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Block Decomposition and Weighted Hausdorff Content

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Abstract. Block decomposition of L^p spaces with weighted Hausdorff content is established for $0 < p \leqslant 1$ and the Fefferman–Stein type inequalities are shown for fractional integral operators and some variants of maximal operators.

1 Introduction

The purpose of this paper is to develop a two-weight theory of dyadic Hausdorff content. Let $\mathcal{D}$ be the set of all dyadic cubes in $\mathbb{R}^n$, that is,

$$\mathcal{D} := \{2^{-k}(m + [0,1)^n) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

We first introduce the weighted Hausdorff content (*cf.* [16, 17]) and Choquet spaces. By weights we will always mean nonnegative and locally integrable functions on $\mathbb{R}^n$. Given a measurable set E and a weight w, $w(E) = \int_E w\,dx$, $|E|$ denotes the Lebesgue measure of E and $\mathbf{1}_E$ denotes the characteristic function of E.

Let w be a weight on $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$ and $0 < d \leq n$, then the d-dimensional weighted Hausdorff content H^d_w of E is defined by

$$H^d_w(E) := \inf \sum_{j=1}^{\infty} \fint_{Q_j} w\,dx\, \ell(Q_j)^d,$$

where the infimum is taken over all coverings of E by countable families of dyadic cubes Q_j, the barred integral $\fint_Q w\,dx$ stands for the usual integral average of w over Q, $w(E)/|E|$, and by $\ell(Q)$ we denote the side length of the cube Q. When $w \equiv 1$, we simply denote by H^d, which is the d-dimensional (dyadic) Hausdorff content. When $d = n$, one has $H^n_w(E) = w(E)$. We emphasize that the set function H^d_w is strong subadditive (*cf.* [11]), that is,

$$H^d_w(E \cup F) + H^d_w(E \cap F) \leq H^d_w(E) + H^d_w(F), \quad E, F \subset \mathbb{R}^n.$$

The Choquet integral of $f \geq 0$ with respect to a set function $\mathcal{C}$ is defined by

$$\int_{\mathbb{R}^n} f\,d\mathcal{C} := \int_0^\infty \mathcal{C}\big(\{x \in \mathbb{R}^n : f(x) > t\}\big)\,dt.$$

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The Choquet space $L^p(\mathcal{C})$, $0 < p < \infty$, is the set of all functions f such that $\int_{\mathbb{R}^n} |f|^p \, d\mathcal{C}$ is finite.

Thanks to the strong subadditivity of the set function H^d_w, one has sublinearity (*cf.* [11]), that is, for nonnegative functions f and g,

$$\int_{\mathbb{R}^n} (f+g) \, dH^d_w \le \int_{\mathbb{R}^n} f \, dH^d_w + \int_{\mathbb{R}^n} g \, dH^d_w,$$

which implies that the quantity

$$\|f\|_{L^p(H^d_w)} := \left(\int_{\mathbb{R}^n} |f|^p \, dH^d_w \right)^{1/p}$$

is the norm when $1 \le p < \infty$ and the quasi-norm when $0 < p < 1$.

We consider the following dyadic maximal operator. For a locally integrable function f on $\mathbb{R}^n$, the dyadic fractional maximal operator M_α, $0 \le \alpha < n$, is defined by

$$M_\alpha f(x) := \sup_{Q \in \mathcal{D}} \mathbf{1}_Q(x) \fint_Q |f| \, dy \, \ell(Q)^\alpha.$$

If $\alpha = 0$, M_0 is the usual dyadic Hardy–Littlewood maximal operator and it will be denoted by M.

In [12], the authors proved the following. For an arbitrary weight w, the Fefferman–Stein type inequality

$$\|M_\alpha f\|_{L^p(H^\delta_w)} \le C_p \|f\|_{L^p(H^d_{M_{\gamma p} w})} \tag{1.1}$$

holds for $0 \le \gamma \le \alpha$, $d/n < p < d/\alpha$ and $\delta = d - (\alpha - \gamma)p$. In both sides of (1.1), the relationship between fractional order and Hausdorff dimension is controlled by the simple equation $\delta + \alpha p = d + \gamma p$. A main ingredient in the proof of (1.1) is the following Sawyer type testing estimate. The testing inequality

$$\int_{\mathbb{R}^n} M_\alpha [\mathbf{1}_Q]^p \, dH^\delta_w \le C_p^p \fint_Q M_{\gamma p} w \, dx \, \ell(Q)^d \tag{1.2}$$

holds for any dyadic cube $Q \in \mathcal{D}$. It wasn't so clear why the testing inequality (1.2) implies the norm inequality (1.1). In this paper, introducing block decomposition with weighted Hausdorff content, we clarify its structure. Once we have had the decomposition, precious two-weight estimates for positive subadditive operators can be proved.

This paper is organized as follows. In Section 2 we introduce block decomposition with weighted Hausdorff content and give a characterization of two-weight inequalities with Hausdorff content for a subadditive operator. In Section 3 we apply this characterization theorem to several maximal operators, including the strong maximal operator and the fractional maximal operator on unweighted Choquet spaces, to reprove some known results for these operators. In Section 4 we discuss the weighted estimates for the composition of maximal operators, the fractional maximal operator, and the fractional integral operator. In appendices, we will verify the Hardy–Littlewood–Sobolev inequality for the fractional integral operator with Hausdorff content and will consider an application for the Riesz potential (usual fractional integral operator).

The letter C will be used for unimportant constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in

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different occurrences. By the notation $A \approx B$ we mean that $C^{-1}B \le A \le CB$ with some positive finite constant C independent of appropriate quantities. We write $X \lesssim Y$, $Y \gtrsim X$ if there is an independent constant C such that $X \le CY$.

