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The linear approach for a nonlinear infiltration equation†

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For the Cauchy problem for the nonlinear infiltration equation

$$\begin{cases} u_t = \frac{1}{m}(u^m)_{xx}, & x \in \mathbb{R}, t > 0, m \geqslant 1, \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}, \end{cases}$$

we use its linear solution $u(x,t,1)$ to approach the nonlinear solution $u(x,t,m)$, and obtain the explicit estimate:

$$\int_0^T \int_{\mathbb{R}} |u(x,t,m) - u(x,t,1)|^2 \, dx \, dt \leqslant (C^*(m-1))^2,$$

where $C^* = O(T^\gamma)$ and $\gamma = \frac{1+m-\alpha}{2(1+m)}$ for any $0 < \alpha < 1$.

1 Introduction

We consider the degenerate parabolic equation

$$u_t = \frac{1}{m}(u^m)_{xx}, \qquad m > 1.$$

Although the equation has been studied by many authors some years ago, there are also some new results for the equation in recent years [6, 8, 9]. In these papers, the qualitative behaviour of the solutions of the equations is studied. But there are few works on the continuous dependence on the nonlinearities of the equations. In 1981, Benilan & Crandall [1] studied a similar problem for degenerate parabolic equations, but their results are not written in terms of explicit estimates. Cockburn & Gripenberg [4] also discussed a similar problem, and obtained a continuous dependence results in the space $L^1(\mathbb{R})$ for any fixed $t > 0$. To the knowledge of the authors, there are no other results on such problems.

Let $u(x,t,m)$ be the solutions for $m \geqslant 1$. it is well-known that $\|u(\cdot,t,m)\|_{L^2}$ is bounded for any given $t > 0$, so $\|u(\cdot,t,m) - u(\cdot,t,m_0)\|_{L^2}$ is bounded also. Therefore, we hope we can obtain the continuous dependence on the nonlinearities in $L^2(Q_T)$, not in $L^2(\mathbb{R})$, where,

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$Q_T = \mathbb{R} \times (0, T)$. So the present paper discusses the problem in $L^2(Q_T)$. We obtain the following result:

$$\|u(x,t,m) - u(x,t,1)\|_{L^2(Q_T)} \leqslant C^*(m-1),$$

where $C^* = O(T^\gamma)$ and $\gamma = \frac{m+1-\alpha}{2(1+m)}$ for any $0 < \alpha < 1$.

We begin by recalling some known results. For the Cauchy problem

$$\begin{cases} u_t = \frac{1}{m}(u^m)_{xx}, & x \in R, t > 0, m \geqslant 1, \\ u|_{t=0} = u_0(x), & x \in R \end{cases} \tag{1.1.1}$$

with

$$0 \leqslant u_0(x) \leqslant M, \quad u_0 \in L^1(R), \quad u_0^m(x) \text{ is Lipschitz continuous on } x \in R, \tag{1.1.2}$$

we can very easily obtain the following results on the existence and uniqueness.

Lemma 1.1 *For any positive constant T, the Cauchy problem* (1.1) *with the conditions* (1.2) *has a unique weak solution $u(x,t,m) \in C(\overline{Q}_T)$ such that $\frac{\partial}{\partial x}u^m(x,t,m) \in C(Q_T)$ and*

$$0 \leqslant u(x,t,m) \leqslant M \qquad \text{for } (x,t) \in \overline{Q}_T,\ m \geqslant 1, \tag{1.1.3}$$

$$u(x,t,m) \in C([0,+\infty); L^1(R)) \qquad \text{for } m \geqslant 1, \tag{1.1.4}$$

$$\int_R u(x,t,m)\,dx = \int_R u_0(x)\,dx \qquad \text{for } t > 0,\ m \geqslant 1 \tag{1.1.5}$$

