entailment logic ought to be metacomplete. [See below for this latter point, where the logic MC is presented.]

Regarding negation, it can be immediately seen that the LEM fails in general, since by the Priming Property, $\vdash A \vee \sim A$ only when either $\vdash A$ or $\vdash \sim A$. Also, the DS, $\sim A, A \vee B \Rightarrow B$, depends on the consistency of the formula A, as does $A \to .B \to A$. So, the two main characteristics of classical logic, i.e., the LEM and the DS, are absent, at least initially, from metacomplete logics. Since, given the logic B, the LEM is deductively equivalent[2] to both the rules, $A \to \sim A \Rightarrow \sim A$ and $A \to B \Rightarrow \sim A \vee B$, the $\to$-forms, $A \to \sim A \to \sim A$ and $A \to B \to \sim A \vee B$, are also not included. Further, reductio arguments in the form $A \to B \& \sim B \Rightarrow \sim A$ are not included, as $A \to B \& \sim B$ is deductively equivalent to $B \vee \sim B \rightarrow \sim A$. In terms of main negation properties, that leaves double negation, $A \leftrightarrow \sim\sim A$, contraposition, $A \rightarrow B \leftrightarrow .\sim B \rightarrow \sim A$, and hence the De Morgan's Laws, $\sim(A\&B) \leftrightarrow \sim A \vee \sim B$ and $\sim(A \vee B) \leftrightarrow \sim A \& \sim B$, all of which can be included in metacomplete logics. So, negation is essentially De Morgan, with the Boolean properties LEM and DS added as appropriate. Regarding negated entailments, these are of course determined according to whether the logic is an M1- or M2-logic. For M1-logics, negated entailments $\sim(A \rightarrow B)$ are never theorems, whilst for M2-logics they must all be built up from the theorems A and $\sim B$.

We can take the logic MC of meaning containment (see below) as a good example of an entailment logic which is M1-metacomplete. Importantly, an entailment needs a concept such as meaning containment to provide substance to it and in turn provide guidance in its application. Further, meaning containment provides justification for entailments that have traditionally been expressed as a necessitated implication.

Historically, the logic DJ^d was introduced and supported in Brady (1996) and further in (2006). [DJ^d is DJ with the addition of the disjunctive meta-rule, if $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$.] This logic was modified over time by the removal of the sentential distribution axiom A8, which was essentially replaced by its rule form, achieved by replacing the above single-premise meta-rule by the two-premise meta-rule, if $A, B \Rightarrow C$ then $D \vee A, D \vee B \Rightarrow D \vee C$ (MR1). This gives us the logic MC of meaning containment, as concluded in Brady and Meinander (2013).

For the quantified logic MCQ, we add the axioms QA1-2,4-5 and rule QR1, removing the quantificational distribution axioms QA3 and QA6. Then, we add the existential version, if $A, B^a/x \Rightarrow C^a/x$ then $A, \exists xB \Rightarrow \exists xC$ (QMR1), of the sentential two-premise meta-rule. This and the corresponding sentential meta-rule have the proviso that QR1 does not generalize on any free variable in the premises of the antecedent rules, $A, B \Rightarrow C$ and $A, B^a/x \Rightarrow C^a/x$, of the respective meta-rules.

This enables us to make an observation about M2-logics. The key difference between M1- and M2-logics is the presence of the rule, $A \Rightarrow \sim(A \rightarrow \sim A)$, or equivalently, $A, \sim B \Rightarrow \sim(A \rightarrow B)$ in the M2-logics. Once metacompleteness is shown then the converse: if $\vdash \sim(A \rightarrow B)$ then $\vdash A$ and $\vdash \sim B$, also holds admissibly. Observe that, using the two-premise disjunctive meta-rule MR1, we can show that the rule $A, \sim B \Rightarrow \sim(A \rightarrow B)$ is derivable, given that the LEM holds for $A \rightarrow B$ and the DS holds for B. To see this, we start with Modus Ponens, $A, A \rightarrow B \Rightarrow B$, and apply MR1 to obtain: $\sim(A \rightarrow B) \vee A$, $\sim(A \rightarrow B) \vee (A \rightarrow B) \Rightarrow \sim(A \rightarrow B) \vee B$. Given A and the LEM, $\sim(A \rightarrow B) \vee B$, and then, given $\sim B$ and the DS, $\sim (A \rightarrow B)$ follows. So, the rule that M2-logics depend on for their definition is provable using specific application of the basic classical properties, the LEM and the DS. And, one can conclude that the M1-logics are purer as entailment logics, whilst the M2-logics involve some classical features not standardly present in such logics.

§3. Applications of metavaluations. Whilst metavaluations can be rather straightforwardly applied to M1- and M2-logics, their inductive presentation can be tweaked in quite a variety of ways to yield some interesting results. We briefly present a number of these, which are inclusive of all applications known to the author, but are by no means exhaustive with regard to their possible uses. (However, see note 1.) We show how metavaluations can be applied (i) to prove the admissibility of γ, which is essentially the DS, in relevant logics, (ii) to prove that every metacomplete logic is characterized by its reduced frames, (iii) to prove that the Cut Rule holds for classical logic, (iv) to show negation completeness for Strenge Peano Arithmetic, (v) to provide a rejection system for first-degree formulae, (vi) to help distinguish cancellation negation from its noncancellation properties, (vii) to prove the simple consistency of Peano arithmetic by finitary methods, and (viii) to prove the simple consistency and also the nontriviality of naive set theory.

This sequence of applications is roughly in chronological order, determined by the lead article when several articles are involved. We group together articles according to their topic and methodology.

3.1. The admissibility of γ in relevant logics. This method uses metavaluations to enhance key aspects of a canonical model standardly used to prove completeness of a logic with respect to its standard model-theoretic semantics. In this respect, it is unlike its other usages, but does represent a type of usage that has prospects for further development. However, even here, canonical modelling is set up in proof-theory, and so this is not so divergent from the other proof-theoretic methodology.

Reporting on Meyer's proof and methods in his (1976a), Dunn firstly in (1986), and secondly in Dunn and Restall (2002), sets out a prime regular R-theory T (see below for definition) such that an arbitrary nontheorem A of the logic R is not derivable in R from the formulae in T. This is called the Way Up Lemma and is based on Henkin's method, which uses an enumeration of the formulae. Then, in order to show that there is a normal prime regular R-theory T′ contained in T, a metavaluation v is introduced for such an R-theory T′:

(i) $v(p) = T$ iff $p \in T$, for all sentential variables p.
(ii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.
(iii) $v({\sim}A) = T$ iff ${\sim}A \in T$ and $v(A) = F$.
(iv) $v(A \to B) = T$ iff $A \to B \in T$ and if $v(A) = T$ then $v(B) = T$.

[$v(A\&B) = T$ follows by the De Morgan definition of &.]

One can see that this metavaluation v, with regard to the connectives $\sim$ and $\to$, picks up the feature of Meyer's quasi-valuation V′ of Section 1. Indeed, this is a hybrid between a classical valuation and Meyer's quasi-valuation.