2 Block Decomposition with Weighted Hausdorff Content

In what follows we introduce block decomposition with weighted Hausdorff content.

Definition 2.1 Let v be a weight on $\mathbb{R}^n$, $0 < p \le 1$ and $0 < d < n$. The block space $B^{p;\infty,d}(v)$ is defined by the set of all measurable functions f on $\mathbb{R}^n$ with the quasi-norm

$$\|f\|_{B^{p;\infty,d}(v)} := \inf\Big\{ \|\{c_j\}\|_{l^p} : f = \sum_j c_j b_j \Big\} < \infty,$$

where b_j is a $(p;\infty,d,v)$-block and

$$\|\{c_j\}\|_{l^p} := \Big(\sum_j |c_j|^p\Big)^{1/p}$$

and the infimum is taken over all possible decompositions of f. We say in addition that a function b on $\mathbb{R}^n$ is a $(p;\infty,d,v)$-block provided that b is supported on a dyadic cube $Q \in \mathcal{D}$ and satisfies

$$\|b\|_{L^\infty}\left(\fint_Q v \,\mathrm{d}x \ell(Q)^d \right)^{1/p} \le 1.$$

Theorem 2.2 *Let v be a weight on $\mathbb{R}^n$, $0 < p \le 1$ and $0 < d < n$. Then $L^p(H_v^d) = B^{p;\infty,d}(v)$ with $\|\cdot\|_{L^p(H_v^d)} \approx \|\cdot\|_{B^{p;\infty,d}(v)}$.*

Proof Assume that the nonnegative function f belongs to $L^p(H_v^d)$. Consider $E_k = \{x \in \mathbb{R}^n : f(x) > 2^k\}$, $k \in \mathbb{Z}$. Then,

$$\int_{\mathbb{R}^n} f^p \,\mathrm{d}H_v^d \approx \sum_k 2^{pk} H_v^d(E_k). \tag{2.1}$$

Indeed, by sublinearity,

$$\int_{\mathbb{R}^n} f^p \,\mathrm{d}H_v^d \lesssim \sum_k 2^{pk} H_v^d(E_k).$$

Conversely,

$$\begin{aligned}
\sum_k 2^{pk} H_v^d(E_k) &= \frac{1}{\log 2} \sum_k 2^{pk} H_v^d(E_k) \int_{2^{k-1}}^{2^k} \frac{\mathrm{d}t}{t} \\
&\le \frac{1}{\log 2} \sum_k 2^{pk} \int_{2^{k-1}}^{2^k} H_v^d\big(\{x \in \mathbb{R}^n : f(x) > t\}\big) \frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^\infty H_v^d\big(\{x \in \mathbb{R}^n : f(x) > t\}\big)\, t^{p-1}\,\mathrm{d}t \\
&\approx \int_{\mathbb{R}^n} f^p \,\mathrm{d}H_v^d.
\end{aligned}$$

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We can select a set of the pairwise disjoint dyadic cubes $\{Q_{k,j}\} \subset \mathcal{D}$ such that $E_k \subset \bigcup_j Q_{k,j}$ and

$$\sum_j \fint_{Q_{k,j}} v \, dx \ell(Q_{k,j})^d \le 2H_v^d(E_k). \tag{2.2}$$

Upon defining

$$\Delta_{k,j} := Q_{k,j} \setminus \bigcup_i Q_{k+1,i},$$

we see that the sets $\Delta_{k,j}$ are pairwise disjoint and $\operatorname{supp} f = \bigcup_{k,j} \Delta_{k,j}$. With this, we obtain

$$f = \sum_{k,j} c_{k,j} b_{k,j},$$

where

$$c_{k,j} = 2^{k+1} \left(\fint_{Q_{k,j}} v \, dx \ell(Q_{k,j})^d \right)^{1/p}$$

and

$$b_{k,j} = \frac{f \chi_{\Delta_{k,j}}}{c_{k,j}}.$$

It is easy to check that each $b_{k,j}$ is a $(p;\infty,d,v)$-block, since we have $f(x) \le 2^{k+1}$ if $x \in \Delta_{k,j}$. To prove that $f \in B^{p;\infty,d}(v)$, it remains to verify that $\{c_{k,j}{}^p\}$ is summable. It follows that

$$\begin{aligned} \|\{c_{k,j}\}\|_{l^p}^p &= 2^p \sum_{k,j} 2^{pk} \fint_{Q_{k,j}} v \, dx \ell(Q_{k,j})^d \\ &\lesssim \sum_k 2^{pk} H_v^d(E_k) \\ &\approx \|f\|_{L^p(H_v^d)}^p, \end{aligned}$$

where we have used (2.1) and (2.2). This proves $L^p(H_v^d) \subset B^{p;\infty,d}(v)$ with $\|\cdot\|_{B^{p;\infty,d}(v)} \lesssim \|\cdot\|_{L^p(H_v^d)}$.