Proof The classical case $m = 1$ is well-known and it is easy to see that $u(x,t,1)$ is also classical. If $m > 1$, the existence and uniqueness of $u(x,t,m)$ are given by Friedman & Kamin [3] and Gilding & Peletier [5], the continuity of generalized derivative $\frac{\partial u^m}{\partial x}$ is given by Theorem 3 in Gilding & Peletier [5], (1.3) is given by Friedman [2] (see p. 34), and (1.4) and (1.5) are given by Vazquez [7]. □

Apart from these results, there are also other conclusions about the problem (1.1), such as the large time behaviour [7]. A function $u(x,t,m)$ will be called a weak solution of (1.1), (1.2) if

(i) u is bounded, continuous and nonnegative in $\overline{Q}_T$;
(ii) u^m has a bounded generalized derivative with respect to x in Q_T;
(iii) u satisfies the identity

$$\int_0^T \int_R \left[\left(\frac{1}{m} u^m \right)_x \phi_x - u\phi_t \right] dx\,dt = \int_R \phi(x,0)u_0(x)\,dx \tag{1.1.6}$$

for all $\phi \in C^1(\overline{Q}_T)$ which vanish for large $|x|$ and for $t = T$.

It is necessary to indicate that the solutions $u(x,t,m)$ are obtained by the following steps [4, p. 132–133].

Construct a sequence of functions $\{u_{0,k}\}$ with $u_{0.k}(x) \in C^\infty[-k,k]$ such that

(1) $u_{0.k} \longrightarrow u_0$ as $k \longrightarrow \infty$, uniformly on bounded intervals;

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(2) $u_{0,k+1} \leqslant u_{0,k}$ for all $k \geqslant 1$;

(3) $k^{-1} \leqslant u_{0,k} \leqslant M_0$ for all $k \geqslant 1$, where $M_0 \geqslant M$;

(4) $u_{0,k} = M_0$ on $[-k, -k+1] \bigcup [k-1, k]$ for all $k \geqslant 1$;

(5) $|\frac{\partial}{\partial x} u_{0,k}^m| \leqslant K$ for all $k \geqslant 1$, where $K_0 \geqslant K$,
in which K is a positive constant such that

$$|u_0^m(x_1) - u_0^m(x_2)| \leqslant K|x_1 - x_2| \qquad \text{for } x_1, x_2 \in \mathbb{R}.$$

For any given $k \geqslant 1$, Lemma 2 of Gilding & Peletier [5] assures the existence and uniqueness of

$$u_k(x,t,m) \in C^\infty(\overline{Q}_T^k), \tag{1.1.7}$$

solution of

$$\begin{cases} u_t = \frac{1}{m}(u^m)_{xx}, & \text{for } (x,t) \in Q_T^k = (-k,k) \times (0,T], \\ u(\pm k, t) = M_0, & \text{for } t \in [0,T], \\ u(x,0) = u_{0,k}(x), & \text{for } x \in [-k,k], \end{cases} \tag{1.1.8}$$

with

$$k^{-1} \leqslant u_k \leqslant M_0 \qquad \text{for } (x,t) \in \overline{Q}_T^k. \tag{1.1.9}$$

Theorem 2 of Gilding & Peletier [5] shows that the limit functions

$$u(x,t,m) = \lim_{k \longrightarrow +\infty} u_k(x,t,m) \qquad \text{for } (x,t) \in \overline{Q}_T \tag{1.1.10}$$

are indeed the weak solutions of (1.1) for $m > 1$.

In this paper, we shall establish the following result.

Theorem *Let $u(x,t,m)$ be the solutions of the Cauchy problem* (1.1) *with the conditions* (1.2) *for $m \in [1,2)$. Then*

$$\int_0^T \int_R |u(x,t,m) - u(x,t,1)|^2 \, dx \, dt \leqslant (C^*(m-1))^2, \tag{1.1.11}$$

where $C^* = O(T^\gamma)$ and $\gamma = \frac{1+m-\alpha}{2(1+m)}$ for any $0 < \alpha < 1$ when T is large enough.