Then, by the usual formula induction, the completeness theorem: if $v(A) = T$ then $A \in T$, and if $v(A) = F$ then ${\sim}A \in T$, was proved. It can then be shown that the metavaluation v is consistent and negation-complete in the respective forms: if $v({\sim}A) = T$ then $v(A) = F$, and if $v(A) = F$ then $v({\sim}A) = T$. By putting $T' = \{A: v(A) = T\}$, the completeness theorem shows that T′ is contained in T and, by this consistency and negation-completeness, T′ is normal. Also, T′ is prime, due to (ii). It remains to show that T′ is a regular R-theory, which is done by checking the axioms and rules of R to show that all theorems of R are in T′, together with the R-theory condition: if $\vdash_R A \rightarrow B$ and $A \in T'$ then $B \in T'$. This yields what is called the Way Down Lemma which, together with the Way Up Lemma, gives us the admissibility of γ (A, ∼A∨B ⇒ B) for the logic R, for which the normality of T′ is a key component, as sketched on pp. 35–36 of their (2002). This is a quicker and neater proof of the admissibility of γ for R than appears in Meyer and Dunn (1969).

What is unusual here, in light of Slaney's work, is that metavaluations are used for a logic such as R containing the LEM. Note that there is no use of Slaney's ⋆-metavaluation, but there is a reason why the classical negation is used. It is because normality is what we are proving here, which is essentially classical negation for T′, and the metavaluation is set up to encapsulate in T′ what we are trying to prove.

Mares and Meyer in (1992) also adopt this method to prove the admissibility of γ for the modal logic R4, which consists of the relevant logic R of Anderson and Belnap (1975), plus the following axioms concerning □, with the definition, ◊A =df ∼□∼A:

1. □A → A.
2. □A → □□A.
3. □(A → B) → .□A → □B.
4. □A & □B → □(A&B).
5. □(A∨B) → ◊A ∨ □B.
6. If A is an axiom then □A is an axiom.

The logic NR of Meyer's (1968) consists of R, plus the above axioms 1-4,6, yielding a necessitated R. However, NR does not contain the modal logic S4, which is standardly used in the capture of entailment as a necessitated implication. The axiom 5 was then added to this logic NR, yielding R4, which enables the modal logic S4 to be contained therein (hence, the symbolism 'R4').

Mares and Meyer use Meyer's above metavaluation v, replacing his T by a prime theory T_i of R4 such that $\Box^{-1}T \subseteq T_i \subseteq \Diamond^{-1}T$, where T is an arbitrary prime theory, $\Box^{-1}T$ =df {A: □A ∈T} and $\Diamond^{-1}T$ =df {A: ◊A ∈T}. They call their metavaluation v_i (for T_i), adding the valuation for □:

$v_i(\Box A) = T$ iff $\forall j$(if ST_iT_j then $v_j(A) = T$), where S is the accessibility relation between prime theories T_i and T_j.

This accessibility relation ensures that the axioms 1-6 are sound. The remainder of the proof is similar that of Meyer (1976a).

Seki in (2011) takes Meyer's process a step further to establish the admissibility of γ for a wider range of relevant modal logics L. He introduces the following metavaluation v as a two-place function defined on formulae and regular prime L-theories s, where L is one of a wide range of relevant modal logics, sententially containing the logic B together with the LEM and the rule, C∨A ⇒ C∨∼(A→∼A).

(i) v(p,s) = T iff p ∈ s, for all sentential variables p.
(ii) v(A&B,s) = T iff v(A,s) = T and v(B,s) = T.

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(iii) $v(A \vee B,s) = T$ iff $v(A,s) = T$ or $v(B,s) = T$.
(iv) $v(\sim A,s) = T$ iff $\sim A \in s$ and $v(A,s) = F$.
(v) $v(A \rightarrow B,s) = T$ iff $A \rightarrow B \in s$ and if $v(A,s) = T$ then $v(B,s) = T$.
(vi) $v(\Box A,s) = T$ iff $\Box A \in s$ and $v(A,s) = T$.
(vii) $v(\Diamond A,s) = T$ iff $\Diamond A \in s^\star$ or $v(A,s) = T$, where $s^\star$ is defined as $\{A: \sim A \notin s\}$.

Seki defines Tr(s) as $\{A: v(A,s) = T\}$ and subsequently proves that:

(1) For any regular prime L-theory s, $s^\star \subseteq Tr(s) \subseteq s$.
(2) If $A \in Tr(s)$, for all regular prime L-theories s, then A is a theorem of L.
(3) For any regular prime theory s, $\sim A \in Tr(s)$ iff $A \notin Tr(s)$.
(4) If A is a theorem of L then, for all regular prime L-theories s, $A \in Tr(s)$.

He goes on to prove the admissibility of γ for any such logic L, which includes the purely sentential logics without modal operators.

3.2. The characterization of metacomplete logics by reduced frames. We now return to more standard usages of metavaluations, based on Slaney's M1- and M2-logics. In his (1987), he generalizes his treatment of metavaluations, using this to ensure that each M1- and M2-logic can be characterized by its reduced Routley-Meyer frames.

We start this generalization for a logic L, which is M1- or M2-metacomplete and contains the basic logic B. Let A' be a nontheorem of L. Instead of the set of theorems, Slaney considers a regular L-theory T, closed under the rules of L, with A' not in T. (An L-theory satisfies the Adjunction rule and preservation of provable entailments, and a regular L-theory is an L-theory containing all the theorems of L.) Such a theory T can be constructed using the Henkin enumeration of formulae, as set out on pp. 399–400 of Slaney's (1987).

Slaney then sets out two pairs of metavaluations, v_1 and $v_1{}^\star$, and v_2 and $v_2{}^\star$, the first pair v_1 and $v_1{}^\star$ being for M1-logics and v_2 and $v_2{}^\star$ being for M2-logics. These metavaluations are set up in such a way as to replace provability in L by what is essentially the membership of T, and are set out as follows: [We use v when v_1 and v_2 are the same, to save space.]

(i) $v(p) = T$ iff $p \in T$; $v^\star(p) = T$ iff $p \in T$ or $\sim p \notin T$, for sentential variables p.
(ii) $v(A\&B) = T$ iff $v(A) = T$ and $v(B) = T$;
$v^\star(A\&B) = T$ iff $v^\star(A) = T$ and $v^\star(B) = T$.
(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$;
$v^\star(A \vee B) = T$ iff $v^\star(A) = T$ or $v^\star(B) = T$.
(iv) $v(\sim A) = T$ iff $v^\star(A) = F$ and $v(A) = F$ and $\sim A \in T$;
$v^\star(\sim A) = T$ iff $v(A) = F$.
(v) $v(A \rightarrow B) = T$ iff $A \rightarrow B \in T$, if $v(A) = T$ then $v(B) = T$, and if $v^\star(A) = T$ then $v^\star(B) = T$.
$v_1{}^\star(A \rightarrow B) = T$.
$v_2{}^\star(A \rightarrow B) = T$ iff, if $v_2(A) = T$ then $v_2{}^\star(B) = T$.

Then, Slaney proves a series of lemmas and corollaries, as follows:

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If $v_1(A) = T$ then $A \in T$, and if $v_2(A) = T$ then $A \in T$. (Completeness)

If $v_1(A) = T$ then $v_1{}^*(A) = T$, and if $v_2(A) = T$ then $v_2{}^*(A) = T$. (Consistency)

If $v_1{}^*(A) = F$ then $\sim A \in T$, and if $v_2{}^*(A) = F$ then $\sim A \in T$, leading to:

$v_1(\sim A) = T$ iff $v_1{}^*(A) = F$, and $v_2(\sim A) = T$ iff $v_2{}^*(A) = F$, which gets the metavaluations of negation back to the earlier shape.

$\{A{:}v_1(A) = T\}$ and $\{A{:}v_2(A) = T\}$ are then regular, prime and consistent L-theories, closed under the rules of L. (Soundness)

These true metavaluations enable the addition of primeness and consistency to the properties of T. Now, $\{A{:}\ v_1(A) = T\}$ and $\{A{:}\ v_2(A) = T\}$ can be used to construct a reduced L-model falsifying A', by the method used in the completeness proof for the Routley-Meyer semantics (see Routley, Meyer, Plumwood and Brady (1982), Chapter 4).

