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GREEN'S FUNCTIONS FOR POWERS OF THE INVARIANT LAPLACIAN

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ABSTRACT. The aim of the present paper is the computation of Green's functions for the powers $\mathbf{\Delta}^m$ of the invariant Laplace operator on rank-one Hermitian symmetric spaces. Starting with the noncompact case, the unit ball in $\mathbb{C}^d$, we obtain a complete result for $m = 1, 2$ in all dimensions. For $m \geq 3$ the formulas grow quite complicated so we restrict ourselves to the case of the unit disc ($d = 1$) where we develop a method, possibly applicable also in other situations, for reducing the number of integrations by half, and use it to give a description of the boundary behaviour of these Green functions and to obtain their (multi-valued) analytic continuation to the entire complex plane. Next we discuss the type of special functions that turn up (hyperlogarithms of Kummer). Finally we treat also the compact case of the complex projective space $\mathbb{P}^d$ (for $d = 1$, the Riemann sphere) and, as an application of our results, use eigenfunction expansions to obtain some new identities involving sums of Legendre ($d = 1$) or Jacobi ($d > 1$) polynomials and the polylogarithm function. The case of Green's functions of powers of weighted (no longer invariant, but only covariant) Laplacians is also briefly discussed.

0. **Introduction.** Let $\mathbb{B}^d = \{z \in \mathbb{C}^d : |z| < 1\}$ be the unit ball in the complex d-space $\mathbb{C}^d$. In [HK] Hayman and Korenblum obtained a formula for the Green function of the polyharmonic operator Δ^m on $\mathbb{B}^d$ with the Dirichlet boundary data ($u = \partial u/\partial n = \cdots = \partial^{m-1}u/\partial n^{m-1} = 0$):

$$G_{m,d}(z,w) = \frac{(-1)^m}{4^m \pi^d (m-1)!} \sum_{j=0}^{\infty} \frac{(j+d-1)!}{(j+m)!} \frac{(1-|z|^2)^{m+j}(1-|w|^2)^{m+j}}{\big|w/|w| - |w|z\big|^{2d+2j}}. \tag{0.1}$$

Their proof rests, more or less, on skillful explicit computations. Subsequently the present authors gave another proof [EP] in the case of dimension $d = 1$, based on Moebius-invariance techniques (Bojarski's theorem, which in this simple case essentially reduces to Bol's lemma). In short, their main idea was to use invariance to reduce to the case $w = 0$, which is essentially a problem in ordinary differential equations and, thus, much easier to handle.

The latter approach is particularly suitable also for the invariant Green's functions $\mathbf{G}_{m,d}$ of the *invariant* polyharmonic operator $\mathbf{\Delta}^m$ on the unit ball $\mathbb{B}^d$. The operator $\mathbf{\Delta}$ is given by the formula (*cf.* [Ru], Theorem 4.1.3)

$$\mathbf{\Delta} u = 4(1-|z|^2) \sum_{i,j=1}^{d} (\delta_{ij} - z_i \overline{z_j}) \frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}. \tag{0.2}$$

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In this case, certain care must be exercised with the boundary conditions, since the boundary of $\mathbb{B}^d$ is a characteristic for the operator $\mathbf{\Delta}$; for this reason, we shall mean by Green's function, informally, the function satisfying $\mathbf{\Delta}^m G = \delta$ and having the least possible growth rate at the boundary. (It should be possible to show that this coincides with the Green function of the operator obtained by defining $\mathbf{\Delta}$ by (0.2) on C_0^∞ functions and taking the Friedrichs extension.) For $d = 1$ (the unit disc) and $m = 1, 2, 3$, the following explicit formulas were obtained in [EP]:

$$
\begin{aligned}
\mathbf{G}_{1,1} &= G_{1,1} = \frac{1}{4\pi}\log t,\\
\mathbf{G}_{2,1} &= \frac{1}{16\pi}\left[\log t \log(1-t) - 2\operatorname{Li}_2(t) + \frac{\pi^2}{3}\right],\\
\mathbf{G}_{3,1} &= \frac{1}{64\pi}\Big[\log t \cdot \Big(\frac{1}{2}\log^2(1-t) + \log(1-t) + \operatorname{Li}_2(t)\Big)\\
&\qquad + 2\big(\operatorname{Li}_2(t)\log(1-t) + M_3(t) + \operatorname{Li}_2(t) - \operatorname{Li}_3(t)\big)\\
&\qquad - 2\big(\zeta(3) + \zeta(2)\big) - \frac{\pi^2}{3}\log(1-t)\Big]
\end{aligned}
\tag{0.3}
$$

where $t = |\frac{z-w}{1-z\bar{w}}|^2$, $\zeta(s) = \operatorname{Li}_s(1)$ is the Riemann's zeta function, Li_s is the polylogarithm

$$
\operatorname{Li}_s(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^s},
\tag{0.4}
$$

and M_3 is Kummer's function

$$
M_3(t) = \int_0^t \frac{\log^2(1-x)}{x}\,dx
\tag{0.5}
$$

which can also be expressed as

$$
M_3(t) = 2\zeta(3) + \log t \log^2(1-t) + 2\operatorname{Li}_2(1-t)\log(1-t) - 2\operatorname{Li}_3(1-t).
\tag{0.6}
$$

In the present paper we continue this program by calculating the invariant Green's functions $\mathbf{G}_{m,d}$ for $m = 2$ and arbitrary d (Section 1) and for the unit disc ($d = 1$) and $m = 4$ (Section 2). For $d = 1$ we further develop a method for generating a rather explicit formula for $\mathbf{G}_m \equiv \mathbf{G}_{m,1}$ for general m, and use it to show that $\mathbf{G}_m(z,w) = O\big((1-t)\log^{m-1}(1-t)\big)$ as $t = |\frac{z-w}{1-w\bar{z}}| \longrightarrow 1$, and to obtain an analytic continuation for $\mathbf{G}_m$; this too is done in Section 2. It also turns out that, in general, the functions $\mathbf{G}_m$ are given by formulas involving Kummer's hyperlogarithms ([We] in [Le2], Chapter 8); this is shown in Section 3, which further contains a brief overview of the transcendental functions which enter into the formulas for $\mathbf{G}_m$, $m \leq 4$. In Section 4, we carry out a similar computation (for $m = 1, 2$) in the compact case of the Riemann sphere, and, as an application of our formulae, use eigenfunction expansions to prove two identities involving sums of Legendre polynomials $P_n(x)$, for $m = 1$:

$$
\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n(x) = \log 2 - 1 - \log(1-x)
\tag{0.7}
$$

and for $m = 2$:

$$(0.8)\quad \sum_{n=1}^{\infty} \frac{2n+1}{\big(n(n+1)\big)^2} P_n(x) = \log\frac{1-x}{1+x}\cdot\log\frac{2}{1+x} - \frac{1}{2}\log^2\frac{2}{1+x} + \mathrm{Li}_2\left(-\frac{1-x}{1+x}\right) + 1.$$

In fact, formula (0.7) is nothing but the special case $y = 1$ of an analogous "bilinear formula" (with the product $P_n(x)P_n(y)$ in place of just $P_n(x)$) which can be found, *e.g.*, in [BE], Section 10.10, formula (53). On the contrary, up to our knowledge (0.8) is new. In principle, similar formulas for higher hyperlogarithms which appear in the expressions for the Green's functions $\mathbf{G}_m$ can be obtained along these lines as well. We observe also that (0.7) and (0.8) can be interpreted as giving the value of a certain Minakshisundaram-Pleijel type zeta function. In conclusion, we discuss (Remark 4.5) also the case of weighted (no longer invariant, but only covariant) Laplacians $\boldsymbol{\Delta}_{\nu*}$ on the sphere, and indicate still other generalizations of (0.7) and (0.8) that can be obtained in this way.

1. **Invariant Green functions on the ball.** Let us put ourselves into the scenario described in the Introduction, *i.e.*, let $\mathbb{B}^d$ be the unit ball in $\mathbb{C}^d$ and consider the differential operator $\boldsymbol{\Delta}$ given by (0.2). It is well known that a great virtue of the latter operator is its invariance under holomorphic mappings: for any holomorphic automorphism ϕ of $\mathbb{B}^d$ one has

$$\boldsymbol{\Delta}(f\circ\phi) = (\boldsymbol{\Delta} f)\circ\phi.$$

A proof of this fact can be found, *e.g.*, in Chapter IV of Rudin's book [Ru]. Our main goal in this section will be the identification of the Green functions (in the sense made clear in the Introduction) $\mathbf{G}_{m,d}$ for the operators $\boldsymbol{\Delta}^m$ where $m = 1$ or 2. In view of the invariance of $\boldsymbol{\Delta}$, these Green functions must satisfy

$$(1.1)\qquad \mathbf{G}_{m,d}(z,w) = \mathbf{G}_{m,d}\big(\phi(z),\phi(w)\big)\quad \forall\phi\in\mathrm{Aut}(\mathbb{B}^d).$$

Since for any point $a \in \mathbb{B}^d$ there exists an automorphism ϕ_a interchanging a and 0 ([Ru], Proposition 2.2.2), it therefore suffices to find the Green function $\mathbf{G}_{m,d}(z,0)$ with the pole at the origin. Further, in view of rotational symmetry, it is clear that the last function must actually depend only on the modulus $|z|$ of z. Thus we may write

$$(1.2)\qquad \mathbf{G}_{m,d}(z,0) = \Lambda_{m,d}(t)$$

for some function $\Lambda_{m,d}$, where we have introduced the variable $\boxed{t = |z|^2}$. By (1.1) and formula 2.2.2(iv) in [Ru], we will then have

$$(1.3)\qquad \begin{aligned}\mathbf{G}_{m,d} &= \Lambda_{m,d}\big(|\phi_w(z)|^2\big)\\ &= \Lambda_{m,d}\left(1 - \frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z,w\rangle|^2}\right).\end{aligned}$$

In order to find $\Lambda_{m,d}$, let us first determine what is the action of $\boldsymbol{\Delta}$ on radial functions. Using (0.2), one has

$$\begin{aligned}
\boldsymbol{\Delta} f(t) &= 4(1-t)\sum_{i,j=1}^{d}(\delta_{ij}-z_i\overline{z_j})\,\partial^2 f/\partial z_i\partial\overline{z_j} \\
&= 4(1-t)\sum_{i,j=1}^{d}(\delta_{ij}-z_i\overline{z_j})(\delta_{ij}f'+\overline{z}_i z_j f'') \\
&= 4(1-t)\sum_{i,j=1}^{d}(\delta_{ij}f'-|z_i|^2 f'+|z_i|^2 f''-|z_i\overline{z_j}|^2 f'') \\
&= 4(1-t)\big[(d-t)f'+(t-t^2)f''\big] \\
&= 4\frac{(1-t)^{d+1}}{t^{d-1}}\left[\frac{t^d}{(1-t)^{d-1}}f'\right]',
\end{aligned}$$

where the prime $'$ stands for the differentiation with respect to the t variable. This suggests putting into play the ordinary differential operator

$$L_d f = \frac{(1-t)^{d+1}}{t^{d-1}}\left[\frac{t^d}{(1-t)^{d-1}}f'\right]', \tag{1.4}$$

which represents the radial part of $\frac{1}{4}\boldsymbol{\Delta}$. The function $\Lambda_{m,d}$ is a solution to the equation

$$L_d^m \Lambda_{m,d} = 0 \text{ on } (0,1). \tag{1.5}$$

Moreover, at the origin it must have the same singularity as the fundamental solution of the ordinary polyharmonic operator Δ^m. Our approach to finding $\mathbf{G}_{m,d}$ will be very simple-minded: we construct a basis of the solutions of equation (1.5) and then seek a linear combination of the basis elements which has the required singularity at the origin and the required boundary behavior.