As an example, we test our result on explicit solutions, namely the Barenblatt solution of the equation $B_t = (B^m)_{xx}$ $(m > 1)$ and the fundamental solution of the heat equation $J_t = J_{xx}$. However, because the solutions B and J are not bounded as t tends to zero, so although we can employ the method of the theorem, the result is different from (1.11).

2 Some lemmas

Lemma 2.1 *Assume $1 < m < 2$. For any given $T > 0$, let $u_k(x,t,m)$ be the solutions of* (1.8) *on $\overline{Q}_T^k$. Then for any $l > 0$ and $2l < k$, there exist positive constants C_0 and C_1 such that*

$$\left|\left(u_k^{\frac{m}{2}}\right)_x\right| \leqslant \left(2\left(\frac{C_1}{l}\right)^2 + \frac{2m}{2-m} C_0 M_* t^{-1}\right)^{\frac{1}{2}} \tag{2.2.1}$$

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for $(x,t) \in [-l,l] \times (0,T]$, *where* $M_* = \max_{(x,t)\in\overline{Q}_T^k} u_k$.

Proof We first perform the change $u_k = V^{\frac{2}{m}}$. In view of $2l < k$, we have

$$V_t = V^{2-\frac{2}{m}} V_{xx} + V^{1-\frac{2}{m}} (V_x)^2 \qquad \text{for } (x,t) \in [-2l,2l] \times (0,T]. \tag{2.2.2}$$

It follows from (1.7) and (1.9) that $V \in C^\infty(\overline{Q}_T^k)$. We differentiate (2.2) with respect to x and multiply through by V_x. Writing $p = V_x$ we obtain

$$\frac{1}{2}(p^2)_t - V^{2-\frac{2}{m}} p p_{xx} = \left(4 - \frac{2}{m}\right) V^{1-\frac{2}{m}} p^2 p_x + \left(1 - \frac{2}{m}\right) V^{-\frac{2}{m}} p^4$$
$$\text{for } (x,t) \in [-2l,2l] \times (0,T]. \tag{2.2.3}$$

Let $f(x)$ be a $C_0^\infty(\mathbb{R})$ function such that $0 \leqslant f(x) \leqslant 1$ and

$$f(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2. \end{cases}$$

For $l > 0$ and $\tau > 0$, we define the functions

$$f_l(x) = f\left(\frac{x}{l}\right) \tag{2.2.4}$$

and

$$g(t) = \begin{cases} f(\frac{t-2\tau}{\tau}), & 0 \leqslant t < \tau, \\ 1, & \tau \leqslant t \leqslant T. \end{cases}$$

Set

$$\zeta(x,t) = f_l(x) g(t).$$

Then

$$0 \leqslant \zeta(x,t) \leqslant 1$$

and

$$\zeta(x,t) = \begin{cases} 1, & \text{for } (x,t) \in [-l,l] \times [\tau,T], \\ 0, & \text{for } (x,t) \in \overline{Q}_T - (2l,2l) \times (0,T] \end{cases}$$

and there exists a positive constant C_0 such that

$$|\zeta_x| \leqslant \frac{C_0}{l}, \quad |\zeta_{xx}| \leqslant \frac{C_0}{l^2}, \quad |\zeta_t| \leqslant \frac{C_0}{\tau}, \qquad \text{for } (x,t) \in Q_T.$$

Let

$$Z = (p\zeta)^2,$$

then $Z \in C(\overline{Q}_T^k)$. Set

$$Z(x_0,t_0,m) = \max_{(x,t)\in\overline{Q}_T^k} Z(x,t,m).$$

It follows from the definition of $\zeta(x,t)$ that $(x_0,t_0) \in (-2l,2l) \times (0,T]$ and

$$Z_x = 0 \qquad \text{and} \qquad Z_t - V^{2-\frac{2}{m}} Z_{xx} \geqslant 0 \qquad \text{at } (x_0,t_0).$$

