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NEIGHBOURHOOD AND THE EXISTENCE OF FRACTIONAL k-FACTORS OF GRAPHS

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Abstract

Let G be a graph, and k a positive integer. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) \mid h(e) > 0\}$. In this paper we use neighbourhoods to obtain a new sufficient condition for a graph to have a fractional k-factor. Furthermore, this result is shown to be best possible in some sense.

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1. Introduction

In this paper we consider only finite undirected graphs which have neither loops nor multiple edges. We refer the readers to [1] for the terminology not defined here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For each $x \in V(G)$, we use $d_G(x)$ to denote the degree of x in G, and $N_G(x)$ to denote the neighbourhood of x in G. For a subset X of $V(G)$, we define the neighbourhood of X as

$$N_G(X) = \bigcup_{x \in X} N_G(x).$$

Note that $N_G(x)$ does not contain x, but it may happen that $N_G(X) \supseteq X$. For any $S \subseteq V(G)$, we use $G[S]$ and $G - S$ to denote the subgraph of G induced by S and $V(G) - S$, respectively. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let S and T be two disjoint subsets of $V(G)$; we denote by $E_G(S, T)$ the set of edges with one end in S and the other end in T, and $e_G(S, T) = |E_G(S, T)|$. We denote the minimum degree of G by $\delta(G)$. Let r be a real number. Recall that $\lfloor r \rfloor$ is the greatest integer such that $\lfloor r \rfloor \leq r$.

Let k be an integer such that $k \geq 1$. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If

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$\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) \mid h(e) > 0\}$.

Many authors have investigated graph factors [2, 5, 6, 8, 12]. Liu and Zhang [3] obtained a necessary and sufficient condition for a graph to have a fractional k-factor. Liu and Zhang [4] gave a toughness condition for a graph to have a fractional k-factor. Zhou [9–11] gave some other sufficient conditions for graphs to have fractional k-factors. Yu *et al.* [7] obtained a degree condition for a graph to have a fractional k-factor.

The following results on fractional k-factors are known.

THEOREM 1.1 [4]. *Let $k \geq 2$ be an integer. A graph G of order n with $n \geq k+1$ has a fractional k-factor if its toughness $t(G) \geq k - 1/k$.*

THEOREM 1.2 [7]. *Let k be an integer with $k \geq 1$, and let G be a connected graph of order n with $n \geq 4k-3$, $\delta(G) \geq k$. If*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for each pair of nonadjacent vertices x, y of G, then G has a fractional k-factor.

THEOREM 1.3 [10]. *Let k be an integer such that $k \geq 1$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2+2}$, and the minimum degree $\delta(G) \geq k$. If*

$$|N_G(x) \cup N_G(y)| \geq \max\left\{\frac{n}{2}, \frac{1}{2}(n+k-2)\right\}$$

for each pair of nonadjacent vertices x, $y \in V(G)$, then G has a fractional k-factor.

THEOREM 1.4 [9]. *Let k be a positive integer and G a graph of order n with $n \geq 4k-6$. Then:*

(a) *if k is even and*

$$|N_G(X)| \geq \frac{(k-1)n + |X| - 1}{2k-1}$$

for every nonempty independent subset X of $V(G)$, and

$$\delta(G) \geq \frac{k-1}{2k-1}(n+2),$$

then G has a fractional k-factor; and

(b) *if k is odd, and*

$$|N_G(X)| > \frac{(k-1)n + |X| - 1}{2k-1}$$

for every nonempty independent subset X of $V(G)$, and

$$\delta(G) > \frac{k-1}{2k-1}(n+2),$$

then G has a fractional k-factor.

In this paper we use neighbourhoods to obtain a new sufficient condition for a graph to have a fractional k-factor. The main result is the following theorem.

THEOREM 1.5. *Let k be an integer with $k \geq 1$, and let G be a graph of order n with $n \geq 6k - 12 + 6/k$. Suppose, for any subset $X \subset V(G)$, that*

$$N_G(X) = V(G) \quad \text{if } |X| \geq \left\lfloor \frac{kn}{2k-1} \right\rfloor; \quad \text{or}$$
$$|N_G(X)| \geq \frac{2k-1}{k}|X| \quad \text{if } |X| < \left\lfloor \frac{kn}{2k-1} \right\rfloor.$$

Then G has a fractional k-factor.

