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THREE NON-TRIVIAL SOLUTIONS FOR NOT NECESSARILY COERCIVE p-LAPLACIAN EQUATIONS

SHOUCHUAN HU[1,2] AND NIKOLAS S. PAPAGEORGIOU[3]

[1] *College of Mathematics, Shandong Normal University, Jinan, Shandong, China*
[2] *Department of Mathematics, Missouri State University, Springfield, MO 65804, USA* (hu@math.smsu.edu)
[3] *Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece* (npapg@math.ntua.gr)

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Abstract We consider the existence of three non-trivial smooth solutions for nonlinear elliptic problems driven by the p-Laplacian. Using variational arguments, coupled with the method of upper and lower solutions, critical groups and suitable truncation techniques, we produce three non-trivial smooth solutions, two of which have constant sign. The hypotheses incorporate both coercive and non-coercive problems in our framework of analysis.

Keywords: non-trivial solutions; truncations; upper and lower solutions; p-Laplacian; nonlinear regularity; critical groups

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1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. We study here the existence of multiple non-trivial smooth solutions for the following nonlinear Dirichlet problem:

$$\left.\begin{aligned} -\Delta_p x(z) &= m(z)|x(z)|^{r-2}x(z) + f(z, x(z)) \quad \text{a.e. on } Z, \\ x|_{\partial Z} &= 0. \end{aligned}\right\} \tag{1.1}$$

Here $1 < r < p < \infty$ and $\Delta_p x = \operatorname{div}(\|Dx\|^{p-2}Dx)$, the p-Laplacian differential operator. Our goal is to prove a 'three-solutions theorem' for problem (1.1). Recently, such theorems were proved by Dancer and Perera [**3**], Liu [**8**], Liu and Liu [**9**], Papageorgiou and Papageorgiou [**10**] and Zhang and co-workers [**12**, **13**]. In all these works the Euler functional of the problem is coercive. In addition, in [**3**, **12**, **13**], the asymptotic limits

$$a_\pm = \lim_{x \to 0\pm} \frac{f(z, x)}{|x|^{p-2}x}$$

play an important role. Additional multiplicity results (two solutions) for coercive problems, using critical groups, can be found in [**4**]. Here the Euler functional need not be coercive. In fact, the hypotheses incorporate both coercive and non-coercive problems in our framework of analysis, since the conditions that we impose on the nonlinearity f concerning its behaviour near infinity are minimal. More precisely, we require only that $x \to f(z,x)$ has subcritical growth. Also, here we do not assume that the limits $a_\pm = \lim_{x\to 0\pm} f(z,x)/(|x|^{p-2}x)$ exist.

2. Preliminaries and hypotheses

In our analysis of problem (1.1), we shall use the Sobolev space $W_0^{1,p}(Z)$ and the subspace

$$C_0^1(\bar{Z}) = \{x \in C^1(\bar{Z}) : x|_{\partial Z} = 0\}.$$

Both $W_0^{1,p}(Z)$ and $C_0^1(\bar{Z})$ are ordered Banach spaces, with order cones given, respectively, by

$$W_+ = \{x \in W_0^{1,p}(Z) : x(z) \geqslant 0 \text{ a.e. on } Z\}$$

and

$$C_+ = \{x \in C_0^1(\bar{Z}) : x(z) \geqslant 0 \text{ for all } z \in Z\}.$$

In fact, C_+ has non-empty interior, given by

$$\operatorname{Int} C_+ = \left\{x \in C_+ : x(z) > 0 \text{ for all } z \in Z,\ \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\right\}.$$

Here we denote by $n(z)$ the outward unit normal at $z \in \partial Z$. In an ordered Banach space X with order cone K, we write $u \leqslant v$ if and only if $v - u \in K$, and $u < v$ if and only if $u \leqslant v$ and $u \neq v$. Also, if $u \leqslant v$, then we define

$$[u,v] = \{y \in W_0^{1,p}(Z) : u(z) \leqslant y(z) \leqslant v(z) \text{ a.e. on } Z\}.$$

Henceforth, by $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$, where $1/p + 1/p' = 1$, we denote the nonlinear operator corresponding to $-\Delta_p$ and defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}\,\mathrm{d}z \quad \text{for all } x, y \in W_0^{1,p}(Z).$$

Here, $\langle\cdot\,,\cdot\rangle$ denotes the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z))$.