Hence, every M1- and M2-logic can be characterized by its reduced frames, i.e., those with a single base world which verifies the theorems of L.

3.3. The proof of the cut rule for classical logic. Dunn and Meyer in (1989) used a metavaluation to prove Gentzen's Cut Rule in the form: if $M \vee A$ and $\sim A \vee N$ are theorems then so is $M \vee N$. This application of metavaluations is even more unusual in that it is used for classical logic itself, with the metavaluation actually being an instance of a classical valuation.

For a prime, rich theory T, they define its metavaluation v:

(i) $v(A) = T$ iff $A \in T$, for atomic formulae A.
(ii) $v(\sim A) = T$ iff $v(A) = F$.
(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.
(iv) $v(\exists x A) = T$ iff $v(A^a/x) = T$, for some free variable a.

They follow the Gentzen separation of free and bound variables:

a, b, c, . . . free variables.

x, y, z, . . . bound variables.

Note that T is a theory if it is closed under deducibility in classical logic, and such a theory T is rich iff, whenever $\sim A^a/x \in T$, for all free variables a, then $\sim \exists x A \in T$. Note too that there is no need to conjoin $\sim A \in T$ in (ii), this then yielding a classical valuation but with the atoms evaluated in accordance with their membership of the theory T.

As in Section 3.1 above, T is constructed in their Way Up Lemma using Henkin's method, and its metavaluation v is used to determine a subset of T satisfying normality. This is encapsulated by the Completeness Lemma in the form: if $v(A) = T$ then $A \in T$, and if $v(A) = F$ then $\sim A \in T$, for such a metavalution v and theory T. However, the Soundness Lemma, if $\vdash A$ then $v(A) = T$, was proved for all valuations v, in the standard classical sense of valuation. Putting all this together, they prove the Soundness and Completeness Theorem, $\vdash A$ iff $v(A) = T$, for all metavaluations v of rich prime theories T. They are then able, using (ii) and (iii), to prove the Cut Theorem: if $M \vee A$ and $\sim A \vee N$ are theorems of classical logic, formulated without Cut, then so is $M \vee N$. This again provides a much quicker and neater proof of the admissibility of Cut for classical logic than Gentzen's original proof.

3.4. Negation completeness for Strenge Peano Arithmetic. Meyer and Restall in (1999) use metavaluations to establish the Priming Property in the form of negation completeness for sentences of Peano Arithmetic based on the logic E (called E#), but with the addition of the infinitary ω-rule. This extended system is called E##. They then extend E## to TE#, which is what they call *strenge true arithmetic*, by ensuring negation completeness for the sentences of E## in the process. TE# is established from E## by using metavaluations that are similar to that used by Meyer in (1971), in that they both employ provability for the intensional connective/operator, leaving the remaining connectives (and quantifier) with classical valuations. However, here, the metavaluation for $\rightarrow$ has the effect of Meyer's quasi-valuation from his (1976).

We set out their metavaluation V for arithmetic sentences as a set TR of truths, as follows:

VAt If A is an atomic sentence u=v, then A $\in$ TR iff A is arithmetically correct.

V$\sim$ $\sim$B $\in$ TR iff B $\notin$ TR.
V& B&C $\in$ TR iff B $\in$ TR and C $\in$ TR.
V$\vee$ B$\vee$C $\in$ TR iff B $\in$ TR or C $\in$ TR.
V$\forall$ $\forall$xBx $\in$ TR iff, for all numerals n, Bn $\in$ TR.
V$\rightarrow$ B$\rightarrow$C $\in$ TR iff both E##$\vdash$ B$\rightarrow$C and if B $\in$ TR then C $\in$ TR.

By letting TE# be the system TR of true sentences on V, they then go on to prove their soundness theorem for E##:

E# $\subseteq$ E## $\subseteq$ TE#.

This establishes the result they aimed for. And, they were also able to prove the admissibility of γ for TE#, through use of this metavaluation.

3.5. A rejection system for first-degree formulae. In Brady (2008), we determine a rejection system for the logic L_1, which is the first-degree entailment system E_{fde} of Anderson and Belnap (1975), together with the following NonEntailment Rules:

1. A, B $\Rightarrow$ A&B.
2. A $\Rightarrow$ A$\vee$B.
3. B $\Rightarrow$ A$\vee$B.
4. A $\Rightarrow$ $\sim\sim$A.
5. $\sim$A $\Rightarrow$ $\sim$(A&B).
6. $\sim$B $\Rightarrow$ $\sim$(A&B).
7. $\sim$A, $\sim$B $\Rightarrow$ $\sim$(A$\vee$B).

As such, L_1 is the *first-degree formula fragment* of B, DW and DJ, and essentially differs from Anderson and Belnap's E_{fdf} (see Section 19 of (1975)) in that the LEM and negated entailments are absent. L_1 is M1-metacomplete, with metacompleteness properties:

(I) If $\vdash$A$\vee$B then $\vdash$A or $\vdash$B.
(II) Not - $\vdash\sim$(A$\rightarrow$B).
(III) If $\vdash$A&B then $\vdash$A and $\vdash$B.
(IV) If $\vdash\sim\sim$A then $\vdash$A.

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(V) If $\vdash\sim(A\&B)$ then $\vdash\sim A \vee \sim B$.
(VI) If $\vdash\sim(A\vee B)$ then $\vdash\sim A \;\&\; \sim B$.

As in Brady (2008), we consider properties (I)–(VI) as an axiom and rules for the purpose of framing the rejection axiom RA2 and rejection rules RR3-9, in what follows:
Rejection-Axioms.

1. $\dashv p\&{\sim}q\&r\&{\sim}r \to \sim p\vee q\vee s\vee{\sim}s$.
2. $\dashv \sim(A\to B)$.

Rejection Rules.

1. $\vdash A\to B,\ \dashv A\to C \Rightarrow\ \dashv B\to C$.
2. $\vdash B\to C,\ \dashv A\to C \Rightarrow\ \dashv A\to B$.
3. $\dashv A,\ \dashv B \Rightarrow\ \dashv A\vee B$.
4. $\dashv A \Rightarrow\ \dashv A\&B$.
5. $\dashv B \Rightarrow\ \dashv A\&B$.
6. $\dashv A \Rightarrow\ \dashv {\sim}{\sim}A$.
7. $\dashv {\sim}A \Rightarrow\ \dashv \sim(A\vee B)$.
8. $\dashv {\sim}B \Rightarrow\ \dashv \sim(A\vee B)$.
9. $\dashv {\sim}A,\ \dashv {\sim}B \Rightarrow\ \dashv \sim(A\&B)$.
10. $\dashv A^{B}/p \Rightarrow\ \dashv A$, where B is uniformly substituted for p in A.

We call this rejection logic L_{1r}.
Some Key Rejection Theorems.

1. $\dashv p$.
2. $\dashv {\sim}p$.