We now prove converse. Suppose that f belongs to $B^{p;\infty,d}(v)$. Thus, $f = \sum_j c_j b_j$ with $\{c_j\} \in l^p$ and each b_j is a $(p;\infty,d,v)$-block. Assume that Q_j is the support cube of b_j. Then we have

$$\int |b_j|^p \, dH_v^d \le \|b_j\|_{L^\infty}^p \fint_{Q_j} v \, dx \ell(Q_j)^d \le 1.$$

Thus, thanks to $p \in (0,1]$ and the sublinearity of dH_v^d,

$$\int |f|^p \, dH_v^d \le \sum_j |c_j|^p \int |b_j|^p \, dH_v^d \le \sum_j |c_j|^p$$

and, hence, $B^{p;\infty,d}(v) \subset L^p(H_v^d)$ with $\|\cdot\|_{L^p(H_v^d)} \le \|\cdot\|_{B^{p;\infty,d}(v)}$. These complete the proof. ∎

Remark Theorem 2.2 for $p = 1$ and $v \equiv 1$ was first verified in [3, Remark 3.4]. Block spaces as the predual of Morrey spaces were first introduced in [18]. The Fatou property of block spaces was shown in [13].

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By the use of Theorem 2.2, we give a characterization of two-weight inequalities with Hausdorff content for a subadditive operator.

Suppose that T is a subadditive operator. Let w and v be arbitrary weights on $\mathbb{R}^n$, and $0 < d, \delta < n$.

Theorem 2.3 *Let $0 < p \le 1$ and $q \ge p$. Then the following statements are equivalent.*

(a) *The two-weight norm inequality*

$$\|Tf\|_{L^q(H_w^\delta)} \le C_1 \|f\|_{L^p(H_v^d)}$$

holds.

(b) *The testing inequality*

$$\|Tb\|_{L^q(H_w^\delta)} \le C_2$$

holds for any $(p; \infty, d, v)$-block b.

Moreover, the least possible constants C_1 and C_2 are comparable.

Proof By testing (a) on $f = b$, we see that (b) follows at once from (a). We shall verify the converse.

Decompose $f \in L^p(H_v^d)$ as $f = \sum_j c_j b_j$, where $\|\{c_j\}\|_{l^p} \approx \|f\|_{L^p(H_v^d)}$ and b_j is a $(p; \infty, d, v)$-block. Thanks to subadditivity of T and the assumption $0 < p \le 1$, we have that

$$|Tf|^p \le \sum_j |c_j|^p \, |Tb_j|^p.$$

By the triangle inequality of the norm $\|\cdot\|_{L^{q/p}(H_w^\delta)}$, we have further that

$$\begin{aligned} \|Tf\|_{L^q(H_w^\delta)}^p &= \left\| |Tf|^p \right\|_{L^{q/p}(H_w^\delta)} \\ &\le \sum_j |c_j|^p \left\| |Tb_j|^p \right\|_{L^{q/p}(H_w^\delta)} \\ &\le {C_2}^p \sum_j |c_j|^p \\ &\approx C_2^p \|f\|_{L^p(H_v^d)}^p, \end{aligned}$$

where we have used (b). This completes the proof. ∎

3 Examples: Unweighted Cases

In what follows we reprove some known results as an application of Theorem 2.3.

Example 3.1 We apply Theorem 2.3 with $T = M$. It was shown in [8, Lemma 1] that

$$\int_{\mathbb{R}^n} M[\mathbf{1}_Q]^p \, \mathrm{d}H^d \lesssim \ell(Q)^d, \quad \frac{d}{n} < p < \infty.$$

This yields (b), and thus (a):

$$\|Mf\|_{L^p(H^d)} \lesssim \|f\|_{L^p(H^d)}, \quad \frac{d}{n} < p \le 1.$$

Especially,

$$\|Mf\|_{L^1(H^d)} \lesssim \|f\|_{L^1(H^d)},$$

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which yields

$$\|M[|f|^p]\|_{L^1(H^d)} \lesssim \big\| |f|^p \big\|_{L^1(H^d)}, \quad p > 1.$$

Since we have $(Mf)^p \le M[|f|^p]$,

$$\|Mf\|_{L^p(H^d)} \lesssim \|f\|_{L^p(H^d)}, \quad p > 1.$$

Thus, we conclude that

$$\|Mf\|_{L^p(H^d)} \lesssim \|f\|_{L^p(H^d)}, \quad \frac{d}{n} < p < \infty.$$

Example 3.2 For a locally integrable function f on $\mathbb{R}^n$, the dyadic strong maximal operator M_S is defined by

$$M_S f(x) := \sup_R \mathbf{1}_R(x) \fint_R |f| \, \mathrm{d}y,$$

where the supremum is taken over all dyadic rectangles $R \subset \mathbb{R}^n$ forming the Cartesian product of the dyadic intervals in $\mathbb{R}$. In [10], it is proved that

$$\int_{\mathbb{R}^n} M_S[\mathbf{1}_Q]^p \, \mathrm{d}H^d \lesssim \ell(Q)^d$$

holds for $\min(1, d) < p < \infty$. Invoking the same argument as in Example 3.1, when $0 < d < 1$, we obtain

$$\|M_S f\|_{L^p(H^d)} \lesssim \|f\|_{L^p(H^d)}, \quad d < p < \infty, \quad 0 < d < 1.$$

Example 3.3 We treat the fractional maximal operator M_α, $0 \le \alpha < n$. We have that the testing inequality

$$\|M_\alpha[\mathbf{1}_Q]\|_{L^q(H^\delta)} \lesssim \ell(Q)^{d/p}, \quad q \ge p, \frac{d}{n} < p < \frac{d}{\alpha}, \frac{\delta}{q} + \alpha = \frac{d}{p}, \tag{3.1}$$

holds for every $Q \in \mathcal{D}$. Indeed, we proved the pointwise expression in [12]

$$M_\alpha[\mathbf{1}_Q](x) = \ell(Q)^\alpha \Big(\mathbf{1}_Q(x) + \sum_{j=1}^{\infty} 2^{-(n-\alpha)j} \mathbf{1}_{\pi^j(Q) \setminus \pi^{j-1}(Q)}(x) \Big), \tag{3.2}$$

