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Smoothness of topological equivalence on the half line for nonautonomous systems

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We study the differentiability properties of the topological equivalence between a uniformly asymptotically stable linear nonautonomous system and a perturbed system with suitable nonlinearities. For this purpose, we construct a homeomorphism inspired in the Palmer's one restricted to the positive half line, studying additional continuity properties and providing sufficient conditions ensuring its C^r–smoothness.

Keywords: topological equivalence; nonautonomous differential equations; nonautonomous hyperbolicity; uniform asymptotic stability; diffeomorphism

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1. Introduction

This work is devoted to study the relation between the solutions of the systems

$$\dot{x} = A(t)x \tag{1.1}$$

and

$$\dot{y} = A(t)y + f(t,y), \tag{1.2}$$

where $A\colon \mathbb{R}^+ \to M(n,\mathbb{R})$ and $f:\mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. Moreover, the following properties are verified:

(P1) $A(t)$ is continuous and $\sup\limits_{t\in\mathbb{R}^+} ||A(t)|| = M > 0$.

(P2) The system (1.1) is uniformly asymptotically stable, namely, there exist constants $K \geqslant 1$ and $\alpha > 0$ such that its transition matrix $\Phi(t,s)$ verifies

$$||\Phi(t,s)|| \leqslant K\mathrm{e}^{-\alpha(t-s)} \quad \text{for any}\, t \geqslant s \geqslant 0. \tag{1.3}$$

(P3) The function f is continuous on (t,y). Moreover, for any $t \geqslant 0$ and any couple $(y,\bar{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ it follows

$$|\, f(t,y) - f(t,\bar{y})\,| \leqslant \gamma\,|\, y - \bar{y}\,| \quad \text{and} \quad |\, f(t,y)\,| \leqslant \mu, \tag{1.4}$$

where $||\cdot||$ and $|\cdot|$ denote a matrix norm and vector norm respectively.

In the autonomous case, Hartman [6] and Grobman [5] found a local homeomorphism between the solutions of a nonlinear system and the solutions of its linearization around a hyperbolic equilibrium point. Pugh in [13] enhanced the previous result by constructing an explicit and global homeomorphism for the particular case of perturbed linear systems, also requiring hyperbolicity of the equilibrium point.

In the nonautonomous case, Palmer in [10] extended the Pugh's result using the exponential dichotomy as a natural version of the hyperbolicity property. In addition, Palmer introduced the concept of topological equivalence, which was generalized by Shi *et al.* in [16] as strongly topological equivalence.

Now, in order to present the aforementioned concepts, we will consider an interval $J \subseteq \mathbb{R}$.

Definition 1.1 ([3, 9]). The linear system (1.1) has an exponential dichotomy property on $J \subseteq \mathbb{R}$ if there exists a projection $P^2 = P$ and constants $\bar{K} \geqslant 1$, $\bar{\alpha} > 0$, such that its fundamental matrix $\Phi(t)$ verifies:

$$\begin{cases} ||\Phi(t)P\Phi^{-1}(s)|| & \leqslant & \bar{K}\mathrm{e}^{-\bar{\alpha}(t-s)} \quad \text{for any} \quad t \geqslant s \quad t,s \in J, \\ ||\Phi(t)(I-P)\Phi^{-1}(s)|| & \leqslant & \bar{K}\mathrm{e}^{-\bar{\alpha}(s-t)} \quad \text{for any} \quad s \geqslant t, \quad t,s \in J. \end{cases} \tag{1.5}$$

Definition 1.2. The systems (1.1) and (1.2) are J-topologically equivalent if there exists a function $H\colon J \times \mathbb{R}^n \to \mathbb{R}^n$ with the properties

(i) If $x(t)$ is a solution of (1.1), then $H[t,x(t)]$ is a solution of (1.2),

(ii) $H(t,u) - u$ is bounded in $J \times \mathbb{R}^n$,

(iii) For each fixed $t \in J$, $u \mapsto H(t,u)$ is a homeomorphism of $\mathbb{R}^n$,

In addition, the function $G(t,u) = H^{-1}(t,u)$ has properties (ii)–(iii) and maps solutions of (1.2) into solutions of (1.1).

Definition 1.3. The systems (1.1) and (1.2) are $J-$strongly topologically equivalent if they are $J-$topologically equivalents and H is a uniform homeomorphism. Namely, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $|u - \tilde{u}| < \delta$ implies $|H(t,u) - H(t,\tilde{u})| < \varepsilon$ and $|G(t,u) - G(t,\tilde{u})| < \varepsilon$ for any $t \in J$.

Notice that definition 1.1 implies uniform asymptotic stability when $P = I$ and $J = \mathbb{R}^+$ as in the property **(P2)**. We point out that definitions 1.2 and 1.3 are slight modifications of the ones introduced by Palmer and Shi *et al.*, since they considered the particular cases $J = \mathbb{R}$ in [10, 16] and $J = \mathbb{R}^+$ in [11].

Notice that the continuity of H and G in any $(t,u) \in J \times \mathbb{R}^n$ is not considered in definitions 1.2 and 1.3. However, a careful review of the topological equivalence literature shows that this property has a disparate treatment. Indeed, it has been

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considered by Palmer [**10**, p. 754] and omitted by Shi *et al.* [**16**, p. 814]. The following definition is inspired in the seminal result of Palmer [**10**]:

Definition 1.4. The systems (1.1) and (1.2) are J-continuously topologically equivalent if there exists a function $H: J \times \mathbb{R}^n \to \mathbb{R}^n$ with the properties

(i) If $x(t)$ is a solution of (1.1), then $H[t, x(t)]$ is a solution of (1.2),

(ii) $H(t, u) - u$ is bounded in $J \times \mathbb{R}^n$,

(iii) For each fixed $t \in J$, $u \mapsto H(t, u)$ is a homeomorphism of $\mathbb{R}^n$,

(iv) H is continuous in any $(t, u) \in J \times \mathbb{R}^n$.

In addition, the function $G(t, u) = H^{-1}(t, u)$ has properties (ii)–(iv) and maps solutions of (1.2) into solutions of (1.1).

To the best of our knowledge, the differentiability of the maps H and G in any $(t, u) \in J \times \mathbb{R}^n$ has not been studied, which prompt us to introduce the following definition:

Definition 1.5. The systems (1.1) and (1.2) are C^r continuously topologically equivalent on J if:

(i) The systems are $J-$continuously topologically equivalent,

(ii) For any fixed $t \in J$; the map $u \mapsto H(t, u)$ is a C^r-diffeomorphism of $\mathbb{R}^n$, with $r \geqslant 1$,

(iii) The partial derivatives of H and G up to order r with respect to u are continuous functions of $(t, u) \in J \times \mathbb{R}^n$.

In § 2 we state our first result, which provides sufficient conditions ensuring the $\mathbb{R}^+-$strongly topological equivalence between systems (1.1) and (1.2). We follow the lines of Palmer [**10**] which constructs the maps H and G by combining the Green's function associated with the exponential dichotomy with a perturbation f satisfying **(P3)**, obtaining a $\mathbb{R}$–topological equivalence, this was improved by Shi *et al.* [**16**] which obtains a $\mathbb{R}$-strong topological equivalence. Palmer and Shi's results are extended in [**8**, **15**, **17**] by considering nonexponential dichotomies and properties more general than **(P3)**. Nevertheless to the best of our knowledge, this Green's function approach has always assumed that (1.1) has an exponential dichotomy in $\mathbb{R}$ rather than in $\mathbb{R}^+$.