Hence,

$$p_x\zeta = -p\zeta_x \qquad \text{at } (x_0, t_0)$$

and

$$p^2(V^{2-\frac{2}{m}}\zeta\zeta_{xx} - 2V^{2-\frac{2}{m}}(\zeta_x)^2 - \zeta\zeta_t) \leqslant \left(\frac{1}{2}(p^2)_t - V^{2-\frac{2}{m}}pp_{xx}\right)\zeta^2 \qquad \text{at } (x_0, t_0). \tag{2.2.5}$$

Using (2.5) in (2.3), we find that

$$\frac{2-m}{m}(p\zeta)^2 \leqslant \frac{2-4m}{m}Vp\zeta\zeta_x + V^2(2\zeta_x^2 - \zeta\zeta_{xx}) + V^{\frac{m}{2}}\zeta\zeta_t \qquad \text{at } (x_0, t_0).$$

Notice that $u_k \leqslant M_*$, so $V^{\frac{m}{2}} \leqslant M_*$. Therefore, for $1 < m < 2$, there exists a positive constant C_1 such that

$$\begin{aligned}(p\zeta)^2|_{(x_0,t_0)} &\leqslant \frac{2-4m}{2-m}Vp\zeta\zeta_x + \frac{m}{2-m}V^2(2(\zeta_x)^2 - \zeta\zeta_{xx}) + \frac{m}{2-m}M_*C_0\tau^{-1} \\ &\leqslant \left(\frac{C_1}{l}\right)^2 + \frac{1}{2}(p\zeta)^2|_{(x_0,t_0)} + \frac{m}{2-m}M_*C_0\tau^{-1}.\end{aligned}$$

Thus,

$$(p\zeta)^2|_{(x_0,t_0)} \leqslant 2\left(\frac{C_1}{l}\right)^2 + \frac{2m}{2-m}M_*C_0\tau^{-1}. \tag{2.2.6}$$

It follows from the definition of $Z(x_0, t_0)$ that (2.6) holds for all $(x, t) \in [-2l, 2l] \times (0, T]$, especially, for $(x, t) \in [-l, l] \times \{\tau\}$. Notice that $\zeta(x, \tau) = 1$ for $x \in [-l, l]$, hence

$$|p(x, \tau, m)| \leqslant \left(2\left(\frac{C_1}{l}\right)^2 + \frac{2m}{2-m}M_*C_0\tau^{-1}\right)^{\frac{1}{2}} \qquad \text{for } x \in [-l, l].$$

This yields (2.1) immediately. □

Lemma 2.2 *For any given $T > 0$, assume $u(x, t, m)$ be the weak solutions of* (1.1) *with the conditions* (1.2) *on Q_T. Then for any given $m \in [1, 2)$, there is a positive constant C_2 such that*

$$|(u^m(x, t, m))_x| \leqslant C_2 t^{-\frac{1}{2}} \qquad \textit{for } (x, t) \in Q_T. \tag{2.2.7}$$

Proof By Lemma 1, for any $x, x' \in [-l, l], 0 < 2l < k$, (2.1) yields

$$\begin{aligned}|u_k^{\frac{m}{2}}(x', t, m) - u_k^{\frac{m}{2}}(x, t, m)| &= |\int_x^{x'} \frac{\partial}{\partial x}u_k^{\frac{m}{2}}(\xi, t, m)d\xi| \\ &\leqslant \left(2\left(\frac{C_1}{l}\right)^2 + \frac{2m}{2-m}M_*C_0t^{-1}\right)^{\frac{1}{2}}|x' - x|,\end{aligned}$$

Letting $k \longrightarrow \infty$, and then letting $l \longrightarrow \infty$, recalling from (1.3) and (1.10), we have

$$\left|u^{\frac{m}{2}}(x', t, m) - u^{\frac{m}{2}}(x, t, m)\right| \leqslant \left(\frac{2m}{2-m}MC_0t^{-1}\right)^{\frac{1}{2}}|x' - x|$$

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for $(x,t),(x',t) \in Q_T$. Letting $x' \longrightarrow x$ yields $|(u^{\frac{m}{2}})_x| \leqslant (\frac{m}{2-m} M C_0 t^{-1})^{\frac{1}{2}}$. This gives (2.7) for $1 < m < 2$.