Following (2008), we start with the following R-soundness theorem, and then proceed to prove the R-completeness theorem using R-metavaluations and the R-metacompleteness theorem.
R-*Soundness Theorem*:
For all formulae A, if $\vdash A$ then not-$\dashv A$.
Thus, each theorem of L_1 is not provable in the rejection system L_{1r}.
We introduce the following R-metavaluations v_r and $v_r{}^{\star}$:

(i) $v_r(p) = T$; $v_r{}^{\star}(p) = F$.
(ii) $v_r(A\&B) = T$ iff $v_r(A) = T$ or $v_r(B) = T$;
$v_r{}^{\star}(A\&B) = T$ iff $v_r{}^{\star}(A) = T$ or $v_r{}^{\star}(B) = T$.
(iii) $v_r(A\vee B) = T$ iff $v_r(A) = T$ and $v_r(B) = T$;
$v_r{}^{\star}(A\vee B) = T$ iff $v_r{}^{\star}(A) = T$ and $v_r{}^{\star}(B) = T$.
(iv) $v_r({\sim}A) = T$ iff $v_r{}^{\star}(A) = F$;
$v_r{}^{\star}({\sim}A) = T$ iff $v_r(A) = F$.
(v) $v_r(A\to B) = T$ iff $\dashv A\to B$.
$v_r{}^{\star}(A\to B) = F$.

R-*metacompleteness Theorem.*
For all formulae A, $v_r(A) = T$ iff $\dashv A$.
This yields a number of properties below, which will enable the axiomatization of L_{1r} to be enhanced.

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(I_r) If $\dashv$ A&B then $\dashv$ A or $\dashv$ B.
(II_r) If $\dashv$ A∨B then $\dashv$ A and $\dashv$ B.
(III_r) If $\dashv$ ∼∼A then $\dashv$ A.
(IV_r) If $\dashv$ ∼(A&B) then $\dashv$ ∼A∨∼B.
(V_r) If $\dashv$ ∼(A∨B) then $\dashv$ ∼A&∼B.

These are admissible rules of L_{1r} and as such we can use them in addition to the primitive rules RR1-10. These rules enable us to use normal forms which, together with the tautological entailments from Anderson and Belnap (1975), yield the proof of:

R-*Completeness Theorem.*

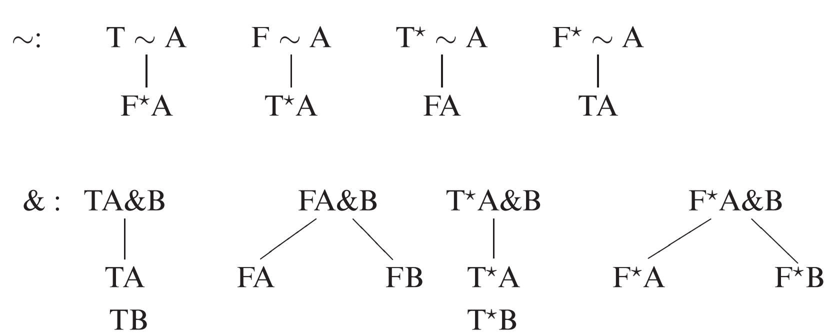
For all formulae A, if not-$\vdash$A then $\dashv$ A.

Thus, each nontheorem of L_1 is provable in the above rejection system L_{1r}.

3.6. Distinguishing cancellation negation. In Brady (2008a), a tree method is used to distinguish cancellation negation by isolating the negation properties all of whose negations cancel out when brought together in an unpacking of the components of the formula. This unpacking takes the form of a tree constructed from the formula with help from the inductive process of metavaluation appropriate to the system of logic involved. It suffices to say that the logic must be metacomplete to start with, of either of the two types.

To set out the tree system for M1- and M2-logics, we place T, T*, F or F* in front of each formula A according to whether v(A) = T, v*(A) = T, v(A) = F, or v*(A) = F, respectively. As for classical trees, we use a reductio argument and assume that the formula under consideration takes the v metavaluation to F, with the aim of showing that each branch closes by having a T and F or a T* and F* in front of the same formula within each branch. In this case, we will say that the M-tree (as they are to be called) *closes*. If each of the metavaluation conditions holds for theorems of a metacomplete logic to start with, this method should work, as the theorems of form $\vdash$A→B occurring in metavaluation for A→B will be theorems anyway, since the theorems of metacomplete logics are all inductively built from such entailment theorems. The upshot is that any negation property expressed as a theorem of such a logic must have this cancellation property, and any such negation is essentially a cancellation negation.

We set out the tree rules as follows:



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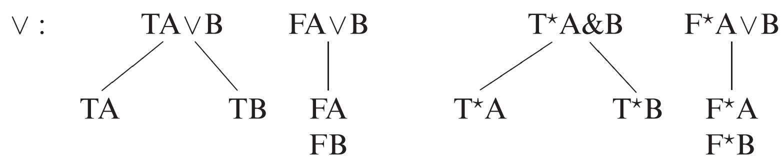
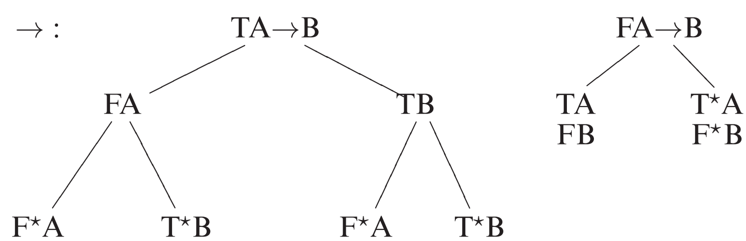


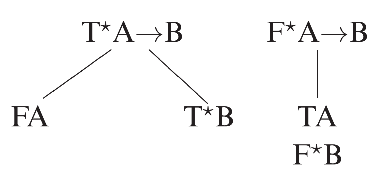


For M1-logics:

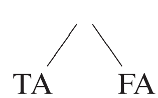
T*A→B
Redundant, since true.

F*A→B
x
Closure, since false.

For M2-logics:



For the sake of the logical rules, we add the following tree rule, to enable a T to be placed in front of each of the premises of each of the logical rules, given that each such premise is closed when F is placed in front:



THEOREM. *The* M-*trees of the theorems of the* M1-*logic* TJ (*and hence all weaker logics*) *are all closed.*

THEOREM. *The* M-*trees of the theorems of the* M2-*logic* RW (*and hence all weaker logics*) *are all closed.*

The negation picked out by this method is De Morgan, determined by double negation and contraposition, with De Morgan's Laws following. However, for M2-logics only, the rule A ⇒ ∼(A→∼A) is included as well. The LEM, A∨∼A, fails, as it should, as does A→∼ A → ∼A and (A → B&∼B) → ∼A, from which the LEM is derivable. The DS also fails, as it should. Both the LEM and the DS are what might be called single negations, as there is no opportunity for these negations to cancel out.

3.7. The simple consistency of Peano Arithmetic using finitary methods. This is a complex proof, but we will pick out the key details from Brady (2012). Though the proof works for a range of metacomplete logics, we focus on the logic of meaning containment MCQ, further reducing it to MCQ–, with the quantificational extension below:

$\exists xA =_{df} \sim\forall x\sim A$.

Quantificational Axioms:

1. $\forall xA \rightarrow A^t/x$, for any term t.

2′. $A \rightarrow \forall xA$. (Note that x cannot occur free in A.)

Quantificational Rule:

1. $A^a/x \Rightarrow \forall xA$, where a is not free in A.

Quantificational Meta-Rule.

1. If A, $B^m/x \Rightarrow C^m/x$ then A, $\exists xB \Rightarrow \exists xC$, with the proviso that, in the derivation A, $B^m/x \Rightarrow C^m/x$, QR1 cannot generalize on any variable free in either of the premises A and B^m/x. Similarly, MR1 is subject to the same proviso concerning the derivation A, B $\Rightarrow$ C.

[m,n,. . . are individual constant (i.e., natural number) schemes.]