Let $\lambda_1 > 0$ denote the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$ and let u_1 denote the L^p-normalized principal eigenfunction. It is known that u_1 does not change its sign, and so we may assume that $u_1 \geqslant 0$. Nonlinear regularity theory implies that $u_1 \in C_+$ and the nonlinear strong maximum principle of Vazquez [**11**] yields that $u_1 \in \operatorname{Int} C_+$.

Let X be a Banach space and $\varphi \in C^1(X)$. The critical groups of φ at an isolated critical point x with $\varphi(x) = c$ are defined by

$$C_k(\varphi, x) = H_k(\varphi^c, \varphi^c \setminus \{x\}) \quad \text{for all } k \geqslant 0,$$

where H_k is the kth singular relative homology group with coefficients in $\mathbb{Z}$ and $\varphi^c = \{x \in X : \varphi(x) \leqslant c\}$.

The hypotheses on the nonlinearity f are the following.

Hypothesis 2.1. $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z, 0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable;

(ii) for almost every $z \in Z$, $x \to f(z, x)$ is continuous;

(iii) for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$|f(z, x)| \leqslant a(z) + c|x|^{q-1},$$

where $a \in L^\infty(Z)_+$, $c > 0$ and

$$p < q < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geqslant N; \end{cases}$$

(iv) there exists $\tau \in (p, p^*)$ such that

$$\limsup_{x \to 0} \frac{f(z, x)}{|x|^{\tau-2}x} < \infty \text{ uniformly for almost every } z \in Z;$$

(v) $f(z, x)x > 0$ for almost every $z \in Z$ and all $x \neq 0$ (strict sign condition).

Hypothesis 2.2. $m \in L^\infty(Z)$, $m \geqslant 0$ and $m \neq 0$.

3. Two constant-sign solutions

We consider the truncated functions, $f_\pm : Z \times \mathbb{R} \to \mathbb{R}$, defined by

$$f_+(z, x) = f(z, x^+) \quad \text{and} \quad f_-(z, x) = f(z, -x^-)$$

We consider the following auxiliary nonlinear Dirichlet problem:

$$\left.\begin{aligned} -\Delta_p x(z) &= m(z)x^+(z)^{r-1} + f_+(z, x(z)) \quad \text{a.e. on } Z, \\ x|_{\partial Z} &= 0. \end{aligned}\right\} \tag{3.1}$$

By an upper solution for problem (3.1), we mean a function $\bar{x} \in W^{1,p}(Z)$ such that $\bar{x}|_{\partial Z} \geqslant 0$ and, for all $y \in W_+$,

$$\int_Z \|D\bar{x}\|^{p-2}(D\bar{x}, Dy)_{\mathbb{R}^N}\,\mathrm{d}z \geqslant \int_Z m(\bar{x}^+)^{r-1}y\,\mathrm{d}z + \int_Z f_+(z, \bar{x})y\,\mathrm{d}z.$$

We say that $\bar{x}$ is a strict upper solution for (3.1), if it is not a solution of (3.1). Next we derive a strict upper solution for problem (3.1).

Proposition 3.1. *If Hypotheses 2.1 and 2.2 hold, then there exists some* $\lambda_+^* > 0$ *such that problem (3.1) has a strict upper solution* $\bar{x} \in \operatorname{Int} C_+$*, provided that* $0 < \|m\|_\infty < \lambda_+^*$.

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Proof. By virtue of Hypothesis 2.1 (iii)–(v), we have, for almost every $z \in Z$ and all $x \geqslant 0$,

$$0 \leqslant m(z)x^{r-1} + f(z,x) \leqslant c_1(\|m\|_\infty^s + x^{\vartheta-1}), \tag{3.2}$$

where $c_1 > 0$, $1 < s$ and $p < \vartheta < p^*$.