where $\pi^0(Q) = Q$ and $\pi^j(Q)$ is the smallest dyadic cube containing $\pi^{j-1}(Q)$ for $j = 1, 2, \ldots$. This and the sublinearity of $\mathrm{d}H^\delta$ give us that

$$\begin{aligned}
\|M_\alpha[\mathbf{1}_Q]\|_{L^q(H^\delta)}^q &= \int_{\mathbb{R}^n} M_\alpha[\mathbf{1}_Q]^q \, \mathrm{d}H^\delta \\
&\le \ell(Q)^{\alpha q} \sum_{j=0}^{\infty} 2^{-q(n-\alpha)j} H^\delta\big(\pi^j(Q)\big) \\
&= \ell(Q)^{\delta + \alpha q} \sum_{j=0}^{\infty} 2^{(\delta - q(n-\alpha))j} \\
&\lesssim \ell(Q)^{\delta + \alpha q} = \ell(Q)^{qd/p},
\end{aligned}$$

where we have used the fact that

$$H^\delta\big(\pi^j(Q)\big) = \ell\big(\pi^j(Q)\big)^\delta = 2^{\delta j} \ell(Q)^\delta$$

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and

$$\delta - q(n-\alpha) = q\Big(\frac{\delta}{q} - n + \alpha\Big) = q\Big(\frac{d}{p} - n\Big) < 0.$$

It follows from Theorem 2.3 and the testing inequality (3.1) that

$$\|M_\alpha f\|_{L^q(H^\delta)} \lesssim \|f\|_{L^p(H^d)}, \quad q \ge p, \frac{d}{n} < p < \min\Big(1, \frac{d}{\alpha}\Big), \frac{\delta}{q} + \alpha = \frac{d}{p}$$

and, when $1 < d/\alpha$, that

$$\|M_\alpha f\|_{L^q(H^\delta)} \lesssim \|f\|_{L^1(H^d)}, \quad q \ge 1, \frac{\delta}{q} + \alpha = d. \tag{3.3}$$

If $1 < p < d/\alpha$, then, in (3.3), letting $f = |g|^p$ and $r = pq$ and using $(M_\beta g)^p \le M_\alpha[|g|^p]$, $\beta p = \alpha$, we have that

$$\|M_\beta g\|_{L^r(H^\delta)} \lesssim \|g\|_{L^p(H^d)}, \quad r \ge p, \frac{\delta}{r} + \beta = \frac{d}{p}.$$

In conclusion, we obtain

$$\|M_\alpha f\|_{L^q(H^\delta)} \lesssim \|f\|_{L^p(H^d)}, \quad q \ge p, \frac{d}{n} < p < \frac{d}{\alpha}, \frac{\delta}{q} + \alpha = \frac{d}{p}.$$

Remark Example 3.3 of $\alpha = 0$ is an extension of Example 3.1, which is a classical theorem due to Orobitg and Verdera in [1, 2, 8]. Especially, we have that

$$\|Mf\|_{L^{pn/d}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(H^d)}, \quad \frac{d}{n} < p < \infty,$$

which also follows from Hardy–Littlewood maximal theorem and [8, Lemma 3].

4 Fefferman–Stein Type Inequalities

In what follows we investigate the weighted estimates of (b) in Theorem 2.3 by the same strategy as in the previous section.

4.1 Composition of the Dyadic Hardy–Littlewood Maximal Operator

A composition of maximal operators plays an important role in the weight theory. In [9], Pérez established the weighted inequalities for singular integrals and their commutators with non-*a priori* assumption on the weights. The proof is based on the fact that the singular integral operators and their commutators concerned in [9] are controlled by the composition of the Hardy–Littlewood maximal operator. The composition of the Orlicz maximal operator and the fractional Orlicz maximal operator also have been studied in [4, 5]. In this subsection we shall show the pointwise behavior of the dyadic Hardy–Littlewood maximal operator acting on $\mathbf{1}_Q$ and will invoke the same strategy as in the previous section.

For the function $\mathbf{1}_Q$, $Q \in \mathcal{D}$, we shall compute the value of $M^k[\mathbf{1}_Q](x)$, where M^k denotes the k-fold composition of the dyadic Hardy–Littlewood maximal operator M.

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Lemma 4.1 *For $k = 1, 2, \ldots$ and $Q \in \mathcal{D}$, we have*

$$M^k[\mathbf{1}_Q](x) = a_0^{(k)}\mathbf{1}_Q(x) + \sum_{j=1}^{\infty} 2^{-nj} a_j^{(k)} \mathbf{1}_{\pi^j(Q)\setminus\pi^{j-1}(Q)}(x),$$

where

$$a_0^{(k)} = a_j^{(1)} = 1$$

and, letting $A = 1 - 2^{-n}$,

$$a_j^{(k)} = 1 + jA + \frac{j(j+1)}{2!}A^2 + \cdots + \frac{j(j+1)\cdots(j+k-2)}{(k-1)!}A^{k-1}.$$

Proof By considering the density of the function, we see that

$$M^1[\mathbf{1}_Q](x) = \mathbf{1}_Q(x) + \sum_{j=1}^{\infty} 2^{-nj} \mathbf{1}_{\pi^j(Q)\setminus\pi^{j-1}(Q)}(x) \tag{4.1}$$

and that, for $k = 2, 3, \ldots$,

$$M^k[\mathbf{1}_Q](x) = \fint_{\pi^j(Q)} M^{k-1}[\mathbf{1}_Q](y)\, \mathrm{d}y \quad \text{for all } x \in \pi^j(Q)\setminus\pi^{j-1}(Q).$$