We now present MC#, based on MCQ–, with the addition of =, ′, +, x:

MC#.

Identity Axioms.

1. a=a.
2. a=b $\rightarrow$ b=a.
3. a=b&b=c $\rightarrow$ a=c.

Identity Rule.

1. s=t, A(s) $\Rightarrow$ A(t), where, for terms s and t, t is substituted for s in a single argument place.

Number-theoretic Axioms.

1. $\sim a' = 0$.
2. $a + 0 = a$.
3. $a + b' = (a+b)'$.
4. a x 0 = 0.
5. a x b′ = (axb) + a.

Number-theoretic Rules.

1. $s=t \Rightarrow s'=t'$.
2. $s'=t' \Rightarrow s=t$.
3. $\sim s=t \Rightarrow \sim s'=t'$.
4. $\sim s'=t' \Rightarrow \sim s=t$.

Number-theoretic Meta-Rule.

If $A(m) \Rightarrow A(m')$ then $A(0) \Rightarrow A(t)$, where t is a term, i.e., an (arbitrary) numerical constant or variable. [Mathematical Induction]

Classicality Axiom.

1. a=b $\vee$ $\sim$a=b. [The LEM for a=b.]

We then introduce the following metavaluations v and v*:

(i) v(s=t) = T iff s=t is a theorem of MC#, for *constant terms* s and t. $v^*(s=t) = v(s=t)$, for *constant terms* s and t. [Ensuring classicality of s=t]

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For the following metavaluations (ii)–(v), we let A and B be sentences.

(ii) $v(A\&B) = T$ iff $v(A) = T$ and $v(B) = T$.
$v^*(A\&B) = T$ iff $v^*(A) = T$ and $v^*(B) = T$.

(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.
$v^*(A \vee B) = T$ iff $v^*(A) = T$ or $v^*(B) = T$.

(iv) $v(\sim A) = T$ iff $v^*(A) = F$.
$v^*(\sim A) = T$ iff $v(A) = F$.

(v) $v(A \to B) = T$ iff $A \to B$ is a theorem of MC#, if $v(A) = T$ then $v(B) = T$, and if $v^*(A) = T$ then $v^*(B) = T$.
$v^*(A \to B) = T$.

We add the following metavaluations v and v* to account for the quantifier ∀, where ∀xA is a sentence and A^n/x is a constant instance of A, obtained by substituting the numerical constant scheme n for x in A.

(vi) $v(\forall xA) = T$ iff $v(A^n/x) = T$, for all numerical constants n, *recursively generated*. (see later)
$v^*(\forall xA) = F$ iff $v^*(A^n/x) = F$, for some numerical constant n, *recursively determined*. (see later)
In the case of vacuous quantification,
$v(\forall xA) = T$ iff $v(A) = T$ and $v^*(\forall xA) = F$ iff $v^*(A) = F$.
Then, to take account of *free variables*, we add the following metavaluation:

(vii) $v(A) = T$ iff $v(A_i) = T$, for all constant instances A_i of A, *recursively generated*.
$v^*(A) = F$ iff $v^*(A_i) = F$, for all constant instances A_i of A, *recursively generated*.

We express '$v(A^n/x) = T$, for all numerical constants n, recursively generated' as the conjunction: $v(A^0/x) = T$ and, for all m, if $v(A^m/x) = T$ then $v(A^{m'}/x) = T$. Since negation is constructive for these logics, the recursive determination for $v^*(\forall xA) = F$ is established by a process, leading to a witness for the existential and, as such, is a complementary (De Morgan) process to that of recursive generation of the universal.

We list the following results:

Lemma 3.1 (Soundness). *If* A *is a theorem of* MC# *then* $v(A) = T$, *and hence*:
if ∼A *is a theorem of* MC# *then* $v^*(A) = F$.

Corollary. *For any identity* s=t, *for constant terms* s *and* t, *not both* s=t *and* ∼s=t *are provable in* MC#.

Lemma 3.2 (Completeness).

(1) *If* $v(A) = T$ *then* A *is a theorem of* MC#, *and*:
(2) *If* $v^*(A) = F$ *then* ∼A *is a theorem of* MC#.

Theorem 3.3. *The system* MC# *is metacomplete, i.e.*, $v(A) = T$ *iff* A *is provable in* MC#, *and also* $v^*(A) = F$ *iff* ∼A *is provable in* MC#.

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Corollaries. *For* MC#:

(1) *If* $\vdash$A$\vee$B *then* $\vdash$A *or* $\vdash$B, *for sentences* A, B. [*Priming Property*]
(2) *If* $\vdash$A$\vee$B *then, for all constant instances* $A_i \vee B_i$ *of* A$\vee$B, $\vdash A_i$ *or* $\vdash B_i$. [*Extended Priming Property for formulae* A *and* B]
(3) *If* $\vdash\exists$xA *then* $\vdash A^m/x$, *for some numerical constant* m, *for sentence* $\exists$xA. [*Satisfaction Property*]
(4) *If* $\vdash\exists$xA *then, for all constant instances* $\exists xA_i$ *of* $\exists$xA, $\vdash {A_i}^m/x$, *for some numerical constant* m. [*Extended Satisfaction Property for formula* $\exists$xA]

Theorem 3.4 (Consistency). *If* $v(A) = T$ *then* $v^*(A) = T$. *Thus,* MC# *is simply consistent.*

The reason this proof is finitary is that the metavaluations are rooted in proof theory and do not involve standard model theory, thus circumventing the familiar argument that the consistency is proved relative to the theory required to set up the modelling. Note too, as shown by (vi), that the universal quantifier is evaluated in terms of the two steps of mathematical induction, which is a finitary process, ensuring in turn that this metavaluational argument is finitary.

Theorem 3.5. *The* DS, $\sim$A, A$\vee$B $\Rightarrow$ B, *is an admissible rule of* MC#.

The DS is then added to MC# and the appropriate cases of LEM are then proved, thus allowing rules of form A $\Rightarrow$ B to be replaced by formulae of form A$\supset$B. This in turn allows mathematical induction to be carried out for these formulae, re-creating classical arithmetic, up to a point, including the familiar primitive recursive arithmetic. Though this procedure can be widely used for specific formulae, it does not generally apply to formula-schemes. Further, the LEM does not hold for the Godel sentence G, on pain of Godel's First Theorem, since, by the Priming Property, if $\vdash$G$\vee\sim$G then $\vdash$G or $\vdash\sim$G, neither of which hold, given the consistency of classical arithmetic. (For more on this, see Section 4(iii) below.) What stops this from happening is the absence of the rule, $\forall$x(A$\vee$B) $\Rightarrow$ A$\vee\forall$xB, since otherwise the fact that the LEM applies to atomic sentences would inductively extend to all $\rightarrow$-free sentences including the quantified ones.

3.8. The simple consistency and non-triviality of naive set theory. We continue with Brady's proof of simple consistency of naive set theory, previously proved by a model-theoretic approach in his (1983) and (2006), but proved using metavaluations in his (2014). Then, we consider Brady's model-theoretic proof of nontriviality of naive set theory in (2006) and briefly indicate how it is adapted for metavaluations in his (2011).

Each of these model-theoretic approaches use a transfinite sequence of transfinite sequences of 3-valued models in the form:

$M_{0,0}, M_{0,1}, \ldots, M_{0,\lambda_0}$.
$M_{1,0}, M_{1,1}, \ldots, M_{1,\lambda_1}$.
... ...
$M_{\tau,0}, M_{\tau,1}, \ldots, M_{\tau,\lambda_\tau}$.
... ...