Contrary to the above context, our construction of H and G have technical differences with the previous works and it induces surprising and interesting consequences from a continuity and differentiability point of view. In fact, our restriction to $\mathbb{R}^+$ allow also to prove our second result; namely; the systems (1.1) and (1.2) are $\mathbb{R}^+$-continuously topologically equivalent.

In § 3 we show that if (1.2) has an equilibrium then is unique, and the assumptions on the nonlinearity imply its uniform asymptotic stability. Moreover, we study the behaviour of the equilibria through of homeomorphisms H and G. We verify that

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if the origin is an equilibrium of the nonlinear system then is a fixed point of H, otherwise if the nonlinear system has an equilibrium different of the origin then H and G maps the corresponding equilibria in an asymptotical way.

Section 4 states our main theorem, which provides additional conditions on the smoothness of f which ensures that (1.1) and (1.2) are C^r continuously topologically equivalent on $\mathbb{R}^+$. The restriction to $\mathbb{R}^+$ allows us to work in a simpler way than [**1**]. It is important to note that, contrary to the autonomous context, in the nonautonomous framework there are few results about the global differentiability of maps H and G and we refer the reader to [**4**] for local results.

2. Strongly and continuous topological equivalence on the half line

The following theorem provides a sufficient condition ensuring the $\mathbb{R}^+-$strongly topological equivalence between the systems (1.1) and (1.2).

THEOREM 2.1. *Assume that* **(P1)–(P3)** *are satisfied and*

$$\frac{K\gamma}{\alpha} < 1, \tag{2.1}$$

then systems (1.1) *and* (1.2) *are* $\mathbb{R}^+-$*strongly topologically equivalent.*

Proof. In order to make a more readable proof, we will decompose it in several steps. Namely, step 1 defines two auxiliary systems whose solutions are used in step 2 to construct the maps H and G. To prove that these maps establish a topological equivalence, the properties (i)–(ii) are verified in step 3, while the uniform continuity is proved in steps 4 and 5.

Step 1: Preliminaries. Let $t \mapsto x(t, \tau, \xi)$ and $t \mapsto y(t, \tau, \eta)$ be solutions of (1.1) and (1.2) passing through ξ and η at $t = \tau$. Now, we will consider the initial value problems

$$\begin{cases} w' = A(t)w - f(t, y(t, \tau, \eta)) \\ w(0) = 0, \end{cases} \tag{2.2}$$

and

$$\begin{cases} z' = A(t)z + f(t, x(t, \tau, \xi) + z) \\ z(0) = 0. \end{cases} \tag{2.3}$$

By using the variation of parameters formula we have that

$$w^*(t; (\tau, \eta)) = -\int_0^t \Phi(t, s) f(s, y(s, \tau, \eta))\, \mathrm{d}s \tag{2.4}$$

is the unique solution of (2.2). Let $BC(\mathbb{R}^+, \mathbb{R}^n)$ be the Banach space of bounded continuous functions with the supremum norm. Now, for any couple $(\tau, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n$, we define the operator $\Gamma_{(\tau,\xi)}\colon BC(\mathbb{R}^+,\mathbb{R}^n)\to BC(\mathbb{R}^+,\mathbb{R}^n)$ as follows

$$\phi\mapsto\Gamma_{(\tau,\xi)}\phi:=\int_0^t\Phi(t,s)f(s,x(s,\tau,\xi)+\phi)\,\mathrm{d}s. \tag{2.5}$$

Since $\gamma K/\alpha<1$ it is easy to see by **(P2)**–**(P3)** that the operator $\Gamma_{(\tau,\xi)}$ is a contraction and by the Banach fixed point theorem it follows that

$$z^*(t;(\tau,\xi))=\int_0^t\Phi(t,s)f(s,x(s,\tau,\xi)+z^*(s;(\tau,\xi)))\,\mathrm{d}s$$

is the unique solution of (2.3).

On the other hand, by uniqueness of solutions it can be proved that

$$z^*(t;(\tau,\xi))=z^*(t;(r,x(r,\tau,\xi)))\quad\text{for any}\quad r\geqslant 0, \tag{2.6}$$

and

$$w^*(t;(\tau,\nu))=w^*(t;(r,y(r,\tau,\nu)))\quad\text{for any}\quad r\geqslant 0. \tag{2.7}$$

Step 2: Construction of the maps H and G. For any $t\geqslant 0$ we define the maps $H(t,\cdot)\colon\mathbb{R}^n\to\mathbb{R}^n$ and $G(t,\cdot)\colon\mathbb{R}^n\to\mathbb{R}^n$ as follows:

$$\begin{aligned}H(t,\xi)&:=\xi+\int_0^t\Phi(t,s)f(s,x(s,t,\xi)+z^*(s;(t,\xi)))\,\mathrm{d}s\\&=\xi+z^*(t;(t,\xi)),\end{aligned}$$

and

$$\begin{aligned}G(t,\eta)&:=\eta-\int_0^t\Phi(t,s)f(s,y(s,t,\eta))\,\mathrm{d}s\\&=\eta+w^*(t;(t,\eta)).\end{aligned} \tag{2.8}$$

By using (2.6), we can verify that

$$\begin{aligned}H[t,x(t,\tau,\xi)]&=x(t,\tau,\xi)+\int_0^t\Phi(t,s)f(s,x(s,t,x(t,\tau,\xi)+z^*(s;(t,x(t,\tau,\xi))))\,\mathrm{d}s\\&=x(t,\tau,\xi)+\int_0^t\Phi(t,s)f(s,x(s,\tau,\xi)+z^*(s;(\tau,\xi)))\,\mathrm{d}s\\&=x(t,\tau,\xi)+z^*(t;(\tau,\xi)).\end{aligned}$$

Step 3: H and G satisfy properties (i)–(ii) of definition 1.2. By (1.1) and (2.3) combined with the above equality, we have that

$$\begin{aligned}\frac{\partial}{\partial t}H[t,x(t,\tau,\xi)]&=\frac{\partial}{\partial t}x(t,\tau,\xi)+\frac{\partial}{\partial t}z^*(t;(\tau,\xi))\\&=A(t)x(t,\tau,\xi)+A(t)z^*(t;(\tau,\xi))+f(t,H[t,x(t,\tau,\xi)])\\&=A(t)H[t,x(t,\tau,\xi)]+f(t,H[t,x(t,\tau,\xi)]),\end{aligned}$$

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then $t \mapsto H[t, x(t, \tau, \xi)]$ is solution of (1.2) passing through $H(\tau, \xi)$ at $t = \tau$. As consequence of uniqueness of solution we obtain

$$H[t, x(t, \tau, \xi)] = y(t, \tau, H(\tau, \xi)), \tag{2.9}$$

similarly, it can be proved that $t \mapsto G[t, y(t, \tau, \eta)]$ is solution of (1.1) passing through $G(\tau, \eta)$ at $t = \tau$ and

$$G[t, y(t, \tau, \eta)] = x(t, \tau, G(\tau, \eta)) = \Phi(t, \tau)G(\tau, \eta), \tag{2.10}$$

and the property (i) follows. Secondly, by using (1.3) and (1.4) it follows that

$$|H(t, \xi) - \xi| \leqslant K\mu \int_0^t \mathrm{e}^{-\alpha(t-s)}\, \mathrm{d}s \leqslant \frac{K\mu}{\alpha}$$

for any $t \geqslant 0$. A similar inequality can be obtained for $|G(t, \eta) - \eta|$ and the property (ii) is verified.