Let $e \in \operatorname{Int} C_+$ be the unique solution of the Dirichlet problem:

$$-\Delta_p e(z) = 1 \text{ a.e. on } Z \quad \text{and} \quad e|_{\partial Z} = 0.$$

Claim 3.2. *There exists $\lambda_+^* > 0$ such that for each $m \in L^\infty(Z)_+$ with $0 < \|m\|_\infty < \lambda_+^*$ we can find some $\eta_1 = \eta_1(m) > 0$ satisfying*

$$c_1\|m\|_\infty^s + c_1(\eta_1\|e\|_\infty)^{\vartheta-1} < \eta_1^{p-1}. \tag{3.3}$$

We argue by contradiction. So, we suppose that the claim is false. Then, we can find $\{m_n\} \subseteq L^\infty(Z)_+$ such that $\|m_n\|_\infty \to 0$ and, for all $\eta > 0$,

$$\eta^{p-1} \leqslant c_1\|m_n\|_\infty + c_1(\eta\|e\|_\infty)^{\vartheta-1}.$$

Hence, we obtain $1 \leqslant c_1\eta^{\vartheta-p}\|e\|_\infty^{\vartheta-1}$ for all $\eta > 0$.

Since $\vartheta > p$, by letting $\eta \downarrow 0$ we have a contradiction. This proves the claim. Now, set $\bar{x} = \eta_1 e \in \operatorname{Int} C_+$. We have

$$\begin{aligned} -\Delta_p \bar{x}(z) &= -\eta_1^{p-1}\Delta_p e(z) \\ &= \eta_1^{p-1} \\ &> c_1\|m\|_\infty^s + c_1(\eta_1\|e\|_\infty)^{\vartheta-1} && \text{(see (3.3))} \\ &\geqslant m(z)\bar{x}(z)^{r-1} + f_+(z,\bar{x}(z)) \quad \text{a.e. on } Z && \text{(see (3.2))}. \end{aligned}$$

This implies that $\bar{x} \in \operatorname{Int} C_+$ is a strict upper solution for problem (3.1). □

We also consider the following auxiliary nonlinear Dirichlet problem:

$$\left.\begin{aligned} -\Delta_p v(z) &= -m(z)v^-(z)^{r-1} + f_-(z,v(z)) \quad \text{a.e. on } Z, \\ v|_{\partial Z} &= 0. \end{aligned}\right\} \tag{3.4}$$

We say that $\underline{v} \in W^{1,p}(Z)$ is a lower solution for problem (3.4) if $\underline{v}|_{\partial Z} \leqslant 0$ and

$$\int_Z \|D\underline{v}\|^{p-2}(D\underline{v}, Dy)_{\mathbb{R}^N}\,\mathrm{d}z \leqslant \int_Z -m(\underline{v})^{r-1}y\,\mathrm{d}z + \int_Z f_-(z,\underline{v})y\,\mathrm{d}z$$

for all $y \in W_+$. We say that $\underline{v}$ is a strict lower solution for (3.4) if it is a lower solution but not a solution of (3.4).

Arguing as in the proof of Proposition 3.1, we obtain the following.

Proposition 3.3. *If Hypotheses 2.1 and 2.2 hold, then there exists $\lambda_-^* > 0$ such that problem (3.4) has a strict lower solution $\underline{v} \in \operatorname{Int} C_+$, provided that $\|m\|_\infty < \lambda_-^*$.*

Next we introduce an additional truncation. So, let

$$\hat{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ m(z)x^{r-1} + f_+(z,x) & \text{if } 0 \leqslant x \leqslant \bar{x}(z), \\ m(z)\bar{x}(z)^{r-1} + f_+(z,\bar{x}(z)) & \text{if } \bar{x}(z) < x. \end{cases}$$

Clearly, $\hat{f}_+$ is a Carathéodory function. We further set

$$\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)\,\mathrm{d}s$$

for all $x \in \mathbb{R}$.

Also, we introduce the functional $\hat{\varphi}_+ : W_0^{1,p}(Z) \to \mathbb{R}$, defined by

$$\hat{\varphi}_+(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \hat{F}_+(z,x(z))\,\mathrm{d}z.$$

Clearly, we have $\hat{\varphi}_+ \in C^1(W_0^{1,p}(Z))$.

Proposition 3.4. *If Hypotheses 2.1 and 2.2 hold and* $0 < \|m\|_\infty < \lambda_+^*$*, then problem (1.1) has a solution* $x_0 \in \operatorname{Int} C_+$.

