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A NOTE ON p-PARTS OF BRAUER CHARACTER DEGREES

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Abstract

Let G be a finite group and p be an odd prime. We show that if $\mathbf{O}_p(G) = 1$ and p^2 does not divide every irreducible p-Brauer character degree of G, then $|G|_p$ is bounded by p^3 when $p \geqslant 5$ or $p = 3$ and A_7 is not involved in G, and by 3^4 if $p = 3$ and A_7 is involved in G.

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1. Introduction

All groups considered in this paper are finite. Let n be a positive integer and p be a prime and write $n = p^a m$, where $p \nmid m$. We use $n_p = p^a$ to denote the p-part of the integer n. Let G be a group. Let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G and let $\mathrm{IBr}(G)$ be the set of irreducible p-Brauer characters of G. We use $\mathbf{O}_p(G)$ to denote the largest normal p-subgroup of G and $\pi(|G|)$ to denote the set of different prime divisors of the order of G. As usual, A_n denotes the alternating group of degree n.

It is interesting to study how the arithmetic conditions on the invariants of a finite group affect the group structure. Let G be a group and P be a Sylow p-subgroup of G. The well-known Ito–Michler theorem [4] asserts that $p \nmid \chi(1)$ for every $\chi \in \mathrm{Irr}(G)$ if and only if P is normal in G and P is abelian. In particular, this implies that $|G : \mathbf{O}_p(G)|_p = 1$. One natural generalisation of the Ito–Michler theorem is the following result of Lewis, Navarro and Wolf [2]: if G is solvable and $p^2 \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, then $|G : \mathbf{O}_p(G)|_p \leqslant p^2$. It was also proved in [2] that if G is an arbitrary finite group such that 2^2 does not divide $\chi(1)$ for every $\chi \in \mathrm{Irr}(G)$, then $|G : \mathbf{O}_2(G)|_2 \leqslant 2^3$. In [1], Lewis, Navarro, Tiep and Tong-Viet studied the similar problem for arbitrary finite groups when p is an odd prime. They showed that if G is finite and $\chi(1)_p \leqslant p$ for all $\chi \in \mathrm{Irr}(G)$, then $|G : \mathbf{O}_p(G)|_p \leqslant p^4$. Recently, this result has been improved to p^3 by Qian in [5].

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It is natural to study the Brauer character degree analogue of the Ito–Michler theorem. This has been investigated by Michler [4] and Manz [3]. They showed that $p \nmid \varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ if and only if G has a normal Sylow p-subgroup. The condition that G has a normal Sylow p-subgroup implies that $|G : \mathbf{O}_p(G)|_p = 1$. In [1], Lewis *et al.* proved that if G is a group with $\mathbf{O}_p(G) = 1$ and $\varphi(1)_p \leqslant p$ for every $\varphi \in \mathrm{IBr}(G)$, then one of the following holds.

(1) If $p = 2$, then $|G|_2 \leqslant 2^9$.
(2) If $p \geqslant 5$ or if $p = 3$ and A_7 is not involved in G, then $|G|_p \leqslant p^4$.
(3) If $p = 3$ and A_7 is involved in G, then $|G|_3 \leqslant 3^5$.

Lewis *et al.* further conjectured in [1] that the correct bounds should be 2^7 in (1), p^2 in (2) and 3^3 in (3). In this paper, we use some new techniques to improve the bounds stated above when p is an odd prime. Our approach is motivated by a recent result of Qian [5]. We prove the following result.

THEOREM 1.1. *Let p be an odd prime and G be a group such that $\mathbf{O}_p(G) = 1$. Suppose that p^2 does not divide $\varphi(1)$ for every $\varphi \in \mathrm{IBr}(G)$.*

(1) *If $p \geqslant 5$ or if $p = 3$ and A_7 is not involved in G, then $|G|_p \leqslant p^3$.*
(2) *If $p = 3$ and A_7 is involved in G, then $|G|_3 \leqslant 3^4$.*

2. Main results

Note that for a group G and a prime p, the assertion 'p^2 does not divide $\varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$' is inherited by factor groups and normal subgroups of G. This property will be used frequently.

