ABC-triangles

JONNY GRIFFITHS

If we talk about the centre of a triangle, what might we be referring to? Any triangle has many different points that could regarded as its centre; in fact, *Encyclopedia of Triangle Centres* lists over 70 000 possibilities. Three of the most famous centres, that every triangle will possess (although they may coincide), are the incentre (where the three angle bisectors meet), the centroid (where the three medians meet) and the orthocentre (where the three altitudes meet). Proofs that these centres are well-defined and exist for every triangle are simple and satisfying, good examples of reasoning (if we are teachers) for our students. Proving the three altitudes of a triangle share a point using the scalar product of vectors is a wonderful demonstration of the power of this idea.

One day it occurred to me to ask: what happens if we try to do a mathematical mash-up of these three centres? Are there triangles where an altitude from A, an angle bisector from B, and a median from C, all meet? This certainly doesn't always happen, but sometimes does (think of the equilateral triangle). Maybe we could call such a triangle an ABC-triangle (the Altitude is through A, the angle Bisector is through B, and the line-through-the-Centroid (the median) is through C). Playing with *Geogebra* suggests that there is a family of such ABC-triangles to explore. Below, in Figure 1, is an example that is not equilateral.

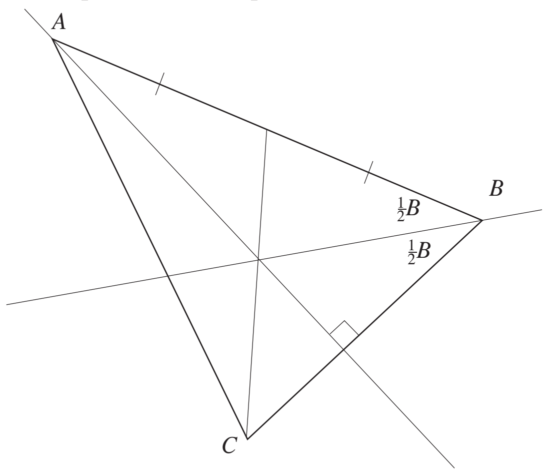

FIGURE 1

Have such triangles been examined before? Searching on https://math.stackexchange.com [1] for ‘altitude, median, angle bisector’ confirms they have, but in the hope that I have something fresh to add, the story of my exploration continues.

https://doi.org/10.1017/mag.2024.13 Published online by Cambridge University Press

Let's take a step back before launching into calculations. What questions does this situation summon up? These are the questions that I find myself asking:

- ABC-triangles are special in some way; what makes them special?
- Are there any other isosceles triangles beyond the equilateral one that are ABC?
- Are there any right-angled triangles that are ABC?
- Are there any integer triangles that are ABC?
- Could a triangle have two distinct ABC centres?
- Can we say anything about obtuse-angled ABC-triangles?
- Might ABC-triangles appear in pairs? Could an ABC-triangle have a dual?

Firstly, we can see that for any triangle, acute or obtuse, the median from one corner must meet the angle bisector of another somewhere inside the triangle.

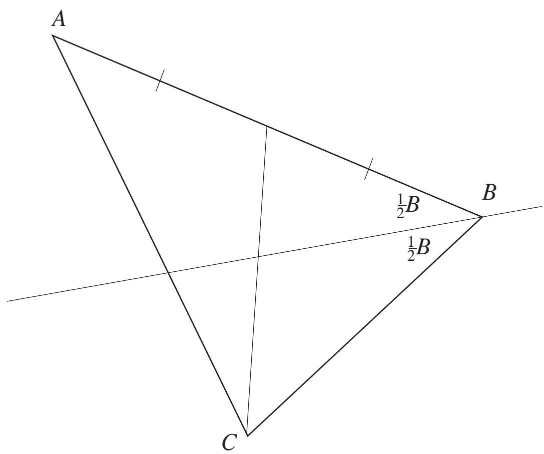

FIGURE 2

For this to be an ABC-triangle, the line from A through this meeting point must be at right angles to BC. So there is just one simple condition to be met; the triangle is going to be special via a single equation. Additionally the ABC-centre of a triangle, if it exists, must be inside the triangle (this is not true for the orthocentre).

Vectors were a nice way to investigate the coincidence of the altitudes; they are a help here too, in Figure 3, with some careful re-labelling.

Note that if B or C is obtuse, then the altitude from A will lie outside the triangle (except for the point A), and thus no ABC-centre can exist. Note also that the magnitude of the vector **a** in what follows is just given as a.