2. The Proof of Theorem 1.5

The proof of Theorem 1.5 relies heavily on the following lemmas.

LEMMA 2.1 [3]. *Let G be a graph. Then a graph G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq 0,$$

where $T = \{x : x \in V(G)\backslash S, d_{G-S}(x) \leq k-1\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

LEMMA 2.2. *Let G be a graph of order n which satisfies the assumption of Theorem 1.5. Then $\delta(G) \geq ((k-1)n + k)/(2k-1)$.*

PROOF. Let x be a vertex of G with degree $\delta(G)$. Set $X = V(G)\backslash N_G(x)$. Obviously, $x \notin N_G(X)$ and $N_G(X) \neq V(G)$. Thus, we obtain

$$n - 1 \geq |N_G(X)| \geq \frac{2k-1}{k}|X|,$$

that is,

$$(2k-1)|X| \leq k(n-1). \tag{2.1}$$

Using (2.1) and $|X| = n - \delta(G)$,

$$(2k-1)(n - \delta(G)) \leq k(n-1).$$

Hence,

$$\delta(G) \geq \frac{(k-1)n + k}{2k-1}.$$

This completes the proof of Lemma 2.2. □

PROOF OF THEOREM 1.5. Let G be a graph satisfying the hypotheses of Theorem 1.5, which has no fractional k-factor. Then by Lemma 2.1, there exists some $S \subseteq V(G)$ such that

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \leq -1 \tag{2.2}$$

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where $T = \{x : x \in V(G)\backslash S, d_{G-S}(x) \le k-1\}$. Obviously, $T \ne \emptyset$ by (2.2). Define

$$h = \min\{d_{G-S}(t) \mid t \in T\}.$$

From the definition of T, we obtain

$$0 \le h \le k-1.$$

Case 1. $2 \le h \le k-1$.
In terms of Lemma 2.2 and the definition of h, we get

$$|S| + h \ge \delta(G) \ge \frac{(k-1)n+k}{2k-1}. \tag{2.3}$$

According to (2.2) and $|S| + |T| \le n$, we obtain

$$\begin{aligned} -1 \ge \delta_G(S, T) &= k|S| + d_{G-S}(T) - k|T| \\ &\ge k|S| + h|T| - k|T| \\ &= k|S| - (k-h)|T| \\ &\ge k|S| - (k-h)(n-|S|) \\ &= (2k-h)|S| - (k-h)n. \end{aligned}$$

This inequality implies that

$$|S| \le \frac{(k-h)n-1}{2k-h}. \tag{2.4}$$

From (2.3) and (2.4),

$$\frac{(k-1)n+k}{2k-1} \le \delta(G) \le |S| + h \le \frac{(k-h)n-1}{2k-h} + h. \tag{2.5}$$

If the left-hand and right-hand sides of (2.5) are denoted by A and B respectively, then (2.5) says that $A - B \le 0$. But, after some rearranging, we find that

$$\begin{aligned} (2k-1)(2k-h)(A-B) = (h-1)(kn-(2k-1)(2k-h)+k-1) \\ -2k^2+5k-2. \end{aligned} \tag{2.6}$$

Since $n \ge 6k - 12 + 6/k$, we obtain

$$kn-(2k-1)(2k-2)+k-1 \ge 2k^2-5k+3 \ge 0. \tag{2.7}$$

Using (2.6), (2.7), $2 \le h \le k-1$ and $n \ge 6k-12+6/k$, we get

$$\begin{aligned} &(2k-1)(2k-h)(A-B) \\ &\quad = (h-1)(kn-(2k-1)(2k-h)+k-1) - 2k^2+5k-2 \\ &\quad \ge (h-1)(kn-(2k-1)(2k-2)+k-1) - 2k^2+5k-2 \\ &\quad \ge kn-(2k-1)(2k-2)+k-1-2k^2+5k-2 \\ &\quad = kn-6k^2+12k-5 \ge 1. \end{aligned}$$

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This inequality implies that

$$A - B > 0,$$

which contradicts $A - B \leq 0$.