Starting from (4.1), we first obtain

$$a_j^{(2)} = 2^{nj}\fint_{\pi^j(Q)} M^1[\mathbf{1}_Q](y)\, \mathrm{d}y = 1 + jA,$$

where we have used

$$|\pi^j(Q)\setminus\pi^{j-1}(Q)| = 2^{jn}|Q|A.$$

Next, we have that

$$\begin{aligned} a_j^{(3)} &= 2^{nj}\fint_{\pi^j(Q)} M^2[\mathbf{1}_Q](y)\, \mathrm{d}y \\ &= 1 + jA + \Big(\sum_{k=1}^{j} k\Big)A^2 \\ &= 1 + jA + \frac{j(j+1)}{2!}A^2. \end{aligned}$$

We have also that

$$\begin{aligned} a_j^{(4)} &= 2^{nj}\fint_{\pi^j(Q)} M^3[\mathbf{1}_Q](y)\, \mathrm{d}y \\ &= 1 + jA + \Big(\sum_{k=1}^{j} k\Big)A^2 + \Big(\sum_{k=1}^{j} \frac{k(k+1)}{2!}\Big)A^3 \\ &= 1 + jA + \frac{j(j+1)}{2!}A^2 + \frac{j(j+1)(j+2)}{3!}A^3. \end{aligned}$$

Continuing these steps, we obtain the desired relation. ∎

Remark 4.2 It follows from the Taylor expansion that

$$(1-t)^{-j} = 1 + jt + \frac{j(j+1)}{2!}t^2 + \frac{j(j+1)(j+2)}{3!}t^3 + \cdots. \tag{4.2}$$

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Surprisingly, the coefficient $a_j^{(k)}$ corresponds to the first k-th sum of (4.2) with $t = A$. Thus,

$$\lim_{k\to\infty} a_j^{(k)} = (1-A)^{-j} = 2^{nj}.$$

Let $0 < d, \delta < n$, $d/n < p \le q < \infty$, $\delta/q = d/p$. Let w be a weight on $\mathbb{R}^n$ and $Q \in \mathcal{D}$. We shall compute

$$\text{(i)} := \|M^k[\mathbf{1}_Q]\|_{L^q(H_w^\delta)}.$$

There holds by Lemma 4.1

$$\begin{aligned}
\text{(i)}^q &= \int_{\mathbb{R}^n} M^k[\mathbf{1}_Q](x)^q \, \mathrm{d}H_w^\delta \\
&\le \sum_{j=0}^{\infty} 2^{-qnj}(a_j^{(k)})^q H_w^\delta\big(\pi^j(Q)\big) \\
&\le \ell(Q)^\delta \sum_{j=0}^{\infty} 2^{(\delta-qn)j}(a_j^{(k)})^q \fint_{\pi^j(Q)} w \, \mathrm{d}x \\
&= \ell(Q)^{qd/p} \sum_{j=0}^{\infty} 2^{-q(n-d/p)j}(a_j^{(k)})^q \fint_{\pi^j(Q)} w \, \mathrm{d}x \\
&= \left\{ \ell(Q)^d \left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j}(a_j^{(k)})^q \fint_{\pi^j(Q)} w \, \mathrm{d}x \right)^{p/q} \right\}^{q/p},
\end{aligned}$$

where we have used

$$H_w^\delta\big(\pi^j(Q)\big) \le \fint_{\pi^j(Q)} w \, \mathrm{d}x \, \ell\big(\pi^j(Q)\big)^\delta = \ell(Q)^\delta 2^{\delta j} \fint_{\pi^j(Q)} w \, \mathrm{d}x.$$

Since we have had

$$\text{(i)} \le \left\{ \ell(Q)^d \left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j}(a_j^{(k)})^q \fint_{\pi^j(Q)} w \, \mathrm{d}x \right)^{p/q} \right\}^{1/p},$$

in order to obtain the estimate (b) of Theorem 2.3, we must seek a weight v that satisfies

$$\left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j}(a_j^{(k)})^q \fint_{\pi^j(Q)} w \, \mathrm{d}x \right)^{p/q} \le \fint_Q v \, \mathrm{d}x \quad \text{for all } Q \in \mathcal{D}.$$

We have the following theorem.

Theorem 4.3 *Let $0 < d, \delta < n$, $d/n < p \le q < \infty$, $\delta/q = d/p$. Let w be a weight on $\mathbb{R}^n$.*

(a) *If $p \le 1$, then there exists a constant C independent of k such that the Fefferman–Stein type inequality*

$$\|M^k f\|_{L^q(H_w^\delta)} \le C \|f\|_{L^p(H_v^d)}$$

holds, where

$$v(x) := \sup_{Q\in\mathcal{D}} \mathbf{1}_Q(x) \left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j}(a_j^{(k)})^q \fint_{\pi^j(Q)} w \, \mathrm{d}y \right)^{p/q}.$$

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(b) *If $p \geq 1$, then there exists a constant C independent of k such that the Fefferman–Stein type inequality*

$$\|M^k f\|_{L^q(H_w^\delta)} \leq C\|f\|_{L^p(H_v^d)},$$

holds, where

$$v(x) := \sup_{Q\in\mathcal{D}} \mathbf{1}_Q(x)\left(\sum_{j=0}^{\infty} 2^{-\frac{q}{p}(n-d/p)j}(a_j^{(k)})^{q/p} \fint_{\pi^j(Q)} w \, \mathrm{d}y\right)^{p/q}.$$