We have that $\mathbf{d} = \mu\left(\frac{\mathbf{a}}{a} + \frac{\mathbf{c}}{c}\right)$ and $\mathbf{d} = \lambda\mathbf{a} + (1 - \lambda)\frac{\mathbf{c}}{2}$ and $(\mathbf{d} - \mathbf{c}).\mathbf{a} = 0$.

https://doi.org/10.1017/mag.2024.13 Published online by Cambridge University Press

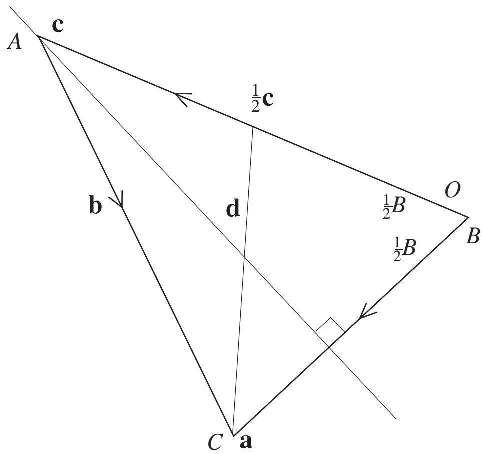

FIGURE 3

The first two equations give $\frac{\mu}{a}\mathbf{a} + \frac{\mu}{c}\mathbf{c} = \lambda\mathbf{a} + \frac{1-\lambda}{2}\mathbf{c}$. Equating the coefficients of $\mathbf{a}$ and then the coefficients of $\mathbf{c}$ yields $\mu = \frac{ac}{2a+c}$, $\lambda = \frac{c}{2a+c}$. Thus $\mathbf{d} = \frac{c}{2a+c}\mathbf{a} + \frac{a}{2a+c}\mathbf{c}$. Now

$$(\mathbf{d} - \mathbf{c}).\mathbf{a} = 0 \Rightarrow \mathbf{a}.\mathbf{d} = \mathbf{a}.\mathbf{c} \Rightarrow \frac{a^2c}{2a+c} + \frac{a^2c\cos B}{2a+c} = ac\cos B \Rightarrow \cos B = \frac{a}{a+c}.$$

The implication works both ways, so this is the necessary and sufficient condition we seek; *ABC* will be an ABC-triangle if, and only if, $\cos B = \frac{a}{a+c}$ – a clean and simple result.

Using the cosine rule in the triangle *ABC* above,

$$b^2 = a^2 + c^2 - 2ac\cos B \Rightarrow b^2 = a^2 + c^2 - \frac{2a^2c}{a+c}.$$

This becomes

$$ab^2 + cb^2 + a^2c = a^3 + ac^2 + c^3.$$

We might call any integer triples (a, b, c) that satisfy this equation 'ABC-triples'. Clearly if (a, b, c) satisfies the equation, then (ka, kb, kc) does also, so it makes sense to search for primitive ABC-triples, where $\gcd(a, b, c) = 1$.

A computer search reveals that such triples are (understandably) rarer than solutions to $a^2 + b^2 = c^2$ (Pythagorean triples); the first three seem to be (15, 13, 12) and (308, 277, 35) and (3193, 26447, 26598). We can conjecture that there are infinitely many such primitive ABC-triples, but that feels hard to prove.

Investigating isosceles triangles reveals straightforwardly that the equilateral triangle is the only ABC-triangle amongst them. Right-angled

https://doi.org/10.1017/mag.2024.13 Published online by Cambridge University Press

triangles prove a little more interesting. It is easy to see that if we are given a right-angled ABC-triangle, then A has to be the right-angle (otherwise the altitude does not cross the triangle).

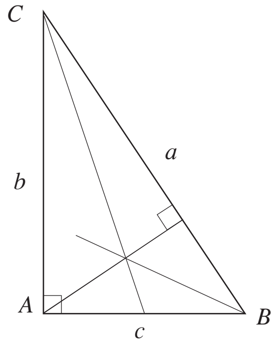

FIGURE 4

Now $\cos B = \dfrac{a}{a + c} = \dfrac{c}{a} \Rightarrow ac + c^2 \Rightarrow \left(\dfrac{c}{a}\right)^2 - \dfrac{a}{c} - 1 = 0$. Thus $\dfrac{c}{a} = \phi$, and the three sides are $\left(1, \sqrt{\phi}, \phi\right)$ or some multiple of this. So there is in effect just one right-angled triangle that is an ABC one, and its sides are in geometric progression.

There is a relevant question from the 4th All-Soviet Union Mathematics Competition [1]. With our labelling, we are asked to show that if ABC is an acute-angled triangle then $\angle ABC > 45°$. This result is true, but our work on right-angled ABC-triangles suggests it is not sharp. In fact we have $\angle ABC > \arccos\left(\dfrac{1}{\phi}\right) = 51.827\ldots°$ as the sharp result.

Once again, for an obtuse ABC-triangle, A must be the obtuse angle for the altitude to cross the triangle, as in Figure 5.

