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ARTICLE

A quantitative Lovász criterion for Property B

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Abstract

A well-known observation of Lovász is that if a hypergraph is not 2-colourable, then at least one pair of its edges intersect at a single vertex. In this short paper we consider the quantitative version of Lovász's criterion. That is, we ask how many pairs of edges intersecting at a single vertex should belong to a non-2-colourable n-uniform hypergraph. Our main result is an *exact* answer to this question, which further characterizes all the extremal hypergraphs. The proof combines Bollobás's two families theorem with Pluhar's randomized colouring algorithm.

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1. Introduction

A hypergraph $\mathcal{H} = (V, E)$ consists of a vertex set V and a set of edges E where each $X \in E$ is a subset of V. If all edges of $\mathcal{H}$ have size n then $\mathcal{H}$ is called an n-uniform hypergraph, or n-graph for short. A hypergraph is 2-colourable if one can assign to each vertex $v \in V$ one of two colours, say *Red*/*Blue*, so that each $X \in E$ contains vertices of both colours. Miller [6], and later Erdős in various papers, referred to this property as *Property B*, after F. Bernstein [2], who introduced it in 1907. Since deciding if a hypergraph is 2-colourable is NP-hard, one cannot hope to find a simple characterization of all 2-colourable hypergraphs. Instead, one looks for general sufficient/necessary conditions for having this property. For example, a famous result of Seymour [8] states that if $\mathcal{H}$ is not 2-colourable (and is minimal with respect to this property) then $|E| \geqslant |V|$. Probably the most well-studied question of this type asks for the smallest number of edges in an n-graph that is not 2-colourable. The study of this quantity, denoted by $m(n)$, was popularized by Erdős; see [1] for a comprehensive treatment. Despite much effort by many researchers, even the asymptotic value of $m(n)$ has not yet been determined.

A pair of edges $X, Y \in E(\mathcal{H})$ is *simple* if $|X \cap Y| = 1$. Let $m_2(\mathcal{H})$ denote the number of ordered simple pairs of edges of $\mathcal{H}$. A well-known observation of Lovász [5] states that if $\mathcal{H}$ is not 2-colourable then $m_2(\mathcal{H}) > 0$. Despite its simplicity, this observation underlies the best known bounds for $m(n)$; see [4, 7]. It is natural to ask if one can obtain a quantitative version of Lovász's observation, that is, estimate how small $m_2(\mathcal{H})$ can be in an n-graph not satisfying property B.

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Our main result in this paper states that (somewhat surprisingly) one can give an exact answer to the above extremal question as well as characterize the extremal n-graphs.

Let K^n_{2n-1} denote the complete n-graph on $2n-1$ vertices. It is easy to see that K^n_{2n-1} is not 2-colourable and that

$$m_2(K^n_{2n-1}) = n \cdot \binom{2n-1}{n}.$$

We first observe that this simple upper bound is tight.

Proposition 1.1. *If an n-graph is not 2-colourable then* $m_2(\mathcal{H}) \geqslant n \cdot \binom{2n-1}{n}$.

As with any extremal problem, one would like to know which graphs or hypergraphs are extremal with respect to this problem. For example, Turán's theorem states that among all n-vertex graphs not containing a complete t-vertex subgraph, there is only one graph maximizing the number of edges. In the setting of our problem, it is easy to see that K^n_{2n-1} is not the only non-2-colourable n-graph satisfying $m_2(\mathcal{H}) = n \cdot \binom{2n-1}{n}$, since one can take a copy of K^n_{2n-1} and add to it more vertices and edges without increasing the number of simple pairs. Our main result in this paper characterizes the extremal n-graphs, by showing that this is in fact the only way to construct an n-graph meeting the bound of Proposition 1.1.

Theorem 1.1. *If a non-2-colourable n-graph* $\mathcal{H}$ *satisfies* $m_2(\mathcal{H}) = n \cdot \binom{2n-1}{n}$, *then it contains a copy of* K^n_{2n-1}.

While the proof of Proposition 1.1 is implicit in Pluhar's [7] argument for bounding $m(n)$, the proof of Theorem 1.1 is more intricate, relying on Bollobás's two families theorem [3] as well as a refined analysis of Pluhar's randomized algorithm for 2-colouring n-graphs.

2. Proof of Proposition 1.1

In this section we describe several preliminary observations regarding a colouring algorithm introduced in [7], and use them to derive Proposition 1.1. The algorithm is the following.

Algorithm $\mathrm{Col}(\mathcal{H}, \pi)$. The input is a hypergraph $\mathcal{H} = (V, E)$ and an ordering $\pi : V \mapsto \{1, \ldots, |V|\}$ (*i.e.* π is a bijection). The output is a 2-colouring of V (not necessarily a proper one). The algorithm runs in $|V|$ steps, where in each time step $1 \leqslant i \leqslant |V|$, the vertex $\pi^{-1}(i)$ is coloured *Blue* if this does not form any monochromatic *Blue* edge. Otherwise, $\pi^{-1}(i)$ is coloured *Red*.