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The models, M_{0,λ_0}, M_{1,λ_1}, …, M_{τ,λ_τ}, …, of these transfinite sequences are all fixed points of those sequences, which are so because of the persistence properties of two of the three values of the models. These fixed points are achieved due to the ordinals outstripping the denumerable set of formulae, forcing two consecutive models to evaluate formulae with the same values. These models themselves have a fixed point, $M_{\kappa,\lambda_\kappa}$, due to a similar type of persistence property. So, the final model structure: M_{0,λ_0}, M_{1,λ_1}, …, M_{τ,λ_τ}, …, $M_{\kappa,\lambda_\kappa}$, is what is used to determine the logics and set theory for which the consistency applies.

The metavaluational approaches also use a transfinite sequence of transfinite sequences, but each of the models $M_{\tau,\upsilon}$ are replaced by a pair of metavaluations $v_{\tau,\upsilon}$ and $v_{\tau,\upsilon}{}^\star$, to deal with negation and thus to capture the effect of the three values taken by the models. These are set out as follows:

$v_{0,0}$, $v_{0,1}$, …, v_{0,λ_0}.
$v_{1,0}$, $v_{1,1}$, …, v_{1,λ_1}.
… … … …
$v_{\tau,0}$, $v_{\tau,1}$, …, v_{τ,λ_τ}.
… … … …
$v_{0,0}{}^\star$, $v_{0,1}{}^\star$, …, $v_{0,\lambda_0}{}^\star$.
$v_{1,0}{}^\star$, $v_{1,1}{}^\star$, …, $v_{1,\lambda_1}{}^\star$.
… … … …
$v_{\tau,0}{}^\star$, $v_{\tau,1}{}^\star$, …, $v_{\tau,\lambda_\tau}{}^\star$.
… … … …

The metavaluations $v_{\tau,\upsilon}$ and $v_{\tau,\upsilon}{}^\star$ in these sequences are each paired off, to produce a combined metavaluation. The metavaluations, v_{0,λ_0}, v_{1,λ_1}, …, v_{τ,λ_τ}, … and $v_{0,\lambda_0}{}^\star$, $v_{1,\lambda_1}{}^\star$, …, $v_{\tau,\lambda_\tau}{}^\star$, … of these transfinite sequences are all fixed points of those sequences, established as such by a persistence theorem and the fact that the formulae are only countably infinite. However, they themselves have respective fixed points, $v_{\kappa,\lambda_\kappa}$ and $v_{\kappa,\lambda_\kappa}{}^\star$, due to a similar persistence property.

For the simple consistency, the axioms and rule of the naive set theory are:

1. $t\in\{x:A\} \leftrightarrow A^t/x$. (CA)
2. $\forall z(z\in s \leftrightarrow z\in t) \Rightarrow \forall w(s\in w \leftrightarrow t\in w)$. (ER)
3. 1.
4. $\sim$0.

Where:
a,b,c,… range over free variables.
w,x,y,z… range over bound variables.
{x:A} is a term, for any bound variable x and formula A.
s,t,… range over terms.
The identity s=t between sets can then be defined as $\forall z(z\in s \leftrightarrow z\in t)$.
For M1-logics, we will strengthen (ER) to the Extensionality Axiom:
$\forall z(z\in s \leftrightarrow z\in t) \rightarrow \forall w(s\in w \leftrightarrow t\in w)$. (EA)

We start by letting v and v⋆ be any such metavaluation and ⋆-metavaluation, respectively, and assign what we can at this level of generality.

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For the atomic formulae, 1 and 0, we put:
$v(1) = T$, $v^\star(1) = T$, $v(0) = F$, and $v^\star(0) = F$.
For atomic formulae of form $t{\in}a$, for the free variable a:
$v(t{\in}a) = F$ and $v^\star(t{\in}a) = T$.
For the connectives (except '$\rightarrow$') and quantifiers, we follow Meyer (1976), Slaney (1984) and (1987) (also in Section 4.2 of Brady (2006)):
$v(A\&B) = T$ iff $v(A) = T$ and $v(B) = T$.
$v^\star(A\&B) = T$ iff $v^\star(A) = T$ and $v^\star(B) = T$.
$v(A{\vee}B) = T$ iff $v(A) = T$ or $v(B) = T$.
$v^\star(A{\vee}B) = T$ iff $v^\star(A) = T$ or $v^\star(B) = T$.
$v({\sim}A) = T$ iff $v^\star(A) = F$.
$v^\star({\sim}A) = T$ iff $v(A) = F$.
$v(\forall xA) = T$ iff $v(A^t/x) = T$, for all terms t.
$v^\star(\forall xA) = T$ iff $v^\star(A^t/x) = T$, for all terms t.
$v(\exists xA) = T$ iff $v(A^t/x) = T$, for some term t.
$v^\star(\exists xA) = T$ iff $v^\star(A^t/x) = T$, for some term t.
It remains to set out the metavaluations and *-metavaluations for atomic formulae of the form $t{\in}\{x{:}A\}$, and for $\rightarrow$-formulae, for each member of the above sequences.

This process is carried out by double transfinite induction, firstly within each of the τ-sequences, $v_{\tau,0}$, $v_{\tau,1}$, $\ldots$, v_{τ,λ_τ}, and $v_{\tau,0}{}^\star$, $v_{\tau,1}{}^\star$, $\ldots$, $v_{\tau,\lambda_\tau}{}^\star$, and then over each of these sequences as a whole for v and $v^\star$. We simply divide this induction into the four cases:

$v_{0,0}$ and $v_{0,0}{}^\star$; $v_{\tau,0}$ and $v_{\tau,0}{}^\star$, for $\tau > 0$; $v_{\tau,\upsilon}$ and $v_{\tau,\upsilon}{}^\star$, for υ a successor ordinal; $v_{\tau,\upsilon}$ and $v_{\tau,\upsilon}{}^\star$, for υ a limit ordinal.

CASE 1. For $v_{0,0}$ and $v_{0,0}{}^\star$:
$v_{0,0}(t{\in}\{x{:}A\}) = F$; $v_{0,0}{}^\star(t{\in}\{x{:}A\}) = T$.
$v_{0,0}(A{\rightarrow}B) = T$ iff $\vdash A{\rightarrow}B$; $v_{0,0}{}^\star(A{\rightarrow}B) = T$, for M1- and M2-logics.

CASE 2. For $v_{\tau,0}$ and $v_{\tau,0}{}^\star$, where $\tau > 0$:
$v_{\tau,0}(t{\in}\{x{:}A\}) = F$; $v_{\tau,0}{}^\star(t{\in}\{x{:}A\}) = T$.
$v_{\tau,0}(A{\rightarrow}B) = T$ iff $\vdash A{\rightarrow}B$ and, if $v_{\rho,\lambda_\rho}(A) = T$ then $v_{\rho,\lambda_\rho}(B) = T$, and if $v_{\rho,\lambda_\rho}{}^\star(A) = T$ then $v_{\rho,\lambda_\rho}{}^\star(B) = T$, for all $\rho < \tau$.
$v_{\tau,0}{}^\star(A{\rightarrow}B) = T$, for M1-logics.
$v_{\tau,0}{}^\star(A{\rightarrow}B) = T$ iff, if $v_{\rho,\lambda_\rho}(A) = T$ then $v_{\rho,\lambda_\rho}{}^\star(B) = T$, for all $\rho < \tau$, for M2-logics.