Proof. Clearly, $\hat{\varphi}_+$ is coercive and sequentially w-lower semicontinuous. So, by the Weierstrass theorem we can find $x_0 \in W_0^{1,p}(Z)$ such that

$$\hat{\varphi}_+(x_0) = \hat{m}_+ = \inf[\hat{\varphi}_+(x) : x \in W_0^{1,p}(Z)];$$

hence, $\hat{\varphi}_+'(x_0) = 0$ and consequently

$$A(x_0) = \hat{N}_+(x_0), \tag{3.5}$$

where $\hat{N}_+(x)(\cdot) = \hat{f}_+(\cdot\,,x(\cdot))$ for all $x \in W_0^{1,p}(Z)$. Since $\bar{x} \in \operatorname{Int} C_+$ is a strict upper solution for problem (3.1), we have

$$A(\bar{x}) > m(\bar{x}^+)^{r-1} + N_+(\bar{x}) \quad \text{in } W^{-1,p'}(Z), \tag{3.6}$$

where $N_+(x)(\cdot) = f_+(\cdot\,,x(\cdot))$ for all $x \in W_0^{1,p}(Z)$. From (3.5) and (3.6), it follows that in $W^{-1,p'}(Z)$ we have

$$A(\bar{x}) - A(x_0) > m\bar{x}^{r-1} + N_+(\bar{x}) - \hat{N}_+(x_0). \tag{3.7}$$

On (3.7) we act with the test function $(x_0 - x)^+ \in W_0^{1,p}(Z)$. Notice that $\hat{f}_+(z,x_0(z)) = m(z)\bar{x}(z)^{r-1} + f_+(z,\bar{x}(z))$ for almost every $z \in \{x_0(z) > \bar{x}(z)\}$. Therefore, we obtain

$$\begin{aligned} 0 &\leqslant \langle A(\bar{x}) - A(x_0), (x_0 - \bar{x})^+\rangle \\ &= \int_{\{x_0 > \bar{x}\}} (\|D\bar{x}\|^{p-2}D\bar{x} - \|Dx_0\|^{p-2}Dx_0, Dx_0 - D\bar{x})_{\mathbb{R}^N}\,\mathrm{d}z. \end{aligned}$$

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Hence, $|\{x_0 > \bar{x}\}|_N = 0$, i.e. $x_0 \leqslant \bar{x}$. Here $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. Also, if on (3.5) we act with the test function $-x_0^- \in W_0^{1,p}(Z)$, then

$$\|Dx_0^-\|_p^p = 0, \text{ i.e. } 0 \leqslant x_0.$$

It follows that $\hat{N}_+(x_0) = mx_0^{r-1} + N_+(x_0)$, which implies that (3.5) becomes $A(x_0) = mx_0^{r-1} + N_+(x_0)$ and, consequently,

$$-\Delta_p x_0(z) = mx_0(z)^{r-1} + f_+(z, x_0(z)) \text{ a.e. on } Z \quad \text{and} \quad x_0|_{\partial Z} = 0. \tag{3.8}$$

Next we show that $x_0 \neq 0$. To this end, for $t > 0$ small we have

$$\begin{aligned}\hat{\varphi}_+(tu_1) &= \frac{t^p}{p}\|Du_1\|_p^p - \frac{t^r}{r}\int_E mu_1^r \,\mathrm{d}z - \int_Z F(z, tu_1)\,\mathrm{d}z \\ &\leqslant \frac{t^p}{p}\lambda_1 - \frac{t^r}{r}\int_Z mu_1^r \,\mathrm{d}z.\end{aligned}$$

Since $r < p$, if we make $t \in (0, 1)$ small enough, then we infer that $\hat{\varphi}_+(tu_1) < 0$, and hence

$$\hat{\varphi}_+(x_0) = \hat{m}_+ < 0 = \hat{\varphi}_+(0), \quad \text{i.e. } x_0 \neq 0.$$