We now state an important property of $\mathrm{Col}(\mathcal{H}, \pi)$. For two disjoint subsets $X, Y \subseteq V$, we use the notation $\pi(X) < \pi(Y)$ whenever $\max_{x \in X} \pi(x) < \min_{y \in Y} \pi(y)$, that is, the elements of X precede all the elements of Y in the ordering π. Suppose (X, Y) is a simple pair of edges in $\mathcal{H}$ with[a] $X \cap Y = y$. We say that π *separates* (X, Y) if $\pi(X \setminus y) < \pi(y) < \pi(Y \setminus y)$.

Claim 1. *If* $\mathrm{Col}(\mathcal{H}, \pi)$ *fails to properly colour* $\mathcal{H}$, *then* π *separates at least one pair of simple edges.*

Proof. We first observe that (by definition) for every ordering π, the algorithm $\mathrm{Col}(H, \pi)$ does not produce monochromatic *Blue* edges. Suppose then it produced a *Red* edge $Y \in E$. Let y be the first vertex of Y according to the ordering π. If y was coloured red, then there must have been an edge X so that $y \in X$, and all other vertices of X were already coloured *Blue* (otherwise the algorithm would colour y *Blue*). This means (X, Y) is simple and that π separates it. □

[a] Here, and in what follows, we slightly abuse notation by writing y instead of the more appropriate $\{y\}$.

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Note that the claim above already shows that if $\mathcal{H}$ is not 2-colourable then $m_2(\mathcal{H}) > 0$. For the proof of Proposition 1.1 we will also need the following simple fact.

Claim 2. *A random permutation separates any given simple pair with probability* $1/n\binom{2n-1}{n}$.

Proof. Let (X, Y) be a simple pair, and let $X \cap Y = y$. A permutation π separates (X, Y) if and only if $\pi(X \setminus y) < \pi(y) < \pi(Y \setminus y)$, and this happens with probability exactly

$$\frac{(n-1)!(n-1)!}{(2n-1)!} = n^{-1}\binom{2n-1}{n}^{-1},$$

as desired. □

The above claims suffice for proving Proposition 1.1.

Proof of Proposition 1.1. Assume

$$m_2(\mathcal{H}) < n\binom{2n-1}{n}.$$

Suppose we pick a uniformly random π. Then by the union bound and Claim 2, we infer that with positive probability π does not separate any simple pair edges. Hence there is a π not separating any simple pair. Claim 1 then implies that $\text{Col}(\mathcal{H}, \pi)$ will produce a legal 2-colouring of $\mathcal{H}$. □

3. Proof of Theorem 1.1

For the rest of this section fix some non-2-colourable n-graph $\mathcal{H} = (V, E)$ satisfying $m_2(\mathcal{H}) = n\binom{2n-1}{n}$. We need to show that $\mathcal{H}$ contains a copy of K^n_{2n-1}. We start with a few preliminary claims regarding $\mathcal{H}$.

First, we show that no π separates more than one simple pair.

Claim 3. *Every ordering* π *separates at most one simple pair.*

Proof. Suppose π separates two simple pairs. By Claim 2, the assumption on $m_2(\mathcal{H})$, and by linearity of expectation, the expected number of simple pairs separated by a random permutation is exactly 1. Hence, if π separates 2 simple pairs, then there must exist a permutation σ which separates less than 1, and therefore 0, simple pairs. Therefore, by Claim 1 we obtain that $\text{Col}(\mathcal{H}, \sigma)$ produces a legal 2-colouring of $\mathcal{H}$, which is a contradiction to the assumption that $\mathcal{H}$ is not 2-colourable. □

Claim 4. *If* (X, Y) *and* (X', Y) *are simple pairs, then* $X \cap Y \neq X' \cap Y$.

Proof. We observe that if $X \cap Y = X' \cap Y = y$, then there is a π that separates both (X, Y) and (X', Y), and this will contradict Claim 3. Indeed, if (X, Y) and (X', Y) are simple pairs and $X \cap Y = X' \cap Y = y$, then $(X \cup X') \setminus y$ and Y are disjoint. Therefore, any π satisfying

$$\pi((X \cup X') \setminus y) < \pi(y) < \pi(Y \setminus y)$$

separates (X, Y) and (X', Y). This completes the proof. □

In addition to the above observations about $\mathcal{H}$, the last ingredient we will need is the following theorem of Bollobás [3].

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Lemma 3.1. *Let I be an index set. For all $i \in I$, let A_i and B_i be subsets of a set V of p elements satisfying the following conditions:*

(i) $A_i \cap B_i = \emptyset$ *for all* $i \in I$, *and*
(ii) $A_j \nsubseteq A_i \cup B_i$ *for all* $i \neq j \in I$.