LEMMA 2.1. *Let p be a prime and G be an almost simple group with nonabelian simple socle S. Suppose that $p \in \pi(|S|)$ and $\varphi(1)_p \leqslant p$ for all $\varphi \in \mathrm{IBr}(G)$. Then the following assertions hold.*

(1) $|S|_p > |G/S|_p$.
(2) *If $p = 2$, then $G = S \simeq M_{22}$ and $|G|_2 = 2^7$.*
(3) *If $p = 3$ and $S = \mathsf{A}_7$, then $|G|_3 = |S|_3 = 3^2$.*
(4) *If $p \geqslant 3$ and $(S, p) \neq (\mathsf{A}_7, 3)$, then $|G|_p = |S|_p = p$.*

PROOF. See [1, Lemma 3.1] for (1) and [1, Lemma 4.3] for (2)–(4). □

LEMMA 2.2. *A group G has a normal Sylow p-subgroup P if and only if $p \nmid \varphi(1)$ for every $\varphi \in \mathrm{IBr}(G)$.*

PROOF. See [4, Theorem 5.5] or [3, page 121]. □

LEMMA 2.3 [5, Proposition B]. *Let p be an odd prime and let W be a normal p'-subgroup of G. Assume that p^3 does not divide $|G/\mathbf{C}_G(W)|$ and that W is a direct product of some nonabelian simple groups. Then there exists $\theta \in \mathrm{Irr}(G)$ such that $I_G(\theta)/\mathbf{C}_G(W)$ is a p'-group.*

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Lemma 2.4 [1, Lemma 3.6]. *Suppose that G is either A_7 or $3 \cdot \mathsf{A}_7$, and that G acts nontrivially on the set Ω. Then there exist disjoint (possibly empty) subsets B_1, B_2, B_3, B_4, B_5 such that 9 divides $|G : \bigcap_{i=1}^{5} \mathrm{Stab}_G(B_i)|$.*

Lemma 2.5 [1, Lemma 4.4]. *Let p be a prime and let G be a solvable group with $\mathbf{O}_p(G) = 1$. If $\varphi(1)_p \leqslant p$ for all $\varphi \in \mathrm{IBr}(G)$, then $|G|_p \leqslant p^2$.*

Let $\mathrm{Sol}(G)$ denote the largest solvable normal subgroup of a group G.

Lemma 2.6. *Let p be an odd prime and G be a group with $\mathrm{Sol}(G) = 1$. Suppose that $\varphi(1)_p \leqslant p$ for all $\varphi \in \mathrm{IBr}(G)$. In addition, assume that A_7 is not involved in G if $p = 3$. Then $|G|_p \leqslant p$.*

Proof. We may assume that $p \in \pi(|G|)$ and $\mathbf{O}^{p'}(G) = G$ by induction. Let U be a maximal normal subgroup of G. Then G/U is a simple group and p divides the order of G/U. By Lemma 2.1(4), $|G/U|_p = p$ and, by induction, $|G|_p \leqslant p^2$. Let W be a minimal normal subgroup of G contained in U. Clearly, W is nonabelian because $\mathrm{Sol}(G) = 1$.

Assume that p divides $|W|$. By the choice of U and since W is contained in U, we see that $|W|_p \le |U|_p \le p$. Since p divides $|W|$, it follows that W is a nonabelian simple group with $|W|_p = p$. Since $G/(W \times \mathbf{C}_G(W)) \lesssim \mathrm{Out}(W)$, by Lemma 2.1(1) and the first paragraph of the proof, $G/(W \times \mathbf{C}_G(W))$ is a p'-group. This implies that $G = W \times \mathbf{C}_G(W)$ because $G = \mathbf{O}^{p'}(G)$. Suppose that p is a prime divisor of $|\mathbf{C}_G(W)|$. Since $\mathbf{O}_p(W)$, $\mathbf{O}_p(\mathbf{C}_G(W)) \le \mathrm{Sol}(G) = 1$, by Lemma 2.2 there exist $\varphi_1 \in \mathrm{IBr}(W)$ and $\varphi_2 \in \mathrm{IBr}(\mathbf{C}_G(W))$ such that p divides $\varphi_1(1)$ and $\varphi_2(1)$. Then $\varphi = \varphi_1\varphi_2 \in \mathrm{IBr}(G)$ and p^2 divides $\varphi(1)$, a contradiction. Hence $\mathbf{C}_G(W)$ is a p'-group and the required result follows.