From (3.8) and the nonlinear regularity theory (see, for example, [**6**, pp. 737–738]), we have $x_0 \in C_+ \setminus \{0\}$. Invoking the nonlinear strong maximum principle of [**11**], we conclude that $x_0 \in \operatorname{Int} C_0$. Moreover,

$$-\Delta_p x_0(z) = m(z)x_0(z)^{r-1} + f(z, x_0(z)) \quad \text{a.e. on } Z;$$

hence, $x_0 \in \operatorname{Int} C_+$ is a solution of problem (1.1). □

Now we execute an analogous process on the negative semi-axis, for which we define

$$\hat{f}_-(z, x) = \begin{cases} m(z)\underline{v}(z)^{r-1} + f_-(z, \underline{v}(z)) & \text{if } x < \underline{v}(z), \\ mx^{r-1} + f_-(z, x) & \text{if } \underline{v}(z) \leqslant x \leqslant 0, \\ 0 & \text{if } 0 < x. \end{cases}$$

Set

$$\hat{F}_-(z, x) = \int_0^x \hat{f}_-(z, s)\,\mathrm{d}s.$$

Also, we consider the C^1-functional $\hat{\varphi}_- : W_0^{1,p}(Z) \to \mathbb{R}$, defined by

$$\hat{\varphi}_-(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \hat{F}_-(z, x(z))\,\mathrm{d}z.$$

Arguing as in the proof of Proposition 3.4, we obtain the following.

Proposition 3.5. *If Hypotheses 2.1 and 2.2 hold and* $0 < \|m\|_\infty < \lambda_-^*$, *then problem (1.1) has a solution* $v_0 \in -\operatorname{Int} C_+$.

4. The three-solutions theorem

In this section we prove the three-solutions theorem for problem (1.1). For this purpose, we introduce the following truncations of the identity map, of the nonlinearity f and of $mx^{r-1} + f(z, x)$:

$$\bar{f}_0(z,x) = \begin{cases} f(z, v_0(z)) & \text{if } x < v_0(z), \\ f(z,x) & \text{if } v_0(z) \leqslant x \leqslant x_0(z), \\ f(z, x_0(z)) & \text{if } x_0(z) < x, \end{cases}$$

$$\bar{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ m(z)x^{r-1} + f(z,x) & \text{if } 0 \leqslant x \leqslant x_0(z), \\ m(z)x_0(z)^{r-1} + f(z, x_0(z)) & \text{if } x_0(z) < x, \end{cases}$$

$$\bar{f}_-(z,x) = \begin{cases} m(z)v_0(z)^{r-1} + f(z, v_0(z)) & \text{if } x < v_0(z), \\ m(z)x^{r-1} + f(z,x) & \text{if } v_0(z) \leqslant x \leqslant 0, \\ 0 & \text{if } 0 < x, \end{cases}$$

and

$$\bar{f}_0^*(z,x) = \begin{cases} m(z)v_0(z)^{r-1} + f(z, v_0(z)) & \text{if } x < v_0(z), \\ m(z)x^{r-1} + f(z,x) & \text{if } v_0(z) \leqslant x \leqslant x_0(z), \\ m(z)x_0(z)^{r-1} + f(z, x_0(z)) & \text{if } x_0(z) < x. \end{cases}$$

Also, we define

$$\bar{F}_\pm(z,x) = \int_0^x \bar{f}_\pm(z,s)\,\mathrm{d}s \quad \text{and} \quad \bar{F}_0^*(z,x) = \int_0^x \bar{f}_0^*(z,s)\,\mathrm{d}s.$$

Finally, we introduce the C^1-functionals $\bar{\varphi}_\pm, \bar{\varphi}_0 : W_0^{1,p}(Z) \to \mathbb{R}$, defined by

$$\bar{\varphi}_\pm(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \bar{F}_\pm(z, x(z))\,\mathrm{d}z$$

and

$$\bar{\varphi}_0(z) = \frac{1}{p}\|Dx\|_p^p - \int_Z \bar{F}_0^*(z, x(z))\,\mathrm{d}z.$$

In the next proposition we will locate the critical points of these three functionals.