Then we have

$$\sum_{i \in I} \binom{p - |B_i|}{|A_i|}^{-1} \leqslant 1,$$

with equality if and only if $B_i = B$ *for all* $i \in I$ *and the sets* A_i *are all the* q*-tuples of the set* $P \setminus B$ *for some value of* q.

Let us now show how to use Lemma 3.1 in order to derive Theorem 1.1. Recall that V is the vertex set of $\mathcal{H}$, and set $p := |V|$. Let $M(\mathcal{H})$ be a collection of simple pairs (X, Y) defined as follows. Out of all the simple pairs (X, Y) with the same 'second' set Y, put one of these pairs in $M(\mathcal{H})$. Observe that by Claim 4 each Y belongs to at most $|Y| = n$ simple pairs of the form (X, Y) (*i.e.* with Y as the second set), implying that

$$t := |M(\mathcal{H})| \geqslant \frac{1}{n} \cdot m_2(\mathcal{H}) = \binom{2n-1}{n}.$$

We now define a collection $\mathcal{F}$ consisting of pairs of subsets of V as follows. For every simple pair $s := (X, Y) \in M(\mathcal{H})$, define $A_s = X \setminus Y$ and $B_s = V \setminus (X \cup Y)$, and let $\mathcal{F} = \{(A_s, B_s) \colon s \in M(\mathcal{H})\}$. For convenience, let us rename the pairs in $\mathcal{F}$ as (A_i, B_i) with $1 \leqslant i \leqslant t$.

Now we wish to show that $\mathcal{F}$ satisfies the conditions in Lemma 3.1. Observe that if it does, then since

$$\sum_{i=1}^{t} \binom{p - |B_i|}{|A_i|}^{-1} = \sum_{i=1}^{t} \binom{2n-1}{n-1}^{-1} \geqslant 1,$$

it follows by the first part of Lemma 3.1 that the last inequality is in fact an equality. Therefore, by the second part of Lemma 3.1, we conclude that all the B_i are the same set B, and the set of all the A_i consists of all $n-1$ subsets of a ground set of size $2n-1$. That is, let $B = B_i$ and $U = V \setminus B$. Then we have that $|U| = 2n-1$, and that the sets A_i are all the $n-1$ subsets of U. Since by construction we have that $U \setminus A_i \in E(\mathcal{H})$ for all i, we conclude that $\mathcal{H}$ restricted to the set U is a copy of K_{2n-1}^{n} as desired. It thus remains to show the following.

Claim 5. *$\mathcal{F}$ satisfies the conditions in Lemma 3.1.*

Proof. The first condition $A_i \cap B_i = \emptyset$ for all i is trivially satisfied by construction. For the second condition, let (A, B) and (A', B') be two elements in $\mathcal{F}$ coming from simple pairs (X, Y) and (X', Y') belonging to $M(\mathcal{H})$, respectively. Recall that, by the way we defined $M(\mathcal{H})$ and $\mathcal{F}$, we have $Y \neq Y'$. Let us use y and y' to denote the unique elements in $X \cap Y$ and $X' \cap Y'$, respectively. We wish to show that $A \nsubseteq A' \cup B'$, which, by construction, is implied by $(X \setminus y) \cap Y' \neq \emptyset$. Assuming $(X \setminus y) \cap Y' = \emptyset$, we will derive a contradiction to Claim 3 by showing that there is a permutation π separating two distinct simple pairs.

Observe that it cannot be that $y \in Y'$. Indeed, if it were the case, then together with the assumption that $(X \setminus y) \cap Y' = \emptyset$ we would infer that (X, Y) and (X, Y') are both simple pairs intersecting at y (and distinct as $Y \neq Y'$), contradicting Claim 4. Assume then that $y \notin Y'$ (so in particular $y \neq y'$). We claim that we can find a π satisfying

$$\pi(X \setminus y) < \pi(y) < \pi((X' \setminus y') \setminus X) < \pi(y') < \pi((Y \cup Y') \setminus (X \cup X')).$$

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Indeed, the only thing that needs to be justified is the ability to place y' as above, which follows from the fact that $y' \in Y'$ and the assumption $(X \setminus y) \cap Y' = \emptyset$, which together imply that $y' \notin X$. Observe that since π first places $X \setminus y$ and then y, the pair (X, Y) is separated by π. Such a π clearly places $X' \setminus y'$ before y' and the assumption $(X \setminus y) \cap Y' = \emptyset$ together with the fact that $y \notin Y'$ imply that such a π places all of $Y' \setminus y'$ after y', so it separates (X', Y') as well, giving us the desired contradiction. □

This completes the proof of Theorem 1.1.

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