Let us start with $m = 1$. The general solution to the equation $L_d u = v$ is given by the integral

$$u = \int \frac{(1-t)^{d-1}}{t^d}\left(\int \frac{t^{d-1}}{(1-t)^{d+1}}\,v\,dt\right)dt. \tag{1.6}$$

Taking in particular $v = 0$ we see that a basis of solutions for the equation $L_d u = 0$ is given by

$$f_0 = 1, \quad g_0 = \int \frac{(1-t)^{d-1}}{t^d}\,dt. \tag{1.7}$$

The integral can be evaluated using the binomial theorem:

$$g_0 = \sum_{j=1}^{d-1}\frac{d-1}{j}(-1)^{d-j}\frac{1}{jt^j} - (-1)^d \log t.$$

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At the boundary point $t = 1$ we have

$$g_0(1) = \sum_{j=1}^{d-1} \frac{d-1}{j} \frac{(-1)^{d-j}}{j} \equiv C_d. \tag{1.8}$$

The function $\Lambda_{1,d}$ must therefore be of the form $\Lambda_{1,d} = \gamma_d(g_0 - C_d)$, for some constant γ_d yet to be determined. To that end, observe that the behavior of g_0 at the origin is

$$g_0 \sim \begin{cases} \log t & \text{if } d = 1, \\ -\frac{1}{(d-1)t^{d-1}} & \text{if } d > 1. \end{cases} \tag{1.9}$$

On the other hand, the fundamental solution for the Laplace operator Δ is well known to be

$$\begin{cases} \frac{1}{4\pi} \log t & \text{for } d = 1, \\ -\frac{(d-2)!}{4\pi^d} t^{1-d} & \text{for } d > 1. \end{cases}$$

Thus we conclude that

$$\begin{aligned} \Lambda_{1,1}(t) &= \frac{1}{4\pi} g_0 \quad \text{(trivial!)}, \\ \text{and } \Lambda_{1,d}(t) &= \frac{(d-1)!}{4\pi^d} (g_0 - C_d) \quad \text{for } d > 1 \end{aligned} \tag{1.10}$$

with the constant C_d given by (1.8). Since $C_1 = 0$, the second formula actually works for all values of d.

Now we take $m = 2$. An obvious choice for the basis elements f_1, g_1 which together with f_0 and g_0 would span the vector space of the solutions to $L_d^2 u = 0$ is

$$\begin{aligned} f_1 &= \int \frac{(1-t)^{d-1}}{t^d} \left(\int \frac{t^{d-1}}{(1-t)^{d+1}} f_0 \, dt \right) dt, \\ g_1 &= \int \frac{(1-t)^{d-1}}{t^d} \left(\int \frac{t^{d-1}}{(1-t)^{d+1}} g_0 \, dt \right) dt. \end{aligned}$$

Integration gives

$$f_1 = \frac{1}{d} \int \frac{(1-t)^{d-1}}{t^d} \cdot \frac{t^d}{(1-t)^d} \, dt = \frac{1}{d} \log \frac{1}{1-t},$$

$$\begin{aligned} g_1 = {} & \frac{(-1)^d}{d} \big(2 \operatorname{Li}_2(t) + \log t \log(1-t) \big) \\ & + \sum_{j=1}^{d-1} \frac{d-1}{j} \frac{(-1)^{d-j}}{jd} \left[\frac{\log(1-t)}{t^j} + 2 \log \frac{t}{1-t} - 2 \sum_{k=1}^{j-1} \frac{1}{kt^k} \right]. \end{aligned}$$

The sought function $\Lambda_{2,d}$ will be a linear combination

$$\Lambda_{2,d} = Af_0 + Bg_0 + Cf_1 + Dg_1,$$

with constants A, B, C, D yet to be determined from the boundary conditions at $t = 0$ and $t = 1$. Let us look first at $t = 0$. One has

$$f_0 \sim 1, \quad f_1 \sim \frac{t}{d}, \quad g_0 \sim \text{ is given by (1.9)},$$

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and it is not difficult to see that

$$g_1 \sim \begin{cases} t \log t & \text{for } d = 1 \\ -\log t & \text{for } d = 2 \\ \frac{1}{(d-1)(d-2)t^{d-2}} & \text{for } d > 2. \end{cases}$$

On the other hand, the fundamental solution for Δ^2 is

$$\sim \begin{cases} \frac{1}{16\pi} t \log t & \text{for } d = 1 \\ \frac{1}{16\pi^2} \cdot (-\log t) & \text{for } d = 2 \\ \frac{(d-3)!}{16\pi^d} t^{2-d} & \text{for } d > 2. \end{cases}$$

By comparison we thus infer that

$$B = 0 \quad \text{and} \quad D = \frac{(d-1)!}{16\pi^d}.$$

Let us now investigate the situation at $t = 1$. This time one has

$$f_0 = 1, \quad f_1 = \frac{1}{d} \log \frac{1}{1-t},$$

g_0 is irrelevant, and

$$g_1 = C_d \cdot \frac{1}{d} \log \frac{1}{1-t} + A_d + o(1),$$

where C_d is the constant from (1.8) and A_d is given by

$$A_d = 2 \sum_{j=1}^{d-1} \frac{d-1}{j} \frac{(-1)^{d-j}}{jd} \Big[\sum_{k=1}^{j-1} \frac{1}{k} \Big] - (-1)^d \frac{\pi^2}{3d}.$$

Therefore we conclude that

$$\Lambda_{2,d} = \frac{(d-1)!}{16\pi^d} (g_1 - C_d f_1 - A_d f_0)$$

for all $d \geq 1$. Supplying all the constants *et cetera*, we can summarize our results in this section as the following theorem.

THEOREM 1.1. *The Green functions* $\mathbf{G}_{1,d}$ *and* $\mathbf{G}_{2,d}$ *for the operators* $\mathbf{\Delta}$ *and* $\mathbf{\Delta}^2$ *are given by the formulas* (1.3) *where*

$$\Lambda_{1,d}(t) = \mathbf{G}_{1,d}(z,0) = \frac{(d-1)!}{4\pi^d} \left[(-1)^{d-1} \log t + \sum_{j=1}^{d-1} \frac{d-1}{j} \frac{(-1)^{d-j}}{j} (t^{-j} - 1) \right],$$

$$\begin{aligned} \Lambda_{2,d}(t) = \mathbf{G}_{2,d}(z,0) = \frac{(d-1)!}{16\pi^d} \Bigg[& \frac{(-1)^d}{d} \left(2\,\mathrm{Li}_2(t) + \log t \log(1-t) - \frac{\pi^2}{3} \right) \\ & + \sum_{j=1}^{d-1} \frac{d-1}{j} \frac{(-1)^{d-j}}{jd} \left((1 - t^{-j}) \log \frac{1}{1-t} + 2 \log t + 2 \sum_{k=1}^{j-1} \frac{1 - t^{-k}}{k} \right) \Bigg]. \end{aligned}$$

COROLLARY 1.2. *For* $m=1$ *and* 2, $\mathbf{G}_{m,d}(z,0)=O\big((1-t)\log^{m-1}(1-t)\big)$ *as* $|z|^2\equiv t$ *approaches* 1.

COROLLARY 1.3. *For* $m=1$ *and* 2, *the functions* $\Lambda_{m,d}$ *extend to multi-valued analytic functions on* $\mathbb{C}\setminus\{0,1\}$ *with logarithmic singularities at the exceptional points* 0 *and* 1.

Already from these two cases $m=1,2$ one gets a feeling what might be happening for general m. By repeated applications of the integral operator (1.6) we create two chains of functions $f_0, f_1, f_1, \ldots$ and $g_0, g_1, g_2, \ldots$ which satisfy $L_d^k f_k = f_0$ and $L_d^k g_k = g_0$; hence, they are linearly independent and

$$f_0, f_1, \ldots, f_{m-1}, \quad g_0, g_1, \ldots, g_{m-1}$$

is a basis of solutions for $L_d^m u = 0$. The function $\Lambda_{m,d}$ is a certain linear combination of these basis elements, and one expects to recover the coefficients at the f_j from the behavior near the point $t=0$ and the coefficients at the g_j from the behavior at the boundary ($t=1$). Due to the increasing complexity of the calculations involved, there seems to be little hope of pursuing this program much further than $m=2$ in the general case; however, we shall see in the next section that, to a certain extent, this can be done for $d=1$, and it turns out that all the observations above come out to be true for all m, and, moreover, so do even Corollaries 1.2 and 1.3.

2. **The case of the unit disc.** With our simple-minded method from the preceding section, solving the equation $L_d u = v$ involves two integrations, so the construction of a basis of solutions to $L_d^m = 0$ requires $4m$ integrations. It turns out that there is a more refined approach by which the number of integrations can be reduced by half, and moreover the functions being integrated will be of simpler form. This device, unfortunately, seems to work only in the case of the unit disc, $d=1$, and so we restrict ourselves to this situation throughout the present section. The radial part of the invariant Laplacian $\mathbf{\Delta}$ takes then the simple form

$$\mathbf{\Delta} f(t) = 4(1-t)^2[tf']' \quad (t=|z|^2). \tag{2.1}$$

We denote this ordinary differential operator by $4L$ (omitting the subscript $d=1$) and (likewise) abbreviate the Green functions $\mathbf{G}_{m,1}$ to $\mathbf{G}_m$.

PROPOSITION 2.1. *Assume that two sequences of functions* $N_0, N_1, N_2, \ldots$ *and* $\tilde N_0$, $\tilde N_1, \tilde N_2, \ldots$ *are given which satisfy, respectively,*

$$\begin{aligned} N_0 &= 1 \\ N'_{2k+1} &= \frac{1}{1-t} N_{2k} \\ N'_{2k+2} &= \frac{1}{t(1-t)} N_{2k+1} \end{aligned} \tag{2.2}$$

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and

$$\tag{2.3} \begin{aligned} \tilde N_0 &= 1 \\ \tilde N'_{2k+1} &= \frac{1}{t(1-t)}\tilde N_{2k} \\ \tilde N'_{2k+2} &= \frac{1}{1-t}\tilde N_{2k+1}. \end{aligned}$$

Define also

$$\tag{2.4} K_k = \tilde N_{k+1} - N_{k+1}.$$

Then

$$\tag{2.5} \begin{aligned} LN_j &= N_{j-1} + N_{j-2} \quad (\textit{with } N_{-1} \equiv N_{-2} \equiv 0), \\ L\tilde N_j &= \tilde N_{j-1} + \tilde N_{j-2} \quad (\textit{with } \tilde N_{-1} \equiv \tilde N_{-2} \equiv 0), \end{aligned}$$

$$\tag{2.6} L^j N_j = L^j \tilde N_j = N_0 (= 1),$$

$$\tag{2.7} L^{j+1} N_j = L^{j+1}\tilde N_j = 0,$$

and

$$\tag{2.8} LK_j = K_{j-1} + K_{j-2} \quad (K_{-1} \equiv K_{-2} \equiv 0),$$

$$\tag{2.9} L^j K_j = K_0,$$

$$\tag{2.10} L^{j+1} K_j = 0.$$

Moreover, for each n, the $2n$ functions

$$\tag{2.11} N_0, N_1, \ldots, N_{n-1}, \quad K_0, K_1, \ldots, K_{n-1},$$

are a complete system of fundamental solutions of the equation $L^n f = 0$.

PROOF. One has

$$\begin{aligned} LN_{2k+1} &= (1-t)^2 \left[t \cdot \frac{1}{1-t} N_{2k} \right]' = (1-t)^2 \left[\frac{1}{(1-t)^2} N_{2k} + \frac{t}{1-t} N'_{2k} \right] \\ &= N_{2k} + t(1-t) \cdot \frac{1}{t(1-t)} N_{2k-1} = N_{2k} + N_{2k-1}, \\ LN_{2k+2} &= (1-t)^2 \left[t \cdot \frac{1}{t(1-t)} N_{2k+1} \right]' = (1-t)^2 \left[\frac{1}{(1-t)^2} N_{2k+1} + \frac{1}{1-t} N'_{2k+1} \right] \\ &= N_{2k+1} + (1-t) \cdot \frac{1}{1-t} N_{2k} = N_{2k+1} + N_{2k}, \end{aligned}$$

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and (2.5) follows; the proof for $\tilde{N}_j$ is completely similar. Iterating (2.5) gives

$$L^k N_j = \sum_{i=0}^{k} \binom{k}{i} N_{j-k-i} \quad (N_j \equiv 0 \text{ if } j < 0) \tag{2.12}$$

and similarly for $\tilde{N}_j$. Taking $k = j$ and $k = j + 1$ gives (2.6) and (2.7). The formulas (2.8) and (2.10) are immediate consequences of (2.5) and (2.7), respectively, and the definition of K_j. Finally, by (2.12)

$$L^j N_{j+1} = N_1 + jN_0$$

and similarly for $L^j \tilde{N}_{j+1}$. Subtracting, we get

$$L^j K_j = (\tilde{N}_1 - N_1) + j(\tilde{N}_0 - N_0) = \tilde{N}_1 - N_1 = K_0,$$

which proves (2.9).