(c) *If $p \geq 1$, then there exists a constant C independent of k such that the Fefferman–Stein type inequality*

$$\|M^k f\|_{L^p(H_w^d)} \leq C\|f\|_{L^p(H_v^d)}$$

holds, where

$$v(x) := \sup_{Q\in\mathcal{D}} \mathbf{1}_Q(x) \sum_{j=0}^{\infty} 2^{-(n-d/p)j} a_j^{(k)} \fint_{\pi^j(Q)} w \, \mathrm{d}y.$$

Proof By the above discussion and Theorem 2.3, we need only verify (b). It follows from (a) that

$$\|M^k f\|_{L^q(H_w^\delta)} \leq C\|f\|_{L^1(H_v^d)}, \quad q \geq 1, \quad \frac{\delta}{q} = d, \tag{4.3}$$

where

$$v(x) := \sup_{Q\in\mathcal{D}} \mathbf{1}_Q(x)\left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j}(a_j^{(k)})^{q} \fint_{\pi^j(Q)} w \, \mathrm{d}y\right)^{1/q}.$$

In (4.3), letting $f = |g|^p$ and $q = pr$ and using $(M^k g)^p \leq M^k[|g|^p]$, we have

$$\|M^k g\|_{L^r(H_w^\delta)} \leq C^{1/p}\|g\|_{L^p(H_v^d)}, \quad r \geq p, \frac{\delta}{r} = \frac{d}{p},$$

where

$$v(x) := \sup_{Q\in\mathcal{D}} \mathbf{1}_Q(x)\left(\sum_{j=0}^{\infty} 2^{-\frac{r}{p}(n-d/p)j}(a_j^{(k)})^{r/p} \fint_{\pi^j(Q)} w \, \mathrm{d}y\right)^{p/r}.$$

This means (b). ∎

4.2 Fractional Maximal Operator

We treat again the fractional maximal operator M_α, $0 \leq \alpha < n$. Let $0 < d, \delta < n$, $q \geq p$, $d/n < p < d/\alpha$, $\delta/q + \alpha = d/p$. Let w be a weight on $\mathbb{R}^n$ and $Q \in \mathcal{D}$. In the same manner as Example 3.3, we shall compute

$$\text{(ii)} := \|M_\alpha[\mathbf{1}_Q]\|_{L^q(H_w^\delta)}.$$

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Using (3.2), we have that

$$\begin{aligned}
(\mathrm{ii})^q &= \int_{\mathbb{R}^n} M_\alpha[\mathbf{1}_Q]^q \, \mathrm{d}H_w^\delta \\
&\le \ell(Q)^{\alpha q} \sum_{j=0}^{\infty} 2^{-q(n-\alpha)j} H_w^\delta\big(\pi^j(Q)\big) \\
&\le \ell(Q)^{\delta+\alpha q} \sum_{j=0}^{\infty} 2^{(\delta-q(n-\alpha))j} \fint_{\pi^j(Q)} w \, \mathrm{d}x \\
&= \ell(Q)^{qd/p} \sum_{j=0}^{\infty} 2^{-q(n-d/p)j} \fint_{\pi^j(Q)} w \, \mathrm{d}x \\
&= \left\{ \ell(Q)^d \left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j} \fint_{\pi^j(Q)} w \, \mathrm{d}x \right)^{p/q} \right\}^{q/p},
\end{aligned}$$

where we have used

$$H_w^\delta(\pi^j(Q)) \le \fint_{\pi^j(Q)} w \, \mathrm{d}x \, \ell\big(\pi^j(Q)\big)^\delta = \ell(Q)^\delta 2^{\delta j} \fint_{\pi^j(Q)} w \, \mathrm{d}x$$

and

$$\delta - q(n-\alpha) = q\left(\frac{\delta}{q} - n + \alpha\right) = -q\left(n - \frac{d}{p}\right).$$

This implies

$$(\mathrm{ii}) \le \left\{ \ell(Q)^d \left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j} \fint_{\pi^j(Q)} w \, \mathrm{d}x \right)^{p/q} \right\}^{1/p}.$$

We have the following theorem.

Theorem 4.4 *Let $0 < d, \delta < n$, $q \ge p$, $d/n < p < d/\alpha$, $\delta/q + \alpha = d/p$. Let w be a weight on $\mathbb{R}^n$.*

(a) *If $p \le 1$, then the Fefferman–Stein type inequality*

$$\|M_\alpha f\|_{L^q(H_w^\delta)} \lesssim \|f\|_{L^p(H_v^d)}$$

holds, where

$$v(x) := \sup_{Q\in\mathcal{D}} \mathbf{1}_Q(x) \left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j} \fint_{\pi^j(Q)} w \, \mathrm{d}y \right)^{p/q}.$$

(b) *If $p \ge 1$, then the Fefferman–Stein type inequality*

$$\|M_\alpha f\|_{L^q(H_w^\delta)} \lesssim \|f\|_{L^p(H_v^d)}$$

holds, where

$$v(x) := \sup_{Q\in\mathcal{D}} \mathbf{1}_Q(x) \left(\sum_{j=0}^{\infty} 2^{-\frac{q}{p}(n-d/p)j} \fint_{\pi^j(Q)} w \, \mathrm{d}y \right)^{p/q}.$$

Proof This can be proved by the same arguments in the proof of Theorem 4.3 and in Example 3.2. ∎

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4.3 Fractional Integral Operator

The dyadic fractional integral operator I_α, $0 < \alpha < n$, is defined by

$$I_\alpha f(x) := \sum_{Q \in \mathcal{D}} \mathbf{1}_Q(x) \fint_Q f \, \mathrm{d}y \ell(Q)^\alpha.$$