Step 4: H is bijective for any $t \geqslant 0$. We will first show that $H(t, G(t, \eta)) = \eta$ for any $t \geqslant 0$. Indeed,

$$\begin{aligned} H[t, G[t, y(t, \tau, \eta)]] &= G[t, y(t, \tau, \eta)] + \int_0^t \Phi(t, s) f(s, x(s, t, G[t, y(t, \tau, \eta)]) \\ &\quad + z^*(s; (t, G[t, y(t, \tau, \eta)]))))\, \mathrm{d}s \\ &= y(t, \tau, \eta) - \int_0^t \Phi(t, s) f(s, y(s, \tau, \eta))\, \mathrm{d}s \\ &\quad + \int_0^t \Phi(t, s) f(s, x(s, t, G[t, y(t, \tau, \eta)]) \\ &\quad + z^*(s; (t, G[t, y(t, \tau, \eta)]))))\, \mathrm{d}s. \end{aligned}$$

Let $\omega(t) = |H[t, G[t, y(t, \tau, \eta)]] - y(t, \tau, \eta)|$. Hence by using **(P2)** and **(P3)** we have that

$$\begin{aligned} \omega(t) &= \Big| \int_0^t \Phi(t, s)\{f(s, x(s, t, G[t, y(t, \tau, \eta)]) \\ &\quad + z^*(s; (t, G[t, y(t, \tau, \eta)]))) - f(s, y(s, \tau, \eta))\}\, \mathrm{d}s| \\ &\leqslant K\gamma \int_0^t \mathrm{e}^{-\alpha(t-s)} |\{x(s, t, G[t, y(t, \tau, \eta)]) \\ &\quad + z^*(s; (t, G[t, y(t, \tau, \eta)])) - y(s, \tau, \eta)\}|\, \mathrm{d}s. \end{aligned}$$

Notice that,

$$x(s, t, G[t, y(t, \tau, \eta)]) + z^*(s; (t, G[t, y(t, \tau, \eta)])) = H[s, x(s, t, G[t, y(t, \tau, \eta)])]$$

and recalling that

$$x(s, t, G[t, y(t, \tau, \eta)]) = x(s, \tau, G(\tau, \eta)) = G[s, y(s, \tau, \eta)],$$

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we can see

$$H[s, x(s, t, G[t, y(t, \tau, \eta)])] = H[s, G[s, y(s, \tau, \eta)]].$$

Therefore, we obtain

$$\omega(t) \leqslant K\gamma \int_0^t \mathrm{e}^{-\alpha(t-s)}\omega(s)\,\mathrm{d}s \leqslant \frac{K\gamma}{\alpha} \sup_{s\in\mathbb{R}^+}\{\omega(s)\} \quad \text{for all} \quad \mathrm{t} \geqslant 0.$$

The supremum is well defined by property (i) from definition 1.2 and the fact that all the solutions of systems (1.1) and (1.2) are bounded on $\mathbb{R}^+$. Now, we take the supremum on the left side above and due to $K\gamma/\alpha < 1$ it follows that $\omega(t) = 0$ for any $t \geqslant 0$. In particular, when we take $t = \tau$ we obtain $H(\tau, G(\tau, \eta)) = \eta$.

Next, we will prove that $G(t, H(t, \xi)) = \xi$. In fact, due to (2.9) we have that

$$\begin{aligned} G[t, H[t, x(t, \tau, \xi)]] &= H[t, x(t, \tau, \xi)] - \int_0^t \Phi(t, s) f(s, y(s, t, H[t, y(x, \tau, \xi)]))\,\mathrm{d}s \\ &= x(t, \tau, \xi) + \int_0^t \Phi(t, s)\{f(s, H[s, x(s, \tau, \xi)]) \\ &\quad - f(s, y(s, \tau, H(\tau, \xi)))\}\,\mathrm{d}s \\ &= x(t, \tau, \xi), \end{aligned}$$

and taking $t = \tau$ leads to $G(\tau, H(\tau, \xi)) = \xi$. In consequence, for any $t \geqslant 0$, H is a bijection and G is its inverse.

Step 5: H and G are uniformly continuous for any fixed t. Firstly, we prove that G is uniformly continuous.

As stated in [**16**, p. 823], it can be proved that $\alpha \leqslant M$. Now, we construct the auxiliary functions $\theta, \theta_0 \colon [0, +\infty) \to [0, +\infty)$ defined by

$$\theta(t) = 1 + K\gamma \left(\frac{\mathrm{e}^{(M+\gamma-\alpha)t} - 1}{M + \gamma - \alpha}\right) \quad \text{and} \quad \theta_0(t) = \begin{cases} K\gamma t & \text{if } \alpha = M, \\ K\gamma \left(\dfrac{\mathrm{e}^{(M-\alpha)t} - 1}{M - \alpha}\right) & \text{if } \alpha < M. \end{cases}$$

Now, given $\varepsilon > 0$, let us define the constants

$$L(\varepsilon) = \frac{1}{\alpha} \ln\left(\frac{4\mu K}{\alpha\varepsilon}\right), \quad \theta_0^* = \max_{t\in[0,L(\varepsilon)]} \theta_0(t) \quad \text{and} \quad \theta^* = \max_{t\in[0,L(\varepsilon)]} \theta(t). \tag{2.11}$$

We will prove the uniform continuity of G by considering two cases:

Case i) $t \in [0, L(\varepsilon)]$. By **(P2)** and **(P3)** we can deduce that

$$|G(t, \eta) - G(t, \bar{\eta})| \leqslant |\eta - \bar{\eta}| + \gamma K \mathrm{e}^{-\alpha t} \int_0^t \mathrm{e}^{\alpha s} |y(s, t, \eta) - y(s, t, \bar{\eta})|\,\mathrm{d}s. \tag{2.12}$$

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Now, by **(P1)** and **(P3)** we obtain that the following inequalities are satisfied for any $0 \leqslant s \leqslant t$:

$$\begin{aligned}|y(s,t,\eta)-y(s,t,\bar{\eta})| &\leqslant |\eta-\bar{\eta}| + \int_s^t ||A(\tau)|||y(\tau,t,\eta)-y(\tau,t,\bar{\eta})|\,\mathrm{d}\tau \\ &\quad + \int_s^t |f(\tau,y(\tau,t,\eta))-f(\tau,y(\tau,t,\bar{\eta}))|\,\mathrm{d}\tau \\ &\leqslant |\eta-\bar{\eta}| + (M+\gamma)\int_s^t |y(\tau,t,\eta)-y(\tau,t,\bar{\eta})|\,\mathrm{d}\tau. \end{aligned} \tag{2.13}$$