To end the proof of the lemma, we notice that

$$u(x,t,1) = \frac{1}{2\sqrt{\pi t}} \int_R u_0(\xi) e^{\frac{-(x-\xi)^2}{4t}} d\xi.$$

It is easy to find that (2.7) also holds for $m = 1$. □

3 Proof of the theorem

Let $u(x,t,m)$ be the solutions of the problem (1.1) with the conditions (1.2) on Q_T. Define

$$\eta = \int_T^t \left(\frac{1}{m} u^m(x,\tau,m) - u(x,\tau,1) \right) d\tau \qquad \text{for } t \in (0,T).$$

It is not difficult to see that $f_l(x)\eta$ are admissible test functions in (1.6), in which $f_l(x)$ is defined by (2.4). Thus, with the choice $\phi = f_l(x)\eta$ in (1.6), we have

$$\int_0^T \int_R (u(x,t,m) - u(x,t,1))\phi_t \, dx \, dt = \int_0^T \int_R H_x \phi_x \, dx \, dt, \tag{3.3.1}$$

in which,

$$H = \frac{1}{m} u^m(x,t,m) - u(x,t,1).$$

Notice that

$$\begin{aligned} \int_0^T \int_R H_x \phi_x \, dx \, dt &= \int_0^T \int_R H_x \eta f_l'(x) \, dx \, dt + \int_0^T \int_R H_x \eta_x f_l(x) \, dx \, dt \\ &\stackrel{\text{def}}{=} I_1 + I_2. \end{aligned}$$

By (1.5),

$$\int_R |H| \, dx \leqslant \left(\frac{1}{m} M^{m-1} + 1 \right) \bar{u}_0,$$

where $\bar{u}_0 = \int_R u_0 \, dx$. Hence Lemma 2.2 implies that there exist positive constants C_3 and C_4, which do not depend on l, such that

$$\begin{aligned} |I_1| &\leqslant \int_0^T \int_{l \leqslant |x| \leqslant 2l} \frac{C_0}{l} |H_x \eta(x,t)| \, dx \, dt \\ &\leqslant \frac{C_0}{l} \int_0^T \int_{l \leqslant |x| \leqslant 2l} |H_x| \left(\int_0^T |H(x,\tau)| d\tau \right) dx \, dt \\ &\leqslant \frac{C_0 C_3}{l} \int_0^T t^{-\frac{1}{2}} \int_{l \leqslant |x| \leqslant 2l} \int_0^T |H(x,\tau)| d\tau \, dx \, dt \\ &\leqslant \frac{C_4}{l}. \end{aligned}$$

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It is easy to see that

$$I_2(T)|_{T=0} = 0,$$
$$\frac{dI_2}{dT} = -\frac{1}{2}\int_R \frac{d}{dT}\left(\int_0^T H_x\,dt\right)^2 f_l(x)\,dx.$$

Thus $I_2 \leqslant 0$ and therefore,

$$\int_0^T \int_R (u(x,t,m) - u(x,t,1))\phi_t\,dx\,dt \leqslant \frac{C_4}{l}. \tag{3.3.2}$$

On the other hand,

$$\begin{aligned}
&\int_0^T \int_R (u(x,t,m) - u(x,t,1))\phi_t\,dx\,dt \\
&\quad = \int_0^T \int_R (u(x,t,m) - u(x,t,1))Hf_l(x)\,dx\,dt \\
&\quad = \frac{1}{m}\int_0^T \int_R (u(x,t,m) - u(x,t,1))(u^m(x,t,m) - u(x,t,m))f_l(x)\,dx\,dt \\
&\qquad + \frac{1}{m}\int_0^T \int_R (u(x,t,m) - u(x,t,1))^2 f_l(x)\,dx\,dt \\
&\qquad + \frac{1-m}{m}\int_0^T \int_R (u(x,t,m) - u(x,t,1))u(x,t,1)f_l(x)\,dx\,dt.
\end{aligned}$$