The functions (2.11) belong to the kernel of L^n, by (2.7) and (2.10); in order to prove that they are a complete system of fundamental solutions, it suffices to show that they are linearly independent. So suppose that for some constants a_k and b_k

$$a_{n-1}N_{n-1} + b_{n-1}K_{n-1} + a_{n-2}N_{n-2} + b_{n-2}K_{n-2} + \cdots + a_0N_0 + b_0K_0 = 0.$$

Applying L^{n-1} to both sides gives

$$a_{n-1}N_0 + b_{n-1}K_0 = 0 \tag{2.13}$$

by (2.6) and (2.9). On the other hand, from (2.2) and (2.3) we have

$$N_1 = \log \frac{1}{1-t} + \gamma_1, \quad \tilde{N}_1 = \log t + \log \frac{1}{1-t} + \gamma_2,$$

so

$$K_0 = \log t + \gamma$$

and (2.13) reads

$$a_{n-1} + b_{n-1}(\log t + \gamma) = 0$$

implying that $a_{n-1} = b_{n-1} = 0$. Proceeding by induction shows that $a_{n-1} = b_{n-1} = a_{n-2} = b_{n-2} = \cdots = a_0 = b_0 = 0$, which proves the linear independence of the functions (2.11) and finishes the proof. ∎

A general solution to the system (2.2) is given recursively by

$$N_{2k+1}(t) = \int_{a_{2k+1}} \frac{N_{2k}(t)}{1-t},$$
$$N_{2k+2}(t) = \int_{a_{2k+2}} \frac{N_{2k+1}(t)}{t(1-t)},$$

for some points $a_j \in [0, 1]$ for which the integrals exist, and similarly for $\tilde{N}_j$. Here we have made the *convention* (to be observed throughout the rest of this paper) of introducing the shorthand $\int_a$ to denote the primitive which vanishes at a; that is,

$$\int_a f(t) \quad \text{is an abbreviation for} \quad \int_a^t f(x)\,dx.$$

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In practice, with the view on constructing the invariant Green's functions $\mathbf{G}_m$, it is convenient to choose the integral limits a_j so as to have control of the behavior of the functions N_j and $\tilde{N}_j$ at the boundary. We achieve this by taking $a_j = 0\ \forall j \geq 0$ and $\tilde{a}_1 = 1/2$, $\tilde{a}_j = 0$ $\forall j \geq 1$.

Thus, we define

$$N_0 = 1, \qquad N_{2k+1} = \int_0 \frac{N_{2k}}{1-t}, \quad N_{2k+2} = \int_0 \frac{N_{2k+1}}{t(1-t)} \quad \text{for } k \geq 0, \tag{2.18}$$

and

$$\tilde{N}_0 = 1, \quad \tilde{N}_1 = \log t + \log \frac{1}{1-t}, \qquad \tilde{N}_{2k+1} = \int_0 \frac{\tilde{N}_{2k}}{t(1-t)} \quad \text{for } k \geq 1, \quad \tilde{N}_{2k+2} = \int_0 \frac{\tilde{N}_{2k+1}}{1-t} \quad \text{for } k \geq 0. \tag{2.19}$$

It is clear that these functions satisfy the conditions (2.2) and (2.3), granted we show that they are well-defined, *i.e.*, that the integrals above exist. This is contained in the following proposition.

PROPOSITION 2.2. *The functions N_j and $\tilde{N}_j$ in* (2.18) *and* (2.19) *are correctly defined and satisfy*

$$\tilde{N}_{k+1} = N_k \log t + t g_{k+1}, \tag{2.20}$$

$$N_{2k} = t^k h_{2k}, \quad N_{2k+1} = t^{k+1} h_{2k+1} \tag{2.21}$$

for any $k \geq 0$, where g_j and h_j are functions holomorphic on the unit disc.

For brevity, we shall employ the notation O_0 for a general function (not necessarily the same one at each occurrence) holomorphic on the unit disc $\mathbb{D}$. The formula (2.20) can then be written as

$$\tilde{N}_{k+1} = N_k \log t + t O_0,$$

and similarly for (2.21).

PROOF. By definition, (2.20) holds for $\tilde{N}_1$. Assume that it holds for $\tilde{N}_{2k+1}$ for some k. Then by (2.19)

$$\tilde{N}_{2k+2} = \int_0 \frac{\tilde{N}_{2k+1}}{1-t} = \int_0 \left(\frac{N_{2k} \log t}{1-t} + t O_0 \right).$$

Observe that for any function h holomorphic on a simply connected domain containing the origin

$$\int_0 h \cdot \log t = \left(\textstyle\int_0 h\right) \log t - \int_0 \frac{\int_0 h}{t}, \tag{2.22}$$

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by integration by parts. Consequently,

$$\tilde{N}_{2k+2} = \log t \cdot \int_0 \frac{N_{2k}}{1-t} - tO_0 + t^2 O_0 = \log t \cdot N_{2k+1} + tO_0,$$

which is (2.20) for $\tilde{N}_{2k+2}$. Further,

$$\tilde{N}_{2k+3} = \int_0 \frac{\tilde{N}_{2k+2}}{t(1-t)} = \int_0 \left(\frac{N_{2k+1} \log t}{t(1-t)} + O_0 \right)$$

and by (2.22) again

$$\tilde{N}_{2k+3} = \log t \cdot \int_0 \frac{N_{2k+1}}{t(1-t)} - tO_0 + tO_0 = \log t \cdot N_{2k+2} + tO_0,$$

which is (2.20) for $\tilde{N}_{2k+3}$. By induction, formula (2.20) follows for each $k \geq 0$.

Similarly, (2.21) trivially holds for N_0. Assume that it holds for N_{2k} for some k. Then

$$N_{2k+1} = \int_0 \frac{N_{2k}}{1-t} = \int_0 \frac{t^k O_0}{1-t} = \int_0 t^k O_0 = t^{k+1} O_0$$

and

$$N_{2k+2} = \int_0 \frac{N_{2k+1}}{t(1-t)} = \int_0 \frac{t^{k+1} O_0}{t(1-t)} = \int_0 t^k O_0 = t^{k+1} O_0,$$

which is (2.21) for N_{2k+1} and N_{2k+2}, respectively. By induction, (2.21) holds for all $k \geq 0$. ∎

REMARK. It is easy to see that

$$h_{2k}(0) = 1/k!^2, \quad h_{2k+1}(0) = 1/k!(k+1)!,$$

so the formulas (2.21) are, in fact, the best possible.

COROLLARY 2.3. *For each* $k \geq 0$,

$$\begin{aligned} K_{2k} &= N_{2k} \log t + O_0 = t^k \log t \cdot O_0 + O_0, \\ K_{2k+1} &= N_{2k+1} \log t + O_0 = t^{k+1} \log t \cdot O_0 + O_0. \end{aligned} \tag{2.23}$$

PROPOSITION 2.4. *For each* $n > 0$, *the functions* $K_0, K_1, \ldots, K_{n-1}$ *are linearly independent modulo* $t^n \log t \cdot O_0 + O_0$. *That is, for any complex numbers* $a_0, \ldots, a_{n-1}$,

$$\sum_{k=0}^{n-1} a_k K_k = t^n \log t \cdot O_0 + O_0$$

is only possible when all $a_k = 0$.

PROOF. Aiming at a contradiction, assume that

$$\sum_{k=0}^{n-1} a_k K_k = t^m \log t \cdot f + O_0, \quad f \in O_0,$$

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where $m \geq n$ and $f(0) \neq 0$. By (2.21), we then also have

$$\sum_{k=0}^{n-1} a_k \tilde{N}_{k+1} = t^m \log t \cdot f + \mathcal{O}_0.$$

Apply to both sides the operator $\mathcal{L}^n$. By (2.6) and (2.7), the right-hand side reduces to $a_{n-1}\tilde{N}_0 = a_{n-1}1$. On the left-hand side, we can use the formulas

$$\mathcal{L}(F \log t) = \log t \cdot \mathcal{L}(F) + \mathcal{O}_0$$

and

$$\mathcal{L}(t^j F) = j^2 F(0) t^{j-1} + t^j \mathcal{O}_0$$

valid for any $F \in \mathcal{O}_0$. Thus we arrive at

$$a_{n-1} 1 = t^{m-n} F \cdot \log t + \mathcal{O}_0$$

where $F \in \mathcal{O}_0$ and $F \not\equiv 0$ since $F(0) = \frac{m!^2}{(m-n)!^2} f(0)$. This is impossible. ∎

COROLLARY 2.5. *For each $n > 0$, there is a unique linear combination $\sum_0^{n-1} a_k K_k$ of the functions $K_0, \dots, K_{n-1}$ such that*

$$\sum_{k=0}^{n-1} a_k K_k = t^{n-1} \log t + t^n \log t \cdot \mathcal{O}_0 + \mathcal{O}_0,$$

i.e., which has precisely the singularity $\sim t^{n-1} \log t$ at the origin.

Our next objective is to get control of the behavior of the functions $N_0, \dots, N_{n-1}$ and $K_0, \dots, K_{n-1}$ at the boundary point $t = 1$.

PROPOSITION 2.6. *For each $k \geq 0$, one has*

$$N_k = P_k\left(\log \frac{1}{1-t}\right) + O\Big((1-t) \log^{k-1} \frac{1}{1-t}\Big), \tag{2.24}$$

$$\tilde{N}_k = \tilde{P}_k\left(\log \frac{1}{1-t}\right) + O\Big((1-t) \log^{k-1} \frac{1}{1-t}\Big), \tag{2.25}$$

$$K_k = R_{k-1}\left(\log \frac{1}{1-t}\right) + O\Big((1-t) \log^{k} \frac{1}{1-t}\Big) \tag{2.26}$$

as $t \nearrow 1$, where P_k, $\tilde{P}_k$ and R_{k-1} are polynomials of degrees k, k and $k-1$, respectively, which satisfy

$$\begin{aligned} P'_{k+1} &= P_k, \quad & \tilde{P}'_{k+1} &= \tilde{P}_k, \quad & R_{k-1} &= \tilde{P}_{k+1} - P_{k+1}, \\ P_0(z) &= 1, & P_1(z) &= z, & P_2(z) &= \frac{1}{2}z^2 + \zeta(2), \\ \tilde{P}_0(z) &= 1, & \tilde{P}_1(z) &= z, & \tilde{P}_2(z) &= \frac{1}{2}z^2 - \zeta(2). \end{aligned}$$

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In particular, the leading terms of $P_k(z)$, $\tilde{P}_k(z)$ *and* $R_{k-1}(z)$ *are* $z^k/k!$, $z^k/k!$ *and* $-2\zeta(2)z^{k-1}/(k-1)!$, *respectively.*

PROOF. Assume first that the assertion (2.24) holds for N_{2k} for some $k \geq 1$:

$$N_{2k} = P_{2k}\left(\log \frac{1}{1-t}\right) + (1-t)f(t), \quad f = O\left(\log^{2k-1} \frac{1}{1-t}\right).$$

Then

$$N_{2k+1} = \int_0 \frac{N_{2k}}{1-t} = (\textstyle\int_0 P_{2k})\left(\log \frac{1}{1-t}\right) + \displaystyle\int_0 f.$$

It is well known (and easily verified by partial integration) that for each $j > 0$ the integral

$$\int_0^1 \log^j \frac{1}{1-t}\, dt$$

is finite, and moreover

$$\int_t^1 \log^j \frac{1}{1-t}\, dt = \left|\int_0^{1-t} \log^j s\, ds\right| \simeq (1-t)\log^j \frac{1}{1-t} \quad \text{as } t \longrightarrow 1.$$

Consequently, as $t \longrightarrow 1$,

$$N_{2k+1} = (\textstyle\int_0 P_{2k})\left(\log \frac{1}{1-t}\right) + C_{2k} - O\big((1-t)\log^{2k-1}(1-t)\big)$$

where $C_{2k} = \int_0^1 f(t)\, dt$, which gives the required assertion for N_{2k+1}, with

$$P_{2k+1} = C_{2k} + \int_0 P_{2k}.$$

Now, similarly, assume that (2.24) holds for N_{2k+1} for some k. Then

$$N_{2k+2} = \int_0 \frac{N_{2k+1}}{t(1-t)} = \int_0 \frac{N_{2k+1}}{t} + \int_0 \frac{N_{2k+1}}{1-t}. \tag{2.27}$$