Assume that p does not divide $|W|$. Clearly $G > W \times \mathbf{C}_G(W)$ and p divides $|G/(W \times \mathbf{C}_G(W))|$ because $G = \mathbf{O}^{p'}(G)$. Hence by induction $|\mathbf{C}_G(W)|_p \le p$ since $\mathbf{C}_G(W)$ is a proper normal subgroup of G satisfying the assumption. By Lemma 2.2 again, there exists $\varphi_3 \in \mathrm{IBr}(\mathbf{C}_G(W))$ such that $|\varphi_3(1)|_p = |\mathbf{C}_G(W)|_p$. By the first paragraph of the proof, $p^3 \nmid |G|$. Since W is a p'-group, $\mathrm{IBr}(W) = \mathrm{Irr}(W)$. Now we can apply Lemma 2.3 to conclude that there exists some $\varphi_4 \in \mathrm{IBr}(W)$ such that $I_G(\varphi_4)/\mathbf{C}_G(W)$ is a p'-group. It is clear that $\varphi_3\varphi_4 \in \mathrm{IBr}(W \times \mathbf{C}_G(W))$ and $I_G(\varphi_3\varphi_4) = I_G(\varphi_3) \cap I_G(\varphi_4)$ since W and $\mathbf{C}_G(W)$ are normal in G. Let θ be an irreducible constituent of $(\varphi_3\varphi_4)^G$. Then, by the hypothesis and the arguments above,

$$\begin{aligned} p \geqslant \theta(1)_p &\geqslant |G : I_G(\varphi_3\varphi_4)|_p \cdot (\varphi_3\varphi_4(1))_p \\ &\geqslant |G : I_G(\varphi_4)|_p \cdot \varphi_3(1)_p = |G : \mathbf{C}_G(W)|_p \cdot |\mathbf{C}_G(W)|_p = |G|_p, \end{aligned}$$

and the proof is complete. □

Lemma 2.7. *Let G be a group such that 3^2 does not divide $\varphi(1)$ for all $\varphi \in \mathrm{IBr}(G)$ ($p = 3$). Assume that $\mathbf{O}^{3'}(G) = G$, $\mathrm{Sol}(G) = 1$ and A_7 is involved in G. Then $G \simeq \mathsf{A}_7$.*

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Proof. Let N be minimal among normal subgroups of G that involve A_7. Let M be a normal subgroup of G contained in N such that N/M is a chief factor of G. Since N/M is nonabelian, $N/M \simeq S^k$ with $S = \mathsf{A}_7$ and k a positive integer. We claim that $k = 1$. If not, by Lemma 2.2, 3 divides $\theta(1)$ for some $\theta \in \mathrm{IBr}(S)$. Since $\theta^k \in \mathrm{IBr}(N/M)$ and 3^k divides $\theta(1)^k$, it follows that p^k divides $\varphi(1)$ for some $\varphi \in \mathrm{IBr}(G)$, contradicting the assumption on G. Hence $N/M \simeq \mathsf{A}_7$. Note that the minimality of N implies that M is the unique normal subgroup of G contained in N so that N/M is a chief factor for G and, in particular, N is perfect. Let L be the solvable residual of M. That is, L is the smallest normal subgroup of M so that M/L is solvable.