It follows from $0 \leqslant u \leqslant M$ that

$$\begin{aligned}
u^m(x,t,m) - u(x,t,m) &= \int_0^1 \left(\frac{d}{ds}(u(x,t,m))^{sm+(1-s)}\right) ds \\
&= (m-1)u^\eta(x,t,m)\ln u(x,t,m)
\end{aligned}$$

with $\eta = m + \theta(1-m)$, $0 < \theta < 1$. Letting $l \longrightarrow \infty$, using (3.2), we have

$$\begin{aligned}
&\int_0^T \int_R (u(x,t,m) - u(x,t,1))^2\,dx\,dt \\
&\quad \leqslant (m-1)\int_0^T \int_R |u(x,t,m) - u(x,t,1)|(|u^\eta(x,t,m)\ln u(x,t,m)| + u(x,t,1))\,dx\,dt.
\end{aligned}$$

Hölder's inequality implies that

$$\begin{aligned}
&\int_0^T \int_R (u(x,t,m) - u(x,t,1))^2\,dx\,dt \\
&\quad \leqslant 2(m-1)^2\int_0^T \int_R (|u^\eta(x,t,m)\ln u(x,t,m)|^2 + u^2(x,t,1))\,dx\,dt.
\end{aligned}$$

By [7] (p.791), there are positive constants $C(m)$ for $m \geqslant 1$ such that

$$u(x,t,m) \leqslant C(m)t^{-\frac{1}{1+m}} \qquad \text{for } (x,t) \in Q_T.$$

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Since $\eta > 1$ and $u \leqslant M$, hence, for any $0 < \alpha < 1$ there exists a positive constant C_5 which does not depend on $m \in [1, 2)$ such that $u^{2\eta-1-\alpha}|\ln u|^2(C(m))^\alpha \leqslant C_5$, and therefore,

$$\begin{aligned}
&\int_0^T \int_R (u(x,t,m) - u(x,t,1))^2 \, dx \, dt \\
&\leqslant 2(m-1)^2 \int_0^T \int_R \left(C_5 t^{-\frac{\alpha}{1+m}} u(x,t,m) + C(1) t^{-\frac{1}{2}} u(x,t,1)\right) dx \, dt \\
&\leqslant 2(m-1)^2 \overline{u}_0 \left(\frac{m+1}{1+m-\alpha} C_5 T^{\frac{1+m-\alpha}{1+m}} + 2C(1) T^{\frac{1}{2}}\right). \qquad (3.3.3)
\end{aligned}$$

Denote

$$(C^*)^2 = 2\overline{u}_0 \left(\frac{m+1}{1+m-\alpha} C_5 T^{\frac{1+m-\alpha}{1+m}} + 2C(1) T^{\frac{1}{2}}\right),$$

Then (3.3) yields (1.11). Clearly, $\frac{1+m-\alpha}{1+m} > \frac{1}{2}$ for all $\alpha \in (0,1)$, so

$$C^* = O(T^\gamma) \qquad \text{with} \quad \gamma = \frac{1+m-\alpha}{2(1+m)}$$

if T is large enough.

4 An example

As an example, we test the present result on explicit solutions, namely the Barenblatt solution and the fundament solution of the heat equation. Let

$$G(s) = [(\beta^2 - c^2 s^2)^+]^{\frac{1}{m-1}},$$

where

$$c^2 = \frac{l(m-1)}{2m}, \qquad l = \frac{1}{1+m}$$

and β is a positive constant such that $\int_R G(x)\,dx = 1$. By Friedman & Kamin [3],

$$B = t^{-l} G\left(\frac{x}{t^l}\right)$$

is a solution of the equation

$$B_t = (B^m)_{xx}, \qquad m > 1$$

and $B(x,0) = \delta(x)$, where $\delta(x)$ is the δ-function. Denote $J = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{t}}$. It is well-known that J is the fundament solution of the heat equation $J_t = J_{xx}$ with $J(x,0) = \delta(x)$.

