Probability in the Engineering and Informational Sciences, **17**, 2003, 571–576. Printed in the U.S.A.

A NOTE ON THE CONDUCTIVITY OF RANDOM TREES

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We prove that no stochastic domination exists between the effective resistance of a spherically symmetric random tree and that of a branching process in a varying environments tree if they grow according to the same law of distribution.

1. INTRODUCTION

All of the trees that we consider in this article are infinite and leafless in the sense that the degree of every vertex (number of edges incident with it) is greater than one. Two types of random trees are considered. The first is the spherically symmetric random tree (SSRT), denoted by T, in which the degree of a vertex at distance n from the root r depends only on n and is denoted by d_n. The degree sequence $\{d_n; n \geq 0\}$ is assumed to consist of independent random variables. For the second type, we consider a doubly-indexed family $\{d_{nk} : n \geq 0; k \geq 1\}$ of independent random variables and they are, for fixed n, identically distributed according to the same distribution of d_n. We interpret d_{nk} as the degree of the kth vertex of the nth level of a branching process in a varying environments tree (BPIVET), denoted by T^*. We call a vertex v a branching vertex if its degree is greater than two. If G is a finite graph (electrical network) and voltage v is applied between two vertices (nodes) a and b such that the voltage at a is v_a and at b it is $v_b = 0$, the effective resistance, R_{eff}, of G is defined as

$$R_{\text{eff}} = \frac{v_a}{i_a},$$

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where i_a is the current flowing into the circuit at node a. See [2]. A flow j on G from a to b is a function defined on the set E of edges of G as follows. If $xy \in E$, then

$$j_{xy} = -j_{yx}, \tag{1}$$

$$\sum_y j_{xy} = 0 \quad \text{if } x \neq a, b, \tag{2}$$

$$j_{kl} = 0 \quad \text{if } kl \notin E. \tag{3}$$

The energy dissipation of a current flow j is defined as

$$\frac{1}{2} \sum_{x,y} j_{xy}^2 r_{xy},$$

where r_{xy} is the resistance assigned to the edge xy.

The principle of conservation of energy [2] states the following: If ω is any function defined on the vertices of a graph G and j is a flow from a vertex a to a vertex b, then

$$(w_a - w_b) j_a = \frac{1}{2} \sum_{x,y} (w_x - w_y) j_{xy}.$$

If a voltage v_a is imposed between a and b with $v_b = 0$, we obtain voltages v_x and currents i_{xy}. The current i defines a flow from a to b and by the above principle, we conclude that

$$v_a i_a = \frac{1}{2} \sum_{x,y} (v_x - v_y) i_{xy},$$

and from Ohm's law, we get

$$v_a i_a = \frac{1}{2} \sum_{x,y} i_{xy}^2 r_{xy}.$$

Since $R_{\text{eff}} = v_a / i_a$, then

$$i_a^2 R_{\text{eff}} = \frac{1}{2} \sum_{x,y} i_{xy}^2 r_{xy}.$$

If a unit resistance is assigned to every edge $xy \in E$ and a voltage is applied between a and b such that $i_a = 1$, then

$$R_{\text{eff}} = \frac{1}{2} \sum_{x,y} i_{xy}^2;$$

that is, the effective resistance is the energy dissipation of the unit current flow when unit resistances are assigned to the edges. See [2].

We consider for a moment a finite tree T_n of height n and short circuit all the leaves into a single vertex. Let R_n be the effective resistance of such a finite tree. Then, the effective resistance of an infinite tree is defined to be the limit, as n goes to infinity, of R_n. The energy dissipation of a flow j on infinite trees is defined similarly.

The following two laws are considerable tools in determining upper and lower bounds for the effective resistances.

Two ways to modify the network that we are interested in bounding its effective resistance so as to get a simpler network are by shorting or cutting. Shorting involves connecting a given set of nodes together with perfectly conducting wires so that the current can pass freely between them. All nodes that were shorted together behave as if they were a single node. However, cutting means deleting some branches of the network. The usefulness of shorting and cutting procedures stems from the following two laws. See [2].

Shorting law: Shorting certain sets of nodes together can only decrease the effective resistance of the network between two given nodes.

Cutting law: Cutting certain branches can only increase the effective resistance between two given nodes.

The following theorem plays a vital role in obtaining an upper bound for the effective resistance.

THOMSON'S THEOREM [2, p. 63]: *If i is the unit current flow between two vertices a and b, then the energy dissipation $\frac{1}{2}\sum_{x,y} i_{xy}^2 r_{xy}$ minimizes the energy dissipation $\frac{1}{2}\sum_{x,y} j_{xy}^2 r_{xy}$ among all unit flows from a to b.*

Berman and Konsowa [1] give a generalization of Thomson's Theorem. It can be easily observed that the two laws above are easy consequences of Thomson's theorem.

2. SSRT AND BPIVET

Let T be the infinite spherically symmetric random tree that grows according to a degree sequence $\{d_n; n \geq 0\}$ and T^* be the branching process in a varying environments tree that grows according to $\{d_{nk} : n \geq 0; k \geq 1\}$. In the rest of the article, all of the edges are assigned one-unit resistances. The following theorem of [4] shows that the mean effective resistance of T dominates that of T^*.