The second integral is susceptible to the same treatment as in the case of N_{2k+1} above, yielding

$$\int_0 \frac{N_{2k+1}}{1-t} = (\textstyle\int_0 P_{2k+1})\left(\log \frac{1}{1-t}\right) + C^*_{2k+1} - O\big((1-t)\log^{2k-1}(1-t)\big)$$

for some constant C^*_{2k+1}; and the first integral in (2.27)–for which there are no problems with the existence at $t = 0$, owing to Proposition 2.2–is, likewise, susceptible to the same treatment as the integral $\int_0 f$ above, yielding

$$\int_0 \frac{N_{2k+1}}{t} = C^{**}_{2k+1} - O\big((1-t)\log^{2k+1}(1-t)\big).$$

Therefore

$$N_{2k+2} = P_{2k+2}\left(\log \frac{1}{1-t}\right) + O\left((1-t)\log^{2k+1} \frac{1}{1-t}\right) \quad \text{as } t \longrightarrow 1,$$

with

$$P_{2k+2} = C^*_{2k+1} + C^{**}_{2k+1} + \int_0 P_{2k+1}.$$

Summing up, we see that if (2.24) holds for some N_k, then it holds also for N_{k+1}, and, moreover,

$$P'_{k+1} = P_k.$$

On the other hand,

$$N_0 = 1, \quad N_1 = \int_0 \frac{1}{1-t} = \log \frac{1}{1-t}, \tag{2.28}$$

and

$$N_2 = \int_0 \left(\frac{1}{t} + \frac{1}{t-1}\right) \log \frac{1}{1-t} = \frac{1}{2} \log^2 \frac{1}{1-t} + \mathrm{Li}_2(t) \tag{2.29}$$

where Li_2 is the familiar *dilogarithm*:

$$\mathrm{Li}_2(t) = \int_0 \frac{1}{t} \log \frac{1}{1-t} = \sum_{n=1}^{\infty} \frac{t^n}{n^2}.$$

For $t = 1$, $\mathrm{Li}_2(1) = \zeta(2) = \pi^2/6$ (the Riemann ζ-function). Using the formula (due to Euler; *cf.* [Le1], p. 5)

$$\mathrm{Li}_2(t) + \mathrm{Li}_2(1-t) = \log t \log \frac{1}{1-t} + \zeta(2), \tag{2.30}$$

we see that

$$N_2 = \frac{1}{2} \log^2 \frac{1}{1-t} + \zeta(2) + O\left((1-t) \log \frac{1}{1-t}\right).$$

Thus, by induction, it follows that (2.24) holds true for all $k \geq 0$, and for $k \geq 2$ we even have a more detailed formula

$$P_k(z) = \frac{1}{k!} z^k + \frac{\zeta(2)}{(k-2)!} z^{k-2} + \cdots, \tag{2.31}$$

so

$$N_k(t) = \frac{1}{k!} \log^k \frac{1}{1-t} + \frac{\zeta(2)}{(k-2)!} \log^{k-2} \frac{1}{1-t} + O\left(\log^{k-3} \frac{1}{1-t}\right).$$

The proof for $\tilde{N}_k$ runs along completely similar lines. This time, for $k = 0, 1, 2$ we obtain

$$\tilde{N}_0 = 1, \quad \tilde{N}_1 = \log t + \log \frac{1}{1-t}, \tag{2.32}$$

$$\tilde{N}_2 = \mathrm{Li}_2(1-t) + \frac{1}{2} \log^2 \frac{1}{1-t} - \zeta(2), \tag{2.33}$$

so (by induction as above) the required assertion (2.25) holds with

$$\tilde{P}_k(z) = \frac{1}{k!} z^k - \frac{\zeta(2)}{(k-2)!} z^{k-2} + \cdots. \tag{2.34}$$

Finally, as $K_k = \tilde{N}_{k+1} - N_{k+1}$, the assertions concerning K_k and R_{k-1} follow from those for N_k, $\tilde{N}_k$, P_k and $\tilde{P}_k$ by subtraction. ■

With the information at hand it is now easy to obtain an expression for the invariant Green function $\mathbf{G}_n$ with the pole at the origin in terms of the functions N_j and K_j. Since the highest order part of the operator $\boldsymbol{\Delta}^n$ is $(1-t)^{2n}\Delta^n$, the behavior of the Green function $\mathbf{G}_n(\cdot, 0)$ at the origin must be the same as that of the fundamental solution of the operator Δ^n; the latter is given by

$$c_n|z|^{2n-2}\log|z|^2 = c_n t^{n-1}\log t,$$

where $c_n = 1/[4^n(n-1)!^2\pi]$. By Corollary 2.5, there exists a unique linear combination $\sum_0^{n-1} a_j K_j$ which is $\sim t^{n-1}\log t$ at the origin. By Proposition 2.6,

$$\sum_{j=0}^{n-1} a_j K_j = Q_{n-2}\left(\log\frac{1}{1-t}\right) + O\left((1-t)\log^{n-1}\frac{1}{1-t}\right)$$

as $t \longrightarrow 1$, where Q_{n-2} is a polynomial of degree $n-2$ defined by $Q_{n-2}(z) = \sum_0^{n-1} a_j R_{j-1}$, with R_{j-1} the polynomials from (2.26). Again by Proposition 2.6, there exist (unique) constants b_k such that the linear combination $\sum_0^{n-1} b_k N_k$ has the same boundary behavior as $t \longrightarrow 1$ (in fact, $b_{n-1} = 0$, *i.e.*, N_{n-1} will be absent!). Since the N_k are holomorphic at the origin (Proposition 2.2), the function

$$\sum_{j=0}^{n-1} a_j K_j - \sum_{k=0}^{n-1} b_k N_k$$

will still have the correct type of singularity at the origin, will vanish at the boundary $t = 1$, and will be annihilated by the operator L^n. Thus we conclude that

$$c_n\Big[\sum_{j=0}^{n-1} a_j K_j - \sum_{k=0}^{n-1} b_k N_k\Big] = \Lambda_{n,1}(t) \tag{2.35}$$

must be the sought Green function for $\boldsymbol{\Delta}^n$ with pole at the origin.

COROLLARY 2.7. $\mathbf{G}_m(z,0) = O\big((1-t)\log^{m-1}(1-t)\big)$ *as* $t \equiv |z|^2 \longrightarrow 1$.

As an illustration of this machinery, we compute the Green functions $\mathbf{G}_m$ for m up to 4. Let us start by identifying the asymptotics at the origin (Proposition 2.2). One has

$$N_1 = \log\frac{1}{1-t} = t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots.$$

Consequently,

$$\begin{aligned} N_2 &= \int_0 \frac{N_1}{t(1-t)} = t + \frac{3}{4}t^2 + \frac{11}{18}t^3 + \frac{25}{48}t^4 + \cdots; \\ N_3 &= \int_0 \frac{N_2}{1-t} = \frac{1}{2}t^2 + \frac{7}{12}t^3 + \frac{85}{144}t^4 + \frac{83}{144}t^5 + \cdots; \\ N_4 &= \int_0 \frac{N_3}{t(1-t)} = \frac{1}{4}t^2 + \frac{13}{36}t^3 + \frac{241}{576}t^4 + \cdots. \end{aligned} \tag{2.36}$$

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The asymptotics at $t = 1$–that is, the polynomials P_k and $\tilde{P}_k$ from Proposition 2.6–are, unfortunately, more difficult to obtain, since to that end it seems to be unavoidable to compute the functions N_k and $\tilde{N}_k$ quite explicitly and then work things out. For $k = 0, 1, 2$ we already know from (2.28)–(2.34) that

$$P_0 = \tilde{P}_0 = 1, \quad P_1 = \tilde{P}_1 = z, \quad P_2 = \frac{1}{2}z^2 + \zeta(2), \quad \tilde{P}_2 = \frac{1}{2}z^2 - \zeta(2).$$

Taking next $k = 3$ we have from (2.29)

$$N_3 = \int_0 \frac{1}{1-t}\left(\frac{1}{2}\log^2\frac{1}{1-t} + \mathrm{Li}_2(t)\right) = \frac{1}{6}\log^3\frac{1}{1-t} + \mathrm{Li}_2(t)\log\frac{1}{1-t} - M_3(t) \tag{2.37}$$

where $M_3(t)$ is the Kummer function

$$M_3(t) = \int_0 \frac{\log^2(1-t)}{t}. \tag{2.38}$$

It can be shown that $M_3(1) = 2\zeta(3)$ (see the next section). Thus

$$P_3(z) = \frac{1}{6}z^3 + \zeta(2)z - 2\zeta(3). \tag{2.39}$$

As for $\tilde{N}_3$, we have from (2.33)

$$\begin{aligned} \tilde{N}_3 &= \int_0 \left(\frac{1}{1-t} + \frac{1}{t}\right)\left(\frac{1}{2}\log^2\frac{1}{1-t} + \mathrm{Li}_2(1-t) - \zeta(2)\right) \\ &= \frac{1}{6}\log^3\frac{1}{1-t} - \mathrm{Li}_3(1-t) + \frac{1}{2}M_3(t) + \mathrm{Li}_2(1-t)\log t \\ &\qquad + M_3(1-t) - \zeta(2)\log\frac{1}{1-t} - \zeta(2)\log t - \zeta(3), \end{aligned} \tag{2.40}$$

and

$$\tilde{P}_3(z) = \frac{1}{6}z^3 - \zeta(2)z. \tag{2.41}$$

Here $\mathrm{Li}_3(t)$ is the *trilogarithm*

$$\mathrm{Li}_3(t) = \int_0 \frac{\mathrm{Li}_2(t)}{t} = \sum_{k=1}^{\infty}\frac{t^k}{k^3}$$

and we have used the fact that $\mathrm{Li}_3(1) = \zeta(3)$ and $M_3(1) = 2\zeta(3)$.

Finally, for $k = 4$ we have by Proposition 2.6

$$P_4 = \frac{1}{24}z^4 + \frac{\zeta(2)}{2}z^2 - 2\zeta(3)z + A, \quad \tilde{P}_4 = \frac{1}{24}z^4 - \frac{\zeta(2)}{2}z^2 + B, \tag{2.42}$$

for some constants A and B whose determination is deferred to the next section. Subtracting, we also get

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$$R_{-1} = 0, \quad R_0 = -2\zeta(2), \quad R_1 = -2\zeta(2)z + 2\zeta(3),$$
$$R_2 = -\zeta(2)z^2 + 2\zeta(3)z + (B - A). \tag{2.43}$$

Let us now proceed to the respective cases $m = 1$ to 4.

THE CASE $m = 1$. This is of course trivial, but let us make the computation for completeness. We have

$$K_0 = \tilde{N}_1 - N_1 = \log t$$

so $a_0 = 1$ and

$$a_0 K_0 = O(1 - t) \quad \text{as } t \longrightarrow 1.$$

Thus

$$\mathbf{G}_1(\cdot, 0) = c_1 K_0 = \frac{1}{4\pi} \log t, \tag{2.44}$$

as it should be.

THE CASE $m = 2$. Now in addition to K_0 we have

$$K_1 = \tilde{N}_2 - N_2 = \mathrm{Li}_2(1 - t) - \mathrm{Li}_2(t) - \zeta(2).$$

By (2.28) and Corollary 2.3,

$$K_1 = N_1 \log t + O_0 \sim t \log t \quad \text{as } t \longrightarrow 0,$$

so $a_0 = 0$ and $a_1 = 1$, and by (2.43)

$$\sum_0^1 a_j K_j = K_1 = -2\zeta(2) + O\left((1 - t) \log \frac{1}{1 - t}\right) \quad \text{as } t \longrightarrow 1.$$

Thus

$$\mathbf{G}_2(\cdot, 0) = c_2\big[K_1 + 2\zeta(2)N_0\big] = \frac{1}{16\pi}\big[\mathrm{Li}_2(1 - t) - \mathrm{Li}_2(t) + \zeta(2)\big].$$

Using formula (2.30) we can rewrite this as

$$\mathbf{G}_2(\cdot, 0) = \frac{1}{16\pi}\left[\log t \log \frac{1}{1 - t} - 2\,\mathrm{Li}_2(t) + 2\zeta(2)\right] \tag{2.45}$$

which is in agreement with the result obtained in [EP] (Theorem 1 in Section 3), as well as with our Theorem 1.1.