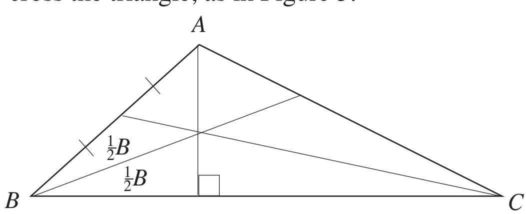

FIGURE 5

Now $c < a$, and $\cos B = \dfrac{a}{a + c} > \dfrac{a}{a + a} = \dfrac{1}{2}$, and so $B < 60°$. This result is also true but not sharp; the sharp version is $B < 51.827\ldots°$.

Could a non-equilateral triangle ABC be an ABC-triangle while BAC (or some other permutation of A, B and C) is too? Perhaps with a different

https://doi.org/10.1017/mag.2024.13 Published online by Cambridge University Press

centre? That could mean, as a possible example, both $\cos B = \dfrac{a}{a + c}$ and $\cos A = \dfrac{b}{b + c}$.

So $b^2 = a^2 + c^2 - 2\dfrac{a^2c}{a + c}$ and $a^2 = b^2 + c^2 - 2\dfrac{b^2c}{b + c}$.

Using a computer algebra package to solve these (and the other permutations) together gives a = b = c as the only solution. So no, it seems that a non-equilateral triangle cannot be ABC in two different ways at once.

We can notice this (using *Geogebra* once more).

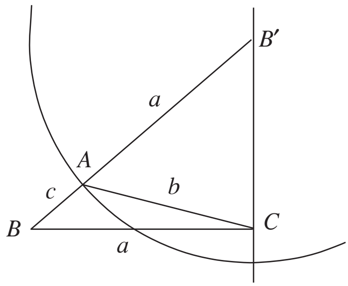

FIGURE 6

Start with a line segment BC, of length a. Draw a perpendicular to BC through c, and pick any point B' on this perpendicular. Now join B' to B. Draw a circle centre B' radius a, and suppose this circle cuts BB' at A (this is always possible, since $a < BB'$). Then ABC is an ABC-triangle, since $\cos B = \dfrac{a}{a + c}$.

Similarly, if we draw a circle centre B, radius $B'C$, then if this circle cuts BB' at E, we have another ABC-triangle, namely $A'B'C$, as in Figure 7.

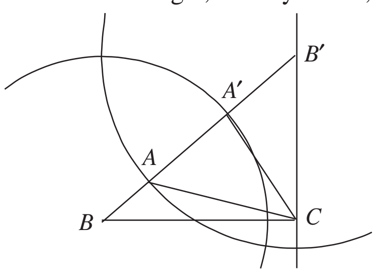

FIGURE 7

Adding the acute angles at B and B' gives 90°, so it makes sense to call these dual ABC-triangles; the dual of the dual is the original. These two triangles never overlap, since $BC + B'C > BB'$.

https://doi.org/10.1017/mag.2024.13 Published online by Cambridge University Press

We might ask if the triangle $AA'C$ is ever an ABC-triangle; playing with *Geogebra* suggests it sometimes is (see Figure 8).

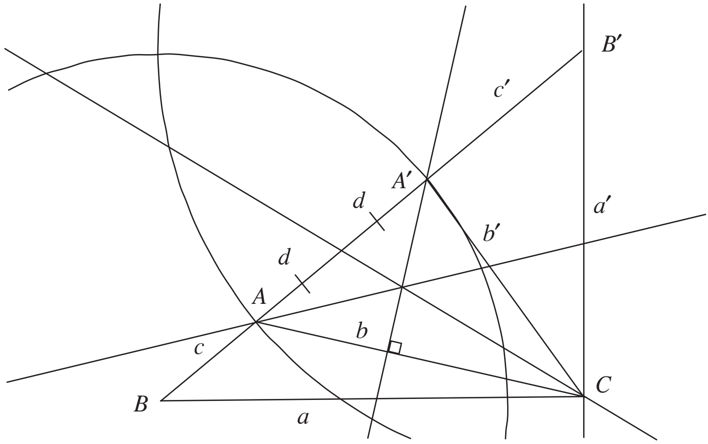

FIGURE 8

If we pick $a = 1$, then we can find in turn (in terms of c) a', $\cos B$, b, c', $2d$, $\cos B'$, b' and $\cos A'AC$. A computer algebra package will be needed for this. In fact, we can find angle $A'AC$ either by using our $\cos B = \dfrac{a}{a + c}$ condition, or by using the cosine rule in $\Delta A'AC$.

Equating these, we find that either $c = 0$, or c is about 0.295. We can find the other sides and angles from this.

Reference

1. https://math.stackexchange.com/questions/914662/the-concurrence-of-angle-bisector-median-and-altitude--in-an-acute-angled-triangle accessed August 2023

10.1017/mag.2024.13
Published by Cambridge University Press on behalf of The Mathematical Association

JONNY GRIFFITHS
11 Great Western Street,
Frome, Somerset BA11 1GA
e-mail: *hello@jonny-griffiths.net*

https://doi.org/10.1017/mag.2024.13 Published online by Cambridge University Press