Clearly, the functions B_m and J_1 are not bounded as $t \longrightarrow 0$, so the condition (1.2) of our theorem is not satisfied. Fortunately, we can also use the above procedure. But certainly, the result is different.

To do this, we also set

$$\eta = \int_T^t (B^m - J) d\tau \qquad \text{for } t \in (0, T).$$

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Because $(B-J)_t = (B^m - J)_{xx}$ and $(B-J) \longrightarrow 0$ as $x \longrightarrow \infty$, so

$$\int_0^T \int_R (B-J)\eta_t \, dx\, dt = \int_0^T \int_R (B-J)_x \eta_x \, dx\, dt.$$

The same procedure yields

$$\int_0^T \int_R (B-J)_x \eta_x \, dx\, dt = -\frac{1}{2} \int_R \left(\int_0^T (B-J)_x \, dt \right)^2 dx.$$

Thus, we also have

$$\int_0^T \int_R (B-J)^2 \, dx\, dt \leqslant 2(m-1)^2 \int_0^T \int_R (|B^\eta \ln B|^2 + J^2) \, dx\, dt, \tag{4.4.1}$$

where $1 < \eta < m$. To estimate the right hand of (4.1), we first find

$$\int_0^T \int_R J^2 \, dx\, dt = T^{\frac{1}{2}}. \tag{4.4.2}$$

Secondly, since $\int_{c|s| \leqslant \beta} (\beta^2 - c^2 s^2)^{\frac{1}{m-1}} ds = 1$, then,

$$\int_0^\beta (\beta^2 - \xi^2)^{\frac{1}{m-1}} d\xi = \frac{c}{2}.$$

Perform the change $\xi = \beta \cos\theta$, thus

$$\beta^{\frac{m+1}{m-1}} \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{m+1}{m-1}} d\theta = \frac{c}{2},$$

or,

$$\beta = \left(\frac{c}{2}\right)^{\frac{m-1}{m+1}} \left(\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{m+1}{m-1}} d\theta \right)^{-\frac{m-1}{m+1}}.$$

Recalling $c = \sqrt{\frac{l(m-1)}{2m}}$ and using L'Hospital's method, we know that there is a positive constant ε such that

$$0 < \left(\frac{c}{2}\right)^{\frac{m-1}{m+1}} < \varepsilon$$

and

$$0 < \left(\int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{m+1}{m-1}} d\theta \right)^{-\frac{m-1}{m+1}} < \varepsilon$$

uniformly for all $m \in [1,2)$. Thus, $0 < \beta < \varepsilon^2$ for all $m \in [1,2)$. Therefore, if we set $\xi = \frac{c}{t^l} x$, then

$$\begin{aligned}\int_0^T \int_R |B^\eta \ln B|^2 \, dx\, dt &= \frac{1}{c} \int_0^T \int_{|\xi| \leqslant \beta} t^{l-2l\eta} (\beta^2 - \xi^2)^{\frac{2\eta}{m-1}} \left[-l \ln t - \ln(\beta^2 - \xi^2)^{\frac{1}{m-1}} \right]^2 \\ &\leqslant \frac{2}{c} \int_0^T \int_{|\xi| \leqslant \beta} t^{l-2l\eta} (\beta^2 - \xi^2)^{\frac{2\eta}{m-1}} \left[(l \ln t)^2 + (\ln(\beta^2 - \xi^2)^{\frac{1}{m-1}})^2 \right].\end{aligned}$$

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Clearly, $(\beta^2-\xi^2)^{\frac{2\eta}{m-1}}$ and $(\beta^2-\xi^2)^{\frac{2\eta}{m-1}}(\ln(\beta^2-\xi^2)^{\frac{1}{m-1}})^2$ are bounded for all $|\xi|\leqslant\beta$, so there is a positive constant k_1 such that

$$\int_0^T\int_R |B^\eta \ln B|^2\,dx\,dt \leqslant \frac{k_1}{c}\int_0^T t^{l-2l\eta}[(\ln t)^2+1)]\,dt.$$