Hence, by Gronwall's lemma we conclude that for any $0 \leqslant s \leqslant t$:

$$|y(s,t,\eta)-y(s,t,\bar{\eta})| \leqslant |\eta-\bar{\eta}|\mathrm{e}^{(M+\gamma)(t-s)}. \tag{2.14}$$

Upon inserting (2.14) in (2.12), we obtain that

$$\begin{aligned}|G(t,\eta)-G(t,\bar{\eta})| &\leqslant \left(1+K\gamma\mathrm{e}^{(M+\gamma-\alpha)t}\int_0^t \mathrm{e}^{-(M+\gamma-\alpha)s}\right)|\eta-\bar{\eta}| \\ &= \left(1+K\gamma\left\{\frac{\mathrm{e}^{(M+\gamma-\alpha)t}-1}{M+\gamma-\alpha}\right\}\right)|\eta-\bar{\eta}| \\ &\leqslant \theta(t)|\eta-\bar{\eta}| \leqslant \theta^*|\eta-\bar{\eta}|.\end{aligned}$$

Case ii) $t > L(\varepsilon)$. By **(P1)**–**(P3)**, we have that

$$\begin{aligned}|G(t,\eta)-G(t,\bar{\eta})| &\leqslant |\eta-\bar{\eta}| + 2\mu K\int_0^{t-L}\mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s \\ &\quad + K\gamma\int_{t-L}^t \mathrm{e}^{-\alpha(t-s)}|y(s,t,\eta)-y(s,t,\bar{\eta})|\,\mathrm{d}s \\ &\leqslant |\eta-\bar{\eta}| + \frac{2\mu K}{\alpha}\mathrm{e}^{-\alpha L} \\ &\quad + K\gamma\int_0^L \mathrm{e}^{-\alpha u}|y(t-u,t,\eta)-y(t-u,t,\bar{\eta})|\,\mathrm{d}u.\end{aligned} \tag{2.15}$$

As in case i), the inequality (2.14) implies that

$$\begin{aligned}K\gamma\int_0^L \mathrm{e}^{-\alpha u}|y(t-u,t,\eta)-y(t-u,t,\bar{\eta})|\,\mathrm{d}u &\leqslant K\gamma\int_0^L \mathrm{e}^{(M+\gamma-\alpha)u}|\eta-\bar{\eta}|\,\mathrm{d}u \\ &= K\gamma\left\{\frac{\mathrm{e}^{(M+\gamma-\alpha)L}-1}{M+\gamma-\alpha}\right\}|\eta-\bar{\eta}|.\end{aligned}$$

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Upon inserting the above inequality in (2.15) and using (2.11), we have that

$$|G(t,\eta)-G(t,\bar{\eta})| \leqslant \left(1+K\gamma\left\{\frac{\mathrm{e}^{(M+\gamma-\alpha)L}-1}{M+\gamma-\alpha}\right\}\right)|\eta-\bar{\eta}|+\frac{2\mu K}{\alpha}\mathrm{e}^{-\alpha L}$$
$$\leqslant \theta^{*}|\eta-\bar{\eta}|+\frac{\varepsilon}{2}.$$

Summarizing, given $\varepsilon>0$, there exists $L(\varepsilon)>0$ and $\theta^{*}>0$ such that:

$$|G(t,\eta)-G(t,\bar{\eta})| \leqslant \begin{cases} \theta^{*}|\eta-\bar{\eta}| & \text{if } t\in[0,L] \\ \theta^{*}|\eta-\bar{\eta}|+\dfrac{\varepsilon}{2} & \text{if } t>L, \end{cases}$$

then it follows that

$$\forall\varepsilon>0\,\exists\delta(\varepsilon)=\frac{\varepsilon}{2\theta^{*}} \quad \text{such that} \quad |\eta-\bar{\eta}|<\delta \Rightarrow |G(t,\eta)-G(t,\bar{\eta})|<\varepsilon$$

and the uniform continuity of G follows.

Finally, we will prove that H is uniformly continuous for any $t\geqslant 0$. As the identity is uniformly continuous, we will only prove that $\xi\mapsto z^{*}(t;(t,\xi))$ is uniformly continuous.

Note that the fixed point $z^{*}(t;(t,\xi))$ can be seen as the uniform limit on $\mathbb{R}^{+}$ of a sequence $z_{j}^{*}(t;(t,\xi))$ defined recursively as follows:

$$\begin{cases} z_{j+1}^{*}(t;(t,\xi))=\displaystyle\int_{0}^{t}\Phi(t,s)f(s,x(s,t,\xi)+z_{j}^{*}(s;(t,\xi)))\,\mathrm{d}s & \text{for any } j\geqslant 1, \\ z_{0}^{*}(t;(t,\xi))=0 \end{cases}$$

The uniform continuity of each map $\xi\mapsto z_{j}^{*}(t;(t,\xi))$ will be proved inductively by following the lines of [**8**, **16**]. First, it is clear that $\xi\mapsto z_{0}^{*}(t;(t,\xi))$ verify this property. Secondly, we will assume the inductive hypothesis

$$\forall\varepsilon>0\,\exists\delta_{j}(\varepsilon)>0 \text{ s.t. } |\xi-\bar{\xi}|<\delta_{j}\Rightarrow|z_{j}^{*}(t;(t,\xi))-z_{j}^{*}(t;(t,\bar{\xi}))|<\varepsilon \quad \text{for any } t\geqslant 0.$$

For the step $j+1$ and given $\varepsilon>0$, we will only consider $\alpha<M$ since the case $\alpha=M$ can be carried out easily. We will use the constants $L(\varepsilon)$ and θ_{0}^{*} defined in (2.11) and introduce the notation

$$\Delta_{j}(t,\xi,\bar{\xi})=z_{j}^{*}(t;(t,\xi))-z_{j}^{*}(t;(t,\bar{\xi})).$$

As before, we will distinguish the cases $t\in[0,L(\varepsilon)]$ and $t>L(\varepsilon)$. First, for $t\in[0,L(\varepsilon)]$ we use **(P1)** combined with the estimation

$$|x(s,t,\xi)-x(s,t,\bar{\xi})|\leqslant|\xi-\bar{\xi}|\mathrm{e}^{M|t-s|}, \tag{2.16}$$

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and we can verify that

$$\begin{aligned}|\Delta_{j+1}(t,\xi,\bar{\xi})| &\leqslant K\gamma \mathrm{e}^{-\alpha t}\int_0^t \mathrm{e}^{\alpha s}\{|x(s,t,\xi)-x(s,t,\bar{\xi})|+|\Delta_j(s,\xi,\bar{\xi})|\}\,\mathrm{d}s\\ &\leqslant K\gamma \mathrm{e}^{-\alpha t}\int_0^t \mathrm{e}^{\alpha s}\{|\xi-\bar{\xi}|\mathrm{e}^{M(t-s)}+||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty\}\,\mathrm{d}s\\ &\leqslant K\gamma\left\{\frac{\mathrm{e}^{(M-\alpha)t}-1}{M-\alpha}\right\}|\xi-\bar{\xi}|+\frac{K\gamma}{\alpha}||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty\\ &\leqslant \theta_0^*|\xi-\bar{\xi}|+\frac{K\gamma}{\alpha}||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty,\end{aligned}$$

where $||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty=\sup_{t\geqslant 0}|\Delta_j(t,\xi,\bar{\xi})|$.