Proposition 4.1. *If Hypotheses 2.1 and 2.2 hold and* $0 < \|m\|_\infty < \lambda_0^* = \min\{\lambda_+^*, \lambda_-^*\}$*, then the critical points of* $\bar{\varphi}_+$ *are in* $[0, x_0]$*, the critical points of* $\bar{\varphi}_-$ *are in* $[v_0, 0]$ *and the critical points of* $\bar{\varphi}_0$ *are in* $[v_0, x_0]$*. Furthermore,* v_0 *and* x_0 *are local minimizers of* $\bar{\varphi}_0$*.*

Proof. We prove the case for $\bar{\varphi}_0$ (the proof for $\bar{\varphi}_\pm$ is similar). So, let $x \in W_0^{1,p}(Z)$ be a critical point of $\bar{\varphi}_0$. Then we have $\bar{\varphi}_0'(x) = 0$; hence,

$$A(x) = \hat{\bar{N}}_0^*(x), \tag{4.1}$$

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where $\hat{\hat{N}}_0^*(x)(\cdot) = \bar{f}_0^*(\cdot\,, x(\cdot))$ for all $x \in W_0^{1,p}(Z)$. Thus,

$$\begin{aligned}\langle A(x), (x-x_0)^+\rangle &= \int_Z m x_0^{r-1}(x-x_0)^+\,\mathrm{d}z + \int_Z f(z,x_0)(x-x_0)^+\,\mathrm{d}z \\ &= \langle A(x_0), (x-x_0)^+\rangle, \end{aligned} \tag{4.2}$$

where the last equality is due to the fact that $x_0 \in \operatorname{Int} C_+$ is a solution of (1.1).

By virtue of the strict monotonicity of the map A, from (4.2) we infer that

$$(x-x_0)^+ = 0,$$

i.e. $x \leqslant x_0$. In a similar fashion we also can show that

$$v_0 \leqslant x.$$

So, indeed the critical points of $\bar{\varphi}_0$ are in the ordered interval $[v_0, x_0]$.

Without loss of generality, we may assume that $x_0 \in \operatorname{Int} C_+$ is the only non-trivial critical point of $\bar{\varphi}_+$ and v_0 is the only non-trivial critical point of $\bar{\varphi}_-$. Otherwise, we already have a third non-trivial solution of (1.1), distinct from x_0 and v_0, which is in fact of constant sign.

As in the proof of Proposition 3.4, we can show that for $t > 0$ small we have $\bar{\varphi}_+(tu_1) < 0$; hence,

$$\bar{m}_+ = \inf[\bar{\varphi}_+(x) : x \in W_0^{1,p}(Z)] < 0 = \bar{\varphi}_+(0).$$

Note that $\bar{\varphi}_+$ is coercive and sequentially w-lower semicontinuous. Therefore, we can find some $\bar{x}_0 \in W_0^{1,p}(Z)$ such that

$$\bar{\varphi}_+(\bar{x}_0) = \bar{m}_+ < 0 = \bar{\varphi}_+(0),$$

i.e. $\bar{x}_0 \neq 0$. It follows that $\bar{x}_0 = x_0$. Because $x_0 \in \operatorname{Int} C_+$, we can find small $r > 0$ such that

$$\bar{\varphi}_+|_{\bar{B}_r^{C_0^1(\bar{Z})}(x_0)} = \varphi_0|_{\bar{B}_r^{C_0^1(\bar{Z})}(x_0)},$$

where

$$\bar{B}_r^{C_0^1(\bar{Z})}(x_0) = \{x \in C_0^1(\bar{Z}) : \|x - x_0\|_{C_0^1(\bar{Z})} \leqslant r\}.$$

Hence, x_0 is a local $C_0^1(\bar{Z})$-minimizer of $\bar{\varphi}_0$. From [**5**], it follows that x_0 is a local $W_0^{1,p}(Z)$-minimizer of $\bar{\varphi}_0$. The argument for $v_0 \in -\operatorname{Int} C_+$ is similar. □

Now we are ready for the multiplicity result.

Theorem 4.2. *If Hypotheses 2.1 and 2.2 hold and* $0 < \|m\|_\infty < \lambda_0^* = \min\{\lambda_+^*, \lambda_-^*\}$, *then problem (1.1) has at least three non-trivial distinct solutions* x_0, v_0 *and* y_0 *such that*

$$x_0 \in \operatorname{Int} C_0, \qquad v_0 \in -\operatorname{Int} C_+, \qquad y_0 \in C_0^1(\bar{Z})$$

and $v_0(z) \leqslant y_0(z) \leqslant x_0(z)$ *for all* $z \in \bar{Z}$.