THEOREM 1: *For T and T^*, we have*

$$E(R_{\text{eff}}^*) \leq E(R_{\text{eff}}),$$

where R_{eff}^* and R_{eff} denote the effective resistances of T^* and T, respectively.

PROOF: Let j be the unit flow, applied at the root r of T^*, that splits equally at every branching vertex and D be its energy dissipation. Let us assume that r has Z children; that is, there are Z vertices each of which is connected to the root r by one edge. At

https://doi.org/10.1017/S0269964803174098 Published online by Cambridge University Press

each of these vertices, we apply a unit flow that divides equally at each branching vertex. Let D_i stand for the energy dissipation of that flow applied at the vertex i. We can easily verify that

$$D = \frac{1}{Z} + \frac{1}{Z^2}\sum_{i=1}^{z} D_i.$$

It follows from Thomson's principle that

$$R^*_{\text{eff}} \le \frac{1}{Z} + \frac{1}{Z^2}\sum_{i=1}^{z} D_i$$

and, hence,

$$\begin{aligned} E(R^*_{\text{eff}}) &\le E\left(\frac{1}{Z} + \frac{1}{Z^2}\sum_{i=1}^{z} D_i\right) \\ &= E\left(E\left(\frac{1}{Z} + \frac{1}{Z^2}\sum_{i=1}^{z} D_i \middle| Z\right)\right) \\ &= E\left(\frac{1}{Z} + \frac{1}{Z} ED_1\right). \end{aligned}$$

The last equation follows because D_i, $i = 1, 2, \ldots, Z$, are identically distributed random variables. On the other hand,

$$E(R_{\text{eff}}) = E(E(R_{\text{eff}}|Z)).$$

The tree T (as an electrical network) consists of Z identical resistances connected in parallel, each of which has mean equals ED_1. For more detail, see [3]. Now,

$$E(R_{\text{eff}}|Z) = \frac{1}{Z} + \frac{1}{Z} ED_1.$$

Then,

$$E(R_{\text{eff}}) = E\left(\frac{1}{Z} + \frac{1}{Z} ED_1\right)$$

and, consequently,

$$E(R^*_{\text{eff}}) \le E(R_{\text{eff}}). \qquad \blacksquare$$

The following theorem gives an affirmative answer to the conjecture of [4] and shows that no stochastic domination exists between R_{eff} and R^*_{eff}. However, we first recall that a tree is called regular if all the vertices have the same degree. We use T^b to denote a regular tree for which every vertex has degree b. The binary tree is an infinite tree such that the degree of the root is 2 and of every other vertex is 3. If the edges of the binary tree are assigned unit resistances and the verices of each level are

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shorted together into a single node, then the effective resistance is easily calculated to be one. Therefore, the effective resistance of any regular tree T^b, $b > 2$, is finite. This follows directly from the cutting law.

THEOREM 2: *If T and T^* are as defined above such that d_n are nondegenerate random variables, then*

$$p(R_{\text{eff}} > R^*_{\text{eff}}) > 0 \quad \text{and} \quad p(R^*_{\text{eff}} > R_{\text{eff}}) > 0.$$

PROOF: Consider two positive integers $a < b$ and a constant $c > 0$ such that for all $n \geq 0$,

$$p(d_n \leq a) > 0 \quad \text{and} \quad p(d_n \geq b) \geq c.$$

Let B_n and B^*_n denote respectively the portions of T and T^* from the root to the nth level. These are balls of radius n centered at the root. If $d(x)$ denotes the degree of an arbitrary vertex x, then for $N \geq 1$,

$$p(d(x) \leq a \text{ for all } x \in B_N) = \prod_{n=1}^{N} p(d_n \leq a) > 0.$$

Let $T^*(v)$ stand for the subtree of T^* that consists of the vertex v and all its descedents in T^* and $R_{\text{eff}}(T^*(v))$ stand for its effective resistance. Let also Z^*_n denote the number of vertices of the nth level of T^* and

$$\Im_n = \sigma(Z^*_k, k \leq n).$$

Then, there is $R_0 < \infty$ such that for $N \geq 1$ and conditioning on $\Im_n$,

$$p(d(x) \geq b \text{ for all } x \in B^*_N \text{ and } R_{\text{eff}}(T^*(v)) \leq R_0 \text{ for all } v \in B^*_N - B^*_{N-1})$$
$$= (p(R_{\text{eff}}(T^*(v)) \leq R_0))^{Z^*_N} \prod_{n=1}^{N-1} (p(d_n \geq b))^{Z^*_n} > 0.$$

It follows consequently that R_{eff} and R^*_{eff} satisfy, for large N, the two inequalities

$$p(R_{\text{eff}} \geq R_{\text{eff}}(T^a)) > 0 \quad \text{and} \quad p(R^*_{\text{eff}} \leq R_{\text{eff}}(T^b)) > 0.$$

Cutting law assures that $R_{\text{eff}}(T^a) > R_{\text{eff}}(T^b)$, from which

$$p(R_{\text{eff}} > R^*_{\text{eff}}) > 0.$$

Changing the roles of T and T^* yields the other inequality:

$$p(R_{\text{eff}} < R^*_{\text{eff}}) > 0. \qquad \blacksquare$$

Acknowledgment

The author is greatly indebted to Professor Harry Kesten for his precise advice in writing this article.

https://doi.org/10.1017/S0269964803174098 Published online by Cambridge University Press

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