On the other hand, when $t>L(\varepsilon)$, we use **(P2)** combined with the boundedness of f in $[0,t-L]$ and Lipschitzness in $[t-L,t)$ to deduce that

$$\begin{aligned}|\Delta_{j+1}(t,\xi,\bar{\xi})| \leqslant 2K\mu\int_0^{t-L}\mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s+K\gamma\int_{t-L}^t \mathrm{e}^{-\alpha(t-s)}\{|x(s,t,\xi)-x(s,t,\bar{\xi})|\\ +|\Delta_j(s,\xi,\bar{\xi})|\}\,\mathrm{d}s.\end{aligned}$$

By **(P1)** combined with $u=t-s$, (2.11) and (2.16), we have that

$$\begin{aligned}|\Delta_{j+1}(t,\xi,\bar{\xi})| &\leqslant \frac{2K\mu}{\alpha}\mathrm{e}^{-\alpha L}+\frac{K\gamma}{\alpha}||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty\\ &\quad +K\gamma\int_0^L \mathrm{e}^{-\alpha u}|x(t-u,t,\xi)-x(t-u,t,\bar{\xi})|\,\mathrm{d}u\\ &\leqslant \frac{2K\mu}{\alpha}\mathrm{e}^{-\alpha L}+\frac{K\gamma}{\alpha}||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty+K\gamma|\xi-\bar{\xi}|\int_0^L \mathrm{e}^{(M-\alpha)u}\,\mathrm{d}u\\ &\leqslant \frac{\varepsilon}{2}+\frac{K\gamma}{\alpha}||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty+K\gamma\left\{\frac{\mathrm{e}^{(M-\alpha)L}-1}{M-\alpha}\right\}|\xi-\bar{\xi}|\\ &\leqslant \frac{\varepsilon}{2}+\frac{K\gamma}{\alpha}||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty+\theta_0^*|\xi-\bar{\xi}|.\end{aligned}$$

Summarizing, for any $t\geqslant 0$ it follows that

$$|\Delta_{j+1}(t,\xi,\bar{\xi})|\leqslant\begin{cases}\theta_0^*|\xi-\bar{\xi}|+\dfrac{K\gamma}{\alpha}||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty & \text{if } t\in[0,L]\\ \dfrac{\varepsilon}{2}+\dfrac{K\gamma}{\alpha}||\Delta_j(\cdot,\xi,\bar{\xi})||_\infty+\theta_0^*|\xi-\bar{\xi}| & \text{if } t>L.\end{cases}$$

Now, for any $\varepsilon>0$ there exists $L(\varepsilon)>0$, $\theta_0^*>0$ and

$$\delta_{j+1}(\varepsilon)=\min\left\{\delta_j(\varepsilon/2),\frac{\varepsilon}{2\theta_0^*}\left(1-\frac{K\gamma}{\alpha}\right)\right\}$$

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such that for any $t \geqslant 0$, we have

$$\forall \varepsilon > 0\, \exists \delta_{j+1}(\varepsilon) > 0 \text{ s.t. } |\xi - \bar{\xi}| < \delta_{j+1} \Rightarrow |z^*_{j+1}(t;(t,\xi)) - z^*_{j+1}(t;(t,\bar{\xi}))| < \varepsilon.$$

and the uniform continuity of $\xi \mapsto z^*_j(t;(t,\xi))$ follows for any $j \in \mathbb{N}$.

In order to finish our proof, we choose $N \in \mathbb{N}$ such that for any $j > N$ it follows that

$$||z^*(\cdot;(\cdot,\xi)) - z^*_j(\cdot;(\cdot,\xi))||_\infty < \varepsilon \quad \text{for any} \quad \xi \in \mathbb{R}^n,$$

and therefore, if $|\xi - \bar{\xi}| < \delta_j$ with $j > N$, it is true that

$$\begin{aligned} |z^*(t;(t,\xi)) - z^*(t;(t,\bar{\xi}))| &\leqslant |z^*(t;(t,\xi)) - z^*_j(t;(t,\xi))| + \Delta_j(t,\xi,\bar{\xi}) \\ &\quad + |z^*(t;(t,\bar{\xi})) - z^*_j(t;(t,\bar{\xi}))| < 3\varepsilon, \end{aligned}$$

and the uniform continuity of $\xi \mapsto z^*(t;(t,\xi))$ and $\xi \mapsto H(t,\xi)$ follows for any fixed $t \geqslant 0$. □

Remark 2.2. As we stated in the introduction, the construction of the homeomorphisms H and G and its uniform continuity is respectively inspired in the works of Palmer [**10**] and Shi et al. [**16**]. Nevertheless, our restriction to $\mathbb{R}^+$ induces some technical difficulties, for example:

(i) We do not have the uniqueness of bounded solutions of (2.2) and (2.3), this fact prompted us to consider a specific couple of initial conditions that allows to construct the maps between solutions of (1.1) and (1.2).

(ii) The proof of the uniform continuity is based in two facts: the continuity of any solution of (1.1)–(1.2) with respect to the initial conditions in a compact interval $[0, L]$ and the smallness of the homeomorphisms on $[L, +\infty[$ when L is big enough. This last condition is easily verified when we have an exponential dichotomy on $\mathbb{R}$ but more technical work is needed when we consider the restriction to $\mathbb{R}^+$.

As we emphasized in the introduction, our restriction to $\mathbb{R}^+$ will allow to prove the following result about the continuity of H and G with respect to both variables:

Theorem 2.3. *Assume that* **(P1)–(P3)** *and* (2.1) *are satisfied, then systems* (1.1) *and* (1.2) *are* $\mathbb{R}^+$*–continuously topologically equivalent.*

Proof. Firstly, by theorem 2.1 we know that (1.1) and (1.2) are $\mathbb{R}^+$–topologically equivalent.

Secondly, by using the continuity of the solutions with respect to the initial time and initial conditions [**7**, Ch.V], we note that for any $\varepsilon_1 > 0$ there exists $\delta_1(t_0, \xi_0, \varepsilon_1) > 0$ such that

$$|x(s,t,\xi) - x(s,t_0,\xi_0)| < \varepsilon_1 \quad \text{when} \quad |t - t_0| + |\xi - \xi_0| < \delta_1 \tag{2.17}$$

for any t and s in a compact interval of $\mathbb{R}^+$ containing t_0.

On the other hand, by using the continuity of the solutions with respect of the parameters [**7**, Ch.V] combined with a Palmer's result [**10**, lemma 1] restricted to $\mathbb{R}^+$, we know that for any $\varepsilon_2 > 0$ there exists $\delta_2(t_0, \xi_0, \varepsilon_2) > 0$ such that

$$|z^*(s;(t,\xi)) - z^*(s;(t_0,\xi_0))| < \varepsilon_2 \quad \text{when} \quad |t - t_0| + |\xi - \xi_0| < \delta_2. \tag{2.18}$$

for any t and s in a compact interval of $\mathbb{R}^+$ containing t_0.