CASE 3. For $v_{\tau,\upsilon}$ and $v_{\tau,\upsilon}{}^\star$, where υ is a successor ordinal:
$v_{\tau,\upsilon}(t{\in}\{x{:}A\}) = T$ iff $v_{\tau,\upsilon-1}(A^t/x) = T$.
$v_{\tau,\upsilon}{}^\star(t{\in}\{x{:}A\}) = T$ iff $v_{\tau,\upsilon-1}{}^\star(A^t/x) = T$.
$v_{\tau,\upsilon}(A{\rightarrow}B) = T$ iff $v_{\tau,0}(A{\rightarrow}B) = T$.
$v_{\tau,\upsilon}{}^\star(A{\rightarrow}B) = T$ iff $v_{\tau,0}{}^\star(A{\rightarrow}B) = T$, for M1- and M2-logics.

CASE 4. For $v_{\tau,\upsilon}$ and $v_{\tau,\upsilon}{}^\star$, where υ is a limit ordinal:
$v_{\tau,\upsilon}(t{\in}\{x{:}A\}) = T$ iff $v_{\tau,\rho}(t{\in}\{x{:}A\}) = T$, for some $\rho < \upsilon$.
$v_{\tau,\upsilon}{}^\star(t{\in}\{x{:}A\}) = T$ iff $v_{\tau,\rho}{}^\star(t{\in}\{x{:}A\}) = T$, for all $\rho < \upsilon$.
$v_{\tau,\upsilon}(A{\rightarrow}B) = T$ iff $v_{\tau,0}(A{\rightarrow}B) = T$.
$v_{\tau,\upsilon}{}^\star(A{\rightarrow}B) = T$ iff $v_{\tau,0}{}^\star(A{\rightarrow}B) = T$, for M1- and M2-logics.

These cases follow the model-theoretic approach of Brady (1983) and (2006), but here we follow the usual metavaluations for A→B instead of using the matrix logics L3 or C3 of (1983) and (2006), respectively.

In order to justify the above metavaluations, we need to establish the fixed points v_{τ,λ_τ} and $v_{\tau,\lambda_\tau}{}^\star$, for the τ-sequences, and also the fixed points, $v_{\kappa,\lambda_\kappa}$ and $v_{\kappa,\lambda_\kappa}{}^\star$.

THEOREM 3.6 (Persistence). *For ordinals* τ, ρ *and* υ, *and a formula* A, *let* $\rho < \upsilon$.

Then, if $v_{\tau,\rho}(A) = T$ *then* $v_{\tau,\upsilon}(A) = T$, *and if* $v_{\tau,\rho}{}^\star(A) = F$ *then* $v_{\tau,\upsilon}{}^\star(A) = F$.

THEOREM 3.7 (Persistence). *For ordinals* μ *and* τ, *and a formula* A→B, *let* $\mu < \tau$. *Then, if* $v_{\mu,0}(A{\to}B) = F$ *then* $v_{\tau,0}(A{\to}B) = F$, *and if* $v_{\mu,0}{}^\star(A{\to}B) = F$ *then* $v_{\tau,0}{}^\star(A{\to}B) = F$.

THEOREM (Completeness). *For all ordinals* τ *and* υ, *if* $v_{\tau,\upsilon}(A) = T$ *then* ⊢A, *and if* $v_{\tau,\upsilon}{}^\star(A) = F$ *then* ⊢∼A. *Hence, if* $v_{\kappa,\lambda_\kappa}(A) = T$ *then* ⊢A, *and if* $v_{\kappa,\lambda_\kappa}{}^\star(A) = F$ *then* ⊢∼A.

THEOREM (Consistency). *For all ordinals* τ *and* υ, *if* $v_{\tau,\upsilon}(A) = T$ *then* $v_{\tau,\upsilon}{}^\star(A) = T$. *Hence, if* $v_{\kappa,\lambda_\kappa}(A) = T$ *then* $v_{\kappa,\lambda_\kappa}{}^\star(A) = T$.

This not only enables a consistency proof to be eventually obtained but also enables the pair of metavaluations to function as a three-valued logic, in a similar way to the model-theoretic approach.

THEOREM (Soundness). *If* ⊢A *then* $v_{\kappa,\lambda_\kappa}(A) = T$, *and hence if* ⊢∼A *then* $v_{\kappa,\lambda_\kappa}{}^\star(A) = F$.

Simple consistency then follows from the Soundness and Consistency Theorems. Completeness is only needed for the soundness proofs of the axioms: A → .B→A (A15), A → .A→B → B (A17), and (A → .B→C) → .B → .A→C (A18).

THEOREM (Metacompleteness). ⊢A *iff* $v_{\kappa,\lambda_\kappa}(A) = T$, *and hence* ⊢∼A *iff* $v_{\kappa,\lambda_\kappa}{}^\star(A) = F$.

COROLLARY. *For any* M1-*metacomplete logic, the above transfinite sequences of transfinite sequences of metavaluations reduce to a single pair of transfinite sequences*:

$v_{0,0}, v_{0,1}, \ldots, v_{0,\lambda_0}; v_{0,0}{}^\star, v_{0,1}{}^\star, \ldots, v_{0,\lambda_0}{}^\star$. [*I.e.*, $\kappa = 0$.]

Note that the last two corollaries in Brady (2014) are incorrect and that this is all that we can claim here.

For nontriviality, the proof using metavaluations in Brady (2011) has the following four key differences with the above simple consistency proof:

(I) There is no Completeness Theorem, the Soundness Theorem being sufficient for the NonTriviality Theorem, thus allowing the separation of the proof-theory from the metavaluations evaluated as T.

(II) One such deviation is the Law of Excluded Middle (A∨∼A), this not being included in the simply consistent theory as it yields inconsistency. Here, we will add this Law as far as we are able, no longer being constrained by the Priming Property (which is not proved due to the lack of Completeness Theorem). In order to extend the Law to include →-formulae for M2-logics only, we alter all the ⋆-metavaluations for

A→B by conjoining $\vdash$A→B as well. It is then shown that Russell's Paradox is derivable from this restricted LEM.

(III) For the M1-logics, as in Brady (2014), the metavaluations will reduce to a single pair of transfinite sequences. However, without completeness, the value of having this facility is somewhat reduced. Note too that Corollary 21 on p. 359 of (2011) is incorrect, though Corollary 20 is fine as it stands.

(IV) In the soundness proof for the Extensionality Rule and elsewhere, the two metavaluations v and v^* together function as a 4-valued logic, as they can independently assign a formula A to be T or F. These four values reduce to three values when the consistency result, if $v(A) = T$ then $v^*(A) = T$, is shown or the negation-completeness result, if $v^*(A) = T$ then $v(A) = T$, is shown. The former result will not hold here and the latter result is only partially shown, yielding a partial Law of Excluded Middle, leaving the logic as essentially 4-valued.

§4. In conclusion. (i) Metavaluations can provide an alternative methodology to that of standard model theory, by-and-large representing proof-theory, whilst still being applicable as model-theoretic constructs that are proof-theoretically defined in part. They are specially applicable where parts of their inductive structure represent properties to be shown, even though other parts may encapsulate completeness by conjunctive or disjunctive insertion of a provability or some membership statement of a theory. The general idea is to create a metavaluation where completeness is easily shown by formula induction, leaving a soundness theorem to be proved by the usual induction on proof steps. As mentioned in the Abstract, and in Sections 3.7 and 3.8, key metavaluational properties can be used to show that a simpler metavaluational modelling can be obtained for naive set theory and that a finitary proof of the consistency of Peano arithmetic is possible.