THE CASE $m = 3$. This time we add $K_2 = \tilde{N}_3 - N_3$. By (2.36) and Proposition 2.2,

$$K_1 = N_1 \log t + O_0 \sim \log t \cdot (t + \frac{1}{2}t^2 + \cdots),$$
$$K_2 = N_2 \log t + O_0 \sim \log t \cdot (t + \frac{3}{4}t^2 + \cdots).$$

It follows that

$$4(K_2 - K_1) \sim t^2 \log t \quad \text{as } t \longrightarrow 0,$$

so $a_0 = 0$, $a_2 = -a_1 = 4$. At the boundary we have from (2.43)

$$K_2 - K_1 = -2\zeta(2)\log\frac{1}{1-t} + 2\big(\zeta(3)+\zeta(2)\big) + o(1) \quad \text{as } t \longrightarrow 1.$$

Also

$$N_1 = \log\frac{1}{1-t},$$

so

$$\mathbf{G}_3(\cdot,0) = 4c_3\Big[K_2 - K_1 + 2\zeta(2)N_1 - 2\big(\zeta(3)+\zeta(2)\big)\Big]. \tag{2.46}$$

Inserting the expressions for $\tilde{N}_3$, N_3 and K_1 obtained above and using the formula (2.30) and the formulas below in Section 3, it can be shown that this agrees with the formula for $\mathbf{G}_3$ in [EP], Theorem 1 in Section 4.

THE CASE $m = 4$. Proceeding as above we see from (2.36) that

$$\begin{aligned}
K_1 &\sim \log t \cdot \left(t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots\right),\\
K_2 &\sim \log t \cdot \left(t + \frac{3}{4}t^2 + \frac{11}{18}t^3 + \cdots\right),\\
K_3 &\sim \log t \cdot \left(\ \frac{1}{2}t^2 + \frac{7}{12}t^3 + \cdots\right)
\end{aligned}$$

as $t \longrightarrow 0$, so

$$36\big(K_3 - 2(K_2 - K_1)\big) \sim t^3 \log t \quad \text{as } t \longrightarrow 0.$$

Thus $a_3 = 36$, $a_1 = -a_2 = 72$. As $t \longrightarrow 1$,

$$\begin{aligned}
K_3 - 2K_2 + 2K_1 = -\zeta(2)\log^2\frac{1}{1-t} &+ \big(2\zeta(3)+4\zeta(2)\big)\log\frac{1}{1-t}\\
&+ \big(B - A - 4\zeta(3) - 4\zeta(2)\big) + o(1).
\end{aligned}$$

On the other hand,

$$N_1 = \log\frac{1}{1-t}, \quad N_2 = \frac{1}{2}\log^2\frac{1}{1-t} + \zeta(2) + o(1),$$

so we arrive at

$$\begin{aligned}
\mathbf{G}_4(\cdot,0) = 36c_4\Big[&K_3 - 2K_2 + 2K_1 + 2\zeta(2)N_2 - \big(2\zeta(3)+4\zeta(2)\big)N_1\\
&- \big(2\zeta(2)^2 + B - A - 4\zeta(3) - 4\zeta(2)\big)\Big].
\end{aligned} \tag{2.47}$$

The expressions for the function K_3 and the constants A, B will be derived in the next section; inserting them into the last right-hand side yields an explicit formula for the Green function $\mathbf{G}_4$. (It is rather unwieldy, so we do not reproduce it here.)

We conclude this section by proving an improved version of Proposition 2.6 which can be used to obtain an analytic continuation of the Green's functions past the boundary circle $|z| = 1$ (Corollary 2.10).

THEOREM 2.8. *For each $k \geq 0$, one has*

$$
\begin{aligned}
N_k &= \frac{1}{k!}\log^k \frac{1}{1-t} + \sum_{j=0}^{k-1} f_{kj}(t)\cdot \log^j \frac{1}{1-t},\\
\tilde N_k &= \frac{1}{k!}\log^k \frac{1}{1-t} + \sum_{j=0}^{k-1} \tilde f_{kj}(t)\cdot \log^j \frac{1}{1-t},
\end{aligned}
\tag{2.48}
$$

where f_{kj} and $\tilde f_{kj}$ are functions holomorphic in the right half-plane.

PROOF. As we did before with O_0, we introduce the notation O_+ for a general function, not necessarily the same one on each occurrence, holomorphic in the right half-plane. For each $g \in O_+$ we have, by integration by parts,

$$\int g(t)\log^k \frac{1}{1-t}\,dt = \log^k \frac{1}{1-t}\cdot \left(\textstyle\int_1 g\right) - \int \frac{k(\int_1 g)}{1-t}\log^{k-1}\frac{1}{1-t}\,dt.$$

Since the function $\frac{1}{1-t}\int_1 g$ is also O_+, we see that

$$\int g(t)\log^k \frac{1}{1-t}\,dt = \sum_{j=0}^{k} O_+\cdot \log^j \frac{1}{1-t}.$$

Combining this with the elementary equality

$$\int \frac{1}{1-t}\log^k \frac{1}{1-t}\,dt = \frac{1}{k+1}\log^{k+1}\frac{1}{1-t},$$

we finally obtain

$$\int \frac{g(t)}{1-t}\log^k \frac{1}{1-t}\,dt = \frac{g(1)}{k+1}\log^{k+1}\frac{1}{1-t} + \sum_{j=0}^{k} O_+\cdot \log^j \frac{1}{1-t}.$$

Consequently, if (2.48) holds for some k, then

$$
\begin{aligned}
N_{k+1} &= \int_0 \frac{1}{1-t} g_k N_k \quad (g_k = 1/t \text{ for } k \text{ odd}, g_k = 1 \text{ for } k \text{ even})\\
&= \int_0 \left(\frac{1}{k!}\frac{g_k}{1-t}\log^k \frac{1}{1-t} + \sum_{j=0}^{k-1} O_+\cdot \log^j \frac{1}{1-t}\right) \quad \text{(by assumption)}\\
&= C + \frac{g_k(1)}{(k+1)!}\log^{k+1}\frac{1}{1-t} + \sum_{j=0}^{k} O_+\cdot \log^j \frac{1}{1-t}\\
&= \frac{1}{(k+1)!}\log^k \frac{1}{1-t} + \sum_{j=0}^{k} O_+\cdot \log^j \frac{1}{1-t},
\end{aligned}
$$

as $g_k(1) = 1$ and the constant of integration C can be absorbed into the summand $j = 0$. Since $N_0 = 1$, it follows by induction that (2.48) holds true for all $k \geq 0$. The proof for $\tilde N_k$ is quite similar and hence omitted. ∎

Splitting off the constant term from each f_{kj},

$$f_{kj}(t) = f_{kj}(1) + (1-t)g_{kj}(t)$$

(and similarly for $\tilde f_{kj}$) and then comparing our last result with Proposition 2.6, we immediately obtain:

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COROLLARY 2.9. *For each $k \geq 0$,*

$$N_k = P_k\left(\log\frac{1}{1-t}\right) + (1-t)\sum_{j=0}^{k-1} g_{kj}(t)\log^j\frac{1}{1-t},$$

$$\tilde{N}_k = \tilde{P}_k\left(\log\frac{1}{1-t}\right) + (1-t)\sum_{j=0}^{k-1} \tilde{g}_{kj}(t)\log^j\frac{1}{1-t},$$

$$K_k = R_{k-1}\left(\log\frac{1}{1-t}\right) + (1-t)\sum_{j=0}^{k} h_{kj}(t)\log^j\frac{1}{1-t},$$

where P_k, $\tilde{P}_k$ and R_{k-1} are the polynomials from Proposition 2.6 and g_{kj}, $\tilde{g}_{kj}$ and h_{kj} are functions holomorphic in the right half-plane.

As a consequence we have also the following amplification of Corollary 2.7.

COROLLARY 2.10. *The invariant Green functions satisfy*

$$\mathbf{G}_m = (1-t)\sum_{j=0}^{m-1} G_{kj}(t)\log^j\frac{1}{1-t},$$

where G_{kj} are functions holomorphic in the right half-plane.

3. **Some transcendental functions. Hyperlogarithms.** In this section we discuss in more detail the transcendental functions which appear in connection with the N_k and $\tilde{N}_k$. Perhaps the most conspicuous among them are the *polylogarithms*

$$\mathrm{Li}_s(t) = \sum_{k=1}^{\infty}\frac{t^k}{k^s}$$

which can be defined recursively by

$$\mathrm{Li}_{s+1}(t) = \int_0 \frac{\mathrm{Li}_s(t)}{t}, \quad \mathrm{Li}_1(t) = \log\frac{1}{1-t}.$$

For $t = 1$,

$$\mathrm{Li}_s(1) = \zeta(s)$$

the Riemann ζ-function. Another important chain are the Kummer functions

$$M_k(t) = \int_0 \frac{\log^{k-1}(1-t)}{t}, \tag{3.1}$$

of which we have seen M_3 to enter into the formulas for N_3 and $\tilde{N}_3$, and M_4 arises if we calculate N_4 or $\tilde{N}_4$:

$$\begin{aligned} N_4 &= \int_0 \left(\frac{1}{t}+\frac{1}{1-t}\right)\left(\frac{1}{6}\log^3\frac{1}{1-t} + \mathrm{Li}_2(t)\log\frac{1}{1-t} - M_3(t)\right) \quad \text{(by (2.37))} \\ &= -\frac{2}{3}M_4(t) + \frac{1}{2}\mathrm{Li}_2(t)^2 - M_4^*(t) + \frac{1}{24}\log^4\frac{1}{1-t} \\ &\quad + \frac{1}{2}\mathrm{Li}_2(t)\log^2\frac{1}{1-t} - M_3(t)\log\frac{1}{1-t}. \end{aligned} \tag{3.2}$$

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Here we have denoted by M_4^* another transcendental function

$$M_4^*(t) = \int_0 \frac{M_3(t)}{t}. \tag{3.3}$$

Similarly, for $\tilde{N}_4$ we have by (2.40)

$$\begin{aligned}
\tilde{N}_4 &= \int_0 \frac{1}{1-t}\Big(\frac{1}{6}\log^3\frac{1}{1-t} - \mathrm{Li}_3(1-t) + \frac{1}{2}M_3(t) + \mathrm{Li}_2(1-t)\log t \\
&\qquad + M_3(1-t) - \zeta(2)\log\frac{1}{1-t} - \zeta(2)\log t - \zeta(3)\Big) \\
&= \frac{1}{24}\log^4\frac{1}{1-t} + \mathrm{Li}_4(1-t) + \frac{1}{2}M_3(t)\log\frac{1}{1-t} + \frac{1}{2}M_4(t) \\
&\qquad + \frac{1}{2}\mathrm{Li}_2(1-t)^2 + M_4^*(1) - M_4^*(1-t) - \frac{\zeta(2)}{2}\log^2\frac{1}{1-t} \\
&\qquad - \zeta(2)\,\mathrm{Li}_2(1-t) - \zeta(3)\log\frac{1}{1-t} - \zeta(4) + \frac{\zeta(2)^2}{2},
\end{aligned} \tag{3.4}$$

and we see that $\tilde{N}_4$ too can be expressed in terms of the functions M_4, M_4^*, M_3, Li_2, Li_3 and Li_4.