Case 2. $h = 1$.

Subcase 2.1. $|T| \geq \lfloor kn/(2k-1) \rfloor + 1$.

In terms of the definition of h and $h = 1$, there exists $t \in T$ such that $d_{G-S}(t) = h = 1$. Thus, we obtain

$$t \notin N_G(T \backslash N_G(t)),$$

which implies that

$$N_G(T \backslash N_G(t)) \neq V(G). \tag{2.8}$$

On the other hand, using $|T| \geq \lfloor kn/(2k-1) \rfloor + 1$ and $d_{G-S}(t) = 1$,

$$|T \backslash N_G(t)| \geq |T| - 1 \geq \left\lfloor \frac{kn}{2k-1} \right\rfloor.$$

Combined with the condition of Theorem 1.5, the inequality above implies that

$$N_G(T \backslash N_G(t)) = V(G),$$

which contradicts (2.8).

Subcase 2.2. $|T| \leq \lfloor kn/(2k-1) \rfloor$.

Since $h = 1$, there exists $u \in T$ such that $d_{G-S}(u) = 1$. Thus, from Lemma 2.2,

$$|S| + 1 = |S| + d_{G-S}(u) \geq d_G(u) \geq \delta(G) \geq \frac{(k-1)n + k}{2k-1},$$

that is,

$$|S| \geq \frac{(k-1)n + k}{2k-1} - 1 = \frac{(k-1)(n-1)}{2k-1}. \tag{2.9}$$

Subcase 2.2.1. $|T| > (k(n-1))/(2k-1)$.

In terms of (2.9) and $|T| > (k(n-1))/(2k-1)$, we get

$$|S| + |T| > \frac{(k-1)(n-1)}{2k-1} + \frac{k(n-1)}{2k-1} = n - 1.$$

Combining this with $|S| + |T| \leq n$, we obtain

$$|S| + |T| = n. \tag{2.10}$$

According to (2.2), (2.10) and $|T| \leq \lfloor kn/(2k-1) \rfloor \leq kn/(2k-1)$,

$$\begin{aligned} -1 \geq \delta_G(S, T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + |T| - k|T| \\ &= k|S| - (k-1)|T| \\ &= k(n - |T|) - (k-1)|T| \\ &= kn - (2k-1)|T| \end{aligned}$$

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$$\geq kn - (2k-1) \cdot \frac{kn}{2k-1}$$
$$= 0,$$

which is a contradiction.

Subcase 2.2.2. $|T| \leq (k(n-1))/(2k-1)$.

Since $k - 1 \geq h = 1$, we obtain $k \geq 2$ in this case. Set

$$p = |\{t : t \in T, d_{G-S}(t) = 1\}|.$$

Clearly, $|T| \geq p$. Combining this with (2.9) and $k \geq 2$ and $|T| \leq (k(n-1))/(2k-1)$, we obtain

$$\begin{aligned}
\delta_G(S, T) &= k|S| + d_{G-S}(T) - k|T| \\
&\geq k|S| + 2(|T| - p) + p - k|T| \\
&= k|S| - (k-2)|T| - p \\
&\geq k \cdot \frac{(k-1)(n-1)}{2k-1} - (k-2) \cdot \frac{k(n-1)}{2k-1} - p \\
&= \frac{k(n-1)}{2k-1} - p \\
&\geq |T| - p \geq 0.
\end{aligned}$$

This contradicts (2.2).

Case 3. $h = 0$.

Let m be the number of vertices x in T such that $d_{G-S}(x) = 0$. Clearly, $m \geq 1$ since $h = 0$. Set $Y = V(G) \backslash S$. Then $N_G(Y) \neq V(G)$ since $h = 0$.

Claim 1. $|Y| < \lfloor kn/(2k-1) \rfloor$.

If $|Y| \geq \lfloor (kn/(2k-1)) \rfloor$, then by the condition of Theorem 1.5 we have $N_G(Y) = V(G)$. This contradicts $N_G(Y) \neq V(G)$ and proves Claim 1.