Since

$$\int_0^T t^{l-2l\eta}dt = \frac{1}{1+l-2\eta l}T^{1+l-2\eta l},$$

$$\int_0^T t^{l-2l\eta}(\ln t)^2 dt = O(T^{1+l-2\eta l}(\ln T)^2) \qquad \text{as } T \text{ is large enough},$$

so recalling $c=\sqrt{\frac{l(m-1)}{2m}}$ again, we know that there is a positive constant k_2 such that

$$\int_0^T\int_R |B^\eta \ln B|^2\,dx\,dt \leqslant \frac{k_2}{2\sqrt{m-1}}T^{1+l-2\eta l}(\ln T)^2, \qquad \text{as } T \text{ is large enough.} \quad (4.4.3)$$

Finally, combining (4.1), (4.2) and (4.3), we can obtain our result.

Corollary If B is the Barenblatt solution and J is the fundament solution of the heat equation. Then there is a positive constant C_* such that

$$\int_0^T\int_R (B-J)^2\,dx\,dt \leqslant C_*(m-1)^{\frac{3}{2}} \tag{4.4.4}$$

where $1<\eta<m$ and $C_*=O(T^{\frac{2+m-2\eta}{1+m}}(\ln T)^2)$ as T is large enough.

5 Conclusion

Set

$$\widetilde{A}=\{u(x,t,m);\ m\geqslant 1\},$$

where $u(x,t,m)$ are the solutions of the Cauchy problem

$$\begin{cases} u_t = \frac{1}{m}(u^m)_{xx}, & x\in\mathbb{R}, t>0,\\ u|_{t=0}=u_0(x), & x\in\mathbb{R}, \end{cases}$$

$u_0(x)$ satisfies (1.2). We prove that in this paper, the classical solution $u(x,t,1)$ is a limit of functions of $\widetilde{A}$ in space $L^2(Q_T)$ as $m\longrightarrow 1$. The convergence speed is controlled by $T^\gamma|m-1|$, where $\gamma=\frac{1+m-\alpha}{2(1+m)}$. But if $u_0(x)$ is not bounded, then the speed is different. As an example of such a case, we show the convergence speed of the Barenblatt solution tends to the fundament solution of the heat equation in the last section of the paper.

References

[1] Benilan, Ph. & Crandall, M. G. (1981) The continuous dependence on ϕ of solution of $u_t - \triangle\,\phi(u)=0$. *J. Indiana Univ. Math.* **30**, 161–177.

[2] Friedman, A. (1964) *Partial Differential Equations of Parabolic Type*. Prentice-Hall.

[3] Friedman, A. & Kamin, S. (1980) The asymptotic behavior of a gas in an n-dimensional porous medium *Trans. Am. Math. Soc.* **262**, 551–563.

[4] Cockburn, B. & Gripenberg, G. (1999) Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. *J. Diff. Eq.* **151**, 231–251.

[5] Gilding, B.H. & Peletier, L. A. (1976) The Cauchy problem for an equation in the theory of infiltration. *Arch. Rat. Mech. Anal.* **61**, 127–140.

[6] Ebmeyer, C. (2005) Regularity in Soblev spaces for the fast diffusion and the porus medium equation. *J. Math. Anal. Appl.* **307**, 134–152

[7] Vazquez, J. L. (1984) Behaviour of the velocity of one-dimensional flows in porous media. *Trans. Am. Math. Soc.* **286**, 787–802.

[8] Vazquez, J. L. (2005) Symmetrization and mass comparison for degenerate nonlinear parobolic and ralated elliptic equations. *Adv. Nonlinear Stud.* **5**, 87–131.

[9] Hui, K. M. (2004) Existence of solutions of the very fast diffusion equation. *Nonlinear Anal.* **58**, 75–101.