There are numerous relations between the various functions just mentioned. An example is Euler's formula (2.30), which we have already used several times and which can be verified easily by differentiation. Another important formula, essentially due to Kummer ([Le1], p. 159), connects $M_3(t)$ with $\mathrm{Li}_3(1-t)$:

$$M_3(t) = M_3(1) + \log t\log^2\frac{1}{1-t} - 2\,\mathrm{Li}_2(1-t)\log\frac{1}{1-t} - 2\,\mathrm{Li}_3(1-t). \tag{3.5}$$

Setting $t = 0$ in (3.5) we obtain the important equality

$$M_3(1) = 2\zeta(3). \tag{3.6}$$

In general one has the formula ([Le1], p. 203)

$$M_k(t) - M_k(1) = (-1)^k \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!}\,\mathrm{Li}_j(1-t)\,\log^{k-j}\frac{1}{1-t} \tag{3.7}$$

which is a generalization of (2.30) and (3.5), and

$$M_k(1) = (-1)^{k+1}(k-1)!\,\zeta(k). \tag{3.8}$$

The function M_4^* is more evasive. Using again integration by parts shows that it satisfies the formula

$$M_4^*(t) + M_4^*(1-t) = M_4^*(1) + M_3(1-t)\log(1-t) + M_3(t)\log t - \frac{1}{2}\log^2 t\log^2(1-t)$$

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which can be regarded as a higher-order analog of (2.30). An explicit expression for M_4^* can be obtained from the formula (7.65) in [Le1], p. 204; the result is

$$
\begin{aligned}
M_4^*(t) = 2&\left[\mathrm{Li}_4\left(\frac{-t}{1-t}\right) + \mathrm{Li}_4(t) - \mathrm{Li}_4(1-t) + \zeta(4)\right] \\
&+ 2\,\mathrm{Li}_3(t)\log\frac{1}{1-t} + \frac{1}{3}\log t\log^3\frac{1}{1-t} + \frac{1}{12}\log^4\frac{1}{1-t} \\
&+ \zeta(2)\log^2\frac{1}{1-t} - 2\zeta(3)\log\frac{1}{1-t}.
\end{aligned} \tag{3.9}
$$

The number $M_4^*(1)$ is best evaluated directly. To that end, use (2.30) to rewrite the formula (3.5) as

$$M_3(t) = M_3(1) - 2\,\mathrm{Li}_3(1-t) - \log t\log^2\frac{1}{1-t} - 2\zeta(2)\log\frac{1}{1-t} + 2\,\mathrm{Li}_2(t)\log\frac{1}{1-t}.$$

Dividing by t and using (3.6) gives

$$\frac{M_3(t)}{t} + \log t\cdot M_3(t)' = 2\frac{\zeta(3) - \mathrm{Li}_3(1-t)}{t} - 2\zeta(2)\,\mathrm{Li}_2(t)' + 2\,\mathrm{Li}_2(t)\cdot\mathrm{Li}_2(t)'.$$

Integrating from 0 to 1, the left-hand side vanishes, and we obtain

$$2\int_0^1 \frac{\zeta(3) - \mathrm{Li}_3(1-t)}{t}\,dt = \zeta(2)^2. \tag{3.10}$$

The last integral can be evaluated by power series expansion:

$$\int_0^1 \frac{\zeta(3) - \mathrm{Li}_3(t)}{1-t}\,dt = \sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1\frac{1-t^n}{1-t}\,dt = \sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}\frac{1}{k}\right) \tag{3.11}$$

On the other hand, from the Taylor series for $\log^2(1-t)$

$$\log^2(1-t) = 2\sum_{n=2}^{\infty}\frac{t^n}{n}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)$$

it follows that

$$M_3(t) = 2\sum_{n=2}^{\infty}\frac{t^n}{n^2}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right), \quad M_4^*(t) = 2\sum_{n=2}^{\infty}\frac{t^n}{n^3}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right),$$

and so

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}\frac{1}{k}\right) = \frac{1}{2}M_4^*(1) + \zeta(4).$$

Substituting this back into (3.11) and (3.10) yields

$$M_4^*(1) = \zeta(2)^2 - 2\zeta(4) = \frac{1}{2}\zeta(4) \tag{3.12}$$

(since $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$).

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This equality, as well as (3.6), have an interesting interpretation as sums of series: (3.6) can be restated as

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \left(\sum_{k=1}^{n-1} \frac{1}{k} \right) = \zeta(3),$$

and (3.12) means that [1]

$$\sum_{n=2}^{\infty} \frac{1}{n^3} \left(\sum_{k=1}^{n-1} \frac{1}{k} \right) = \frac{1}{4}\zeta(4).$$

We finish this discussion by evaluating the constants A, B in (2.42). From (3.2) we have

$$\begin{aligned} N_4 - \frac{1}{24}\log^4 \frac{1}{1-t} - \frac{\zeta(2)}{2}\log^2 \frac{1}{1-t} + 2\zeta(3)\log \frac{1}{1-t} \\ = -\frac{2}{3}M_4(t) + \frac{1}{2}\operatorname{Li}_2(t)^2 - M_4^*(t) + \frac{\operatorname{Li}_2(t) - \zeta(2)}{2}\log^2 \frac{1}{1-t} \\ - \big(M_3(t) - M_3(1)\big)\log \frac{1}{1-t}. \end{aligned}$$

In view of (2.30) and (3.5) the last two terms are of order $(1-t)\log^3 \frac{1}{1-t}$ as $t \longrightarrow 1$. Letting t tend to 1 we therefore get

$$A = -\frac{2}{3}M_4(1) + \frac{1}{2}\zeta(2)^2 - M_4^*(1) = \frac{19}{4}\zeta(4). \tag{3.13}$$

Similarly from (3.4)

$$\begin{aligned} \tilde{N}_4 - \frac{1}{24}\log^4 \frac{1}{1-t} + \frac{\zeta(2)}{2}\log^2 \frac{1}{1-t} \\ = \operatorname{Li}_4(1-t) + \frac{1}{2}\big(M_3(t) - M_3(1)\big)\log \frac{1}{1-t} + \frac{1}{2}M_4(t) + \frac{1}{2}\operatorname{Li}_2(1-t)^2 \\ + M_4^*(1) - M_4^*(1-t) - \zeta(2)\operatorname{Li}_2(1-t) - \zeta(4) + \frac{\zeta(2)^2}{2}. \end{aligned}$$

The second term on the right-hand side is again of order $(1-t)\log^3 \frac{1}{1-t}$ by (3.5), so letting $t \longrightarrow 1$ yields

$$B = \frac{1}{2}M_4(1) + M_4^*(1) - \zeta(4) + \frac{1}{2}\zeta(2)^2 = -\frac{9}{4}\zeta(4). \tag{3.14}$$

Thus

$$B - A = -8\zeta(4) + 2M_4^*(1) = -7\zeta(4)$$

---

[1] More generally,

$$\sum_{n=2}^{\infty} \frac{1}{n^{m-1}} \left(\sum_{k=1}^{n-1} \frac{1}{k} \right) = \frac{m-1}{2}\zeta(m) - \frac{1}{2}\sum_{k=2}^{m-2} \zeta(k)\zeta(m-k),$$

and also

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m-1}} \left(\sum_{k=1}^{n} \frac{1}{k} \right) = \frac{1}{2}\sum_{k=2}^{2m-2} (-1)^k \zeta(k)\zeta(2m-k).$$

These formulas are due to N. Nielsen (see [Ni], p. 198).

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and the constant term in the square brackets in the formula (2.47) for $\mathbf{G}_4$ is equal to

$$-\Big(\zeta(2)^2-4\zeta(4)+M_4^*(1)-2\zeta(3)-2\zeta(2)\Big)=\zeta(4)+2\zeta(3)+2\zeta(2). \tag{3.15}$$

The formulas above can be used to obtain—as we have promised near the end of the preceding section—various expressions for the functions K_j, $j\leq 4$. For instance, we have

$$K_0=\log t;$$

$$K_1=\operatorname{Li}_2(1-t)-\operatorname{Li}_2(t)-\zeta(2)=\log t\log\frac{1}{1-t}-2\operatorname{Li}_2(t);$$

$$\begin{aligned}K_2=&-4\operatorname{Li}_3(1-t)-2\operatorname{Li}_3(t)+\frac{1}{2}\log t\log^2\frac{1}{1-t}\\&-2\Big[\operatorname{Li}_2(1-t)+\zeta(2)\Big]\log\frac{1}{1-t}+\operatorname{Li}_2(t)\log t+4\zeta(3);\end{aligned}$$

$$\begin{aligned}K_3=\,&8\operatorname{Li}_4(1-t)+M_4^*(t)-M_4^*(1-t)+4\operatorname{Li}_3(1-t)\log\frac{1}{1-t}\\&+\frac{1}{2}\operatorname{Li}_2(1-t)\log^2\frac{1}{1-t}-\frac{1}{2}\operatorname{Li}_2(t)\log^2\frac{1}{1-t}+\frac{1}{2}\operatorname{Li}_2(1-t)^2\\&-\frac{1}{2}\operatorname{Li}_2(t)^2+\frac{1}{3}\log t\log^3\frac{1}{1-t}-\zeta(2)\operatorname{Li}_2(1-t)\\&-\frac{1}{2}\zeta(2)\log^2\frac{1}{1-t}+2\zeta(3)\log\frac{1}{1-t}-\frac{25}{4}\zeta(4),\ \textit{etc.}\end{aligned}$$

In particular, feeding this information into the formulas for $\mathbf{G}_2$, $\mathbf{G}_3$ and $\mathbf{G}_4$ in Section 2, one obtains explicit formulas for the latter in terms of the polylogarithms Li_2, Li_3 and Li_4.

In principle, the Green functions $\mathbf{G}_m$ can be computed by the method of Section 2 for any m, but for $m\geq 5$ the results become immensely complicated and also new transcendental functions pop up; for this reason, we won't pursue these matters any further and, instead, will be content with stating a simple result of general nature.

Recall that, for a $2n$-tuple of complex numbers $a_1,\dots,a_n$, $b_1,\dots,b_n$, the *hyperlogarithm* of Kummer is defined as

$$F_n\begin{pmatrix}a_1 & a_2 & \cdots & a_n\\ b_1 & b_2 & \cdots & b_n\end{pmatrix}(t)=\int_{b_n}\frac{1}{t-a_n}\int_{b_{n-1}}\frac{1}{t-a_{n-1}}\cdots\int_{b_1}\frac{1}{t-a_1}.$$

This is in general a multi-valued analytic function on $\mathbb{C}$ which may have (and usually has) various logarithmic singularities at the points $a_1,a_2,\dots,a_n$. It is, however, easy to see (by considerations akin to our proof of Proposition 2.2 above) that a single-valued holomorphic branch can always be selected on any simply connected domain Ω as long as, for each k, either $a_k=b_{k-1}$, or $a_k\notin\Omega$ and $b_k\neq a_k$. As a rather straightforward application of the discussion in the preceding section, we then have the following result.

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THEOREM 3.1. *For each $k \geq 1$ the function N_k is a linear combination of the hyperlogarithms*

$$F_k \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{3.16}$$

where $a_j = 1$ for j odd and $a_j \in \{0,1\}$ for j even. Similarly, the functions $\tilde{N}_k$ are linear combinations of

$$F_k \begin{pmatrix} 0 & a_2 & a_3 & \cdots & a_k \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad F_k \begin{pmatrix} 1 & a_2 & a_3 & \cdots & a_k \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{3.17}$$

where $a_j = 1$ for j even and $a_j \in \{0,1\}$ for j odd, $j \neq 1$. Consequently, the Green function $\mathbf{G}_k(\cdot, 0)$ too is a linear combination of the functions (3.16) *and* (3.17).

Hyperlogarithms were first studied by Kummer in the third part of his paper [Ku]. For a recent exposition, see, *e.g.*, G. Wechsung's article [We] in Lewin's book [Le2].

4. **The dual spaces.** In this section we consider the compact duals of the previous symmetric domains (ball, disc).

In the first place let us examine the case of the Riemann sphere S^2 ($\approx$ the complex projective line $\mathbb{P}^1$). We let its diameter be 1 (radius $\frac{1}{2}$). Removing a base point denoted ∞ (the point at infinity) we map the remainder of S^2 onto the complex plane $\mathbb{C}$. We can thus use the generic point z of $\mathbb{C}$ as a local coordinate on $S^2 \setminus \{\infty\}$. We put also, as before, $t = r^2 = |z|^2$.