Proof. From Propositions 3.4 and 3.5, we already have two solutions of constant sign: $x_0 \in \operatorname{Int} C_+$ and $v_0 \in -\operatorname{Int} C_+$. By Proposition 4.1, we know that both x_0 and v_0 are local minimizers of $\bar{\varphi}_0$. So, as in [**1**, Proposition 29], we can find $r > 0$ small enough such that

$$\bar{\varphi}_0(x_0) < \inf[\bar{\varphi}_0(x) : \|x - x_0\| = r]$$

and

$$\bar{\varphi}_0(v_0) < \inf[\bar{\varphi}_0(v) : \|v - v_0\| = r].$$

Without loss of generality, we may assume that $\varphi_0(v_0) \leqslant \varphi_0(x_0)$. Then, the sets $E_0 = \{v_0, x_0\}$, $E = [v_0, x_0]$ and

$$D = \partial B_r(x_0) = \{x \in W_0^{1,p}(Z) : \|x - x_0\| = r\}$$

are linking in $W_0^{1,p}(Z)$ (see, for example, [**6**, p. 642]). Also, $\bar{\varphi}_0$ being coercive, we can easily verify that it satisfies the Palais–Smale condition. So, we can apply the linking theorem (see, for example, [**6**, p. 644]) and obtain some $y_0 \in W_0^{1,p}(Z)$, a critical point of $\bar{\varphi}_0$ of mountain-pass type, $y_0 \neq x_0$, $y_0 \neq v_0$. Hence [**2**],

$$C_1(\bar{\varphi}_0, y_0) \neq 0. \tag{4.3}$$

On the other hand, by Hypothesis 2.1 (iv), we can find some $\beta > 0$ and $\delta > 0$ such that

$$0 \leqslant f(z, x)x \leqslant \beta|x|^{\tau}$$

for all $z \in Z$ and all $|x| \leqslant \delta$. Now, let $|x| \leqslant \delta$. If $x \in [v_0(z), x_0(z)]$, then $\bar{f}_0(z, x) = f(z, x)$, and so

$$0 \leqslant \bar{f}_0(z, x)x \leqslant \beta|x|^{\tau}. \tag{4.4}$$

If $x > x_0(z)$ (respectively, $x < v_0(z)$), then

$$\bar{f}_0(z, x) = f(z, x_0(z))$$

(respectively, $\bar{f}_0(z, x) = f(z, v_0(z))$).

If $\mu \in (r, p)$, then for almost every $z \in Z$ and all $|x| \leqslant \delta$, $x \in [v_0(z), x_0(z)]$, we have

$$\left(\frac{\mu}{r} - 1\right)|x|^r + \mu\bar{F}_0(z, x) - \bar{f}_0(z, x)x \geqslant \left(\frac{\mu}{r} - 1\right)|x|^r - \beta|x|^{\tau} \tag{4.5}$$

since $\bar{F}_0 \geqslant 0$, and due to (4.4).

Since $r < \tau$ and $|x| \leqslant \delta < 1$, from (4.5) it follows that

$$\left(\frac{\mu}{r} - 1\right)|x|^r + \mu\bar{F}_0(z, x) - \bar{f}_0(z, x)x \geqslant 0 \tag{4.6}$$

for almost all $z \in Z$ and all $|x| \leqslant \delta$, $x \in [v_0(z), x_0(z)]$.

If $x > x_0(z)$, then

$$\left(\frac{\mu}{r} - 1\right) x_0(z)^r - f(z, x_0(z)) x_0(z) \geqslant \left(\frac{\mu}{r} - 1\right) x_0(z)^r - \beta x_0(z)^\tau \geqslant 0.$$

A similar result is obtained if $x < v_0(z)$.

Invoking [**7**, Proposition 2.1], by (4.6) we have

$$C_k(\bar{\varphi}_0, 0) = 0 \quad \text{for all } k \geqslant 0. \tag{4.7}$$

If we compare (4.3) and (4.7), it is clear that $y_0 \neq 0$. Finally, the nonlinear regularity theory implies that $y_0 \in C_0^1(\bar{Z})$. Since $y_0 \in [v_0, x_0]$, we conclude that y_0 is a non-trivial smooth solution of problem (1.1), distinct from x_0 and v_0. □

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