(ii) As indicated in Section 3.8, M1- and M2-logics are closely aligned to logics which solve the set-theoretic paradoxes, especially as the LEM, A∨∼A, and the Modus Ponens axiom, A&(A→B) → B, cannot occur in such logics. As discussed in Section 2, M1-logics are essentially entailment logics. So, it is worth considering the relationship between entailment and the paradoxes. Each of the set-theoretic paradoxes are expressed in terms of what Russell called "contextual definitions", i.e., the introduced set is defined in terms of its generating predicate. [E.g., $\forall x(x \in R \leftrightarrow x \notin x)$, with the predicate '$x \notin x$' introducing the Russell set R.] Now, a definition is really a meaning identity between the definiens and the definiendum and an ideal connective to use would be one representing meaning equivalence, expressible as the conjunction of two meaning containments.

Brady has introduced and worked upon the logic MC of meaning containment in (1996) and (2006), making adjustments in Brady and Meinander (2013) concerning the forms of distribution. As such, MC seems to be about as close as one is likely to get to capturing an appropriate logic for meaning containment. And, MC is an M1-logic and is certainly a good entailment logic, being based on meaning containment. So, such a logic would be ideal as a logic to express the sort of definition used in the set-theoretic paradoxes (and the Liar Paradox below), and it is an M1-entailment logic. And, this widening out of the scope of logics would provide quite some wriggle room if there is a need to alter the logic of meaning containment.

In order to convert such an entailment $A \leftrightarrow B$ into a classical equivalence $A \equiv B$, the direct way would be to use the entailment rules, $A \rightarrow B \Rightarrow {\sim}A \vee B$ and $B \rightarrow A \Rightarrow {\sim}B \vee A$, which are deductively equivalent to ${\sim}A \vee A$ and ${\sim}B \vee B$, respectively, given the logic B with MR1. From a constructive point of view, appropriate to metacomplete logics, these instances of the LEM in the context of paradoxes would be very dubious. Indeed, it can be shown for the logic DJ that neither $R \in R$ nor $R \notin R$ (nor their disjunction $R \in R \vee R \notin R$) is derivable, through examination of simply consistent models of naive set theory (c.f. Sections 5.3 and 5.4 of Brady (2006)). This sets up a barricade between the co-entailment $A \leftrightarrow B$ and its classical equivalence $A \equiv B$, preventing any clear passage from $A \leftrightarrow B$ to $A \equiv B$. Thus, the logic one needs to express these paradoxes is an entailment logic, and not classical logic. What this then amounts to is that a human-made definition does not guarantee that the LEM holds for the definiens or for the definiendum.

These same considerations apply to the semantic paradoxes that also use key definitions. E.g., For the Liar Paradox, p: Fp introduces the sentence p, which is defined by the expression 'p is false'. Again, the appropriate connective is a co-entailment '$\leftrightarrow$', representing meaning identity in a logic such as MC. Thus, this yields the formal expression $p \leftrightarrow T{\sim}p$, and, since $T{\sim}p \leftrightarrow {\sim}p$ by Tarski's formal definition of truth, we derive $p \leftrightarrow {\sim}p$, where the above arguments again apply. (More on this topic can be found in Brady (2015).)

(iii) We briefly examine Godel's First and Second Theorems in the light of the results of Section 3.7 above. In the case of both of these, the classically-based results are not in dispute.

Regarding the Second Theorem, Peano Arithmetic, based on MCQ–, is consistent, proved by finitary methods, as indicated above in Section 3.7. However, since the Peano Arithmetic is based on MCQ–, with the absence of the rule, $\forall x(A \vee B) \Rightarrow A \vee \forall xB$, the full classical Peano Arithmetic is not provable.

Regarding the First Theorem, however, unprovability in classical Peano Arithmetic is going to translate into unprovability in MC#, maintaining the First Theorem for MC#. This can be simply shown by replacing each '$\rightarrow$' in MC# by '$\supset$' in classical Peano Arithmetic, and so any theorem of MC# is also a theorem of classical Peano Arithmetic, under this replacement. Thus, neither G nor ${\sim}G$ can be theorems of MC#, G being the Godel sentence, subject of course to the consistency of classical Peano Arithmetic. And, neither is $G \vee {\sim}G$ a theorem, this being due to the Priming Property for the sentence G: if $\vdash G \vee {\sim}G$ then $\vdash G$ or $\vdash {\sim}G$.

(iv) As flagged by Slaney in his (1984) articles, there may be other metavaluations, i.e., other than M1- and M2-valuations, determined by

3- or 4-valued matrix logics, especially as they provide different characterizations for negated entailments. These matrices are obtained by removal of the provability $\vdash A \rightarrow B$ from Slaney's metavaluation of $A \rightarrow B$, as occurs in Section 1. This then yields a 4-valued logic with the following values, t, b, n and f, for the formula A:

t. $v(A) = T$ and $v^{\star}(A) = T$.
b. $v(A) = T$ and $v^{\star}(A) = F$.
n. $v(A) = F$ and $v^{\star}(A) = T$.
f. $v(A) = F$ and $v^{\star}(A) = F$.

Given the metacompleteness theorem, $v(A) = T$ iff $\vdash A$ and $v^{\star}(A) = F$ iff $\vdash \sim A$, the value t represents $\vdash A$ and not-$\vdash \sim A$, value b represents $\vdash A$ and $\vdash \sim A$, value n represents not-$\vdash A$ and not-$\vdash \sim A$, and value f represents not-$\vdash A$ and $\vdash \sim A$. So, we can think of value t being for classical proof, value b being for an inconsistency, value n being for a negation incompleteness, and value f being for classical nonproof.

Since the simple consistency result, if $v(A) = T$ then $v^{\star}(A) = T$, is easily proved by induction on formulae, for both M1- and M2-logics, the value b above can thereby be removed, creating a 3-valued logic with just values t, n and f. Such 3-valued logics are discussed on p. 167 of Slaney's (1984). Another reason for removing the value b is that the metavaluation of all sentential variables is n, from which the value t can be obtained from either of the $\rightarrow$-matrices below and the value f can be then obtained from the $\sim$-matrix. There is no such passage to reach the value b.

We consider the 4-valued matrix logics that correspond to M1- and M2-metavaluations. The matrices below for $\sim$, & and $\vee$ are common:

$\sim$	
t	f
b	b
n	n
f	t

&	t	b	n	f
t	t	b	n	f
b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

v	t	b	n	f
t	t	t	t	t
b	t	b	t	b
n	t	t	n	n
f	t	b	n	f

The $\rightarrow$-matrices differ for M1- and M2-metavaluations, as follows:

(For M1−logics)

$\rightarrow$	t	b	n	f
t	t	n	n	n
b	t	t	n	n
n	t	n	t	n
f	t	t	t	t

(For M2−logics)

$\rightarrow$	t	b	n	f
t	t	f	n	f
b	t	b	n	f
n	t	n	t	n
f	t	t	t	t

The M2-matrix for $\rightarrow$ is the matrix logic BN4 of Brady (1982), whilst the M1-matrix replaces b by t and f by n, in the four respective positions, creating an $\rightarrow$-matrix consisting entirely of t and n. This phenomenon is produced by the constant assignment of T to $v^{\star}(A \rightarrow B)$.

At the moment, it is unclear what the relation is between metavaluations and their corresponding matrix valuations or what other $\rightarrow$-matrices might be suitable to yield other metavaluations. This is subject to further research.

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PHILOSOPHY
LA TROBE UNIVERSITY
MELBOURNE, VICTORIA 3086, AUSTRALIA
E-mail: Ross.Brady@Latrobe.edu.au

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