Below we present in table form some relevant quantities associated with S^2 along with, for comparison, their counterparts in the dual case of the disk $\mathbb{D}$; for bookkeeping reasons the former are equipped with a subscript in the form of a star $*$.

	sphere	disc
metric	$ds_* = \frac{\lvert dz\rvert}{1+\lvert z\rvert^2}$	$ds = \frac{\lvert dz\rvert}{1-\lvert z\rvert^2}$
area element	$dA_* = \frac{dx\,dy}{(1+\lvert z\rvert^2)^2}$	$dA = \frac{dx\,dy}{(1-\lvert z\rvert^2)^2}$
total area	π	∞
Laplace operator	$\boldsymbol{\Delta}_* = 4(1+\lvert z\rvert^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}}$	$\boldsymbol{\Delta} = 4(1-\lvert z\rvert^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}}$
radial part $\cdot \frac{1}{4}$	$L_* = (1+t)^2 \left[t \frac{d^2}{dt^2} + \frac{d}{dt}\right]$	$L = (1-t)^2 \left[t \frac{d^2}{dt^2} + \frac{d}{dt}\right]$

TABLE 1.

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The main difference is thus that everywhere the factor $1-|z|^2$ (respectively $1-t$) in the previous situation has been replaced by $1+|z|^2$ (respectively $1+t$).

There is a simple connection between the operators L_* and L. Indeed, let us set for $f = f(t)$, any radial function,

$$f^* = f^*(t) = f(-t).$$

Then we have

$$L(f^*) = -(L_* f)^*. \tag{4.1}$$

PROOF OF (4.1). We have by definition (see Table 1)

$$\begin{aligned} L(f^*) &= (1-t)^2\big[tf''(-t) - f'(-t)\big];\\ L_* f &= (1+t)^2\big[tf''(t) + f'(t)\big]. \end{aligned}$$

The second of these relations gives

$$\begin{aligned} (L_* f)^* = (L_* f)(-t) &= (1-t)^2\big[-tf''(-t) + f'(-t)\big]\\ &= -(1-t)^2\big[tf''(-t) - f'(-t)\big] = -L(f^*). \end{aligned}$$

This proves the desired equality. ∎

Iterating (4.1) gives

$$L^m(f^*) = (-1)^m (L_*^m f)^*. \tag{4.2}$$

From (4.2) we may draw the following conclusion.

LEMMA 4.1. *Let $f_1, \ldots, f_m, g_1, \ldots, g_m$ be the basis for the solutions of the differential equation $L^m f = 0$ indicated in Section 1 (cf. also [EP], Section 4). Then a basis for the solutions of $L_*^m f = 0$ is constituted by the functions*

$$f_1(-t), \ldots, f_m(-t), g_1(-t), \ldots, g_m(-t).$$

We now turn our attention to the Green's functions of the iterated operators $\mathbf{\Delta}_*^m$. First we must, however, make precise what is meant by Green's function in the present compact situation.

Consider quite generally the inhomogeneous equation $\mathbf{\Delta}_*^m u = v$ on S^2. Let us multiply by dA_* and integrate, yielding

$$\int_{S^2} \mathbf{\Delta}_*^m u \, dA_* = \int_{S^2} v \, dA_*.$$

If we use the fact that the operator $\mathbf{\Delta}_*^m$ is selfadjoint and that, in addition, $\mathbf{\Delta}_*^m 1 = 0$, it follows that the left-hand side is zero so we obtain the following necessary condition for the existence of a (global) solution:

$$\int_{S^2} v \, dA_* = 0. \tag{4.3}$$

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It may be proved[2] that, conversely, (4.3) implies the existence of a solution. Finally, we obtain a unique solution u if we impose the additional hypothesis

$$\int_{S^2} u\, dA_* = 0. \tag{4.4}$$

Accordingly, we define the m-th order invariant Green's function on S^2 with pole at the point w to be the unique function $\mathbf{G}_{m*} = \mathbf{G}_{m*}(z) = \mathbf{G}_{m*}(z, w)$ such that

1° $\mathbf{\Delta}_*^m \mathbf{G}_{m*} = \delta_w - \frac{1}{\pi}$ where δ_w is the Dirac delta function at w;

2° $\int_{S^2} \mathbf{G}_{m*}\, dA_* = 0$.

The constant $\frac{1}{\pi}$ appears because in view of our normalization of the area (see Table 1) precisely then the condition (4.3) is formally satisfied.

As usual, we may take $w = 0$.

Let us begin with the case $\boxed{m = 1}$. Then there are two independent radial solutions: the functions 1 and $\log t$. Furthermore, the equation $L_* f = 1$ is satisfied by the function $\log(1 + t)$ (this follows from (4.1) and (2.28), or also from Lemma 4.1 and the results of [EP]; see Scholium 1, Section 4 there). This leads the following expression for $\mathbf{G}_{1*}$:

$$\mathbf{G}_{1*}(t) = \frac{1}{4\pi}\big(\log t - \log(1 + t) + A\big),$$

where the constant A has to be chosen in such a way that condition 2° is met. Observe that the sign in front of the second term helps us to take care of the singularity at ∞ which otherwise would have occurred.

Before passing to the actual computation of A it is handy to collect some salient facts about integration of radial functions, which we do as a lemma.

LEMMA 4.2. *Let $f = f(t)$ be any radial function on S^2. Then we have*

$$\int_{S^2} f\, dA_* = \pi \int_0^\infty \frac{f(t)}{(1 + t)^2}\, dt.$$

Moreover, the differential $\frac{dt}{(1+t)^2}$ is invariant if we make the substitution $t \longmapsto \frac{1}{t}$ (inversion). In particular, we have

$$\int_0^\infty \frac{f(\frac{1}{t})}{(1 + t)^2}\, dt = \int_0^\infty \frac{f(t)}{(1 + t)^2}\, dt.$$

PROOF. All there is to do is to observe that if $z = re^{i\theta}$, $t = r^2$ then it follows that

$$dA_*(z) = \pi \frac{dt\, d\theta}{(1 + t)^2}. \qquad \blacksquare$$

From the first half of Lemma 4.2 it follows that we must have

$$\int_0^\infty \frac{\log t}{(1 + t)^2}\, dt - \int_0^\infty \frac{\log(1 + t)}{(1 + t)^2}\, dt + A = 0.$$

Using the symmetry property in the second half we see that the first integral must vanish. On the other hand, a direct computation reveals that the second integral has the value 1. Thus we conclude that $\boxed{A = 1}$. In summary, we have now proved the following result.

[2] For instance, using the spectral resolution for $\mathbf{\Delta}_*$.

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THEOREM 4.3. *The radial part of the invariant Green's function* $\mathbf{G}_{1*}$ *is given by the formula*

$$\mathbf{G}_{1*}(t) = -\frac{1}{4\pi}\big(\log t - \log(1+t) + 1\big). \tag{4.5}$$

We can combine this with the expression for $\mathbf{G}_{1*}$ obtained by using the eigenfunction expansion of $\boldsymbol{\Delta}_*$. We recall that the n-th eigenvalue of $\boldsymbol{\Delta}_*$ is $-4n(n+1)$ and occurs with multiplicity $2n+1$ ($n = 0, 1, 2, \ldots$). So choosing for each n an orthonormal basis of eigenfunctions (spherical harmonics) $Y_{n\alpha}$ ($\alpha = 1, \ldots, 2n+1$) we see that we must also have

$$\mathbf{G}_{1*}(z,w) = -\sum_{n=1}^{\infty} \frac{1}{4n(n+1)} \sum_{\alpha=1}^{2n+1} Y_{n\alpha}(z) Y_{n\alpha}(w).$$

Because of the invariance the inner sum depends only of the distance between the points z and w (with respect to the metric ds_*). In other words, it must be proportional to $P_n(\cos\theta)$, where P_n stands for the n-th Legendre polynomial and θ is the angle between z and w. The proportionality constant can be found using that, in view of the normalization $P_n(1) = 1$, it must equal the integral of the expression obtained by putting $z = w$, divided by the integral of the function 1; that is, using also the orthonormalization of the eigenfunctions, it turns out to be

$$\frac{1}{\pi}\int_{S^2} \sum_{\alpha=1}^{2n+1} \big(Y_{n\alpha}(z)\big)^2 \, dA_* = \frac{1}{\pi} \sum_{\alpha=1}^{2n+1} \int_{S^2} \big(Y_{n\alpha}(z)\big)^2 \, dA_* = \frac{1}{\pi} \sum_{\alpha=1}^{2n+1} 1 = \frac{2n+1}{\pi}.$$

So we end up with the formula

$$\mathbf{G}_{1*}(z,w) = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n(\cos\theta). \tag{4.6}$$

REMARK 4.1. Alternatively, we could have used the fact that the sum

$$\sum_{\alpha=1}^{2n+1} Y_{n\alpha}(z) Y_{n\alpha}(w)$$

is nothing but the reproducing kernel of the n-th eigenspace. In this way we could have stayed entirely in the complex domain without having to pass to the real (cf. [PZ], p. 231, the last remark in Section 2).

Let us now recall the relation between the angle θ and the parameter t: As the inverse image (pullback) of z under stereographic projection is the point with coordinates

$$\left(\frac{\operatorname{Re} z}{1+|z|^2}, \frac{\operatorname{Im} z}{1+|z|^2}, \frac{1}{2}\,\frac{1-|z|^2}{1+|z|^2}\right)$$

in $\mathbb{R}^3$ and $t = |z|^2$, we must have

$$\cos\theta = \frac{1-t}{1+t}.$$

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This again shows that, putting $x = \cos\theta$,

$$t = \frac{1-x}{1+x}, \quad 1+t = \frac{2}{1+x}, \quad \frac{t}{1+t} = \frac{1-x}{2}.$$

Finally, juxtaposing (4.5) and (4.6) we see that

$$\boxed{\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n(x) = \log 2 - 1 - \log(1-x)} \tag{4.7}$$

which formula is formally valid for $-1 \le x < 1$.[3]

Formula (4.7) is however not new. Indeed, it is the special case of a result which appears in [BE], Section 10.10, as the bilinear formula (53) on p. 183:

$$\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} P_n(x) P_n(y) = 2\log 2 - 1 - \log(1-x)(1+y) \tag{4.8}$$

with $-1 < x \le y < 1$.

REMARK 4.2. Actually, (4.8) can be obtained from its special case (4.7) by invoking the multiplication theorem for Legendre polynomials, which we found in [V], p. 141:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_n\Big(xy - \sqrt{1-x^2}\sqrt{1-y^2}\cos\phi\Big)\, d\phi = P_n(x) P_n(y).$$

Indeed, if we replace x by $xy - \sqrt{1-x^2}\sqrt{1-y^2}\cos\phi$ in (4.7) and integrate, then the left-hand side of the resulting formula agrees with the left-hand side of (4.8). In order to be able to reduce the right-hand side into the right-hand side of (4.8) we must therefore show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log\Big(1 - xy + \sqrt{1-x^2}\sqrt{1-y^2}\cos\phi\Big)\, d\phi = -\log 2 + \log(1-x)(1+y), \tag{4.9}$$

or, upon setting $z = -\sqrt{1-x^2}\sqrt{1-y^2}/(1-xy)$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(1 - z\cos\phi)\, d\phi = \log \frac{1+\sqrt{1-z^2}}{2},$$

which is easily done by power series expansion.