If one can prove that $L = 1$, then M is trivial by the condition that $\mathrm{Sol}(G) = 1$. It follows that $N \simeq \mathsf{A}_7$ is a minimal normal subgroup of G. Since $G/(N \times \mathbf{C}_G(N))$ is isomorphic to a subgroup of $\mathrm{Out}(N)$ and $\mathbf{O}^{3'}(G) = G$, we see that $G = N \times \mathbf{C}_G(N)$. If $\mathbf{C}_G(N)$ is not a $3'$-group, then by Lemma 2.2, 3^2 divides $\varphi(1)$ for some $\varphi \in \mathrm{IBr}(G)$, a contradiction. Hence $3 \notin \pi(|\mathbf{C}_G(N)|)$ and so $G = N$ because $\mathbf{O}^{3'}(G) = G$.

From now on, we assume that $L \neq 1$ and let K be a normal subgroup of G contained in L such that L/K is a chief factor of G. Observe that $L/K \simeq T_1 \times \cdots \times T_n$, where $T_i \simeq T$ is a nonabelian simple group. Let $D/K = \mathbf{C}_{G/K}(L/K)$. Then $D \cap L = K$. Therefore $D \cap N < N$ and, by the uniqueness of M, we see that $D \cap N \leqslant M$. Since $G/DL \lesssim \mathrm{Aut}(L/K)$, if $n = 1$, then $ND/DL \simeq N/(D \cap N)L$ is solvable, a contradiction because $(D \cap N)L \leqslant M$ and $N/M \simeq \mathsf{A}_7$. It follows that $n > 1$. If $3 \in \pi(|T|)$, then arguing as above, we get a contradiction. Hence 3 does not divide the order of T and so $T \simeq {}^2B_2(2^{2m+1})$ for some $m \geqslant 1$. This implies that T has five distinct nonlinear irreducible 3-Brauer character degrees, say $\tau_1, \tau_2, \tau_3, \tau_4$ and τ_5.

Observe that N/L acts on the set $\{T_1, \ldots, T_n\}$ and, since $D \cap N \leqslant M$, we conclude that N/L acts nontrivially. By Lemma 2.4, we can find disjoint subsets $B_1, \ldots, B_5 \subseteq \{T_1, \ldots, T_n\}$ such that $\bigcap_{i=1}^{5} \mathrm{Stab}_{N/L}(B_i)$ has index in N divisible by 9. Define $\phi \in \mathrm{IBr}(L/K) = \mathrm{Irr}(L/K) = \mathrm{Irr}(T_1 \times \cdots \times T_n)$ by $\phi = \phi_1 \times \cdots \times \phi_n$ with $\phi_i = \tau_j$ if $T_i \in B_j$ for $j \in \{1, \ldots, 5\}$ and $\phi_i = 1_T$ otherwise. It is easy to see that $I_N(\phi)/L \leqslant \bigcap_{i=1}^{5} \mathrm{Stab}_{N/L}(B_i)$ and so $9 \mid |N : I_N(\phi)|$. Therefore 9 divides $\varphi(1)$ for some $\varphi \in \mathrm{IBr}(G)$. This contradiction shows that $L = 1$ and thus the proof is complete by the argument in the second paragraph of the proof. □

Proof of Theorem 1.1. We may assume that $G = \mathbf{O}^{p'}(G)$ by appealing to induction. First, we assume that $p \geqslant 5$, or that $p = 3$ and A_7 is not involved in G. By Lemma 2.6, $|G : \mathrm{Sol}(G)|_p \leqslant p$. Since $\mathbf{O}_p(\mathrm{Sol}(G)) \leqslant \mathbf{O}_p(G) = 1$ by the hypothesis, we have $|\mathrm{Sol}(G)|_p \leqslant p^2$ by Lemma 2.5 and consequently $|G|_p \leqslant p^3$. If $p = 3$ and A_7 is involved in G, then $|G/\mathrm{Sol}(G)|_3 = 3^2$ by Lemma 2.7. Since $|\mathrm{Sol}(G)|_3 \leqslant 3^2$ as above, we conclude that $|G|_3 \leqslant 3^4$. This completes the proof. □

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