Additionally, we know that for any $\varepsilon_3 > 0$ there exists $\delta_3(\varepsilon_3, t_0)$ such that

$$||\Phi(t,s) - \Phi(t_0,s)|| < \varepsilon_3 \quad \text{when} \quad |t - t_0| < \delta_3 \tag{2.19}$$

for any t and s in a compact interval of $\mathbb{R}^+$ containing t_0.

From now on, we will assume that t,s and t_0 are in a compact interval $I \subset \mathbb{R}^+$ and we denote

$$\omega_1(s,t,t_0,\xi,\xi_0) = f(s, x(s,t,\xi) + z^*(s;(t,\xi))) - f(s, x(s,t_0,\xi_0) + z^*(s;(t_0,\xi_0))).$$

Without loss of generality, we will assume that $t > t_0$. Now, notice that

$$\begin{aligned} H(t,\xi) - H(t_0,\xi_0) &= \xi - \xi_0 + \int_0^t \Phi(t,s) f(s, x(s,t,\xi) + z^*(s;(t,\xi)))\,\mathrm{d}s \\ &\quad - \int_0^{t_0} \Phi(t_0,s) f(s, x(s,t_0,\xi_0) + z^*(s;(t_0,\xi_0)))\,\mathrm{d}s \\ &= \xi - \xi_0 + \int_0^{t_0} \{\Phi(t,s) - \Phi(t_0,s)\} f(s, x(s,t,\xi) + z^*(s;(t,\xi)))\,\mathrm{d}s \\ &\quad + \int_0^{t_0} \Phi(t_0,s)\omega_1(s,t,t_0,\xi,\xi_0)\,\mathrm{d}s \\ &\quad + \int_{t_0}^t \Phi(t,s) f(s, x(s,t,\xi) + z^*(s;(t,\xi)))\,\mathrm{d}s. \end{aligned}$$

Let $C = \max\{||\Phi(u,s)||\colon u, s \in I\}$. By using **(P3)** combined with (2.17), (2.18) and (2.19), we deduce that

$$\begin{aligned} |H(t,\xi) - H(t_0,\xi_0)| &\leqslant |\xi - \xi_0| + \mu \int_0^{t_0} ||\Phi(t,s) - \Phi(t_0,s)||\,\mathrm{d}s + \mu \int_{t_0}^t ||\Phi(t,s)||\,\mathrm{d}s \\ &\quad + \int_0^{t_0} ||\Phi(t_0,s)||\,|\omega_1(s,t,t_0,\xi,\xi_0)|\,\mathrm{d}s \\ &\leqslant |\xi - \xi_0| + \mu t_0 \varepsilon_3 + C|t - t_0|\mu + C\gamma(\varepsilon_1 + \varepsilon_2)t_0, \end{aligned}$$

and we conclude that H is continuous in any $(t_0, \xi_0) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Finally, the result follows by verifying that G is continuous in any $(t_0, \eta_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, which can be proved in a similar way.

□

Corollary 2.4. *Assume that* **(P1)**–**(P3)** *and* (2.1) *are satisfied, then systems* (1.1) *and* (1.2) *are* $\mathbb{R}^+$*-topologically equivalent.*

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Proof. The proof is a direct consequence of either theorem 2.1 or 2.3. By one hand, if we follow the steps of theorem 2.1 the proof is the same but in step 5 it can be proved that

$$|G(t,\eta)-G(t,\bar{\eta})|\leqslant C(t)|\eta-\bar{\eta}|$$

where

$$C(t)=\begin{cases}1+K\gamma\dfrac{1-\mathrm{e}^{(-\alpha+M+\gamma)t}}{\alpha-M-\gamma} & \text{if } \alpha\neq M+\gamma\\ 1+K\gamma t & \text{if } \alpha=M+\gamma,\end{cases}$$

and the $\mathbb{R}^{+}$–topologically equivalence follows.

On the other hand, as definition 1.2 is a particular case of definition 1.4, the conclusion follows from theorem 2.3.

□

3. Consequences of the topological equivalence and stability issues

It is known that $\bar{y}$ is an equilibrium of (1.2) if

$$A(t)\bar{y}+f(t,\bar{y})=0 \quad \text{for any } t\geqslant 0. \tag{3.1}$$

On the other hand, it is important to emphasize that the $\mathbb{R}^{+}$–strong topological equivalence between (1.1) and (1.2) does not necessarily imply the existence of an equilibrium for (1.2). Indeed, we adapt the example introduced by Jiang in [**8**, p. 487]

$$y'=-y+\frac{1}{5}\left(\frac{\pi}{2}-\arctan(|t|+|y|)\right).$$

It is easy to see that **(P1)**–**(P3)** are satisfied with $K=M=\alpha=1$, $\gamma=1/5$ and $\mu=\pi/5$. We can see that for any $t_0\geqslant 0$ there exists $y(t_0)$ such that $f(t_0,y(t_0))=0$, however the above equation has no equilibria in the sense of (3.1).

The next results show some properties of the equilibria of the system (1.2) when they exist.

THEOREM 3.1. *Assume that* **(P1)**–**(P3)** *and condition* (2.1) *are fulfilled, then*

(i) *If the system* (1.2) *has an equilibrium* $\bar{y}$*, then it is unique.*

(ii) *If* $\bar{y}=0$*, namely* $f(t,0)=0$ *for any* $t\geqslant 0$*. Then:*

$$H(t,0)=G(t,0)=0 \quad \text{for any} \quad t\geqslant 0.$$

(iii) *If* $\bar{y}\neq 0$*, then*

$$\lim_{t\to+\infty}H(t,0)=\bar{y} \quad \text{and} \quad \lim_{t\to+\infty}G(t,\bar{y})=0.$$

(iv) *The equilibrium* $\bar{y}$ *is globally uniformly asymptotically stable for the system* (1.2).

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Proof. Let η and $\overline{\eta}$ be two initial conditions of (1.2) at $t = s$. By the Gronwall's lemma it follows that

$$|y(t,s,\eta) - y(t,s,\overline{\eta})| \leqslant K\mathrm{e}^{-(\alpha - K\gamma)(t-s)}|y(s,s,\eta) - y(s,s,\overline{\eta})| \quad \text{for } 0 \leqslant s \leqslant t. \tag{3.2}$$

Let $\bar{y}$ be an equilibrium of (1.2), in order to prove its uniqueness we will assume that $\bar{\xi} \neq \bar{y}$ is also an equilibrium of (1.2), then it follows that

$$y(t,s,\bar{y}) = \bar{y} \quad \text{and} \quad y(t,s,\bar{\xi}) = \bar{\xi} \quad \text{for any } t \geqslant 0.$$

Now, by (3.2) and $|\bar{y} - \bar{\xi}| \neq 0$ we have

$$1 \leqslant K\mathrm{e}^{-(\alpha - K\gamma)(t-s)} \quad \text{for } 0 \leqslant s \leqslant t$$

and (i) follows by (2.1) when $t \to +\infty$.

If $f(t,0) = 0$ for any $t \geqslant 0$ then the system (2.2) becomes (1.1) and $w^{*}(t;(\tau,0)) = 0$ for any $t \geqslant 0$ and by the definition of $G(t,\cdot)$, we have that $G(t,0) = 0$ and (ii) follows.