In terms of Claim 1 and the condition of Theorem 1.5, we obtain

$$n - m \geq |N_G(Y)| \geq \frac{2k-1}{k}|Y| = \frac{2k-1}{k}(n - |S|).$$

This inequality implies that

$$|S| \geq \frac{(k-1)n + km}{2k-1}. \tag{2.11}$$

From (2.2), (2.11), $m \geq 1$ and the fact that $|S| + |T| \leq n$,

$$\begin{aligned}
-1 \geq \delta_G(S, T) &= k|S| + d_{G-S}(T) - k|T| \\
&\geq k|S| + |T| - m - k|T| \\
&= k|S| - (k-1)|T| - m \\
&\geq k|S| - (k-1)(n - |S|) - m \\
&= (2k-1)|S| - (k-1)n - m
\end{aligned}$$

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$$\geq (2k-1)\cdot\frac{(k-1)n+km}{2k-1}-(k-1)n-m$$
$$=(k-1)m\geq k-1\geq 0.$$

This is a contradiction.

In all the cases above, we deduced contradictions. Hence, G has a fractional k-factor. This completes the proof of Theorem 1.5. □

REMARK 2.3. Let us show that the condition in Theorem 1.5 cannot be replaced by the condition that $N_G(X)=V(G)$ or $|N_G(X)|\geq((2k-1)/k)|X|$ for all $X\subseteq V(G)$. Let k be an odd integer with $k\geq 2$. Let m be any odd positive integer. We construct a graph G of order n as follows. Let $V(G)=S\cup T$ (disjoint union), $|S|=(k-1)m$ and $|T|=km+1$, and put $T=\{t_1,t_2,\ldots,t_{2l}\}$, where $2l=km+1$. For each $s\in S$, define $N_G(s)=V(G)\backslash\{s\}$, and for any $t\in T$, define $N_G(t)=S\cup\{t'\}$, where $\{t,t'\}=\{t_{2i-1},t_{2i}\}$ for some i, $1\leq i\leq l$. Obviously, $n=(2k-1)m+1$. We first show that the condition that $N_G(X)=V(G)$ or $|N_G(X)|\geq((2k-1)/k)|X|$ for all $X\subseteq V(G)$ holds. Let any $X\subseteq V(G)$. It is obvious that if $|X\cap S|\geq 2$, or $|X\cap S|=1$ and $|X\cap T|\geq 1$, then $N_G(X)=V(G)$. Of course, if $|X|=1$ and $X\subseteq S$, then

$$|N_G(X)|=|V(G)|-1=n-1>\frac{n-1}{km}=\frac{(2k-1)m}{km}=\frac{2k-1}{k}=\frac{2k-1}{k}|X|.$$

Hence, we may assume that $X\subseteq T$. Since $|N_G(X)|=|S|+|X|=(k-1)m+|X|$, $|N_G(X)|\geq((2k-1)/k)|X|$ holds if and only if $(k-1)m+|X|\geq((2k-1)/k)|X|$. This inequality is equivalent to $|X|\leq km$. Thus if $X\neq T$ and $X\subset T$, then $|N_G(X)|\geq((2k-1)/k)|X|$ holds for all $X\subseteq V(G)$. If $X=T$, then $N_G(X)=V(G)$. Consequently, $N_G(X)=V(G)$ or $|N_G(X)|\geq((2k-1)/k)|X|$ for all $X\subseteq V(G)$ follows. In the following, we show that G has no fractional k-factor. For above S and T, obviously, $d_{G-S}(t)=1$ for each $t\in T$. Thus, we obtain

$$\begin{aligned}\delta_G(S,T)&=k|S|+d_{G-S}(T)-k|T|\\&=k|S|+|T|-k|T|\\&=k|S|-(k-1)|T|\\&=k(k-1)m-(k-1)(km+1)\\&=-(k-1)\leq -1.\end{aligned}$$

In terms of Lemma 2.1, G has no fractional k-factor. In the above sense, the condition in Theorem 1.5 is best possible.

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