Encouraged by this partial success we next make an assault on the case $\boxed{m = 2}$. Looking again at (2.45) and (2.29), or at the formulae in [EP], especially Scholium 1 in Section 4 there, and using our Lemma 4.1, we see that the radial part of $\mathbf{G}_{2*}$ must be of the form

$$\mathbf{G}_{2*}(t) = \frac{1}{16\pi}\left[\log t \cdot \log(1+t) - \frac{1}{2}\log^2(1+t) + \mathrm{Li}_2(-t) + B\right], \tag{4.10}$$

[3] We do not enter here into the subtleties connected with convergence but we assure the reader that everything can be fixed up with ease.

where B is a constant. Alternatively, it is easy to see directly that the expression within brackets in the last formula satisfies the differential equation $L_*^2 f = -1$. Moreover, the singularity at $t = 0$ clearly is the right one and there is no singularity at $t = \infty$. The last fact follows from the following result for the dilogarithm (see [Le1], formula (1.7), p. 4):

$$\mathrm{Li}_2(-t) = -\frac{1}{2}\log^2 t - \mathrm{Li}_2\left(-\frac{1}{t}\right) - \frac{\pi^2}{6} \tag{4.11}$$

valid for $t \in (0,\infty)$ (all the log-terms cancel). So what remains is the determination of the constant B in (4.10) so as to meet the requirement that the integral of $\mathbf{G}_{2*}$ be zero (see Condition 2° *ultra*). To this end we invoke once more Lemma 4.2. Let the expression within brackets in (4.10) be denoted by $\star$. If we replace there t by $\frac{1}{t}$ we obtain the expression

$$\star\star = -\log t \cdot \bigl(\log(1+t) - \log t\bigr) - \frac{1}{2}\bigl(\log(1+t) - \log t\bigr)^2 + \mathrm{Li}_2\left(-\frac{1}{t}\right) + B.$$

Adding up gives after various simplifications, invoking especially (4.11),

$$\begin{aligned}\star + \star\star &= \log t \cdot \log(1+t) - \log^2(1+t) - \frac{\pi^2}{6} + 2B\\ &= \log(1+t)\cdot\log\left(1 - \frac{1}{1+t}\right) - \frac{\pi^2}{6} + 2B.\end{aligned}$$

According to Lemma 4.2 the integral of this quantity has to vanish. In other words, we must have

$$\int_0^\infty \frac{\log(1+t)\cdot\log\left(1-\frac{1}{1+t}\right)}{(1+t)^2}\,dt - \frac{\pi^2}{6} + 2B = 0.$$

The value of the last integral is $\frac{\pi^2}{6} - 2$. (To see this we make the substitution $t = \frac{1}{p} - 1$. Then we obtain the integral

$$-\int_0^1 \log p \cdot \log(1-p)\,dp.$$

Again, using the power series expansion of the function $\log(1-p)$ we are lead to summing the series

$$-\sum_{n=1}^\infty \frac{1}{n(n+1)^2},$$

which is easily achieved.) Thus we find that $\boxed{B = 1}$. Summing up, we have now found the following result.

THEOREM 4.4. *The radial part of the invariant Green's function $\mathbf{G}_{2*}$ is given by the formula*

$$\mathbf{G}_{2*}(t) = \frac{1}{16\pi}\left[\log t\cdot\log(1+t) - \frac{1}{2}\log^2(1+t) + \mathrm{Li}_2(-t) + 1\right]. \tag{4.12}$$

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If we now use the spectral resolution of $\mathbf{\Delta}_*$ in a similar way as in connection with Theorem 4.3 we are led to the following analog of formula (4.7):

$$\boxed{\sum_{n=1}^{\infty}\frac{2n+1}{\big(n(n+1)\big)^2}P_n(x)=\log\frac{1-x}{1+x}\cdot\log\frac{2}{1+x}-\frac{1}{2}\log^2\frac{2}{1+x}+\mathrm{Li}_2\left(-\frac{1-x}{1+x}\right)+1}$$

where $-1\le x\le 1$.

REMARK 4.3. Using the multiplication theorem for Legendre polynomials one can also formally write down a bilinear formula analogous to (4.8), thus involving the sum

$$\sum_{n=1}^{\infty}\frac{2n+1}{\big(n(n+1)\big)^2}P_n(x)P_n(y).$$

However, so far we have not been able to evaluate the integrals arising in the right-hand side so there is no point in writing it down here.

We have now discussed in detail the cases $m=1$ and $m=2$. It is clear that in principle the previous analysis carries over to the case of any integer m. (For instance, using (2.46) and (2.37), one can after some work produce the result for $m=3$, and from (2.47) and (3.4) for $m=4$.) In particular, we are thus led to consider the sums

$$\sum_{n=1}^{\infty}\frac{2n+1}{\big(n(n+1)\big)^m}P_n(x)\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{2n+1}{\big(n(n+1)\big)^m}P_n(x)P_n(y)$$

and we arrive at the conviction that at least the former can be expressed in closed form in terms of hyperlogarithmic functions.

REMARK 4.4. The above can also interpreted in terms of *zeta values*. Indeed, let us write

$$Z(s,x)=\sum_{n=1}^{\infty}\frac{2n+1}{\big(n(n+1)\big)^s}P_n(x)\quad\text{and}\quad Z(s,x,y)=\sum_{n=1}^{\infty}\frac{2n+1}{\big(n(n+1)\big)^s}P_n(x)P_n(y),$$

where s is a complex variable. The functions $Z(s,x)$ and $Z(s,x,y)$ are related to the *Minakshisundaram-Pleijel zeta function* (see [MP]) for the operator $\mathbf{\Delta}_*$. Our results can thus be expressed by saying that we have computed the values of these functions for the two integer arguments $s=m=1$ and $s=m=2$.

We see that it is of considerable interest to extend this investigation also to the case of other compact Hermitean symmetric spaces. We limit ourselves to pointing out that in the rank one case, namely, the complex d-dimensional projective space $\mathbb{P}^d$, which is the dual of the complex unit ball $\mathbb{B}^d$, one has, instead of the Legendre polynomials $P_n(x)$, the Jacobi polynomials $P_n^{(d-1,0)}(x)$ (see [BE], Section 10.8), and the formula (for $m=1$)

$$\boxed{\sum_{n=1}^{\infty}\frac{2n+d}{n(n+d)}\binom{n+d-1}{n}P_n^{(d-1,0)}(x)=\sum_{j=1}^{d-1}\frac{1}{j}\left(\frac{1+x}{1-x}\right)^j-\sum_{j=1}^{d}\frac{1}{j}-\log\frac{1-x}{2}}$$

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of which (4.7) is a special case ($d = 1$).

REMARK 4.5 (ON THE CASE OF WEIGHTED GROUP ACTIONS). Throughout this paper, up to this moment, we have been concerned with unweighted group actions, that is, functions f are acted upon by composition: $f \longmapsto f \circ \phi$ if ϕ is an automorphism of the manifold under consideration. Now we say also a few words on the case when also a weight (multiplier) is involved. For simplicity, let us fix our attention to the present situation of the sphere S^2. Recall that the isometries ϕ of S^2 are induced by unimodular unitary 2×2 matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where thus $\gamma = -\overline{\beta}$, $\delta = \overline{\alpha}$, $\alpha\delta - \beta\gamma = 1$: If $z \in S^2$ then its image under ϕ is given by $\phi(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. Accordingly we let functions f transform according to the rule

$$f(z) \longmapsto f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)(\gamma z + \delta)^{\nu} \qquad \Bigl(= f\bigl(\phi(z)\bigr)\bigl(\phi'(z)\bigr)^{-\frac{\nu}{2}}\Bigr),$$

where ν is a fixed integer, $\nu = 0, 1, 2, \ldots$. (Such objects should really not be viewed as functions but as forms of negative degree $-\frac{\nu}{2}$, and written as $f(z)(dz)^{-\frac{\nu}{2}}$.) The expression for the corresponding Laplacian can be found in [PZ] (p. 226, beginning of Section 1):[4]

$$\begin{aligned} \mathbf{\Delta}_{\nu*} &= 4(1 + |z|^2)^2 \frac{\partial^2}{\partial z \partial \overline{z}} - 4\nu(1 + |z|^2)\overline{z}\frac{\partial}{\partial \overline{z}} \\ &\quad (= \mathbf{\Delta}_* - \nu Z_* \text{where } Z_* \text{ is the Zhang correction).} \end{aligned} \tag{4.13}$$

The corresponding radial operator is

$$L_{\nu*} = (1 + t)^2 \left[t\frac{d^2}{dt^2} + \frac{d}{dt}\right] - \nu(1 + t)t\frac{d}{dt} = (1 + t)^{2+\nu} \frac{d}{dt}\left[t(1 + t)^{-\nu}\frac{d}{dt}\right].$$

The operator $\mathbf{\Delta}_{\nu*}$ is selfadjoint with respect to the metric

$$\|f\|_{\nu}^2 = \int_{S^2} |f(z)|^2 \frac{dA_*}{(1 + |z|^2)^{\nu}}. \tag{4.14}$$

The kernel of $\mathbf{\Delta}_{\nu*}$ consists precisely of the analytic functions with finite $\|\cdot\|_{\nu}$-norm, that is, of polynomials of degree not exceeding ν (see [PZ], p. 226); in view of selfadjointness, the closure of the range of $\mathbf{\Delta}_{\nu*}$ is the orthogonal complement (with respect to (4.14)) of this kernel. Also, the total area of the Riemann sphere with respect to the measure in (4.14) is now $\pi/(\nu+1)$. Thus we are led to postulate the following definition of the corresponding Green's functions $\mathbf{G}^{\nu}_{1*}$:

1° For any w,

$$\mathbf{\Delta}_{\nu*}\mathbf{G}^{\nu}_{\mathbf{1}*}(z, w) = \delta(z, w) - \frac{\nu + 1}{\pi}(1 + z\overline{w})^{\nu},$$

where it is assumed that the differential operator acts on the z-variable.

[4] Following [EP], Section 3, we ought to consider such objects not as invariant differential operators, but covariant ones.

2° Orthogonality relations:

$$\int_{S^2} \mathbf{G}^\nu_{\mathbf{1}*}(z,w)\overline{z}^k \frac{dA_*(z)}{(1+|z|^2)^\nu} = 0$$

for all monomials z^k with $k = 0, 1, \ldots, \nu$.

(Here the index 1 is intended as a reminder of the fact that it is a first order Green's function.)

In order to determine the Green's function $\mathbf{G}^\nu_{1*}$ we have to solve the ordinary differential equation $L_{\nu*} f = -(\nu+1)$. It is seen, in one way or other, that we have the particular solution $f = \log(1+t) - \log t$. This function has the right behavior both at $t = 0$ and $t = \infty$. Thus we end up the following formula (generalizing (4.5)) for the radial Green's function:

$$\mathbf{G}^\nu_{1*}(t) = \frac{1}{4\pi}\big(\log t - \log(1+t) + A_\nu\big). \tag{4.15}$$

where the constant A_ν has to be determined so that $\mathbf{G}^\nu_{1*}$ has a vanishing mean value with respect to the measure $\frac{dA_*}{(1+|z|^2)^\nu}$. Using Lemma 4.2 we thus obtain

$$\int_0^\infty \frac{\log t}{(1+t)^{\nu+2}}\,dt - \int_0^\infty \frac{\log(1+t)}{(1+t)^{\nu+2}}\,dt + A_\nu = 0$$

which yields

$$A_\nu = \sum_{j=1}^{\nu+1} \frac{1}{j}\,. \tag{4.16}$$

It is now also easy to state the expression for the Green's function when the pole is at an arbitrary point $w \in S^2$:

$$\mathbf{G}^\nu_{1*}(z,w) = \frac{1}{4\pi}\left[\log \frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)} + A_\nu\right] \cdot (1+z\overline{w})^\nu$$

with A_ν again given by (4.16); from this formula the covariance is clearly visible.

Next, in order to determine the spectral expansion of $\mathbf{G}^\nu_{1*}$ we observe that, instead of the Legendre polynomials P_n in the case $\nu = 0$, we shall now have the hypergeometric functions ${}_2F_1(\nu+n+1, -n; 1; \frac{t}{t+1})$, that is, the Jacobi polynomials $P^{(0,\nu)}_n(\frac{1-t}{1+t})$. Indeed, the n-th eigenvalue is $-4n(\nu+n+1)$, and occurs with multiplicity $\nu+2n+1$ (*cf.* [PZ]), so if we make the change of variable $x = \frac{1-t}{1+t}$ the equation for radial eigenfunctions

$$L_{\nu*} f + n(\nu+n+1) f = 0$$

transforms into

$$(1-x^2)\frac{d^2 f}{dx^2} + \big[\nu - (\nu+2)x\big]\frac{df}{dx} + n(n+\nu+1) f = 0,$$

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which is the defining equation for the Jacobi polynomials $P_n^{(0,\nu)}(x)$ (*cf.* [BE], formula (14) in Section 10.8). Proceeding as before, it follows that

$$\mathbf{G}^{\nu}_{1*}(t) = -\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{(\nu+2n+1)\,P_n^{(0,\nu)}(\frac{1-t}{1+t})}{n(\nu+n+1)}. \tag{4.17}$$

Thus if we equate the right hand members of (4.15) and (4.17), using the value of A_ν given by (4.16), we end up with the following formula generalizing (4.7):

$$\boxed{\sum_{n=1}^{\infty}\frac{(2n+\nu+1)}{n(n+\nu+1)}P_n^{(0,\nu)}(x) = -\sum_{j=1}^{\nu+1}\frac{1}{j} - \log\frac{1-x}{2}}. \tag{4.18}$$

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