If $\bar{y} \neq 0$ is the unique equilibrium of (1.2), then $y(t,0,\bar{y}) = \bar{y}$ for any $t \geqslant 0$. By using (2.8) and (2.10) we can deduce that $G(t,\bar{y}) = \Phi(t,0)\bar{y}$ and it follows by **(P2)** that $\lim\limits_{t \to +\infty} G(t,\bar{y}) = 0$.

Now, as $t \mapsto H(t,0)$ and $t \mapsto y(t,0,\bar{y}) = \bar{y}$ are solutions of (1.2), by using (3.2) with $s = 0$, we have that

$$|H(t,0) - \overline{y}| \leqslant K\mathrm{e}^{-(\alpha - K\gamma)t}|H(0,0) - \overline{y}| \quad \text{for } t \geqslant 0.$$

The statement (iii) follows since the right side of the above inequality tends to zero as $t \to +\infty$.

Finally, in order to prove (iv) we recall that $\bar{y}$ is globally uniformly asymptotically stable if

$$\begin{cases} \forall \varepsilon > 0 \wedge \forall c > 0 \quad \exists T : T(\varepsilon, c) > 0 \quad \text{such that} \\ |y(t,s,\eta) - \bar{y}| < \varepsilon \quad \forall t > s + T \;\wedge\; \forall\, |\eta - \bar{y}| < c. \end{cases} \tag{3.3}$$

Upon inserting the identity $y(t,s,\bar{y}) = \bar{y}$ for any $0 \leqslant s \leqslant t$ in (3.2) we can see that

$$|y(t,s,\eta) - \bar{y}| \leqslant K\mathrm{e}^{(\alpha - K\gamma)(t-s)}|\eta - \bar{y}| \quad \text{for } 0 \leqslant s \leqslant t.$$

By using (2.1) combined with $|\eta - \bar{y}| < c$, it easy to see that (3.3) is verified with $T = 1/\alpha - K\gamma \ln(Kc/\varepsilon)$.

□

Remark 3.2. A simple consequence of theorem 3.1 is that any equilibrium of (1.2) must be in a closed ball centred at the origin with radius $K\mu/\alpha$.

Remark 3.3. The preservation of the uniform asymptotic stability of the equilibrium $\bar{y}$ can also be directly proved by using the uniform continuity of $x \mapsto H(t,x)$ for any $t \in \mathbb{R}^{+}$.

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4. Smoothness of the Homeomorphisms H and G

Throughout this section we will introduce the following property:

(D) The function $f(t, x)$ and its derivatives with respect to x up to order r-th are continuous functions of (t, x).

A direct consequence of the above property (see *e.g.* [**2**, Chap. 2]) is that $\partial y(t, \tau, \eta)/\partial\eta$ satisfies the matrix differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\dfrac{\partial y}{\partial\eta}(t, \tau, \eta) = \{A(t) + Df(t, y(t, \tau, \eta))\}\dfrac{\partial y}{\partial\eta}(t, \tau, \eta), \\ \dfrac{\partial y}{\partial\eta}(\tau, \tau, \eta) = I, \end{cases} \tag{4.1}$$

where Df is the Jacobian matrix of f.

The following result provides sufficient conditions ensuring that the maps H and G; constructed in § 2; satisfy definition 1.5.

Theorem 4.1. *Assume that* **(P1)–(P3)**,**(D)** *and* (2.1) *are satisfied, then* (1.1) *and* (1.2) *are C^r continuously topologically equivalent on $\mathbb{R}^+$.*

The proof of this result will be a consequence of the following lemmas and remarks.

Lemma 4.2. *Under the assumptions of theorem 2.1, if f satisfies* **(D)** *with $r = 1$, then $\eta \mapsto G(t, \eta)$ is a diffeomorphism for any fixed $t \geqslant 0$. Moreover, its Jacobian matrix is*

$$\frac{\partial G}{\partial\eta}(t, \eta) = \Phi(t, 0)\frac{\partial y(0, t, \eta)}{\partial\eta}, \tag{4.2}$$

which is continuous in any $(t, \eta) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Proof. It is well known that (see *e.g.*, theorem 4.1 from [**7**, Ch.V]) if $y \mapsto f(t, y)$ is C^r with $r \geqslant 1$, then the map $\eta \mapsto y(t, \tau, \eta)$ is also C^r for any fixed couple (t, τ). Then, as $y \mapsto f(t, y)$ is C^1, it follows that $y \mapsto Df(t, y)$ and $\eta \mapsto \partial y/\partial\eta$ are continuous. This allows to calculate the first partial derivatives of the map $\eta \mapsto G(t, \eta)$ for any $t \geqslant 0$ as follows

$$\frac{\partial G}{\partial\eta_i}(t, \eta) = e_i - \int_0^t \Phi(t, s)Df(s, y(s, t, \eta))\frac{\partial y}{\partial\eta_i}(s, t, \eta)\,\mathrm{d}s \quad (i = 1, \ldots, n), \tag{4.3}$$

which implies that the partial derivatives exist and are continuous for any fixed $t \geqslant 0$, then $\eta \mapsto G(t, \eta)$ is C^1.

By using the identity $\Phi(t,s)A(s) = -\partial/\partial s\Phi(t,s)$ combined with (4.1) we can deduce that for any $t \geqslant 0$, the Jacobian matrix is given by

$$\begin{aligned}\frac{\partial G}{\partial \eta}(t,\eta) &= I - \int_0^t \Phi(t,s)Df(s,y(s,t,\eta))\frac{\partial y}{\partial \eta}(s,t,\eta)\,\mathrm{d}s \\ &= I - \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left\{\Phi(t,s)\frac{\partial y}{\partial \eta}(s,t,\eta)\right\}\,\mathrm{d}s \\ &= \Phi(t,0)\frac{\partial y(0,t,\eta)}{\partial \eta}, \end{aligned} \tag{4.4}$$

and theorems 7.2 and 7.3 from [**2**, Ch. 1] imply that $Det\partial G(t,\eta)/\partial\eta > 0$ for any $t \geqslant 0$.

Summarizing, we have that $\eta \mapsto G(t,\eta)$ is C^1 and its Jacobian matrix has a nonvanishing determinant. In addition, let us recall that

$$G(t,\eta) = \eta + w^*(t;(t,\eta)),$$

where $w^*(t;(t,\eta))$ is given by (2.4), we can deduce that $|G(t,\eta)| \to +\infty$ as $|\eta| \to +\infty$, due to $|w^*(t;(t,\eta))| \leqslant K\mu/\alpha$ for any (t,η).

Therefore, by Hadamard's theorem (see *e.g* [**12**, **14**]), we conclude that $\eta \mapsto G(t,\eta)$ is a global diffeomorphism for any fixed $t \geqslant 0$.