This was first introduced by Sawyer and Wheeden [15], and one can prove that (*cf.* [6]) a finite number of family of dyadic fractional integral operators control the Riesz potential (usual fractional integral operator), which is given by

$$\mathcal{I}_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}y.$$

It is easy to see that

$$M_\alpha f(x) \le I_\alpha f(x) \text{ a.e. } x \in \mathbb{R}^n.$$

In general, there is no pointwise inequality in the reverse direction, but the two quantities are comparable in L^p sense, that is,

$$\|I_\alpha f\|_{L^p(\mathbb{R}^n)} \le C_{\alpha,p} \|M_\alpha f\|_{L^p(\mathbb{R}^n)}, \quad 0 < \alpha < n, \quad 0 < p < \infty.$$

For $Q \in \mathcal{D}$, we shall compute $I_\alpha[\mathbf{1}_Q](x)$. It suffices to consider the point x in $E := \bigcup_{j=0}^\infty \pi^j(Q)$. If $x \in Q$, we see that

$$\begin{aligned} \sum_{x \in P, P \subset Q : P \in \mathcal{D}} \fint_P \mathbf{1}_Q \, \mathrm{d}y \ell(P)^\alpha &= \sum_{x \in P, P \subset Q : \, P \in \mathcal{D}} \ell(P)^\alpha \\ &\approx \ell(Q)^\alpha \end{aligned}$$

and that

$$\sum_{j=0}^\infty \frac{|Q|}{|\pi^j(Q)|} \ell(\pi^j(Q))^\alpha \approx \ell(Q)^\alpha.$$

Thus,

$$I_\alpha[\mathbf{1}_Q](x) \approx \ell(Q)^\alpha, \quad x \in Q. \tag{4.4}$$

If $x \in E$ and $x \notin Q$, then, letting j_0 be the first integer that satisfies $x \in \pi^{j_0}(Q)$, we have that

$$I_\alpha[\mathbf{1}_Q](x) = \sum_{j=j_0}^\infty \frac{|Q|}{|\pi^j(Q)|} \ell\big(\pi^j(Q)\big)^\alpha \approx \ell(Q)^\alpha 2^{-(n-\alpha)j_0}, \quad x \in \pi^{j_0}(Q) \setminus \pi^{j_0-1}(Q). \tag{4.5}$$

By (3.2), (4.4) and (4.5), we obtain

$$I_\alpha[\mathbf{1}_Q](x) \approx M_\alpha[\mathbf{1}_Q](x), \quad x \in \mathbb{R}^n.$$

We have the following theorem.

Theorem 4.5 *Let $0 < d, \delta < n$, $q \ge p$, $d/n < p < d/\alpha$, $p \le 1$, $\delta/q + \alpha = d/p$. Let w be a weight on $\mathbb{R}^n$. Then, the Fefferman–Stein type Inequality*

$$\|I_\alpha f\|_{L^q(H_w^\delta)} \lesssim \|f\|_{L^p(H_v^d)}$$

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holds, where

$$v(x) := \sup_{Q \in \mathcal{D}} \mathbf{1}_Q(x) \left(\sum_{j=0}^{\infty} 2^{-q(n-d/p)j} \fint_{\pi^j(Q)} w \, \mathrm{d}y \right)^{p/q}.$$

Remark One can not expect the following inequality:

$$|I_\alpha g|^p \lesssim I_\alpha[|g|^p], \quad p > 1.$$

5 Appendices

In what follows we give two appendices.

5.1 The Hardy–Littlewood–Sobolev inequality

We shall verify the Hardy–Littlewood–Sobolev inequality for I_α, $0 < \alpha < n$, with Hausdorff content.

Theorem 5.1 *Let $0 < d < n$, $d/n < p < d/\alpha$ and q be defined by $1/q = 1/p - \alpha/d$. Then we have*

$$\|I_\alpha f\|_{L^q(H^d)} \lesssim \|f\|_{L^p(H^d)}.$$

Proof Choose a dyadic cube $P \in \mathcal{D}$ so that $x \in P$ and

$$\frac{I_\alpha f(x)}{2} \le \begin{cases} \sum_{x \in Q, Q \subset P : Q \in \mathcal{D}} \fint_Q f \, \mathrm{d}y \ell(Q)^\alpha, \\ \sum_{x \in Q, Q \supset P : Q \in \mathcal{D}} \fint_Q f \, \mathrm{d}y \ell(Q)^\alpha. \end{cases}$$

A calculus of geometric series gives

$$\sum_{x \in Q, Q \subset P : Q \in \mathcal{D}} \fint_Q f \, \mathrm{d}y \ell(Q)^\alpha \lesssim \ell(P)^\alpha M f(x).$$

Since $d/n < p$ implies $1 < \frac{np}{d}$,

$$\begin{aligned} \fint_Q f \, \mathrm{d}y \ell(Q)^\alpha &\le \ell(Q)^\alpha \left(\fint_Q f^{\frac{np}{d}} \, \mathrm{d}x \right)^{\frac{d}{np}} \\ &= \ell(Q)^{\alpha - d/p} \left(\int_Q f^{\frac{np}{d}} \, \mathrm{d}x \right)^{\frac{d}{np}} \\ &\lesssim \ell(Q)^{\alpha - d/p} \left(\int_Q f^p \, \mathrm{d}H^d \right)^{1/p} \\ &\lesssim \ell(Q)^{\alpha - d/n} \|f\|_{L^p(H^d)}, \end{aligned}$$

where we have used the formula, [8, Lemma 3],

$$\int g \, \mathrm{d}x \le \frac{n}{d} \left(\int g^{d/n} \, \mathrm{d}H^d \right)^{n/d}.$$