Finally, by using again the property **(D)** combined with theorem 4.1 from [**7**, Ch.V] we also can conclude that $(t,\eta) \mapsto \partial y(0,t,\eta)/\partial\eta$ is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and by the identity (4.2) we have that $(t,\eta) \mapsto \partial G/\partial\eta(t,\eta)$ is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and the lemma follows. □

LEMMA 4.3. *Under the assumptions of theorem 2.1, if f satisfies* **(D)** *with $r > 1$, then the partial derivatives of $\eta \mapsto G(t,\eta)$ up to order $r-$th are continuous for any $(t,\eta) \in \mathbb{R}^+ \times \mathbb{R}^n$.*

Proof. In particular, if f satisfies property **(D)** with $r = 2$, we can verify that the second partial derivatives $\partial^2 y(s,\tau,\eta)/\partial\eta_j\partial\eta_i$ satisfy the system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial^2 y}{\partial\eta_j\partial\eta_i} = \{A(t) + Df(t,y)\}\dfrac{\partial^2 y}{\partial\eta_j\partial\eta_i} + D^2 f(t,y)\dfrac{\partial y}{\partial\eta_j}\dfrac{\partial y}{\partial\eta_i} \\ \dfrac{\partial^2 y}{\partial\eta_j\partial\eta_i} = 0, \end{cases} \tag{4.5}$$

for any $i,j = 1,\ldots,n$, where $D^2 f$ is the formal second derivative of f and $y = y(t,\tau,\eta)$. By using (4.3) and (4.5) we have

$$\begin{aligned}\frac{\partial^2 G}{\partial\eta_j\partial\eta_i}(t,\eta) = &-\int_0^t \Phi(t,s)D^2 f(s,y(s,t,\eta))\frac{\partial y}{\partial\eta_j}(s,t,\eta)\frac{\partial y}{\partial\eta_i}(s,t,\eta)\,\mathrm{d}s \\ &-\int_0^t \Phi(t,s)Df(s,y(s,t,\eta))\frac{\partial^2 y(s,t,\eta)}{\partial\eta_j\partial\eta_i}\,\mathrm{d}s\end{aligned}$$

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$$= -\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left\{\Phi(t,s)\frac{\partial^2 y(s,t,\eta)}{\partial\eta_j\partial\eta_i}\right\}\mathrm{d}s$$

$$= \Phi(t,0)\frac{\partial^2 y(0,t,\eta)}{\partial\eta_j\partial\eta_i}.$$

Notice that we can obtain the same expression for the second partial derivatives by using directly (4.2).

Now, by using **(D)** with $r \geqslant 2$ we can easily conclude (see for example theorem 4.1 from [7, Ch. V]) that $\eta \mapsto y(0,t,\eta)$ is C^r and the partial derivatives

$$(t,\eta) \mapsto \frac{\partial^{|m|}y(0,t,\eta)}{\partial\eta_1^{m_1}\cdots\partial\eta_n^{m_n}}, \quad \text{where } |m| = m_1 + \cdots + m_n \leqslant r,$$

are continuous for any $(t,\eta) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Moreover, this fact combined with (4.2) shows that the partial derivatives up to order r–th of G with respect to η

$$(t,\eta) \mapsto \frac{\partial^{|m|}G(t,\eta)}{\partial\eta_1^{m_1}\cdots\partial\eta_n^{m_n}} = \Phi(t,0)\frac{\partial^{|m|}y(0,t,\eta)}{\partial\eta_1^{m_1}\cdots\partial\eta_n^{m_n}}, \quad \text{where } |m| = m_1 + \cdots + m_n \leqslant r,$$

are continuous in $\mathbb{R}^+ \times \mathbb{R}$. □

Remark 4.4. A direct consequence of the above result, in particular, the map $\eta \mapsto G(t,\eta)$ is C^r for any fixed $t \geqslant 0$.

Lemma 4.5. *Under the assumptions of theorem 2.1, if f satisfies* **(D)** *with $r \geqslant 1$, then the partial derivatives of $\xi \mapsto H(t,\xi)$ up to order $r-$th are continuous for any $(t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^n$.*

Proof. By using the previous results combined with the identity $\xi = G(t,H(t,\xi))$ for any fixed $t \in \mathbb{R}^+$, the Jacobian matrix of the identity map on $\mathbb{R}^n$ can be seen as

$$DG(t,H(t,\xi))DH(t,\xi) = I \quad \text{for any fixed}\, t \in \mathbb{R}^+. \tag{4.6}$$

By lemma 4.3, we have that $\eta \mapsto G(t,\eta)$ is a diffeormorphism of class C^1 for any fixed $t \in \mathbb{R}^+$, which implies that

$$DH(t,\xi) = [DG(t,H(t,\xi))]^{-1} \quad \text{for any}\, t \in \mathbb{R}^+ \tag{4.7}$$

is well defined. In addition, note that $(t,\xi) \mapsto DH(t,\xi)$ is continuous since the maps $A \mapsto A^{-1}$ and $(t,\xi) \mapsto DG(t,H(t,\xi))$ are continuous for any $A \in Gl_n(\mathbb{R})$ and $(t,\xi) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Now, differentiating again with respect to the second variable, we have the formal computation

$$D^2G(t,H(t,\xi))DH(t,\xi)DH(t,\xi) + DG(t,H(t,\xi))D^2H(t,\xi) = 0$$

and the identity (4.7) implies that

$$D^2H(t,\xi) = -DH(t,\xi)D^2G(t,H(t,\xi))DH(t,\xi)DH(t,\xi).$$

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It is easy to see that $D^2H(t,\xi)$ is continuous with respect to (t,ξ) due to it is a composition of maps that are continuous with respect to (t,ξ). Finally, by using **(D)**, the higher formal derivatives of H up to order $r-$th and its continuity on $\mathbb{R}^+ \times \mathbb{R}^n$ can be deduced in a recursive way.

□

REMARK 4.6. A direct consequence of the above result is that the map $\xi \mapsto H(t,\xi)$ is C^r with $r \geqslant 1$ for any fixed $t \geqslant 0$.

Proof of theorem 4.1: We will see that the systems (1.1) and (1.2) are C^r continuously topologically equivalent on $\mathbb{R}^+$. In fact, as the hypotheses of theorem 2.3 are verified, we can conclude that the property (i) of definition 1.5 is satisfied.

The property (ii) of definition 1.5 is verified as a consequence of lemma 4.2 combined with remarks 4.4 and 4.6.

The property (iii) of definition 1.5 is verified as a direct consequence of lemmas 4.3 and 4.5. □

REMARK 4.7. It is interesting to point out that the computation of the partial derivatives of G with respect to η is remarkably simple when considering $\mathbb{R}^+ \times \mathbb{R}^n$ as domain. In the case when the domain is $\mathbb{R} \times \mathbb{R}^n$, we refer to [**1**] for details.

5. Conclusions

In this work, we have obtained a sufficient condition for the $\mathbb{R}^+-$ strong topological equivalence between systems (1.1) and (1.2). The restriction to $\mathbb{R}^+$ also allow us to prove that the maps H and G are continuous in $\mathbb{R}^+ \times \mathbb{R}^n$, which prompted us to introduce the notion of J-continuous topological equivalence between systems (1.1) and (1.2). Finally, this paper can be seen as a progress report about smooth linearization for nonautonomous systems. Indeed, we have proved the C^r-differentiability of the homeomorphism constructed between systems (1.1) and (1.2), remarking the simpleness of the proof.

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