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If we assume $d/n < p < d/\alpha$, a calculus of geometric series again gives

$$\sum_{x\in Q, Q\supset P: Q\in\mathcal{D}} \fint_Q f \,\mathrm{d}y \ell(Q)^\alpha \lesssim \ell(P)^{\alpha-d/p} \|f\|_{L^p(H^d)}.$$

Thus,

$$\begin{aligned} I_\alpha f(x) &\lesssim \sup_{t>0} \min\left\{ t^\alpha Mf(x), t^{\alpha-d/p}\|f\|_{L^p(H^d)}\right\} \\ &\approx \|f\|_{L^p(H^d)}^{\frac{\alpha p}{d}} \cdot [Mf(x)]^{1-\frac{\alpha p}{d}}. \end{aligned}$$

If we choose $q > p$ so that $\frac{p}{q} = 1 - \frac{\alpha p}{d}$, then

$$\begin{aligned} \|I_\alpha f\|_{L^q(H^d)} &\lesssim \|f\|_{L^p(H^d)}^{\frac{\alpha p}{d}} \cdot \left(\int_{\mathbb{R}^n} (Mf)^p \,\mathrm{d}H^d \right)^{1/q} \\ &\lesssim \|f\|_{L^p(H^d)}^{\frac{\alpha p}{d}} \cdot \|f\|_{L^p(H^d)}^{\frac{p}{q}} \\ &\approx \|f\|_{L^p(H^d)}. \end{aligned}$$

This completes the proof. ∎

Remark 5.2 Since $M_\alpha f(x) \le I_\alpha f(x)$ holds for any positive f, we observe that the corresponding results hold for the fractional maximal operator as well.

5.2 The Riesz Potential

Recall that the Riesz potential (usual fractional integral operator) $\mathcal{I}_\alpha$, $0 < \alpha < n$, is defined by

$$\mathcal{I}_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \,\mathrm{d}y.$$

Let $\mathcal{T} := \{0, \pm 1/3\}^n$. For $\tau \in \mathcal{T}$, we define the τ translate dyadic cubes by

$$\mathcal{D}^\tau := \{2^{-k}(m + \tau + [0,1)^n) : k \in \mathbb{Z},\ m \in \mathbb{Z}^n\}$$

and define the dyadic fractional integral operator $I_\alpha[\mathcal{D}^\tau]$ by

$$I_\alpha[\mathcal{D}^\tau]f(x) := \sum_{Q\in\mathcal{D}^\tau} \mathbf{1}_Q(x) \fint_Q f \,\mathrm{d}y \ell(Q)^\alpha.$$

It was shown in [6] that the inequality

$$\mathcal{I}_\alpha f(x) \lesssim \sum_{\tau\in\mathcal{T}} I_\alpha[\mathcal{D}^\tau] f(x), \quad x \in \mathbb{R}^n \tag{5.1}$$

holds for the nonnegative locally integrable function f.

Let w be a weight on $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$ and $0 < d \le n$, then the d-dimensional weighted Hausdorff content $H_w^d[\mathcal{D}^\tau]$ of E with respect to $\mathcal{D}^\tau$ is defined by

$$H_w^d[\mathcal{D}^\tau](E) := \inf \sum_{j=1}^\infty \fint_{Q_j} w \,\mathrm{d}x \ell(Q_j)^d,$$

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where the infimum is taken over all coverings of E by countable families of dyadic cubes $\{Q_j\} \subset \mathcal{D}^\tau$. It was shown in [17, Proposition 3.4.2] that the relation

$$H_w^d[\mathcal{D}^\tau](E) \approx H_w^d[\mathcal{Q}](E) \tag{5.2}$$

holds for any $E \subset \mathbb{R}^n$, $\tau \in \mathcal{T}$ and doubling weight w. Here, $\mathcal{Q}$ stands for the family of all cubes in $\mathbb{R}^n$ which have their sides parallel to the coordinate axes. The relation (5.2) entails the norm equivalences

$$\|f\|_{L^p(H_w^d[\mathcal{D}^\tau])} \approx \|f\|_{L^p(H_w^d[\mathcal{Q}])}, \tag{5.3}$$

when the weight w is restricted to doubling weight.

Let now assume that two-weight norm inequality

$$\|I_\alpha[\mathcal{D}^\tau]f\|_{L^q(H_w^\delta[\mathcal{D}^\tau])} \lesssim \|f\|_{L^p(H_{v_\tau}^d[\mathcal{D}^\tau])}, \quad \tau \in \mathcal{T} \tag{5.4}$$

holds for appropriate parameters. Then, by (5.1), (5.3) and (5.4) we have that, for any doubling weight w,

$$\begin{aligned}
\|\mathcal{I}_\alpha f\|_{L^q(H_w^\delta[\mathcal{Q}])} &\lesssim \sum_{\tau\in\mathcal{T}} \|I_\alpha[\mathcal{D}^\tau]f\|_{L^q(H_w^\delta[\mathcal{D}^\tau])} \\
&\lesssim \sum_{\tau\in\mathcal{T}} \|f\|_{L^p(H_{v_\tau}^d[\mathcal{D}^\tau])} \\
&\lesssim \sum_{\tau\in\mathcal{T}} \|f\|_{L^p(H_v^d[\mathcal{D}^\tau])} \\
&\lesssim \|f\|_{L^p(H_v^d[\mathcal{Q}])},
\end{aligned}$$

where

$$v = \sum_{\tau\in\mathcal{T}} v_\tau$$

and in the last inequality we need the further assumption that v is doubling.

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